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Research article

Space-time decay rates of a nonconservative compressible two-phase flow model with common pressure

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Abstract: In this paper, we study space-time decay rates of a nonconservative compressible two-phase flow model with common pressure in the whole space $\mathbb{R}^3$. Based on previous temporal decay results, we establish the space-time decay rate of the strong solution. The main analytical techniques involve delicate weighted energy estimates and interpolation.

Keywords: nonconsevative; compressible; two-phase flow model; space-time decay rate; weighted Sobolev space

1. Introduction

1.1. Background and motivation

As is well-known, multi-fluids are very common in nature as well as in various industry applications such as nuclear power, chemical processing, oil and gas manufacturing. The classic approach to simplify the complexity of multi-phase flows and satisfy the engineer's need of some modeling tools is the well-known volume-averaging method. This approach leads to so-called averaged multi-phase models, see [1–3] for details. As a result of such a procedure, one can obtain the following generic compressible two-phase fluid model:

$$\begin{cases} \partial_t (\alpha^\pm \rho^\pm) + \operatorname{div} (\alpha^\pm \rho^\pm u^\pm) = 0, \\ \partial_t (\alpha^\pm \rho^\pm u^\pm) + \operatorname{div} (\alpha^\pm \rho^\pm u^\pm \otimes u^\pm) + \alpha^\pm \nabla P = \operatorname{div} (\alpha^\pm \tau^\pm) + \sigma^\pm \alpha^\pm \rho^\pm \nabla \Delta (\alpha^\pm \rho^\pm), \\ P = P^\pm (\rho^\pm) = A^\pm (\rho^\pm)^{\bar{\gamma}^\pm}, \end{cases} \tag{1.1}$$

where $0 \le \alpha^\pm \le 1$ are the volume fractions of the fluid + and fluid −, satisfying $\alpha^+ + \alpha^- = 1$; $\rho^\pm(x,t) \ge 0, u^\pm(x,t)$ and $P^\pm(\rho^\pm) = A^\pm (\rho^\pm)^{\bar{\gamma}^\pm}$ denote the densities, velocities of each phase, and the two pressure functions, respectively; $\sigma^\pm \ge 0$ denote the capillary coefficients of each phase; $\bar{\gamma}^\pm \ge 1, A^\pm > 0$

are positive constants. In what follows, we set $A^+ = A^- = 1$ without loss of any generality. Moreover, $\tau^\pm$ are the viscous stress tensors

$$\tau^\pm := \mu^\pm \left(\nabla u^\pm + \nabla^t u^\pm\right) + \lambda^\pm \operatorname{div} u^\pm \mathrm{Id}, \tag{1.2}$$

where the constants $\mu^\pm$ and $\lambda^\pm$ are shear and bulk viscosity coefficients satisfying the physical condition: $\mu^\pm > 0$ and $2\mu^\pm + 3\lambda^\pm \geq 0$, which implies that $\mu^\pm + \lambda^\pm > 0$. For more information about this model, we refer to [1–5] and references therein. In particular, a nice summary of the model was given in the introduction of [6]. However, it is well-known that as far as the mathematical analysis of two-fluid is concerned, there are many technical challenges. Some of them involve, for example:

- The corresponding linear system of the model has multiple eigenvalues, which makes the mathematical analysis (well-posedness and stability) of the model quite difficult and complicated;
- Transition to single-phase regions, i.e., regions where the mass $\alpha^+\rho^+$ or $\alpha^-\rho^-$ becomes zero, may occur when the volume fractions $\alpha^\pm$ or the densities $\rho^\pm$ become zero;
- The system is non-conservative, since the non-conservative terms $\alpha^\pm\nabla P^\pm$ are involved in the momentum equations. This brings various mathematical difficulties for us to employ methods used for single phase models to the two-fluid model.

Bresch et al. [6] investigated the generic two-fluid model (1.1) with the following density-dependent viscosities:

$$\mu^\pm(\rho^\pm) = \mu^\pm\rho^\pm, \quad \lambda^\pm(\rho^\pm) = 0. \tag{1.3}$$

Under the assumption that $1 < \overline{\gamma}^\pm < 6$, they obtained the global weak solutions for the 3D periodic domain problem. Later, Bresch et al. [7] established the global existence of weak solutions in one space dimension without capillary effects (i.e., $\sigma^\pm = 0$) when $\overline{\gamma}^\pm > 1$ by taking advantage of the one space dimension. Under the assumption that

$$\mu^\pm(\rho^\pm) = \nu\rho^\pm, \quad \lambda^\pm(\rho^\pm) = 0, \quad \sigma^+ = \sigma^- = \sigma, \tag{1.4}$$

Cui et al. [8] proved the time-decay rates of global small strong solutions for model (1.1). Recently, Li et al. [9] extended this result to the general constant viscosities as in (1.2). Very recently, Wu et al. [10] proved global existence and large time behavior of global classic solutions for the model (1.1) without capillary effects (i.e., $\sigma^\pm = 0$). We also mention the seminal work by Evje et al. [11], who considered the two-fluid model (1.1) with unequal pressures. More precisely, they made the following assumptions on pressures:

$$P^+\left(\rho^+\right) - P^-\left(\rho^-\right) = \left(\rho^+\right)^{\bar{\gamma}^+} - \left(\rho^-\right)^{\bar{\gamma}^-} = f(\alpha^-\rho^-), \tag{1.5}$$

where f is so-called capillary pressure which belongs to $C^3([0,\infty))$, and is a strictly decreasing function near the equilibrium, i.e., $f'(1) < 0$. When the initial data is sufficiently small, they established global existence and large time behavior of global strong solutions. After that, this model has been studied by several authors. We refer to [12–14] and references therein.

The space-time decay rate of strong solution has attracted more and more attention. In the following, we will state the progress on the topic about the space-time decay in weighted Sobolev space H^N_γ.

Takahashi first established the space-time decay of strong solutions to the Navier-Stokes equations in [15]. In [16, 17], Kukavica et al. used the parabolic interpolation inequality to obtain sharp decay rates of the higher-order derivatives for the solutions in weighted Lebesgue space L^2_γ. In [18, 19], Kukavica et al. also established the strong solution's space-time decay rate in $L^p_\gamma(2 \leq p \leq \infty)$ and extended the result to $n(n \geq 2)$ dimensions.

However, to the best of our knowledge, up to now, there is no result on the space-time decay rate of the nonconservative compressible two-phase flow model (1.1). The main motivation of this paper is to give a definite answer to this issue. More precisely, we establish space-time decay rate of the $k(0 \leq k \leq N)$–order derivative of strong solution to the Cauchy problem of the model (1.1) in weighted Lebesgue space L^2_γ .

In this subsection, we devote ourselves to reformulating the system (1.1) and stating the main results. The relations between the pressures of $(1.1)_3$ imply

$$\mathrm{d}P = s_+^2 \mathrm{d}\rho^+ = s_-^2 \mathrm{d}\rho^-, \quad \text{where} \quad s_\pm := \sqrt{\frac{\mathrm{d}P}{\mathrm{d}\rho^\pm}(\rho^\pm)}. \tag{1.6}$$

Here $s_\pm$ represent the sound speed of each phase respectively. As in [6], we introduce the fraction densities

$$R^\pm = \alpha^\pm \rho^\pm, \tag{1.7}$$

which together with the relation $\alpha^+ + \alpha^- = 1$ leads to

$$\mathrm{d}\rho^+ = \frac{1}{\alpha_+}(\mathrm{d}R^+ - \rho^+ \mathrm{d}\alpha^+), \quad \mathrm{d}\rho^- = \frac{1}{\alpha_-}(\mathrm{d}R^- + \rho^- \mathrm{d}\alpha^+). \tag{1.8}$$

From (1.6)–(1.7), we finally get

$$\mathrm{d}\alpha^+ = \frac{\alpha^- s_+^2}{\alpha^- \rho^+ s_+^2 + \alpha^+ \rho^- s_-^2} \mathrm{d}R^+ - \frac{\alpha^+ s_-^2}{\alpha^- \rho^+ s_+^2 + \alpha^+ \rho^- s_-^2} \mathrm{d}R^-.$$

Substituting the above equality into (1.8) yields

$$\mathrm{d}\rho^+ = \frac{s_-^2}{\alpha^- \rho^+ s_+^2 + \alpha^+ \rho^- s_-^2}(\rho^- \mathrm{d}R^+ + \rho^+ \mathrm{d}R^-),$$

and

$$\mathrm{d}\rho^- = \frac{s_+^2}{\alpha^- \rho^+ s_+^2 + \alpha^+ \rho^- s_-^2}(\rho^- \mathrm{d}R^+ + \rho^+ \mathrm{d}R^-),$$

which together with (1.6) imply for the pressure differential dP

$$\mathrm{d}P = C^2(\rho^- \mathrm{d}R^+ + \rho^+ \mathrm{d}R^-), \tag{1.9}$$

where

$$C^2 := \frac{s_+^2 s_-^2}{\alpha^- \rho^+ s_+^2 + \alpha^+ \rho^- s_-^2}, \quad \text{and} \quad s_\pm^2 = \frac{\mathrm{d}P(\rho^\pm)}{\mathrm{d}\rho^\pm} = \tilde{\gamma}^\pm \frac{P(\rho^\pm)}{\rho^\pm}.$$

Next, by using the relation: $\alpha^+ + \alpha^- = 1$ again, we can get

$$\frac{R^+}{\rho^+} + \frac{R^-}{\rho^-} = 1, \quad \text{and therefore} \quad \rho^- = \frac{R^- \rho^+}{\rho^+ - R^+}. \tag{1.10}$$

By virtue of $(1.1)_3$, we have

$$\varphi(\rho^+, R^+, R^-) := P(\rho^+) - P\left(\frac{R^-\rho^+}{\rho^+ - R^+}\right) = 0.$$

Consequently, for any given two positive constants $\tilde{R}^+$, $\tilde{R}^-$, there exists $\tilde{\rho}^+ > \tilde{R}^+$ such that

$$\varphi(\tilde{\rho}^+, \tilde{R}^+, \tilde{R}^-) = 0.$$

Differentiating φ with respect to ρ^+, we get

$$\frac{\partial\varphi}{\partial\rho^+}(\rho^+, R^+, R^-) = s_+^2 + s_-^2 \frac{R^- R^+}{(\rho^+ - R^+)^2},$$

which implies

$$\frac{\partial\varphi}{\partial\rho^+}(\tilde{\rho}^+, \tilde{R}^+, \tilde{R}^-) > 0.$$

Thus, this together with implicit function theorem implies that the unknowns $\rho^\pm$, $\alpha^\pm$ and C can be given by

$$\rho^\pm = \varrho^\pm(R^+, R^-), \qquad \alpha^\pm = \alpha^\pm(R^+, R^-), \quad \text{and therefore} \quad C = C(R^+, R^-).$$

We refer to [6] for the details.

Therefore, we can rewrite system (1.1) into the following equivalent form:

$$\begin{cases} \partial_t R^\pm + \operatorname{div}(R^\pm u^\pm) = 0, \\ \partial_t (R^+ u^+) + \operatorname{div}(R^+ u^+ \otimes u^+) + \alpha^+ C^2 \left[\rho^- \nabla R^+ + \rho^+ \nabla R^-\right] \\ \qquad = \operatorname{div}\left\{\alpha^+ \left[\mu^+ \left(\nabla u^+ + (\nabla u^+)^T\right) + \lambda^+ \operatorname{div} u^+ \operatorname{Id}\right]\right\} + \sigma^+ R^+ \nabla\Delta R^+, \\ \partial_t (R^- u^-) + \operatorname{div}(R^- u^- \otimes u^-) + \alpha^- C^2 \left[\rho^- \nabla R^+ + \rho^+ \nabla R^-\right] \\ \qquad = \operatorname{div}\left\{\alpha^- \left[\mu^- \left(\nabla u^- + (\nabla u^-)^T\right) + \lambda^- \operatorname{div} u^- \operatorname{Id}\right]\right\} + \sigma^- R^- \nabla\Delta R^-. \end{cases} \tag{1.11}$$

In the present paper, we consider the Cauchy problem of (1.11) subject to the initial condition

$$(R^+, u^+, R^-, u^-)(x, 0) = (R_0^+, u_0^+, R_0^+, u_0^-)(x) \to (\bar{R}^+, \overrightarrow{0}, \bar{R}^-, \overrightarrow{0}) \quad \text{as} \quad |x| \to \infty \in \mathbb{R}^3, \tag{1.12}$$

where two positive constants $\bar{R}^+$ and $\bar{R}^-$ denote the background doping profile, and in the present paper are taken as 1 for simplicity.

Before presenting our results, let us provide a brief explanation of the notation used in this paper.

1.2. Notation

We use L^p and H^ℓ to denote the usual Lebesgue space $L^p\left(\mathbb{R}^3\right)$ and Sobolev spaces $H^\ell\left(\mathbb{R}^3\right) = W^{\ell,2}\left(\mathbb{R}^3\right)$ with norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^\ell}$ respectively. We denote $\|(f,g)\|_X := \|f\|_X + \|g\|_X$ for simplicity. The notation $f \lesssim g$ means that $f \leq Cg$ for a generic positive constant $C > 0$ that only depends on the parameters coming from the problem.

We often drop x-dependence of differential operators, that is $\nabla f = \nabla_x f = (\partial_{x_1} f, \partial_{x_2} f, \partial_{x_3} f)$ and ∇^k denotes any partial derivative ∂^α with multi-index $\alpha, |\alpha| = k$. Furthermore, $\nabla\nabla^\alpha g = \nabla\partial^\alpha g = (\partial_{x_1}(\partial^\alpha g), \partial_{x_2}(\partial^\alpha g), \partial_{x_3}(\partial^\alpha g))$.

For any $\gamma \in \mathbb{R}$, we denote the weighted Lebesgue space by $L^p_\gamma\left(\mathbb{R}^3\right)(2 \le p < +\infty)$ with respect to the spatial variables:

$$L^p_\gamma\left(\mathbb{R}^3\right) := \left\{ f(x) : \mathbb{R}^3 \to \mathbb{R}, \|f\|^p_{L^p_\gamma(\mathbb{R}^3)} := \int_{\mathbb{R}^3} |x|^{p\gamma} |f(x)|^p dx < +\infty \right\}.$$

Then, we can define the weighted Sobolev space:

$$H^s_\gamma\left(\mathbb{R}^3\right) \triangleq \left\{ f \in L^2_\gamma\left(\mathbb{R}^3\right) \mid \|f\|^2_{H^s_\gamma(\mathbb{R}^3)} := \sum_{k \le s} \left\|\nabla^k f\right\|^2_{L^2_\gamma(\mathbb{R}^3)} < +\infty \right\}.$$

Let Λ^s be the pseudo differential operator defined by

$$\Lambda^s f = \mathfrak{F}^{-1}\left(|\xi|^s \widehat{f}\right), \text{ for } s \in \mathbb{R},$$

where $\widehat{f}$ and $\mathfrak{F}(f)$ are the Fourier transform of f. The homogenous Sobolev space $\dot{H}^s\left(\mathbb{R}^3\right)$ with norm given by

$$\|f\|_{\dot{H}^s} \triangleq \|\Lambda^s f\|_{L^2}.$$

Schwartz class $\mathcal{S}$ consists of function f, which is infinitely differentiable and all of its derivative decrease rapidly at infinity, such that

$$\sup_x \left| x^\alpha D^\beta f(x) \right| < \infty,$$

for all $\alpha, \beta \in \mathbb{N}^3$.

1.3. Main results

Theorem 1.1. *Let* (R^+, u^+, R^-, u^-) *be the strong solution to the Cauchy problem* (1.11)–(1.12) *with initial data* $(R^+_0 - 1, u^+_0, R^-_0 - 1, u^-_0)$ *belonging to the Schwartz class* $\mathcal{S}$. *In addition, assume that* $\left(R^+_0 - 1, u^+_0, R^-_0 - 1, u_0\right) \in H^N\left(\mathbb{R}^3\right) \cap H^N_\gamma\left(\mathbb{R}^3\right) \cap L^1\left(\mathbb{R}^3\right)$ *for an integer* $N \ge 2$. *Then if there is a small constant* $\delta_0 > 0$ *such that*

$$\left\|(R^+_0 - 1, R^-_0 - 1)\right\|_{H^{N+1} \cap L^1} + \left\|(u^+_0, u^-_0)\right\|_{H^N \cap L^1} \le \delta_0, \tag{1.13}$$

then there exists a large enough T *and for any* $0 \le k \le N$ *such that*

$$\begin{aligned} &\left\|(\nabla^k u^+, \nabla^k u^-, \nabla\nabla^k R^+, \nabla\nabla^k R^-)\right\|_{L^2_\gamma} \lesssim t^{\gamma - \frac{3}{4} - \frac{k}{2}}, \\ &\left\|(\nabla^k (R^+ - 1), \nabla^k (R^- - 1), \frac{\beta_1}{\beta_2} \nabla^k (R^+ - 1) + \sqrt{\frac{\beta_4}{\beta_3}} \nabla^k (R^- - 1))\right\|_{L^2_\gamma} \lesssim t^{\gamma - \frac{1}{4} - \frac{k}{2}}, \end{aligned} \tag{1.14}$$

for all $t > T$.

Remark 1.2. Applying the Gagliardo-Nirenberg-Sobolev inequality, we can obtain the space-time decay rates of smooth solution in weighted normed linear space (Banach space) L^p_γ as follows. For any $f \in L^2\left(\mathbb{R}^3\right) \cap \dot{H}^2\left(\mathbb{R}^3\right)$, we have $\|f\|_{L^\infty(\mathbb{R}^3)} \leq \|f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}} \|f\|_{\dot{H}^2(\mathbb{R}^3)}^{\frac{3}{4}}$. So we can obtain the estimate $\||x|^\gamma \nabla^k (R^+ - 1, u^+, R^- - 1, u^-)(t)\|_{L^\infty} (k \in [0, N-2])$ from the estimates $\left\||x|^\gamma \nabla^k (R^+ - 1, u^+, R^- - 1, u^-)(t)\right\|_{L^2}$ and $\||x|^\gamma \nabla^k (R^+ - 1, u^+, R^- - 1, u^-)(t)\|_{\dot{H}^2}$. Using the interpolation inequality, we can show that there exists a large enough T such that

$$\left\|\nabla^k (R^+ - 1, u^+, R^- - 1, u^-)(t)\right\|_{L^p_\gamma} \leq C t^{-\frac{3}{2}\left(1 - \frac{1}{p}\right) - \frac{k}{2} + \gamma},$$

for $t > T$, $2 \leq p \leq \infty$ and $0 \leq k \leq N - 2$, where C is a positive constant independent of t.

Now, let's briefly describe the proof process of the main results and the difficulties encountered during this process. For the proof of Theorem 1.1, we employ sophisticated weighted energy estimates, interpolation inequality, and inductive strategies. The proof mainly involves the following three steps.

Firstly, using several lemmas in Section 2 and energy methods, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} E_k(t) \lesssim{} & E_k(t)^{1 - \frac{1}{2\gamma}} (1 + t)^{-\frac{1}{\gamma}\left(\frac{3}{4} + \frac{k}{2}\right)} + E_k(t)^{1 - \frac{1}{\gamma}} (1 + t)^{-\frac{2}{\gamma}\left(\frac{3}{4} + \frac{k}{2}\right)} + E_k(t)^{\frac{1}{2}} \\
& + E_k(t)^{\frac{1}{2} - \frac{1}{2\gamma}} (1 + t)^{-\frac{1}{\gamma}\left(\frac{3}{4} + \frac{k}{2}\right)} \sum_{l=0}^{k} \left(E_l(t)\right)^{\frac{1}{2}} (1 + t)^{-\left(\frac{3}{2} + \frac{k-l}{2}\right)} \\
& + \left(E_k(t)\right)^{\frac{1}{2}} \sum_{l=0}^{k} \left(E_l(t)\right)^{\frac{1}{2}} (1 + t)^{-\left(1 + \frac{k-l}{2}\right)},
\end{aligned} \tag{1.15}$$

for t large enough, and according to the different values of k, the value of γ also varies, $E_k(t) := \left\|\left(\nabla^k n^+, \nabla^k u^+, \nabla^k n^-, \nabla^k u^-, \sqrt{\frac{\beta_1}{\beta_2}} \nabla^k n^+ + \sqrt{\frac{\beta_4}{\beta_3}} \nabla^k n^-\right)\right\|_{L^2_\gamma}^2$ and the range of values for k is from 0 to N.

Secondly, for the case of $k = 0$, the fourth and fifth terms on the left side of inequality (1.15) can be directly written as $E_0(t)^{1 - \frac{1}{2\gamma}} (1 + t)^{-\frac{3}{2} - \frac{1}{\gamma}\frac{3}{4}}$ and $E_0(t)(1 + t)^{-1}$. If we want to use Lemma 2.9 to obtain the result for the case $k = 0$, the main difficulty is to handle the term $E_0(t)(1 + t)^{-1}$. We will multiply $(1 + t)^{-1}$ on both sides of (1.15) simultaneously, and then apply Lemma 2.9 to control $E_0(t)(1 + t)^{-1}$.

Thirdly, using the similar method as $k = 0$ and the decay estimate already obtained by $E_0(t)$, we can show that the Theorem 1.1 holds for $k = 1$, and according to the strategy of induction, we prove that Theorem 1.1 holds for $2 \leq k \leq N$. The main difficulties come from those terms like

$$\left\langle \nabla^k F_2, |x|^{2\gamma} \frac{1}{\beta_2} \nabla^k u^+ \right\rangle, \left\langle \nabla^k F_4, |x|^{2\gamma} \frac{1}{\beta_3} \nabla^k u^- \right\rangle$$

$$\int_{\mathbb{R}^3} |x|^{2\gamma} \nabla^k u^+ \nabla \nabla^k n^- dx, \text{and } \int_{\mathbb{R}^3} |x|^{2\gamma} \nabla^k u^- \nabla \nabla^k n^+ dx$$

which involve three main difficulties. In section 3, we will provide detailed proofs and explain the methods and processes for dealing with these difficulties.

2. Reformulation

2.1. Reformulation

In this subsection, we reformulate the Cauchy problem (1.11)–(1.12). Setting $n^\pm = R^\pm - 1$, we have

$$\partial_t R^\pm + \operatorname{div}(R^\pm u^\pm) = \partial_t n^\pm + \operatorname{div}(n^\pm u^\pm) + \operatorname{div}(u^\pm) = 0 \tag{2.1}$$

From this, we can directly obtain $F_1 = -\operatorname{div}(n^+u^+)$, $F_3 = -\operatorname{div}(n^-u^-)$.
According to the left of $(1.11)_2$, we have

$$\begin{aligned}
&\partial_t(R^+u^+) + \operatorname{div}(R^+u^+\otimes u^+) + \alpha^+C^2(\rho^-\nabla R^+ + \rho^+\nabla R^-)\\
=&R^+\partial_t u^+ - (R^+\operatorname{div} u^+ + \nabla R^+\cdot u^+)u^+ + R^+u^+\nabla u^+\\
&+ (R^+\operatorname{div} u^+)u^+ + (\nabla R^+\cdot u^+)u^+ + R^+(C^2\frac{\rho^-}{\rho^+}\nabla R^+ + C^2\nabla R^-)\\
=&R^+\partial_t u^+ + R^+u^+\nabla u^+ + R^+(C^2\frac{\rho^-}{\rho^+}\nabla R^+ + C^2\nabla R^-)\\
=&R^+\partial_t u^+ + R^+u^+\nabla u^+ + R^+(C^2\frac{\rho^-}{\rho^+}\nabla n^+ + C^2\nabla n^-)
\end{aligned} \tag{2.2}$$

Before discussing the right of $(1.11)_2$, make a transformation to α^+ first:

$$\begin{aligned}
\nabla\alpha^+ &= \frac{\alpha^- s_+^2}{\alpha^-\rho^+s_+^2 + \alpha^+\rho^-s_-^2}\nabla R^+ - \frac{\alpha^+ s_-^2}{\alpha^-\rho^+s_+^2 + \alpha^+\rho^-s_-^2}\nabla R^-\\
&= \frac{\alpha^-C^2}{s_-^2}\nabla R^+ - \frac{\alpha^+C^2}{s_+^2}\nabla R^-\\
&= \frac{\alpha^-C^2}{s_-^2}\nabla n^+ - \frac{\alpha^+C^2}{s_+^2}\nabla n^-
\end{aligned} \tag{2.3}$$

And then, according to the right of $(1.11)_2$ and (2.3), we have

$$\begin{aligned}
&\operatorname{div}(\alpha^+(\mu^+(\nabla u^+ + (\nabla u^+)^T) + (\lambda^+\operatorname{div} u^+)I_3)) + \sigma^+R^+\nabla\Delta R^+\\
=&\alpha^+\mu^+\Delta u^+ + \alpha^+(\mu^+ + \lambda^+)\nabla\operatorname{div} u^+ + \mu^+(\nabla\alpha^+\nabla u^+)\\
&+ \mu^+(\nabla\alpha^+(\nabla u^+)^T) + \lambda^+(\nabla\alpha^+(\operatorname{div} u^+)I_3) + \sigma^+R^+\nabla\Delta R^+\\
=&\alpha^+\mu^+\Delta u^+ + \alpha^+(\mu^+ + \lambda^+)\nabla\operatorname{div} u^+ + \mu^+\frac{\alpha^-C^2}{s_-^2}\nabla n^+\nabla u^+\\
&- \mu^+\frac{\alpha^+C^2}{s_+^2}\nabla n^-\nabla u^+ + \mu^+\frac{\alpha^-C^2}{s_-^2}\nabla n^+(\nabla u^+)^T - \mu^+\frac{\alpha^+C^2}{s_+^2}\nabla n^-(\nabla u^+)^T\\
&+ \lambda^+\frac{\alpha^-C^2}{s_-^2}\operatorname{div} u^+\nabla n^+ - \lambda^+\frac{\alpha^+C^2}{s_+^2}\operatorname{div} u^+\nabla n^- + \sigma^+R^+\nabla\Delta n^+
\end{aligned} \tag{2.4}$$

Divide both sides by R^+, we can get

$$\begin{aligned}
&\partial_t u^+ + u^+\nabla u^+ + (C^2\frac{\rho^-}{\rho^+}\nabla n^+ + C^2\nabla n^-)\\
=&\frac{\mu^+}{\rho^+}\Delta u^+ + \frac{\mu^+}{\rho^+}(\mu^+ + \lambda^+)\nabla\operatorname{div} u^+ + \mu^+\frac{\alpha^-C^2}{(n^+ + 1)s_-^2}\nabla n^+\nabla u^+\\
&- \mu^+\frac{C^2}{\rho^+s_+^2}\nabla n^-\nabla u^+ + \mu^+\frac{\alpha^-C^2}{(n^+ + 1)s_-^2}\nabla n^+(\nabla u^+)^T\\
&- \mu^+\frac{C^2}{\rho^+s_+^2}\nabla n^-(\nabla u^+)^T + \lambda^+\frac{\alpha^-C^2}{(n^+ + 1)s_-^2}\operatorname{div} u^+\nabla n^+\\
&- \lambda^+\frac{C^2}{\rho^+s_+^2}\operatorname{div} u^+\nabla n^- + \sigma^+\nabla\Delta n^+
\end{aligned} \tag{2.5}$$

Finally, by adding some initial terms of ρ on both sides of the equation and shifting the terms, we can obtain F_2, F_4.

After the above operations, the Cauchy problem (1.11)–(1.12) can be rewritten as

$$\begin{cases} \partial_t n^+ + \operatorname{div} u^+ = F_1, \\ \partial_t u^+ + \beta_1 \nabla n^+ + \beta_2 \nabla n^- - \nu_1^+ \Delta u^+ - \nu_2^+ \nabla \operatorname{div} u^+ - \sigma^+ \nabla \Delta n^+ = F_2, \\ \partial_t n^- + \operatorname{div} u^- = F_3, \\ \partial_t u^- + \beta_3 \nabla n^+ + \beta_4 \nabla n^- - \nu_1^- \Delta u^- - \nu_2^- \nabla \operatorname{div} u^- - \sigma^- \nabla \Delta n^- = F_4, \\ (n^+, u^+, n^-, u^-)(x, 0) = (n_0^+, u_0^+, n_0^-, u_0^-)(x) \to (0, \vec{0}, 0, \vec{0}), \quad \text{as} \quad |x| \to +\infty, \end{cases} \tag{2.6}$$

where $\nu_1^\pm = \frac{\mu^\pm}{\bar\rho^\pm}$, $\nu_2^\pm = \frac{\mu^\pm + \lambda^\pm}{\bar\rho^\pm} > 0$, $\beta_1 = \frac{C^2(1,1)\bar\rho^-}{\bar\rho^+}$, $\beta_2 = \beta_3 = C^2(1,1)$, $\beta_4 = \frac{C^2(1,1)\bar\rho^+}{\bar\rho^-}$ (which imply $\beta_1\beta_4 = \beta_2\beta_3 = \beta_2^2 = \beta_3^2$), and the nonlinear terms are given by

$$F_1 = -\operatorname{div}(n^+ u^+), \tag{2.7}$$

$$\begin{aligned} F_2^i = &- g_+(n^+, n^-)\partial_i n^+ - \bar g_+(n^+, n^-)\partial_i n^- - (u^+ \cdot \nabla) u_i^+ \\ &+ \mu^+ h_+(n^+, n^-)\partial_j n^+ \partial_j u_i^+ + \mu^+ k_+(n^+, n^-)\partial_j n^- \partial_j u_i^+ \\ &+ \mu^+ h_+(n^+, n^-)\partial_j n^+ \partial_i u_j^+ + \mu^+ k_+(n^+, n^-)\partial_j n^- \partial_i u_j^+ \\ &+ \lambda^+ h_+(n^+, n^-)\partial_i n^+ \partial_j u_j^+ + \lambda^+ k_+(n^+, n^-)\partial_i n^- \partial_j u_j^+ \\ &+ \mu^+ l_+(n^+, n^-)\partial_j^2 u_i^+ + (\mu^+ + \lambda^+) l_+(n^+, n^-)\partial_i \partial_j u_j^+, \end{aligned} \tag{2.8}$$

$$F_3 = -\operatorname{div}(n^- u^-), \tag{2.9}$$

$$\begin{aligned} F_4^i = &- g_-(n^+, n^-)\partial_i n^- - \bar g_-(n^+, n^-)\partial_i n^+ - (u^- \cdot \nabla) u_i^- \\ &+ \mu^- h_-(n^+, n^-)\partial_j n^+ \partial_j u_i^- + \mu^- k_-(n^+, n^-)\partial_j n^- \partial_j u_i^- \\ &+ \mu^- h_-(n^+, n^-)\partial_j n^+ \partial_i u_j^- + \mu^- k_-(n^+, n^-)\partial_j n^- \partial_i u_j^- \\ &+ \lambda^- h_-(n^+, n^-)\partial_i n^+ \partial_j u_j^- + \lambda^- k_-(n^+, n^-)\partial_i n^- \partial_j u_j^- \\ &+ \mu^- l_-(n^+, n^-)\partial_j^2 u_i^- + (\mu^- + \lambda^-) l_-(n^+, n^-)\partial_i \partial_j u_j^-, \end{aligned} \tag{2.10}$$

where

$$\begin{cases} g_+(n^+, n^-) = \frac{(C^2\rho^-)(n^+ + 1, n^- + 1)}{\rho^+(n^+ + 1, n^- + 1)} - \frac{(C^2\rho^-)(1,1)}{\rho^+(1,1)}, \\ g_-(n^+, n^-) = \frac{(C^2\rho^+)(n^+ + 1, n^- + 1)}{\rho^-(n^+ + 1, n^- + 1)} - \frac{(C^2\rho^+)(1,1)}{\rho^-(1,1)}, \end{cases} \tag{2.11}$$

$$\begin{cases} \bar g_+(n^+, n^-) = C^2(n^+ + 1, n^- + 1) - C^2(1,1) \\ \bar g_-(n^+, n^-) = C^2(n^+ + 1, n^- + 1) - C^2(1,1), \end{cases} \tag{2.12}$$

$$\begin{cases} h_+(n^+, n^-) = \frac{(C^2\alpha^-)(n^+ + 1, n^- + 1)}{(n^+ + 1)s_-^2(n^+ + 1, n^- + 1)}, \\ h_-(n^+, n^-) = -\frac{(C^2)(n^+ + 1, n^- + 1)}{(\rho^- s_-^2)(n^+ + 1, n^- + 1)}, \end{cases} \tag{2.13}$$

$$\begin{cases} k_+(n^+, n^-) = -\frac{C^2(n^+ + 1, n^- + 1)}{(n^+ + 1)(s_+^2\rho^+)(n^+ + 1, n^- + 1)}, \\ k_-(n^+, n^-) = -\frac{(\alpha^+ C^2)(n^+ + 1, n^- + 1)}{(n^- + 1)s_+^2(n^+ + 1, n^- + 1)}, \end{cases} \tag{2.14}$$

$$l_\pm(n^+, n^-) = \frac{1}{\rho_\pm(n^+ + 1, n^- + 1)} - \frac{1}{\rho_\pm(1,1)}. \tag{2.15}$$

3. Preliminaries

3.1. Preliminaries

In the following, we recall several useful tools, which will be frequently used throughout this paper.

Lemma 3.1. *(Gagliardo-Nirenberg inequality) Let* $1 \le q \le +\infty$*, j and m be non-negative integers such that j<m. Let* $1 \le r \le +\infty$*,* $p \ge 1$*and* $\theta \in [0,1]$ *then*

$$\left\|\nabla^j f\right\|_{L^p} \le C \left\|\nabla^m f\right\|_{L^r}^{\theta} \left\|f\right\|_{L^q}^{1-\theta} \qquad (C = C(j,m,n,q,r,\theta)),$$

where θ *satisfies*

$$\frac{1}{p} - \frac{j}{n} = \left(\frac{1}{r} - \frac{m}{n}\right)\theta + \frac{1}{q}(1-\theta).$$

It is worth noting that there are additional requirements when taking some special values for the coefficients.

1) If $j = 0, q = +\infty$ *and rm<n, then an additional assumption is needed either* $u \to 0(|x| \to +\infty)$ *or* $u \in L^s$*for some finite of s.*

2) If $r>1$ *and* $m - j - \frac{n}{r}$*is a non-negative integer, then* $\frac{j}{m} \le \theta<1$ *is needed.*

3) Notice that p usually assumed to be finite. However,there are sharper formulations in which $p = +\infty$ *is considered,but other values maybe excluded* $j = 0$*.*

4) Setting $f = \nabla^l u$*, we have*

$$\left\|\nabla^J u\right\|_{L^p} \le C \left\|\nabla^M u\right\|_{L^r}^{\theta} \left\|\nabla^l u\right\|_{L^q}^{1-\theta} \qquad (C = C(j,m,n,q,r,\theta)),$$

where θ *satisfies*

$$\frac{1}{p} - \frac{J}{n} = \left(\frac{1}{r} - \frac{M}{n}\right)\theta + (\frac{1}{r} - \frac{l}{n})(1-\theta) \qquad (J = j + l, M = m + l).$$

Proof. This is a special case of [15]. □

Lemma 3.2. *Let f and g be smooth functions belonging to* $H^k \cap L^\infty$ *for any integer* $k \ge 1$*, then*

$$\left\|\nabla^k(fg)\right\|_{L^2} \lesssim \|f\|_{L^\infty} \left\|\nabla^k g\right\|_{L^2} + \|g\|_{L^\infty} \left\|\nabla^k f\right\|_{L^2},$$

$$\left\|\nabla^k(fg)\right\|_{L^1} \lesssim \|f\|_{L^2} \left\|\nabla^k g\right\|_{L^2} + \|g\|_{L^2} \left\|\nabla^k f\right\|_{L^2},$$

$$\left\|\nabla^{k-1}(fg)\right\|_{L^{\frac{3}{2}}} \lesssim \|f\|_{L^2} \left\|\nabla^{k-1} g\right\|_{L^6} + \|g\|_{L^6} \left\|\nabla^{k-1} f\right\|_{L^2}.$$

Proof. The proof can be found in [[20], Lemma 3.1]. □

Lemma 3.3. *Let f and g be smooth functions belonging to* $H^k \cap L^\infty$ *for any integer* $k \ge 1$ *and define commutator* $[\nabla^k, f]g = \nabla^k(fg) - f\nabla^k g$*, then*

$$\left\|[\nabla^k, f]g\right\|_{L^2} \lesssim \left\|\nabla^1 f\right\|_{L^\infty} \left\|\nabla^{k-1} g\right\|_{L^2} + \left\|\nabla^k f\right\|_{L^2} \|g\|_{L^\infty}.$$

Proof. The proof can be found in [[20], Lemma 3.1]. □

Lemma 3.4. *Let $F(f)$ be a smooth function of f with bounded derivatives of any order and f belong to H^k for any integer $k \geq 3$, then*

$$\left\|\nabla^k(F(f))\right\|_{L^2} \lesssim \sup_{0\leq i\leq k}\left\|F^{(i)}(f)\right\|_{L^\infty}(\sum_{m=2}^{k}\|f\|_{L^2}^{m-1-\frac{n(m-1)}{2k}}\left\|\nabla^k f\right\|_{L^2}^{1+\frac{n(m-1)}{2k}}+\left\|\nabla^k f\right\|_{L^2}). \tag{3.1}$$

Proof. By direct calculation, we have

$$\begin{aligned}\nabla^k(F(f)) &= \sum_{i=1}^{k}(F^{(i)}(f)\cdot\sum_{\sum_{j=1}^n s_j=i,0\leq s_j\leq i}\nabla^{k-i}((\partial_1 f)^{s_1}(\partial_2 f)^{s_2}\cdots(\partial_n f)^{s_n})))\\ &\leq \sup_{0\leq i\leq k}\left\|F^{(i)}(f)\right\|_{L^2}\sum_{\sum_{d=1}^m h_d=k}C(s_1,s_2\cdots s_n)\nabla^{h_1}f\nabla^{h_2}f\cdots\nabla^{h_m}f.\end{aligned} \tag{3.2}$$

Using Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}&\left\|\nabla^{h_1}f\nabla^{h_2}f\cdots\nabla^{h_m}f\right\|_{L^2} \leq \left\|\nabla^{h_1}f\right\|_{L^2}\left\|\nabla^{h_2}f\right\|_{L^\infty}\cdots\left\|\nabla^{h_m}f\right\|_{L^\infty}\\ &\leq \left\|\nabla^k f\right\|_{L^2}^{1-a_1}\|f\|_{L^2}^{a_1}\left\|\nabla^k f\right\|_{L^2}^{1-a_2}\|f\|_{L^2}^{a_2}\cdots\left\|\nabla^k f\right\|_{L^2}^{1-a_m}\|f\|_{L^2}^{a_m},\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}&\frac{1}{2}-\frac{h_1}{n}=\frac{1}{2}a_1+(\frac{1}{2}-\frac{k}{n})(1-a_1),\\ &-\frac{h_2}{n}=\frac{1}{2}a_2+(\frac{1}{2}-\frac{k}{n})(1-a_2),\\ &\cdots,\\ &-\frac{h_m}{n}=\frac{1}{2}a_m+(\frac{1}{2}-\frac{k}{n})(1-a_m),\\ &a_1+a_2+\cdots+a_m=m-1-\frac{n(m-1)}{2k},\\ &(1-a_1)+(1-a_2)+\cdots+(1-a_m)=1+\frac{n(m-1)}{2k}.\end{aligned} \tag{3.4}$$

□

Moreover, according to $f\in H^k(\|\nabla^p f\|_{L^2}\leq M_p(0\leq p\leq k))$,we have

$$\left\|\nabla^k(F(f))\right\|_{L^2} \lesssim \left\|\nabla^k f\right\|_{L^2}.$$

Lemma 3.5. *Let $F(f)$ be a smooth function of vector function $f=(f_1,f_2,\cdots,f_N)$ with bounded partial derivatives of any order and $f_i=f_i(x_1,x_2,\cdots,x_n)(1\leq i\leq N)$ belonging to H^k for any integer $k\geq 3$, then*

$$\left\|\nabla_x^k(F(f))\right\|_{L^2} \lesssim \sup_{0\leq l\leq k}\left\|\nabla_y^l F\right\|_{L^\infty}\|f\|_{L^2}^{N-1-\frac{n(N-1)}{2k}}\left\|\nabla^k f\right\|_{L^2}^{1+\frac{n(N-1)}{2k}}.$$

Proof. By direct calculation, we have

$$\begin{aligned}|\nabla_x^k(F(f))| &= |\nabla^{k-1}(\nabla_y F\cdot\nabla^1 f)|\\ &\lesssim \sup_{1\leq l\leq k}\left\|\nabla_y^l F\right\|_{L^\infty}\sum_{\sum_{d=1}^N h_d=k}|\nabla^{h_1}f_1\nabla^{h_2}f_2\cdots\nabla^{h_N}f_N|.\end{aligned} \tag{3.5}$$

Using Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}\left\|\nabla^{h_1} f_1 \nabla^{h_2} f_2 \cdots \nabla^{h_N} f_N\right\|_{L^2} &\leq \left\|\nabla^{h_1} f_1\right\|_{L^2} \left\|\nabla^{h_2} f_2\right\|_{L^\infty} \cdots \left\|\nabla^{h_N} f_N\right\|_{L^\infty} \\ &\leq \left\|\nabla^{h_1} f\right\|_{L^2} \left\|\nabla^{h_2} f\right\|_{L^\infty} \cdots \left\|\nabla^{h_N} f\right\|_{L^\infty} \\ &\leq \left\|\nabla^{k} f\right\|_{L^2}^{1-a_1} \|f\|_{L^2}^{a_1} \left\|\nabla^{k} f\right\|_{L^2}^{1-a_2} \|f\|_{L^2}^{a_2} \cdots \left\|\nabla^{k} f\right\|_{L^2}^{1-a_N} \|f\|_{L^2}^{a_N}.\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}&\frac{1}{2} - \frac{h_1}{n} = \frac{1}{2}a_1 + (\frac{1}{2} - \frac{k}{n})(1-a_1), \\ &-\frac{h_2}{n} = \frac{1}{2}a_2 + (\frac{1}{2} - \frac{k}{n})(1-a_2), \\ &\cdots, \\ &-\frac{h_N}{n} = \frac{1}{2}a_N + (\frac{1}{2} - \frac{k}{n})(1-a_N), \\ &a_1 + a_2 + \cdots + a_N = N - 1 - \frac{n(N-1)}{2k}, \\ &(1-a_1) + (1-a_2) + \cdots + (1-a_N) = 1 + \frac{n(N-1)}{2k}.\end{aligned} \tag{3.7}$$

□

Moreover, according to $f_i \in H^k (1 \leq i \leq N, \|\nabla^p f\|_{L^2} \leq M_p (0 \leq p \leq k))$, we have

$$\left\|\nabla^k (F(f))\right\|_{L^2} \lesssim \left\|\nabla^k f\right\|_{L^2} \lesssim \sum_{i=1}^{N} \left\|\nabla^k f_i\right\|_{L^2}.$$

Lemma 3.6. *For any vector function $f \in C_0^\infty\left(\mathbb{R}^3\right)$ and bounded scalar function g, it holds that*

$$\left|\int_{\mathbb{R}^3} \left(\nabla |x|^{2\gamma}\right) \cdot f g \,\mathrm{d}x\right| \lesssim \|g\|_{L^2_\gamma} \|f\|_{L^2_{\gamma-1}}. \tag{3.8}$$

Proof. The left side of the above inequality can be rewritten as

$$\left|2\gamma \int_{\mathbb{R}^3} |x|^{2\gamma-2} x_j \partial_i x_j g f_i dx\right|.$$

Using Hölder's inequality, we have

$$\left|\int_{\mathbb{R}^3} \left(\nabla |x|^{2\gamma}\right) \cdot f g \,\mathrm{d}x\right| \lesssim \|g\|_{L^2_\gamma} \|f\|_{L^2_{\gamma-1}}.$$

□

Lemma 3.7. *(Interpolation inequality with weights) If $p, r \geqslant 1, s + n/r, \alpha + n/p, \beta + n/q > 0$, and $0 \leqslant \theta \leqslant 1$, then*

$$\|f\|_{L^r_s} \leq \|f\|_{L^p_\alpha}^{\theta} \|f\|_{L^q_\beta}^{1-\theta},$$

for $f \in C_0^\infty(\mathbb{R}^n)$, where

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q},$$

and

$$s = \theta\alpha + (1-\theta)\beta.$$

Especially, when $s = p = q = 2, \theta = \frac{\gamma-1}{\gamma}, s = \gamma - 1, \alpha = \gamma, \beta = 0$, *we have*

$$\|f\|_{L^2_{\gamma-1}} \le \|f\|_{L^2_\gamma}^{\frac{\gamma-1}{\gamma}} \|f\|_{L^2}^{\frac{1}{\gamma}}. \tag{3.9}$$

Proof. We compute

$$\begin{aligned}\int_U |x|^{sr}|f|^r dx &= \int_U |x|^{\alpha\theta r}|f|^{\theta r}|x|^{\beta(1-\theta)r}|f|^{(1-\theta)r}dx \\ &\le \left(\int_U \left(|x|^{\alpha\theta r}|f|^{\theta r}\right)^{\frac{p}{\theta r}} dx\right)^{\frac{\theta r}{p}} \left(\int_U \left(|x|^{\beta(1-\theta)r}|f|^{(1-\theta)r}\right)^{\frac{q}{(1-\theta)r}} dx\right)^{\frac{(1-\theta)r}{q}}.\end{aligned}$$

Thus, we complete the proof of Lemma 3.7. □

Lemma 3.8. *(Gronwall-type Lemma) Let* $\alpha_0 > 1, \alpha_1 < 1, \alpha_2 < 1$, *and* $\beta_1 < 1, \beta_2 < 2$. *Assume that a continuously differential function* $F : [1,\infty) \to [0,\infty)$ *satisfies*

$$\begin{aligned}&\frac{d}{dt}F(t) \le C_0 t^{-\alpha_0}F(t) + C_1 t^{-\alpha_1}F(t)^{\beta_1} + C_2 t^{-\alpha_2}F(t)^{\beta_2} + C_3 t^{\gamma_1-1}, t \ge 1 \\ &F(1) \le K_0,\end{aligned} \tag{3.10}$$

where $C_0, C_1, C_2, C_3, K_0 \ge 0$ *and* $\gamma_i = \frac{1-\alpha_i}{1-\beta_i} > 0$ *for* $i = 1, 2$. *Assume that* $\gamma_1 \ge \gamma_2$, *then there exists a constant* C^* *depending on* $\alpha_0, \alpha_1, \beta_1, \alpha_2, \beta_2, K_0, C_i, i = 1, 2, 3$, *such that*

$$F(t) \le C^* t^{\gamma_1},$$

for all $t \ge 1$.

Proof. This is Lemma 2.1 of [21]. □

We will generalize this lemma into the following one.

Lemma 3.9. *Let* $\alpha_0 > 1, 0 < \alpha_i, \beta_i < 1 (i = 1, 2, \cdots, n)$. *Assume that a continuously differential function* $F : [1,\infty) \to [0,\infty)$ *satisfies*

$$\begin{aligned}&\frac{d}{dt}F(t) \le C_0 t^{-\alpha_0}F(t) + \sum_{i=1}^{n} C_i t^{-\alpha_i}F(t)^{\beta_i} + \sum_{i=1}^{n} \overline{C_i} t^{\gamma_i - 1}, t \ge 1, \\ &F(1) \le K_0,\end{aligned} \tag{3.11}$$

where $C_0, C_i, \overline{C_i}, K_0 \ge 0$ *and* $\gamma_i = \frac{1-\alpha_i}{1-\beta_i} > 0$ *for* $i = 1, 2, \cdots, n$. *Assume that* $\gamma_1 \ge \gamma_i (2 \le i \le n)$, *then there exists a constant* C^* *depending on* $\alpha_0, \alpha_i, \beta_i, K_0, C_i, \overline{C_i} (i = 1, 2, \cdots, n)$, *such that*

$$F(t) \le C^* t^{\gamma_1},$$

for all $t \ge 1$.

Proof. For any $t \geq 1$, according to the conditions provided by lemma, it can be concluded that

$$\begin{aligned}\frac{d}{dt}F(t) &\leq C_0 t^{-\alpha_0}F(t) + \sum_{i=1}^{n} C_i t^{-\alpha_i}F(t)^{\beta_i} + \sum_{i=1}^{n}\overline{C_i}t^{\gamma_i-1}\\ &\leq C_0F(t) + \sum_{i=1}^{n} C_i((1-\beta_i)t^{-\frac{\alpha_i}{1-\beta_i}} + \beta_iF(t)) + \sum_{i=1}^{n}\overline{C_i}t^{\gamma_i-1}\\ &\leq CF(t) + \sum_{i=1}^{n} C_i(1-\beta_i)t^{-\frac{\alpha_i}{1-\beta_i}} + \sum_{i=1}^{n}\overline{C_i}t^{\gamma_i-1}(C = C(C_0, C_i, \beta_i)).\end{aligned}$$

Multiplying both sides of the equation by $e^{C(t-1)}$ and integrating the resulting equation from 1 to t, we have

$$\begin{aligned}F(t) &\leq e^{C(t-1)}(F(1) + \int_1^t e^{C(t-1)}\sum_{i=1}^{n}(C_i(1-\beta_i)s^{\frac{\alpha_i}{1-\beta_i}} + \overline{C_i}s^{\gamma_i-1})ds)\\ &\leq e^{C(t-1)}(K_0 + \int_1^t \sum_{i=1}^{n}(C_i(1-\beta_i)s^{\frac{\alpha_i}{1-\beta_i}} + \overline{C_i}s^{\gamma_i-1})ds)\\ &\leq e^{C(t-1)}(K_0 + (1-t)\sum_{i=1}^{n} C_i(1-\beta_i) + \sum_{i=1}^{n}\frac{\overline{C_i}}{\gamma_i}(t^{\gamma_i-1}-1)).\end{aligned}$$

Setting $t_0 = (\frac{\gamma_1}{2C})^{-\frac{1}{\alpha_0-1}}$, then we have

$$F(t_0) \leq e^{C(t_0-1)}(K_0 + (1-t_0)\sum_{i=1}^{n} C_i(1-\beta_i) + \sum_{i=1}^{n}\frac{\overline{C_i}}{\gamma_i}(t_0^{\gamma_i-1}-1)) = K_1.$$

Choosing

$$K \geq max_{1\leq i\leq n}\{(nC_i2^{\beta_i+2}\gamma_1^{-1})^{\frac{1}{1-\beta_i}}, \frac{4n\overline{C_i}}{\gamma_1}, K_1\},$$

and considering the set $R = \{t \geq t_0 | F(t) \leq 2Kt^{\gamma_1}\}$, we clearly have $F(t_0) \leq K_1 \leq K$. It's easy for us to know $t_0 \in R$. Therefore, R is not empty. Since $F(t) - 2Kt^{\gamma_1}$ is continuous function, if there exists maximal interval $[t_0, b) \subset R$ (if the maximal interval does not exist, the proof is completed), then $F(b) = 2Kb^{\gamma_1}$,$(F(t) - 2Kt^{\gamma_1})'|_{t=b} \geq 0$. Through the above discussion, we can conclude that

$$\begin{aligned}2K\gamma_1 b^{\gamma_1-1} \leq F'(b) &\leq C_0 b^{-\alpha_0}F(b) + \sum_{i=1}^{n} C_i b^{-\alpha_i}F(b)^{\beta_i} + \sum_{i=1}^{n}\overline{C_i}b^{\gamma_i-1}\\ &= C_0 2Kb^{\gamma_1-\alpha_0} + \sum_{i=1}^{n} C_i b^{-\alpha_i}(2Kb^{\gamma_1})^{\beta_i} + \sum_{i=1}^{n}\overline{C_i}b^{\gamma_i-1}\\ &\leq K\gamma_1 b^{\gamma_1-1}(2C_0\gamma_1^{-1}b^{1-\alpha_0} + \gamma_1^{-1}\sum_{i=1}^{n} C_i 2^{\beta_i}K^{\beta_i-1}\\ &+ (K\gamma_1)^{-1}\sum_{i=1}^{n}\overline{C_i})\\ &\leq K\gamma_1 b^{\gamma_1-1}(1 + n\frac{1}{4n} + n\frac{1}{4n}) = \frac{3}{2}K\gamma_1 b^{\gamma_1-1}.(contradiction!)\end{aligned}$$

□

Lemma 3.10. *(Gronwall's inequality of differential form). Let $\eta(\cdot)$ be a nonnegative, absolutely continuous function on $[0, T]$, which satisfies for a.e. t the differential inequality*

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t),$$

where $\phi(t)$ and $\psi(t)$ are nonnegative and summable functions on $[0, T]$. Then

$$\eta(t) \leq e^{\int_0^t \phi(s)ds}\left[\eta(0) + \int_0^t e^{-\int_0^s \phi(\tau)d\tau}\psi(s)ds\right], \tag{3.12}$$

for all $0 \leq t \leq T$.

Proof. The proof can be found in [22]. □

4. Proof of Theorem 1.1

Based on the time decay results of [12] and our hypothesis in Theorem 1.1, it is clear that there exists a large enough T such that for any $0 \leq k \leq N$ and $t > T$,

$$\begin{aligned} \left\|\nabla^k u^{\pm}\right\|_{L^2} &\lesssim (1+t)^{-\frac{3}{4}-\frac{k}{2}} \\ \left\|\nabla^k n^{\pm}\right\|_{L^2} &\lesssim (1+t)^{-\frac{1}{4}-\frac{k}{2}} \end{aligned} \tag{4.1}$$

In the following, we will prove Theorem 1.1. The proof mainly involves four steps.

Step 1: *k*-order the energy estimates.

By multiplying $\nabla^k(2.6)_1$, $\nabla^k(2.6)_2$, $\nabla^k(2.6)_3$ and $\nabla^k((2.6)_4$ by $|x|^{2\gamma}\frac{\beta_1}{\beta_2}\nabla^k n^+$, $|x|^{2\gamma}\frac{1}{\beta_2}\nabla^k u^+$, $|x|^{2\gamma}\frac{\beta_4}{\beta_3}\nabla^k n^-$ and $|x|^{2\gamma}\frac{1}{\beta_3}\nabla^k u^-$ respectively, summing up and then integrating the resultant equation over $\mathbb{R}^3$ by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\frac{1}{2}\left\|\sqrt{\frac{\beta_1}{\beta_2}}\nabla^k n^+ + \sqrt{\frac{\beta_4}{\beta_3}}\nabla^k n^-\right\|_{L^2_\gamma}^2 + \frac{\sigma+}{\beta_2}\left\|\nabla\nabla^k n^+\right\|_{L^2_\gamma}^2 + \frac{\sigma-}{\beta_3}\left\|\nabla\nabla^k n^-\right\|_{L^2_\gamma}^2 \\
&+ \frac{\beta_1}{4\beta_2}\left\|\nabla^k n^+\right\|_{L^2_\gamma}^2 + \frac{\beta_3}{4\beta_4}\left\|\nabla^k n^-\right\|_{L^2_\gamma}^2 + \frac{1}{\beta_2}\left\|\nabla^k u^+\right\|_{L^2_\gamma}^2 + \frac{1}{\beta_3}\left\|\nabla^k u^-\right\|_{L^2_\gamma}^2) \\
&+ \frac{1}{\beta_2}(\nu_1^+\left\|\nabla\nabla^k u^+\right\|_{L^2_\gamma}^2 + \nu_2^+\left\|\nabla^k \operatorname{div} u^+\right\|_{L^2_\gamma}^2) + \frac{1}{\beta_3}(\nu_1^-\left\|\nabla\nabla^k u^-\right\|_{L^2_\gamma}^2 + \nu_2^-\left\|\nabla^k \operatorname{div} u^-\right\|_{L^2_\gamma}^2) \\
=&\left\langle\nabla^k F_1, |x|^{2\gamma}\frac{\beta_1}{\beta_2}\nabla^k n^+\right\rangle + \left\langle\nabla^k F_2, |x|^{2\gamma}\frac{1}{\beta_2}\nabla^k u^+\right\rangle \\
&+ \left\langle\nabla^k F_3, |x|^{2\gamma}\frac{\beta_4}{\beta_3}\nabla^k n^-\right\rangle + \left\langle\nabla^k F_4, |x|^{2\gamma}\frac{1}{\beta_3}\nabla^k u^-\right\rangle \\
&- \left\langle\frac{\sigma^+}{\beta_1}\nabla^k F_1, |x|^{2\gamma}\nabla^k\Delta n^+\right\rangle - \left\langle\frac{\sigma^-}{\beta_3}\nabla^k F_3, |x|^{2\gamma}\nabla^k\Delta n^-\right\rangle - \left\langle\frac{\sigma^-}{\beta_3}\nabla^k F_1, \nabla(|x|^{2\gamma})\nabla\nabla^k n^+\right\rangle \\
&- \left\langle\frac{\sigma^+}{\beta_1}\nabla^k F_3, \nabla(|x|^{2\gamma})\nabla\nabla^k n^-\right\rangle + \left\langle\frac{1}{2}\nabla^k F_1, |x|^{2\gamma}\nabla^k n^-\right\rangle + \left\langle\frac{1}{2}\nabla^k F_3, |x|^{2\gamma}\nabla^k n^+\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\beta_1}{\beta_2}\int_{\mathbb{R}^3}\nabla^k n^+\nabla(|x|^{2\gamma})\nabla^k u^+dx+\frac{\beta_4}{\beta_3}\int_{\mathbb{R}^3}\nabla^k n^-\nabla(|x|^{2\gamma})\cdot\nabla^k u^-dx \qquad (4.2)\\
&-\frac{\nu_1^+}{\beta_2}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\nabla\nabla^k u^+\nabla^k u^+dx-\frac{\nu_1^-}{\beta_3}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\nabla\nabla^k u^-\nabla^k u^-dx\\
&-\frac{\nu_2^+}{\beta_2}\int_{\mathbb{R}^3}\nabla^k \operatorname{div} u^+\nabla(|x|^{2\gamma})\nabla^k u^+dx-\frac{\nu_2^-}{\beta_3}\int_{\mathbb{R}^3}\nabla^k \operatorname{div} u^-\nabla(|x|^{2\gamma})\nabla^k u^-dx\\
&-\frac{\sigma^+}{\beta_2}\int_{\mathbb{R}^3}\nabla^k\Delta n^+\nabla(|x|^{2\gamma})\nabla^k u^+dx-\frac{\sigma^-}{\beta_3}\int_{\mathbb{R}^3}\nabla^k\Delta n^-\nabla(|x|^{2\gamma})\nabla^k u^-dx\\
&+\frac{\sigma^+}{\beta_2}\int_{\mathbb{R}^3}\nabla^k \operatorname{div} u^+\nabla(|x|^{2\gamma})\nabla\nabla^k n^+dx+\frac{\sigma^-}{\beta_3}\int_{\mathbb{R}^3}\nabla^k \operatorname{div} u^-\nabla(|x|^{2\gamma})\nabla\nabla^k n^-dx\\
&+\frac{1}{2}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\nabla^k u^+\nabla^k n^-dx+\frac{1}{2}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\nabla^k u^-\nabla^k n^+dx\\
&-\frac{1}{2}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k u^+\nabla\nabla^k n^-dx-\frac{1}{2}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k u^-\nabla\nabla^k n^+dx\\
:=&\sum_{i=1}^{24}J_i.
\end{aligned}
$$

We set

$$
\begin{aligned}
E_k(t)=&\frac{1}{2}\left\|\sqrt{\frac{\beta_1}{\beta_2}}\nabla^k n^+ +\sqrt{\frac{\beta_4}{\beta_3}}\nabla^k n^-\right\|_{L^2_\gamma}^2+\frac{\sigma+}{\beta_2}\left\|\nabla\nabla^k n^+\right\|_{L^2_\gamma}^2+\frac{\sigma-}{\beta_2}\left\|\nabla\nabla^k n^-\right\|_{L^2_\gamma}^2\\
&+\frac{\beta_1}{4\beta_2}\left\|\nabla^k n^+\right\|_{L^2_\gamma}^2+\frac{\beta_3}{4\beta_4}\left\|\nabla^k n^-\right\|_{L^2_\gamma}^2+\frac{1}{\beta_2}\left\|\nabla^k u^+\right\|_{L^2_\gamma}^2+\frac{1}{\beta_3}\left\|\nabla^k u^-\right\|_{L^2_\gamma}^2.
\end{aligned}
\tag{4.3}
$$

Next, we will discuss the items on the right separately.

Applying Hölder's inequality, we have

$$
\begin{aligned}
|J_{11}|+|J_{21}|&\lesssim\left\|\nabla^k n^+\nabla(|x|^{2\gamma})\nabla^k u^+\right\|_{L^1}+\left\|\nabla^k n^-\nabla(|x|^{2\gamma})\nabla^k u^+\right\|_{L^1}\\
&\lesssim\left\||x|^{2\gamma-1}|\nabla^k u^+||\nabla^k n^+|\right\|_{L^1}+\left\||x|^{2\gamma-1}|\nabla^k u^+||\nabla^k n^-|\right\|_{L^1}\\
&\lesssim(\left\|\nabla^k n^+\right\|_{L^2_\gamma}+\left\|\nabla^k n^-\right\|_{L^2_\gamma})\left\|\nabla^k u^+\right\|_{L^2_{\gamma-1}}\\
&\lesssim(E_k(t))^{\frac{1}{2}}\left\|\nabla^k u^+\right\|_{L^2_{\gamma-1}}\\
&\lesssim(E_k(t))^{\frac{1}{2}}\left\|\nabla^k u^+\right\|_{L^2_\gamma}^{\frac{\gamma-1}{\gamma}}\left\|\nabla^k u^+\right\|_{L^2}^{\frac{1}{\gamma}}\\
&\lesssim(E_k(t))^{\frac{2\gamma-1}{2\gamma}}(1+t)^{-\frac{1}{\gamma}(\frac{3}{4}+\frac{k}{2})}.
\end{aligned}
\tag{4.4}
$$

Similarly, we can obtain

$$
|J_{12}|+|J_{22}|\lesssim(E_k(t))^{\frac{2\gamma-1}{2\gamma}}(1+t)^{-\frac{1}{\gamma}(\frac{3}{4}+\frac{k}{2})}. \tag{4.5}
$$

Applying Hölder's inequality and mean value theorem, we have

$$
|J_{13}|+|J_{14}|\le\frac{\nu_1^+}{\beta_2}\left\|\nabla(|x|^{2\gamma})\nabla\nabla^k u^+\nabla^k u^+\right\|_{L^1}+\frac{\nu_1^-}{\beta_3}\left\|\nabla(|x|^{2\gamma})\nabla\nabla^k u^-\nabla^k u^-\right\|_{L^1}
$$

$$
\begin{aligned}
&\le 2\gamma\frac{\nu_1^+}{\beta_2}\left\||x|^{2\gamma-1}|\nabla^k u^+||\nabla\nabla^k u^+|\right\|_{L^1} + 2\gamma\frac{\nu_1^-}{\beta_3}\left\||x|^{2\gamma-1}|\nabla^k u^-||\nabla\nabla^k u^-|\right\|_{L^1} \\
&\le 2\gamma\frac{\nu_1^+}{\beta_2}\left\|\nabla\nabla^k u^+\right\|_{L^2_\gamma}\left\|\nabla^k u^+\right\|_{L^2_{\gamma-1}} + 2\gamma\frac{\nu_1^-}{\beta_2}\left\|\nabla\nabla^k u^-\right\|_{L^2_\gamma}\left\|\nabla^k u^-\right\|_{L^2_{\gamma-1}} \\
&\le 2\gamma\frac{\nu_1^+}{\beta_2}\left\|\nabla\nabla^k u^+\right\|_{L^2_\gamma}\left\|\nabla^k u^+\right\|_{L^2_\gamma}^{\frac{\gamma-1}{\gamma}}\left\|\nabla^k u^+\right\|_{L^2}^{\frac{1}{\gamma}} \\
&+ 2\gamma\frac{\nu_1^-}{\beta_2}\left\|\nabla\nabla^k u^-\right\|_{L^2_\gamma}\left\|\nabla^k u^-\right\|_{L^2_\gamma}^{\frac{\gamma-1}{\gamma}}\left\|\nabla^k u^-\right\|_{L^2}^{\frac{1}{\gamma}} \\
&\le \varepsilon((2\gamma\frac{\nu_2^+}{\beta_2})^2 + (2\gamma\frac{\nu_2^-}{\beta_3})^2)(\left\|\nabla\nabla^k u^+\right\|_{L^2_\gamma}^2 + \left\|\nabla\nabla^k u^-\right\|_{L^2_\gamma}^2) \\
&+ \frac{1}{\varepsilon}(\left\|\nabla^k u^+\right\|_{L^2_\gamma}^{2(\frac{\gamma-1}{\gamma})}\left\|\nabla^k u^+\right\|_{L^2}^{\frac{2}{\gamma}} + \left\|\nabla^k u^-\right\|_{L^2_\gamma}^{2(\frac{\gamma-1}{\gamma})}\left\|\nabla^k u^-\right\|_{L^2}^{\frac{2}{\gamma}}) \\
&\le \varepsilon((2\gamma\frac{\nu_1^+}{\beta_2})^2 + (2\gamma\frac{\nu_1^-}{\beta_3})^2)(\left\|\nabla\nabla^k u^+\right\|_{L^2_\gamma}^2 + \left\|\nabla\nabla^k u^-\right\|_{L^2_\gamma}^2) \\
&+ \frac{1}{\varepsilon}(\beta_2^{\frac{\gamma-1}{\gamma}} + \beta_3^{\frac{\gamma-1}{\gamma}})C_1(E_k(t))^{\frac{\gamma-1}{\gamma}}(1+t)^{-\frac{2}{\gamma}(\frac{3}{4}+\frac{k}{2})}.
\end{aligned}
\tag{4.6}
$$

Employing similar methods used in estimating J_{15} and J_{16}, we can get

$$
\begin{aligned}
|J_{15}| + |J_{16}| &\le \frac{\nu_2^+}{\beta_2}\left\|\nabla^k \operatorname{div} u^+\nabla(|x|^{2\gamma})\nabla^k u^+\right\|_{L^1} + \frac{\nu_2^-}{\beta_3}\left\|\nabla^k \operatorname{div} u^-\nabla(|x|^{2\gamma})\nabla^k u^-\right\|_{L^1} \\
&\le \varepsilon((2\gamma\frac{\nu_2^+}{\beta_2})^2 + (2\gamma\frac{\nu_2^-}{\beta_3})^2)(\left\|\operatorname{div}\nabla^k u^+\right\|_{L^2_\gamma}^2 + \left\|\operatorname{div}\nabla^k u^-\right\|_{L^2_\gamma}^2) \\
&+ \frac{1}{\varepsilon}(\left\|\nabla^k u^+\right\|_{L^2_\gamma}^{2(\frac{\gamma-1}{\gamma})}\left\|\nabla^k u^+\right\|_{L^2}^{\frac{2}{\gamma}} + \left\|\nabla^k u^-\right\|_{L^2_\gamma}^{2(\frac{\gamma-1}{\gamma})}\left\|\nabla^k u^-\right\|_{L^2}^{\frac{2}{\gamma}}) \\
&\le \varepsilon((2\gamma\frac{\nu_2^+}{\beta_2})^2 + (2\gamma\frac{\nu_2^-}{\beta_3})^2)(\left\|\nabla^k \operatorname{div} u^+\right\|_{L^2_\gamma}^2 + \left\|\nabla^k \operatorname{div} u^-\right\|_{L^2_\gamma}^2) \\
&+ \frac{1}{\varepsilon}(\beta_2^{\frac{\gamma-1}{\gamma}} + \beta_3^{\frac{\gamma-1}{\gamma}})C_2(E_k(t))^{\frac{\gamma-1}{\gamma}}(1+t)^{-\frac{2}{\gamma}(\frac{3}{4}+\frac{k}{2})}.
\end{aligned}
\tag{4.7}
$$

As for J_{19} and J_{20}, we have

$$
\begin{aligned}
|J_{19}| + |J_{20}| &\le \frac{\sigma^+}{\beta_2}\left\|\nabla^k \operatorname{div} u^+\nabla(|x|^{2\gamma})\nabla\nabla^k n^+\right\|_{L_1} \\
&+ \frac{\sigma^-}{\beta_3}\left\|\nabla^k \operatorname{div} u^-\nabla(|x|^{2\gamma})\nabla\nabla^k n^-\right\|_{L_1} \\
&\le 2\gamma\frac{\sigma^+}{\beta_2}\left\||\nabla^k \operatorname{div} u^+||x|^{2\gamma-1}|\nabla\nabla^k n^+|\right\|_{L_1} \\
&+ 2\gamma\frac{\sigma^-}{\beta_3}\left\||\nabla^k \operatorname{div} u^-||x|^{2\gamma-1}|\nabla\nabla^k n^-|\right\|_{L_1} \\
&\le 2\gamma\frac{\sigma^+}{\beta_2}\left\|\nabla^k \operatorname{div} u^+\right\|_{L^2_\gamma}\left\|\nabla\nabla^k n^+\right\|_{L^2_{\gamma-1}} \\
&+ 2\gamma\frac{\sigma^-}{\beta_3}\left\|\nabla^k \operatorname{div} u^-\right\|_{L^2_\gamma}\left\|\nabla\nabla^k n^-\right\|_{L^2_{\gamma-1}}
\end{aligned}
\tag{4.8}
$$

$$\leq 2\gamma\frac{\sigma^+}{\beta_2}\left\|\nabla^k \operatorname{div} u^+\right\|_{L^2_\gamma}\left\|\nabla\nabla^k n^+\right\|_{L^2_\gamma}\left\|\nabla\nabla^k n^+\right\|_{L^2}^{\frac{1}{\gamma}}$$

$$+ 2\gamma\frac{\sigma^-}{\beta_3}\left\|\nabla^k \operatorname{div} u^-\right\|_{L^2_\gamma}\left\|\nabla\nabla^k n^-\right\|_{L^2_\gamma}^{\frac{\gamma-1}{\gamma}}\left\|\nabla\nabla^k n^-\right\|_{L^2}^{\frac{1}{\gamma}}$$

$$\leq \varepsilon((2\gamma\frac{\sigma^+}{\beta_2})^2 + (2\gamma\frac{\sigma^-}{\beta_3})^2)(\left\|\nabla^k \operatorname{div} u^+\right\|_{L^2_\gamma}^2 + \left\|\nabla^k \operatorname{div} u^-\right\|_{L^2_\gamma}^2)$$

$$+ \frac{1}{\varepsilon}(E_k(t))^{\frac{\gamma-1}{\gamma}}(\left\|\nabla\nabla^k n^+\right\|_{L^2}^{\frac{1}{\gamma}} + \left\|\nabla\nabla^k n^-\right\|_{L^2}^{\frac{1}{\gamma}})$$

$$\leq \varepsilon((2\gamma\frac{\sigma^+}{\beta_2})^2 + (2\gamma\frac{\sigma^-}{\beta_3})^2)(\left\|\nabla^k \operatorname{div} u^+\right\|_{L^2_\gamma}^2 + \left\|\nabla^k \operatorname{div} u^-\right\|_{L^2_\gamma}^2)$$

$$+ \frac{3}{\varepsilon}((\frac{\beta_2}{\sigma^+})^{\frac{\gamma-1}{\gamma}} + (\frac{\beta_3}{\sigma^-})^{\frac{\gamma-1}{\gamma}})C_3(E_k(t))^{\frac{\gamma-1}{\gamma}}(1+t)^{-\frac{2}{\gamma}(\frac{1}{4}+\frac{k+1}{2})}.$$

The terms J_{17} and J_{18} are more complicated. To begin with, we use integration by parts to get

$$\begin{aligned}
|J_{17}| + |J_{18}| = & \frac{\sigma^+}{\beta_2}\Big|\int_{\mathbb{R}^3} \operatorname{div}(\nabla^k\nabla n^+(\nabla(|x|^{2\gamma})\nabla^k u^+))dx \\
& - \int_{\mathbb{R}^3} \nabla^k\nabla n^+\nabla(\nabla(|x|^{2\gamma})\nabla^k u^+)dx\Big| \\
& + \frac{\sigma^-}{\beta_3}\Big|\int_{\mathbb{R}^3} \operatorname{div}(\nabla^k\nabla n^-(\nabla(|x|^{2\gamma})\nabla^k u^-))dx \\
& - \int_{\mathbb{R}^3} \nabla^k\nabla n^-\nabla(\nabla(|x|^{2\gamma})\nabla^k u^-)dx\Big| \\
= & \frac{\sigma^+}{\beta_2}\Big|\int_{\mathbb{R}^3} \nabla^k\nabla n^+\nabla(\nabla(|x|^{2\gamma})\nabla^k u^+)dx\Big| \\
& + \frac{\sigma^-}{\beta_3}\Big|\int_{\mathbb{R}^3} \nabla^k\nabla n^-\nabla(\nabla(|x|^{2\gamma})\nabla^k u^-)dx\Big| \\
\leq & \frac{\sigma^+}{\beta_2}\int_{\mathbb{R}^3} |\nabla^k\nabla n^+||\nabla(\nabla(|x|^{2\gamma})\nabla^k u^+)|dx \\
& + \frac{\sigma^-}{\beta_3}\int_{\mathbb{R}^3} |\nabla^k\nabla n^-||\nabla(\nabla(|x|^{2\gamma})\nabla^k u^-)|dx \\
\leq & \frac{\sigma^+}{\beta_2}\int_{\mathbb{R}^3} |\nabla^k\nabla n^+||\sum_{i=1}^{3}[\nabla\partial_i(|x|^{2\gamma})\nabla^k u_i^+ + \partial_i(|x|^{2\gamma})\nabla\nabla^k u_i^+]|dx \\
& + \frac{\sigma^-}{\beta_3}\int_{\mathbb{R}^3} |\nabla^k\nabla n^-||\sum_{i=1}^{3}[\nabla\partial_i(|x|^{2\gamma})\nabla^k u_i^- + \partial_i(|x|^{2\gamma})\nabla\nabla^k u_i^-]|dx \qquad (4.9)\\
\leq & 2\gamma\frac{\sigma^+}{\beta_2}\int_{\mathbb{R}^3} |\nabla^k\nabla n^+|\sum_{i=1}^{3}[|\nabla(|x|^{2\gamma-2}x_i)\nabla^k u_i^+| + ||x|^{2\gamma-2}x_i\nabla\nabla^k u_i^+|]dx \\
& + 2\gamma\frac{\sigma^-}{\beta_3}\int_{\mathbb{R}^3} |\nabla^k\nabla n^+|\sum_{i=1}^{3}[|\nabla(|x|^{2\gamma-2}x_i)\nabla^k u_i^-| + ||x|^{2\gamma-2}x_i\nabla\nabla^k u_i^-|]dx \\
\leq & 2\gamma\frac{\sigma^+}{\beta_2}\int_{\mathbb{R}^3} |\nabla^k\nabla n^+|\sum_{i=1}^{3}[2\sqrt{3}|x|^{2\gamma-2}|\nabla^k u_i^+| + |x|^{2\gamma-1}|\nabla\nabla^k u_i^+|]dx
\end{aligned}$$

$$
\begin{aligned}
&+ 2\gamma\frac{\sigma^-}{\beta_3}\int_{\mathbb{R}^3}|\nabla^k\nabla n^+|\sum_{i=1}^{3}[2\sqrt{3}|x|^{2\gamma-2}|\nabla^k u_i^-| + |x|^{2\gamma-1}|\nabla\nabla^k u_i^-|]dx\\
&\leq 12\sqrt{3}\gamma\frac{\sigma^+}{\beta_2}\left\|\nabla\nabla^k n^+\right\|_{L^2_\gamma}\left\|\nabla^k u^+\right\|_{L^2_{\gamma-2}}\\
&+ 6\gamma\frac{\sigma^+}{\beta_2}\left\|\nabla\nabla^k n^+\right\|_{L^2_{\gamma-1}}\left\|\nabla\nabla^k u^+\right\|_{L^2_\gamma}\\
&+ 12\sqrt{3}\gamma\frac{\sigma^-}{\beta_3}\left\|\nabla\nabla^k n^-\right\|_{L^2_\gamma}\left\|\nabla^k u^-\right\|_{L^2_{\gamma-2}}\\
&+ 6\gamma\frac{\sigma^-}{\beta_3}\left\|\nabla\nabla^k n^-\right\|_{L^2_{\gamma-1}}\left\|\nabla\nabla^k u^-\right\|_{L^2_\gamma}\\
&\leq 12\sqrt{3}\gamma\frac{\sigma^+}{\beta_2}\left\|\nabla\nabla^k n^+\right\|_{L^2_\gamma}\left\|\nabla^k u^+\right\|_{L^2_\gamma}^{\frac{\gamma-2}{\gamma}}\left\|\nabla^k u^+\right\|_{L^2}^{\frac{2}{\gamma}}\\
&+ 6\gamma\frac{\sigma^+}{\beta_2}\left\|\nabla\nabla^k n^+\right\|_{L^2_\gamma}^{\frac{\gamma-1}{\gamma}}\left\|\nabla\nabla^k n^+\right\|_{L^2}^{\frac{1}{\gamma}}\left\|\nabla\nabla^k u^+\right\|_{L^2_\gamma}\\
&+ 12\sqrt{3}\gamma\frac{\sigma^-}{\beta_3}\left\|\nabla\nabla^k n^-\right\|_{L^2_\gamma}\left\|\nabla^k u^-\right\|_{L^2_\gamma}^{\frac{\gamma-2}{\gamma}}\left\|\nabla^k u^-\right\|_{L^2}^{\frac{2}{\gamma}}\\
&+ 6\gamma\frac{\sigma^-}{\beta_3}\left\|\nabla\nabla^k n^-\right\|_{L^2_\gamma}^{\frac{\gamma-1}{\gamma}}\left\|\nabla\nabla^k n^-\right\|_{L^2}^{\frac{1}{\gamma}}\left\|\nabla\nabla^k u^-\right\|_{L^2_\gamma}\\
&\leq 12\sqrt{3}\gamma(\sqrt{\sigma^+}(\frac{1}{\beta_2})^{\frac{1}{\gamma}} + \sqrt{\sigma^-}(\frac{1}{\beta_3})^{\frac{1}{\gamma}})C_4(E_k(t))^{\frac{\gamma-1}{\gamma}}(1+t)^{-\frac{2}{\gamma}(\frac{3}{4}+\frac{k}{2})}\\
&+ \frac{1}{\varepsilon}((\frac{\beta_2}{\sigma^+})^{\frac{\gamma-1}{\gamma}} + (\frac{\beta_2}{\sigma^-})^{\frac{\gamma-1}{\gamma}})C_5(E_k(t))^{\frac{\gamma-1}{\gamma}}(1+t)^{-\frac{2}{\gamma}(\frac{1}{4}+\frac{k+1}{2})}\\
&+ \varepsilon 36\gamma^2((\frac{\sigma^+}{\beta_2})^2 + (\frac{\sigma^-}{\beta_3})^2)(\left\|\nabla\nabla^k u^+\right\|_{L^2_\gamma}^2 + \left\|\nabla\nabla^k u^-\right\|_{L^2_\gamma}^2).
\end{aligned}
$$

It can be inferred from (1.15) that

$$
|J_{23}| + |J_{24}| \lesssim \left\|\nabla^k u^+\right\|_{L^2_\gamma}\left\|\nabla\nabla^k n^-\right\|_{L^2_\gamma} + \left\|\nabla^k u^-\right\|_{L^2_\gamma}\left\|\nabla\nabla^k n^+\right\|_{L^2_\gamma} \lesssim (E_k(t))^{\frac{1}{2}}. \tag{4.10}
$$

By choosing ε small enough, we can obtain

$$
\begin{aligned}
&\frac{d}{dt}E_k(t) + C'(\left\|\nabla\nabla^k u^+\right\|_{L^2_\gamma}^2 + \left\|\nabla^k \operatorname{div} u^+\right\|_{L^2_\gamma}^2 + \left\|\nabla\nabla^k u^-\right\|_{L^2_\gamma}^2 + \left\|\nabla^k \operatorname{div} u^-\right\|_{L^2_\gamma}^2)\\
&\lesssim (E_k(t))^{\frac{2\gamma-1}{2\gamma}}(1+t)^{-\frac{1}{\gamma}(\frac{3}{4}+\frac{k}{2})} + (E_k(t))^{\frac{\gamma-1}{\gamma}}(1+t)^{-\frac{2}{\gamma}(\frac{3}{4}+\frac{k}{2})} + (E_k(t))^{\frac{1}{2}}\\
&+ \left|\left\langle\nabla^k F_2, |x|^{2\gamma}\frac{1}{\beta_2}\nabla^k u^+\right\rangle\right| + \left|\left\langle\nabla^k F_4, |x|^{2\gamma}\frac{1}{\beta_3}\nabla^k u^-\right\rangle\right|\\
&+ \left|\left\langle\frac{\sigma^+}{\beta_1}\nabla^k F_1, |x|^{2\gamma}\nabla^k\Delta n^+\right\rangle\right| + \left|\left\langle\frac{\sigma^-}{\beta_3}\nabla^k F_3, |x|^{2\gamma}\nabla^k\Delta n^-\right\rangle\right|\\
&+ \left|\left\langle\frac{\sigma^+}{\beta_1}\nabla^k F_1, \nabla(|x|^{2\gamma})\nabla\nabla^k n^+\right\rangle\right| + \left|\left\langle\frac{\sigma^-}{\beta_3}\nabla^k F_3, \nabla(|x|^{2\gamma})\nabla\nabla^k n^-\right\rangle\right|\\
&+ \left|\left\langle\nabla^k F_1, |x|^{2\gamma}\nabla^k n^+\right\rangle\right| + \left|\left\langle\nabla^k F_1, |x|^{2\gamma}\nabla^k n^-\right\rangle\right|\\
&+ \left|\left\langle\nabla^k F_3, |x|^{2\gamma}\nabla^k n^-\right\rangle\right| + \left|\left\langle\nabla^k F_3, |x|^{2\gamma}\nabla^k n^+\right\rangle\right|
\end{aligned} \tag{4.11}
$$

$$\lesssim (E_k(t))^{\frac{2\gamma-1}{2\gamma}}(1+t)^{-\frac{1}{\gamma}(\frac{3}{4}+\frac{k}{2})}+(E_k(t))^{\frac{\gamma-1}{\gamma}}(1+t)^{-\frac{2}{\gamma}(\frac{3}{4}+\frac{k}{2})}+(E_k(t))^{\frac{1}{2}}$$
$$+\sum_{i=2,i\neq 3}^{8}|J_i|+|J_1|+|J_9|+|J_3|+|J_{10}|.$$

From direct observation, it can be inferred that due to good symmetry, we only need to calculate $|J_1|+|J_9|, |J_5|, |J_7|$ and $|J_2|$.

Let's first consider $|J_1+J_9|, |J_5|$ and $|J_7|$. Applying Lemma 2.1 and (3.1), we have

$$\begin{aligned}
|J_1|+|J_9| &\lesssim \left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k n^+\nabla^k \operatorname{div}(n^+u^+)dx\right| \\
&+\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k n^-\nabla^k \operatorname{div}(n^+u^+)dx\right| \\
&\lesssim (\left\|\nabla^k n^+\right\|_{L^2_\gamma}+\left\|\nabla^k n^-\right\|_{L^2_\gamma})(\left\|\nabla^k(n^+\operatorname{div}u^+)\right\|_{L^2_\gamma}+\left\|\nabla^k(\nabla n^+u^+)\right\|_{L^2_\gamma}) \\
&\lesssim (E_k(t))^{\frac{1}{2}}(\sum_{l=0}^{k}\left\|\nabla^l n^+\nabla^{k-l}\operatorname{div}u^+\right\|_{L^2_\gamma}+\sum_{l=0}^{k}\left\|\nabla^l\nabla n^+\nabla^{k-l}u^+\right\|_{L^2_\gamma})) \\
&\lesssim (E_k(t))^{\frac{1}{2}}(\sum_{l=0}^{k}\left\|\nabla^l n^+\right\|_{L^2_\gamma}\left\|\nabla^{k-l}\operatorname{div}u^+)\right\|_{L^\infty} \qquad (4.12)\\
&+\sum_{l=0}^{k}\left\|\nabla^l\nabla n^+\right\|_{L^2_\gamma}\left\|\nabla^{k-l}u^+\right\|_{L^\infty}) \\
&\lesssim (E_k(t))^{\frac{1}{2}}\sum_{l=0}^{k}(E_l(t))^{\frac{1}{2}}(\left\|\nabla^{k-l}\operatorname{div}u^+\right\|_{L^\infty}+\left\|\nabla^{k-l}u^+\right\|_{L^\infty}) \\
&\lesssim (E_k(t))^{\frac{1}{2}}\sum_{l=0}^{k}(E_l(t))^{\frac{1}{2}}(\left\|\nabla^{k+3}u^+\right\|_{L^2}^{\frac{k+1.5-l}{k+2}}\left\|\nabla^1 u^+\right\|_{L^2}^{\frac{l+0.5}{k+2}} \\
&+\left\|\nabla^{k+2}u^+\right\|_{L^2}^{\frac{k+1.5-l}{k+2}}\left\|u^+\right\|_{L^2}^{\frac{l+0.5}{k+2}}) \\
&\lesssim (E_k(t))^{\frac{1}{2}}\sum_{l=0}^{k}(E_l(t))^{\frac{1}{2}}(1+t)^{-(\frac{3}{2}+\frac{k-l}{2})}.
\end{aligned}$$

$$\begin{aligned}
|J_5| &\lesssim \left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k(n^+\operatorname{div}u^+)\nabla^k\Delta n^+dx\right| \\
&+\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^k(\nabla n^+u^+)\nabla^k\Delta n^+dx\right| \\
&\lesssim \left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla\nabla^k(n^+\operatorname{div}u^+)\nabla\nabla^k n^+dx\right| \\
&+\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla\nabla^k(\nabla n^+u^+)\nabla\nabla^k n^+dx\right| \\
&+\left|\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\nabla^k(n^+\operatorname{div}u^+)\nabla\nabla^k n^+dx\right| \qquad (4.13)
\end{aligned}$$

$$
\begin{aligned}
&+\left|\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\nabla^k(\nabla n^+u^+)\nabla^k\nabla n^+dx\right|\\
&\lesssim \left\|\nabla\nabla^k(n^+\operatorname{div}u^+)\right\|_{L^2_\gamma}\left\|\nabla^k\nabla n^+\right\|_{L^2_\gamma}+\left\|\nabla\nabla^k(\nabla n^+u^+)\right\|_{L^2_\gamma}\left\|\nabla^k\nabla n^+\right\|_{L^2_\gamma}\\
&+\left\|\nabla^k(n^+\operatorname{div}u^+)\right\|_{L^2_\gamma}\left\|\nabla^k\nabla n^+\right\|_{L^2_{\gamma-1}}+\left\|\nabla^k(\nabla n^+u^+)\right\|_{L^2_\gamma}\left\|\nabla^k\nabla n^+\right\|_{L^2_{\gamma-1}}\\
&\lesssim [\left\|\nabla\nabla^k(n^+\operatorname{div}u^+)\right\|_{L^2_\gamma}+\left\|\nabla\nabla^k(\nabla n^+u^+)\right\|_{L^2_\gamma}]\left\|\nabla^k\nabla n^+\right\|_{L^2_\gamma}\\
&+(\left\|\nabla^k(n^+\operatorname{div}u^+)\right\|_{L^2_\gamma}+\left\|\nabla^k(\nabla n^+u^+)\right\|_{L^2_\gamma})\left\|\nabla^k\nabla n^+\right\|_{L^2_\gamma}^{\frac{\gamma-1}{\gamma}}\left\|\nabla^k\nabla n^+\right\|_{L^2}^{\frac{1}{\gamma}}\\
&\lesssim [\left\|\nabla\nabla^k(n^+\operatorname{div}u^+)\right\|_{L^2_\gamma}+\left\|\nabla\nabla^k(\nabla n^+u^+)\right\|_{L^2_\gamma}](E_k(t))^{\frac12}\\
&+(\left\|\nabla^k(n^+\operatorname{div}u^+)\right\|_{L^2_\gamma}+\left\|\nabla^k(\nabla n^+u^+)\right\|_{L^2_\gamma})(E_k(t))^{\frac{\gamma-1}{2\gamma}}(1+t)^{-(\frac14+\frac{k+1}{2})\frac{1}{\gamma}}\\
&\lesssim [\left\|\nabla\nabla^k(n^+\operatorname{div}u^+)\right\|_{L^2_\gamma}+\left\|\nabla\nabla^k(\nabla n^+u^+)\right\|_{L^2_\gamma}](E_k(t))^{\frac12}\\
&+(E_k(t))^{\frac{\gamma-1}{2\gamma}}(1+t)^{-(\frac14+\frac{k+1}{2})\frac{1}{\gamma}}\sum_{l=0}^{k}(E_l(t))^{\frac12}(1+t)^{-(\frac32+\frac{k-l}{2})}\\
&\lesssim [\sum_{l=0}^{k}(\left\|\nabla\nabla^l n^+\nabla^{k-l}\operatorname{div}u^+\right\|_{L^2_\gamma}+\left\|\nabla^l n^+\nabla\nabla^{k-l}\operatorname{div}u^+\right\|_{L^2_\gamma})\\
&+\sum_{l=0}^{k}(\left\|\nabla^{k-l+1}\nabla n^+\nabla^l u^+\right\|_{L^2_\gamma}+\left\|\nabla^l\nabla n^+\nabla^{k-l+1}u^+\right\|_{L^2_\gamma})](E_k(t))^{\frac12}\\
&+(E_k(t))^{\frac{\gamma-1}{2\gamma}}(1+t)^{-(\frac14+\frac{k+1}{2})\frac{1}{\gamma}}\sum_{l=0}^{k}(E_l(t))^{\frac12}(1+t)^{-(\frac32+\frac{k-l}{2})}\\
&\lesssim (E_k(t))^{\frac12}\sum_{l=0}^{k}(E_k(t))^{\frac12}(\left\|\nabla^{k-l}\operatorname{div}u^+\right\|_{L^\infty}+\left\|\nabla\nabla^{k-l}\operatorname{div}u^+\right\|_{L^\infty}\\
&+\left\|\nabla^{k-l+1}\nabla n^+\right\|_{L^\infty}+\left\|\nabla^{k-l+1}u^+\right\|_{L^\infty})\\
&+(E_k(t))^{\frac{\gamma-1}{2\gamma}}(1+t)^{-(\frac14+\frac{k+1}{2})\frac{1}{\gamma}}\sum_{l=0}^{k}(E_l(t))^{\frac12}(1+t)^{-(\frac32+\frac{k-l}{2})}\\
&\lesssim (E_k(t))^{\frac12}\sum_{l=0}^{k}(E_l(t))^{\frac12}(\left\|\nabla^{k-l+1}u^+\right\|_{L^\infty}+\left\|\nabla^{k-l+2}u^+\right\|_{L^\infty}+\left\|\nabla^{k-l+2}n^+\right\|_{L^\infty})\\
&+(E_k(t))^{\frac{\gamma-1}{2\gamma}}(1+t)^{-(\frac14+\frac{k+1}{2})\frac{1}{\gamma}}\sum_{l=0}^{k}(E_l(t))^{\frac12}(1+t)^{-(\frac32+\frac{k-l}{2})}\\
&\lesssim (E_k(t))^{\frac12}\sum_{l=0}^{k}(E_l(t))^{\frac12}(\left\|\nabla^{k+3}u^+\right\|_{L^2}^{\frac{k-l+1.5}{k+2}}\left\|\nabla^1u^+\right\|_{L^2}^{\frac{l+0.5}{k+2}}\\
&+\left\|\nabla^{k+4}u^+\right\|_{L^2}^{\frac{k-l+2.5}{k+3}}\left\|\nabla^1u^+\right\|_{L^2}^{\frac{l+0.5}{k+3}}+\left\|\nabla^{k+4}n^+\right\|_{L^2}^{\frac{k-l+2.5}{k+3}}\left\|\nabla^1n^+\right\|_{L^2}^{\frac{l+0.5}{k+3}})\\
&+(E_k(t))^{\frac{\gamma-1}{2\gamma}}(1+t)^{-(\frac34+\frac{k}{2})\frac{1}{\gamma}}\sum_{l=0}^{k}(E_l(t))^{\frac12}(1+t)^{-(\frac32+\frac{k-l}{2})}
\end{aligned}
$$

$$\lesssim (E_k(t))^{\frac{1}{2}} \sum_{l=0}^{k} (E_l(t))^{\frac{1}{2}} (1+t)^{-(2+\frac{k-l}{2})}$$

$$+ (E_k(t))^{\frac{\gamma-1}{2\gamma}} (1+t)^{-(\frac{3}{4}+\frac{k}{2})\frac{1}{\gamma}} \sum_{l=0}^{k} (E_l(t))^{\frac{1}{2}} (1+t)^{-(\frac{3}{2}+\frac{k-l}{2})}.$$

$$\begin{aligned}
|J_7| \lesssim & \left| \int_{\mathbb{R}^3} \nabla(|x|^{2\gamma}) \nabla^k (n^+ \operatorname{div} u^+) \nabla^k \nabla n^+ dx \right| \\
& + \left| \int_{\mathbb{R}^3} \nabla(|x|^{2\gamma}) \nabla^k (\nabla n^+ u^+) \nabla^k \nabla n^+ dx \right| \\
& \lesssim \int_{\mathbb{R}^3} |x|^{2\gamma-1} \left| \nabla^k (n^+ \operatorname{div} u^+) \right| \left| \nabla^k \nabla n^+ \right| dx \\
& + \int_{\mathbb{R}^3} |x|^{2\gamma-1} \left| \nabla^k (\nabla n^+ u^+) \right| \left| \nabla^k \nabla n^+ \right| dx \\
& \lesssim \left\| \nabla^k \nabla n^+ \right\|_{L^2_{\gamma-1}} \left\| \nabla^k (n^+ \operatorname{div} u^+) \right\|_{L^2_\gamma} + \left\| \nabla^k \nabla n^+ \right\|_{L^2_{\gamma-1}} \left\| \nabla^k (\nabla n^+ u^+) \right\|_{L^2_\gamma} \\
& \lesssim \left\| \nabla^k \nabla n^+ \right\|_{L^2_\gamma}^{\frac{\gamma-1}{\gamma}} \left\| \nabla^k \nabla n^+ \right\|_{L^2}^{\frac{1}{\gamma}} (\left\| \nabla^k (n^+ \operatorname{div} u^+) \right\|_{L^2_\gamma} + \left\| \nabla^k (\nabla n^+ u^+) \right\|_{L^2_\gamma}) \\
& \lesssim (E_k(t))^{\frac{\gamma-1}{2\gamma}} (1+t)^{-\frac{1}{\gamma}(\frac{1}{4}+\frac{k+1}{2})} \sum_{l=0}^{k} (E_l(t))^{\frac{1}{2}} (1+t)^{-(\frac{3}{2}+\frac{k-l}{2})} \\
& + (E_k(t))^{\frac{\gamma-1}{2\gamma}} (1+t)^{-\frac{1}{\gamma}(\frac{1}{4}+\frac{k+1}{2})} \sum_{l=0}^{k} (E_l(t))^{\frac{1}{2}} (1+t)^{-(2+\frac{k-l}{2})} \\
& \lesssim (E_k(t))^{\frac{\gamma-1}{2\gamma}} (1+t)^{-\frac{1}{\gamma}(\frac{1}{4}+\frac{k+1}{2})} \sum_{l=0}^{k} (E_l(t))^{\frac{1}{2}} (1+t)^{-(\frac{3}{2}+\frac{k-l}{2})}.
\end{aligned} \tag{4.14}$$

Combining the above relations, we can conclude that

$$\begin{aligned}
& \frac{d}{dt} E_k(t) + \widetilde{C}(\left\| \nabla \nabla^k u^+ \right\|_{L^2_\gamma}^2 + \left\| \nabla \nabla^k u^- \right\|_{L^2_\gamma}^2 + \left\| \nabla^k \operatorname{div} u^+ \right\|_{L^2_\gamma}^2 + \left\| \nabla^k \operatorname{div} u^- \right\|_{L^2_\gamma}^2) \\
\lesssim & (E_k(t))^{1-\frac{1}{2\gamma}} (1+t)^{-\frac{1}{\gamma}(\frac{3}{4}+\frac{k}{2})} + (E_k(t))^{1-\frac{1}{\gamma}} (1+t)^{-\frac{2}{\gamma}(\frac{3}{4}+\frac{k}{2})} + (E_k(t))^{\frac{1}{2}} \\
& + (E_k(t))^{\frac{1}{2}} \sum_{l=0}^{k} (E_l(t))^{\frac{1}{2}} (1+t)^{-(\frac{3}{2}+\frac{k-l}{2})} \\
& + (E_k(t))^{\frac{\gamma-1}{2\gamma}} (1+t)^{-\frac{1}{\gamma}(\frac{1}{4}+\frac{k+1}{2})} \sum_{l=0}^{k} (E_l(t))^{\frac{1}{2}} (1+t)^{-(\frac{3}{2}+\frac{k-l}{2})} \\
& + |J_2| + |J_4|.
\end{aligned} \tag{4.15}$$

Next, we will calculate $|J_2|$. Applying the mean value theorem of binary functions, we have

$$\begin{aligned}
|J_2| \lesssim & \left\| \nabla^k u^+ \right\|_{L^2_\gamma} (\left\| \nabla^k (g_+ \nabla n^+) \right\|_{L^2_\gamma} + \left\| \nabla^k (\overline{g}_+ \nabla n^-) \right\|_{L^2_\gamma} + \sum_{j=1}^{3} \left\| \nabla^k (u_j^+ \partial_j u^+) \right\|_{L^2_\gamma} \\
& + \sum_{i=1}^{3} \left\| \nabla^k (h_+ \nabla n^+ \nabla u_i^+) \right\|_{L^2_\gamma} + \sum_{i=1}^{3} \left\| \nabla^k (k_+ \nabla n^- \nabla u_i^+) \right\|_{L^2_\gamma}
\end{aligned} \tag{4.16}$$

$$
\begin{aligned}
&+\sum_{i=1}^{3}\left\|\nabla^k(h_+\nabla n^+\partial_i u^+)\right\|_{L^2_\gamma}+\sum_{i=1}^{3}\left\|\nabla^k(k_+\nabla n^-\partial_i u^+)\right\|_{L^2_\gamma}\\
&+\left\|\nabla^k(h_+(\operatorname{div}u^+)\nabla n^+)\right\|_{L^2_\gamma}+\left\|\nabla^k(k_+(\operatorname{div}u^+)\nabla n^-)\right\|_{L^2_\gamma}\\
&+\left\|\nabla^k(l_+\Delta u^+)\right\|_{L^2_\gamma}+\left\|\nabla^k(l_+\nabla\operatorname{div}u^+)\right\|_{L^2_\gamma})\\
&\lesssim (E_k(t))^{\frac{1}{2}}(\sum_{j=0}^{k}\left\|\nabla^l\nabla n^+\right\|_{L^2_\gamma}\left\|\nabla^{k-l}g_+\right\|_{L^\infty}+\sum_{j=0}^{k}\left\|\nabla^l\nabla n^-\right\|_{L^2_\gamma}\left\|\nabla^{k-l}\overline{g}_+\right\|_{L^\infty}\\
&+\sum_{l=0}^{k}\left\|\nabla^l u^+\right\|_{L^2_\gamma}\left\|\nabla^{k-l+1}u^+\right\|_{L^\infty}+\sum_{i=1}^{3}\sum_{l=0}^{k}\left\|\nabla^l\nabla n^+\right\|_{L^2_\gamma}\left\|\nabla^{k-l}(h_+\nabla u_i^+)\right\|_{L^\infty}\\
&+\sum_{i=1}^{3}\sum_{l=0}^{k}\left\|\nabla^l\nabla n^-\right\|_{L^2_\gamma}\left\|\nabla^{k-l}(k_+\nabla u_i^+)\right\|_{L^\infty}+\sum_{i=1}^{3}\sum_{l=0}^{k}\left\|\nabla^l\nabla n^+\right\|_{L^2_\gamma}\left\|\nabla^{k-l}(h_+\partial_i u^+)\right\|_{L^\infty}\\
&+\sum_{i=1}^{3}\sum_{l=0}^{k}\left\|\nabla^l\nabla n^-\right\|_{L^2_\gamma}\left\|\nabla^{k-l}(k_+\partial_i u^+)\right\|_{L^\infty}+\sum_{l=0}^{k}\left\|\nabla^l\nabla n^+\right\|_{L^2_\gamma}\left\|\nabla^{k-l}(h_+(\operatorname{div}u^+))\right\|_{L^\infty}\\
&+\sum_{l=0}^{k}\left\|\nabla^l\nabla n^-\right\|_{L^2_\gamma}\left\|\nabla^{k-l}(k_+\operatorname{div}u^+)\right\|_{L^\infty}+\left\|\nabla^k(\partial_1(l_+)(\xi_1,\xi_2)n^+\Delta u^+)\right\|_{L^2_\gamma}\\
&+\left\|\nabla^k(\partial_2(l_+)(\xi_1,\xi_2)n^-\Delta u^+)\right\|_{L^2_\gamma}+\left\|\nabla^k(\partial_1(l_+)(\xi_1,\xi_2)n^+\nabla\operatorname{div}u^+)\right\|_{L^2_\gamma}\\
&+\left\|\nabla^k(\partial_2(l_+)(\xi_1,\xi_2)n^-\nabla\operatorname{div}u^+)\right\|_{L^2_\gamma})\\
&\lesssim (E_k(t))^{\frac{1}{2}}(\sum_{l=0}^{k}(E_l(t))^{\frac{1}{2}}(\left\|\nabla^{k-l}g_+\right\|_{L^\infty}+\left\|\nabla^{k-l}\overline{g}_+\right\|_{L^\infty}\\
&+\left\|\nabla^{k-l+1}u^+\right\|_{L^\infty}+\sum_{i=1}^{3}\left\|\nabla^{k-l}(h_+\nabla u_i^+)\right\|_{L^\infty}\\
&+\sum_{i=1}^{3}\left\|\nabla^{k-l}(k_+\nabla u_i^+)\right\|_{L^\infty}+\sum_{i=1}^{3}\left\|\nabla^{k-l}(h_+\partial_i u^+)\right\|_{L^\infty}\\
&+\sum_{i=1}^{3}\left\|\nabla^{k-l}(k_+\partial_i u^+)\right\|_{L^\infty}+\left\|\nabla^{k-l}(h_+(\operatorname{div}u^+)\right\|_{L^\infty}\\
&+\left\|\nabla^{k-l}(k_+\operatorname{div}u^+)\right\|_{L^\infty}))\\
&+(E_k(t))^{\frac{1}{2}}(\left\|\nabla^k(\partial_1(l_+)(\xi_1,\xi_2)n^+\Delta u^+)\right\|_{L^2_\gamma}+\left\|\nabla^k(\partial_2(l_+)(\xi_1,\xi_2)n^-\Delta u^+)\right\|_{L^2_\gamma}\\
&+\left\|\nabla^k(\partial_1(l_+)(\xi_1,\xi_2)n^+\nabla\operatorname{div}u^+)\right\|_{L^2_\gamma}+\left\|\nabla^k(\partial_2(l_+)(\xi_1,\xi_2)n^-\nabla\operatorname{div}u^+)\right\|_{L^2_\gamma}),
\end{aligned}
$$

where $\xi_1=\xi_1(n^+,n^-),\xi_2=\xi_2(n^+,n^-)$.

Next, we will discuss the above items separately.

$$
\begin{aligned}
\left\|\nabla^{k-l}g_+\right\|_{L^\infty}&\lesssim\left\|\nabla^{k+2}g_+\right\|_{L^2}^{\frac{k-l+0.5}{k+1}}\left\|\nabla^1 g_+\right\|_{L^2}^{\frac{l+0.5}{k+1}}\\
&\lesssim(\left\|\nabla^{k+2}n^+\right\|+\left\|\nabla^{k+2}n^-\right\|)^{\frac{k-l+0.5}{k+1}}(\left\|\nabla^1 n^+\right\|+\left\|\nabla^1 n^-\right\|)^{\frac{l+0.5}{k+1}}\\
&\lesssim(1+t)^{-(\frac{1}{4}+\frac{k+2}{2})\frac{k-l+0.5}{k+1}}(1+t)^{-(\frac{1}{4}+\frac{1}{2})\frac{l+0.5}{k+1}}
\end{aligned}\tag{4.17}
$$

$$\lesssim (1+t)^{-(1+\frac{k-l}{2})}.$$

Similarly, we have

$$\left\|\nabla^{k-l}\overline{g}_+\right\|_{L^\infty} \lesssim (1+t)^{-(1+\frac{k-l}{2})}, \tag{4.18}$$

$$\begin{aligned}\left\|\nabla^{k-l+1}u^+\right\|_{L^\infty} &\lesssim \left\|\nabla^{k+3}u^+\right\|_{L^2}^{\frac{k-l+1.5}{k+2}} \left\|\nabla^1 u^+\right\|_{L^2}^{\frac{l+0.5}{k+2}} \\ &\lesssim (1+t)^{-(\frac{3}{4}+\frac{k+3}{2})\frac{k-l+1.5}{k+2}}(1+t)^{-(\frac{3}{4}+\frac{1}{2})\frac{l+0.5}{k+2}} \\ &\lesssim (1+t)^{-(2+\frac{k-l}{2})}.\end{aligned} \tag{4.19}$$

Applying Lemma 2.1, Lemma 2.2 and Lemma 2.5, we have

$$\begin{aligned}\sum_{i=1}^{3}\left\|\nabla^{k-l}(h_+\nabla u_i^+)\right\|_{L^\infty} &\lesssim \sum_{i=1}^{3}\left\|\nabla^{k+2}(h_+\nabla u_i^+)\right\|_{L^2}^{\frac{k-l+0.5}{k+1}} \left\|\nabla^1(h_+\nabla u_i^+)\right\|_{L^2}^{\frac{l+0.5}{k+1}} \\ &\lesssim \sum_{i=1}^{3}(\left\|\nabla^{k+2}\nabla u_i^+\right\|_{L^2}\|h_+\|_{L^\infty} + \left\|\nabla^{k+2}h_+\right\|_{L^2}\left\|\nabla u_i^+\right\|_{L^\infty})^{\frac{k-l+0.5}{k+1}} \\ &\quad (\left\|\nabla^1\nabla u_i^+\right\|_{L^2}\|h_+\|_{L^\infty} + \left\|\nabla^1 h_+\right\|_{L^2}\left\|\nabla u_i^+)\right\|_{L^\infty})^{\frac{l+0.5}{k+1}} \\ &\lesssim \sum_{i=1}^{3}(\left\|\nabla^{k+2}\nabla u_i^+\right\|_{L^2} + \left\|\nabla^{k+2}h_+\right\|_{L^2})^{\frac{k-l+0.5}{k+1}} \\ &\quad (\left\|\nabla^1\nabla u_i^+\right\|_{L^2} + \left\|\nabla^1 h_+\right\|_{L^2})^{\frac{l+0.5}{k+1}} \\ &\lesssim (\left\|\nabla^{k+2}\nabla u^+\right\|_{L^2} + \left\|\nabla^{k+2}n^+\right\|_{L^2} + \left\|\nabla^{k+2}n^-\right\|_{L^2})^{\frac{k-l+0.5}{k+1}} \\ &\quad (\left\|\nabla^1\nabla u^+\right\|_{L^2} + \left\|\nabla^1 n^+\right\|_{L^2} + \left\|\nabla^1 n^-\right\|_{L^2})^{\frac{l+0.5}{k+1}} \\ &\lesssim (1+t)^{-(\frac{1}{4}+\frac{k+2}{2})\frac{k-l+0.5}{k+1}}(1+t)^{-(\frac{1}{4}+\frac{1}{2})\frac{l+0.5}{k+1}} \\ &\lesssim (1+t)^{-(1+\frac{k-l}{2})}.\end{aligned} \tag{4.20}$$

Similarly, we have

$$\begin{aligned}&\sum_{i=1}^{3}\left\|\nabla^{k-l}(k_+\nabla u_i^+)\right\|_{L^\infty} \lesssim (1+t)^{-(1+\frac{k-l}{2})}, \\ &\sum_{i=1}^{3}\left\|\nabla^{k-l}(h_+\partial_i u^+)\right\|_{L^\infty} \lesssim (1+t)^{-(1+\frac{k-l}{2})}, \\ &\sum_{i=1}^{3}\left\|\nabla^{k-l}(k_+\partial_i u^+)\right\|_{L^\infty} \lesssim (1+t)^{-(1+\frac{k-l}{2})}.\end{aligned} \tag{4.21}$$

For $\left\|\nabla^{k-l}(h_+(\operatorname{div} u^+))\right\|_{L^\infty}$ and $\left\|\nabla^{k-l}(k_+(\operatorname{div} u^+))\right\|_{L^\infty}$, we only need to make simple transformations, and then follow the same process as above to get

$$\left\|\nabla^{k-l}(h_+(\operatorname{div} u^+))\right\|_{L^\infty} \lesssim \left\|\nabla^{k-l}(h_+\nabla^1 u^+)\right\|_{L^\infty} \lesssim (1+t)^{-(1+\frac{k-l}{2})}, \tag{4.22}$$

$$\left\|\nabla^{k-l}(k_+(\operatorname{div} u^+))\right\|_{L^\infty} \lesssim \left\|\nabla^{k-l}(k_+\nabla^1 u^+)\right\|_{L^\infty} \lesssim (1+t)^{-(1+\frac{k-l}{2})}. \tag{4.23}$$

For the last four items, we use the previous techniques to deal with.

$$
\begin{aligned}
&\left\|\nabla^k(\partial_1(l_+)(\xi_1,\xi_2)n^+\Delta u^+)\right\|_{L^2_\gamma}+\left\|\nabla^k(\partial_2(l_+)(\xi_1,\xi_2)n^-\Delta u^+)\right\|_{L^2_\gamma}\\
\lesssim&\sum_{l=0}^{k}\left\|\nabla^l n^+\nabla^{k-l}(\partial_1(l_+)(\xi_1,\xi_2)\Delta u^+)\right\|_{L^2_\gamma}+\sum_{l=0}^{k}\left\|\nabla^l n^-\nabla^{k-l}(\partial_2(l_+)(\xi_1,\xi_2)\Delta u^+)\right\|_{L^2_\gamma}\\
\lesssim&\sum_{l=0}^{k}\left\|\nabla^l n^+\right\|_{L^2_\gamma}\left\|\nabla^{k-l}(\partial_1(l_+)(\xi_1,\xi_2)\Delta u^+)\right\|_{L^\infty}+\sum_{l=0}^{k}\left\|\nabla^l n^-\right\|_{L^2_\gamma}\left\|\nabla^{k-l}(\partial_2(l_+)(\xi_1,\xi_2)\Delta u^+)\right\|_{L^\infty}\\
\lesssim&\sum_{l=0}^{k}(E_l(t))^{\frac12}(\left\|\nabla^{k+2}(\partial_1(l_+)(\xi_1,\xi_2)\Delta u^+)\right\|_{L^2}^{\frac{k-l+0.5}{k+1}}\left\|\nabla^{1}(\partial_1(l_+)(\xi_1,\xi_2)\Delta u^+)\right\|_{L^2}^{\frac{l+0.5}{k+1}}\\
&+\left\|\nabla^{k+2}(\partial_2(l_+)(\xi_1,\xi_2)\Delta u^+)\right\|_{L^2}^{\frac{k-l+0.5}{k+1}}\left\|\nabla^{1}(\partial_2(l_+)(\xi_1,\xi_2)\Delta u^+)\right\|_{L^2}^{\frac{l+0.5}{k+1}})\\
\lesssim&\sum_{l=0}^{k}(E_l(t))^{\frac12}[(\left\|\nabla^{k+2}\partial_1(l_+)(\xi_1,\xi_2)\right\|_{L^2}\left\|\Delta u^+\right\|_{L^\infty}+\left\|\nabla^{k+2}\Delta u^+\right\|_{L^2}\|\partial_1(l_+)(\xi_1,\xi_2)\|_{L^\infty})^{\frac{k-l+0.5}{k+1}}\\
&(\left\|\nabla^{1}\partial_1(l_+)(\xi_1,\xi_2)\right\|_{L^2}\left\|\nabla^1\Delta u^+)\right\|_{L^\infty}+\left\|\nabla^{1}\Delta u^+\right\|_{L^2}\|\partial_1(l_+)(\xi_1,\xi_2)\|_{L^\infty})^{\frac{l+0.5}{k+1}}\\
&+(\left\|\nabla^{k+2}\partial_2(l_+)(\xi_1,\xi_2)\right\|_{L^2}\left\|\Delta u^+\right\|_{L^\infty}+\left\|\nabla^{k+2}\Delta u^+\right\|_{L^2}\|\partial_2(l_+)(\xi_1,\xi_2)\|_{L^\infty})^{\frac{k-l+0.5}{k+1}}\\
&(\left\|\nabla^{1}\partial_2(l_+)(\xi_1,\xi_2)\right\|_{L^2}\left\|\nabla^1\Delta u^+)\right\|_{L^\infty}+\left\|\nabla^{1}\Delta u^+\right\|_{L^2}\|\partial_2(l_+)(\xi_1,\xi_2)\|_{L^\infty})^{\frac{l+0.5}{k+1}}]\\
\lesssim&\sum_{l=0}^{k}(E_l(t))^{\frac12}(\left\|\nabla^{k+2}n^+\right\|_{L^2}+\left\|\nabla^{k+2}n^-\right\|_{L^2}+\left\|\nabla^{k+4}u^+\right\|_{L^2})^{\frac{k-l+0.5}{k+1}}\\
&(\left\|\nabla^{1}n^+\right\|_{L^2}+\left\|\nabla^{1}n^-\right\|_{L^2}+\left\|\nabla^{3}u^+\right\|_{L^2})^{\frac{l+0.5}{k+1}}\\
\lesssim&\sum_{l=0}^{k}(E_l(t))^{\frac12}((1+t)^{-(\frac14+\frac{k+2}{2})}+(1+t)^{-(\frac34+\frac{k+4}{2})})^{\frac{k-l+0.5}{k+1}}((1+t)^{-(\frac14+\frac12)}+(1+t)^{-(\frac34+\frac32)})^{\frac{l+0.5}{k+1}}\\
\lesssim&\sum_{l=0}^{k}(E_l(t))^{\frac12}(1+t)^{-(\frac14+\frac{k+2}{2})\frac{k-l+0.5}{k+1}}(1+t)^{-(\frac14+\frac12)\frac{l+0.5}{k+1}}\\
\lesssim&\sum_{l=0}^{k}(E_l(t))^{\frac12}(1+t)^{-(1+\frac{k-l}{2})},
\end{aligned}
\tag{4.24}
$$

Like (4.24), by using the same operation, we can obtain

$$
\begin{aligned}
&\left\|\nabla^k(\partial_1(l_+)(\xi_1,\xi_2)n^+\nabla\operatorname{div}u^+)\right\|_{L^2_\gamma}+\left\|\nabla^k(\partial_2(l_+)(\xi_1,\xi_2)n^-\nabla\operatorname{div}u^+)\right\|_{L^2_\gamma}\\
\lesssim&\sum_{l=0}^{k}(E_l(t))^{\frac12}(1+t)^{-(1+\frac{k-l}{2})}.
\end{aligned}
\tag{4.25}
$$

Combining the relations (4.18)–(4.25), we have

$$
|J_2|\lesssim(E_k(t))^{\frac12}(\sum_{l=0}^{k}(E_l(t))^{\frac12}((1+t)^{-(1+\frac{k-l}{2})}+(1+t)^{-(2+\frac{k-l}{2})}))
$$

$$
\begin{aligned}
&+(E_k(t))^{\frac{1}{2}}(\sum_{l=0}^{k}(E_l(t))^{\frac{1}{2}}(1+t)^{-(1+\frac{k-l}{2})}) \\
&\lesssim (E_k(t))^{\frac{1}{2}}(\sum_{l=0}^{k}(E_l(t))^{\frac{1}{2}}(1+t)^{-(1+\frac{k-l}{2})}).
\end{aligned} \tag{4.26}
$$

By relying on good symmetry and combining (4.11)–(4.14) and (4.26), for $0 \leq k \leq N$, we finally conclude that

$$
\begin{aligned}
\frac{d}{dt}E_k(t) &\lesssim (E_k(t))^{\frac{2\gamma-1}{2\gamma}}(1+t)^{-\frac{1}{\gamma}(\frac{3}{4}+\frac{k}{2})} + (E_k(t))^{\frac{\gamma-1}{\gamma}}(1+t)^{-\frac{2}{\gamma}(\frac{3}{4}+\frac{k}{2})} + (E_k(t))^{\frac{1}{2}} \\
&+ (E_k(t))^{\frac{1}{2}}\sum_{l=0}^{k}(E_l(t))^{\frac{1}{2}}(1+t)^{-(2+\frac{k-l}{2})} \\
&+ (E_k(t))^{\frac{\gamma-1}{2\gamma}}(1+t)^{-\frac{1}{\gamma}(\frac{3}{4}+\frac{k}{2})}\sum_{l=0}^{k}(E_l(t))^{\frac{1}{2}}(1+t)^{-(\frac{3}{2}+\frac{k-l}{2})} \\
&+ (E_k(t))^{\frac{\gamma-1}{2\gamma}}(1+t)^{-\frac{1}{\gamma}(\frac{3}{4}+\frac{k}{2})}\sum_{l=0}^{k}(E_l(t))^{\frac{1}{2}}(1+t)^{-(\frac{3}{2}+\frac{k-l}{2})} \\
&+ (E_k(t))^{\frac{1}{2}}(\sum_{l=0}^{k}(E_l(t))^{\frac{1}{2}}(1+t)^{-(1+\frac{k-l}{2})}) \\
&\lesssim (E_k(t))^{\frac{2\gamma-1}{2\gamma}}(1+t)^{-\frac{1}{\gamma}(\frac{3}{4}+\frac{k}{2})} + (E_k(t))^{\frac{\gamma-1}{\gamma}}(1+t)^{-\frac{2}{\gamma}(\frac{3}{4}+\frac{k}{2})} + (E_k(t))^{\frac{1}{2}} \\
&+ (E_k(t))^{\frac{\gamma-1}{2\gamma}}(1+t)^{-\frac{1}{\gamma}(\frac{3}{4}+\frac{k}{2})}\sum_{l=0}^{k}(E_l(t))^{\frac{1}{2}}(1+t)^{-(\frac{3}{2}+\frac{k-l}{2})} \\
&+ (E_k(t))^{\frac{1}{2}}\sum_{l=0}^{k}(E_l(t))^{\frac{1}{2}}(1+t)^{-(1+\frac{k-l}{2})}.
\end{aligned} \tag{4.27}
$$

Step 2: Proof of Theorem 1.1 with $k = 0$.

When $k = 0$, we have

$$
\begin{aligned}
&\frac{d}{dt}E_0(t) + \widetilde{C}(\left\|\nabla u^+\right\|_{L^2_\gamma}^2 + \left\|\nabla u^-\right\|_{L^2_\gamma}^2 + \left\|\operatorname{div} u^+\right\|_{L^2_\gamma}^2 + \left\|\operatorname{div} u^-\right\|_{L^2_\gamma}^2) \\
\lesssim &(E_0(t))^{1-\frac{1}{2\gamma}}(1+t)^{-\frac{3}{4}\frac{1}{\gamma}} + (E_0(t))^{1-\frac{1}{\gamma}}(1+t)^{-\frac{3}{2}\frac{1}{\gamma}} + (E_k(t))^{\frac{1}{2}} \\
+&(E_0(t))^{1-\frac{1}{2\gamma}}(1+t)^{-\frac{3}{4}\frac{1}{\gamma}}(1+t)^{-\frac{3}{2}} + E_0(t)(1+t)^{-1}.
\end{aligned} \tag{4.28}
$$

Multiplying $C(1+Ct)^{-1}$(C is the coefficient of $E_0(t)(1+t)^{-1}$) on both sides simultaneously, and noticing that t is large enough ($1+t\sim 1+Ct$), we have

$$\begin{aligned}\frac{d}{dt}(1+t)^{-1}E_0(t) &= (1+t)^{-1}\frac{d}{dt}E_0(t)-(1+t)^{-2}E_0(t)\\ &\lesssim (E_0(t))^{1-\frac{1}{2\gamma}}(1+t)^{-\frac{3}{4}\frac{1}{\gamma}-1}+(E_0(t))^{1-\frac{1}{\gamma}}(1+t)^{-\frac{3}{2}\frac{1}{\gamma}-1}\\ &+(E_k(t))^{\frac{1}{2}}(1+t)^{-1}+(E_0(t))^{1-\frac{1}{2\gamma}}(1+t)^{-\frac{3}{4}\frac{1}{\gamma}}(1+t)^{-\frac{5}{2}}\\ &\lesssim ((1+t)^{-1}E_0(t))^{1-\frac{1}{2\gamma}}(1+t)^{-\frac{5}{4}\frac{1}{\gamma}}+((1+t)^{-1}E_0(t))^{1-\frac{1}{\gamma}}(1+t)^{-\frac{7}{4}\frac{1}{\gamma}}\\ &+((1+t)^{-1}E_k(t))^{\frac{1}{2}}(1+t)^{-\frac{1}{2}}\\ &\lesssim ((1+t)^{-1}E_0(t))^{1-\frac{1}{2\gamma}}(1+t)^{-\frac{5}{4}\frac{1}{\gamma}}+((1+t)^{-1}E_0(t))^{1-\frac{1}{\gamma}}(1+t)^{-\frac{7}{4}\frac{1}{\gamma}}\\ &+((1+t)^{-1}E_0(t))^{\frac{1}{2}}(1+t)^{-\frac{1}{2}}+E_k(0)(1+t)^{-\alpha_0}.\end{aligned}\tag{4.29}$$

If $\gamma>\frac{7}{4}$, then we can apply Lemma 2.8 with any $\alpha_0>1, \alpha_1=\frac{5}{4}\frac{1}{\gamma}, \beta_1=1-\frac{1}{2\gamma}, \gamma_1=2\gamma-\frac{5}{2}; \alpha_2=\frac{7}{4}\frac{1}{\gamma}, \beta_2=1-\frac{1}{\gamma}, \gamma_2=\gamma-\frac{7}{4}; \alpha_3=\frac{1}{2}, \beta_3=\frac{1}{2}, \gamma_3=1$, γ_1 is the largest of them. An obvious fact is that when t is large enough, $t\sim(1+t)$. Thus, we have

$$E_0(t)\lesssim t^{2\gamma-\frac{3}{2}},\tag{4.30}$$

which directly implies (1.14) with $k=0$.

Step 3: Proof of Theorem 1.1 with $k=1$.
When $k=1$, we have

$$\begin{aligned}\frac{d}{dt}E_1(t) &\lesssim (E_1(t))^{1-\frac{1}{2\gamma}}(1+t)^{-\frac{5}{4}\frac{1}{\gamma}}+(E_1(t))^{1-\frac{1}{\gamma}}(1+t)^{-\frac{5}{4}\frac{2}{\gamma}}+(E_1(t))^{\frac{1}{2}}\\ &+(E_1(t))^{1-\frac{1}{2\gamma}}(1+t)^{-\frac{5}{4}\frac{1}{\gamma}}(1+t)^{-\frac{3}{2}}+(E_1(t))^{\frac{1}{2}-\frac{1}{2\gamma}}(1+t)^{-\frac{5}{4}\frac{1}{\gamma}}(E_0(t))^{\frac{1}{2}}(1+t)^{-2}\\ &+E_1(t)(1+t)^{-1}+(E_1(t))^{\frac{1}{2}}(E_0(t))^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}\\ &\lesssim (E_1(t))^{1-\frac{1}{2\gamma}}(1+t)^{-\frac{5}{4}\frac{1}{\gamma}}+(E_1(t))^{1-\frac{1}{\gamma}}(1+t)^{-\frac{5}{4}\frac{2}{\gamma}}+(E_1(t))^{\frac{1}{2}}\\ &+(E_1(t))^{\frac{1}{2}-\frac{1}{2\gamma}}(1+t)^{-\frac{5}{4}\frac{1}{\gamma}}(E_0(t))^{\frac{1}{2}}(1+t)^{-2}\\ &+(E_1(t))^{\frac{1}{2}}(E_0(t))^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}+E_1(t)(1+t)^{-1}.\end{aligned}\tag{4.31}$$

Employing similar arguments used in estimating $E_0(t)$, we have

$$\begin{aligned}\frac{d}{dt}(1+t)^{-1}E_1(t) &\lesssim ((1+t)^{-1}E_1(t))^{1-\frac{1}{2\gamma}}(1+t)^{-\frac{7}{4}\frac{1}{\gamma}}+((1+t)^{-1}E_1(t))^{1-\frac{1}{\gamma}}(1+t)^{-\frac{7}{2}\frac{1}{\gamma}}\\ &+((1+t)^{-1}E_1(t))^{\frac{1}{2}}(1+t)^{-\frac{1}{2}}+((1+t)^{-1}E_1(t))^{1-\frac{1}{2\gamma}}(1+t)^{-\frac{7}{4}\frac{1}{\gamma}}(1+t)^{-\frac{3}{2}}\\ &+((1+t)^{-1}E_1(t))^{\frac{1}{2}-\frac{1}{2\gamma}}(1+t)^{-\frac{7}{4}\frac{1}{\gamma}}(E_0(t))^{\frac{1}{2}}(1+t)^{-\frac{5}{2}}\\ &+((1+t)^{-1}E_1(t))^{\frac{1}{2}}(E_0(t))^{\frac{1}{2}}(1+t)^{-2}\\ &\lesssim ((1+t)^{-1}E_1(t))^{1-\frac{1}{2\gamma}}(1+t)^{-\frac{7}{4}\frac{1}{\gamma}}+((1+t)^{-1}E_1(t))^{1-\frac{1}{\gamma}}(1+t)^{-\frac{7}{2}\frac{1}{\gamma}}\\ &+((1+t)^{-1}E_1(t))^{\frac{1}{2}}(1+t)^{-\frac{1}{2}}\\ &+((1+t)^{-1}E_1(t))^{\frac{1}{2}-\frac{1}{2\gamma}}(1+t)^{-\frac{7}{4}\frac{1}{\gamma}}(E_0(t))^{\frac{1}{2}}(1+t)^{-\frac{5}{2}}\end{aligned}\tag{4.32}$$

$$
\begin{aligned}
&+((1+t)^{-1}E_1(t))^{\frac{1}{2}}(E_0(t))^{\frac{1}{2}}(1+t)^{-2}\\
&\lesssim((1+t)^{-1}E_1(t))^{1-\frac{1}{2\gamma}}(1+t)^{-\frac{7}{4}\frac{1}{\gamma}}+((1+t)^{-1}E_1(t))^{1-\frac{1}{\gamma}}(1+t)^{-\frac{7}{2}\frac{1}{\gamma}}\\
&+((1+t)^{-1}E_1(t))^{\frac{1}{2}}(1+t)^{-\frac{1}{2}}+((1+t)^{-1}E_1(t))^{\frac{1}{2}-\frac{1}{2\gamma}}(1+t)^{-\frac{7}{4}\frac{1}{\gamma}}(1+t)^{\gamma-\frac{13}{4}}\\
&+((1+t)^{-1}E_1(t))^{\frac{1}{2}}(1+t)^{\gamma-\frac{11}{4}}\\
&\lesssim((1+t)^{-1}E_1(t))^{1-\frac{1}{2\gamma}}(1+t)^{-\frac{7}{4}\frac{1}{\gamma}}+((1+t)^{-1}E_1(t))^{1-\frac{1}{\gamma}}(1+t)^{-\frac{7}{4}\frac{1}{\gamma}}\\
&+((1+t)^{-1}E_1(t))^{\frac{1}{2}}(1+t)^{-\frac{1}{2}}+((1+t)^{-1}E_1(t))^{\frac{1}{2}-\frac{1}{2\gamma}}(1+t)^{-\frac{7}{4}\frac{1}{\gamma}}\\
&+((1+t)^{-1}E_1(t))^{\frac{1}{2}}(1+t)^{\gamma-\frac{11}{4}}.
\end{aligned}
$$

Set $\alpha_1=\frac{7}{4}\frac{1}{\gamma},\beta_1=1-\frac{1}{2\gamma}$; $\alpha_2=\frac{7}{4}\frac{1}{\gamma},\beta_2=1-\frac{1}{\gamma}$; $\alpha_3=\frac{1}{2}-\frac{1}{2\gamma},\beta_3=\frac{7}{4}\frac{1}{\gamma}$;$\alpha_4=\frac{1}{2},\beta_4=\frac{1}{2}$;$\alpha_5=\gamma-\frac{11}{4},\beta_5=\frac{1}{2}$, and then $\gamma_1=2\gamma-\frac{7}{2}$, $\gamma_2=\gamma-\frac{7}{4}$, $\gamma_3=\frac{1-\frac{7}{4}\frac{1}{\gamma}}{\frac{1}{2}+\frac{1}{2\gamma}}$, $\gamma_4=1$, $\gamma_5=\frac{15}{2}-2\gamma$. Applying Lemma 2.9, we have

$$
E_1(t)\lesssim t^{2\gamma-\frac{5}{2}}, \tag{4.33}
$$

which directly implies (1.14) with $k=1$.

Step 4: Proof of Theorem 1.1 with $2\le k\le N$.

When $2\le k\le N$, we have

$$
\begin{aligned}
\frac{d}{dt}E_k(t)&\lesssim(E_k(t))^{1-\frac{1}{2\gamma}}(1+t)^{-\frac{3+2k}{4}\frac{1}{\gamma}}+(E_k(t))^{1-\frac{1}{\gamma}}(1+t)^{-\frac{3+2k}{2}\frac{1}{\gamma}}+(E_k(t))^{\frac{1}{2}}\\
&+(E_k(t))^{1-\frac{1}{2\gamma}}(1+t)^{-\frac{3+2k}{4}\frac{1}{\gamma}-\frac{3}{2}}\\
&+(E_k(t))^{\frac{1}{2}-\frac{1}{2\gamma}}(1+t)^{-\frac{3+2k}{4}\frac{1}{\gamma}}\sum_{l=0}^{k-1}(E_l(t))^{\frac{1}{2}}(1+t)^{-(\frac{3}{2}+\frac{k-l}{2})}\\
&+E_k(t)(1+t)^{-1}+(E_k(t))^{\frac{1}{2}}\sum_{l=0}^{k-l}(1+t)^{-(1+\frac{k-l}{2})}\\
&\lesssim(E_k(t))^{1-\frac{1}{2\gamma}}(1+t)^{-\frac{3+2k}{4}\frac{1}{\gamma}}+(E_k(t))^{1-\frac{1}{\gamma}}(1+t)^{-\frac{3+2k}{2}\frac{1}{\gamma}}+(E_k(t))^{\frac{1}{2}}\\
&+(E_k(t))^{\frac{1}{2}-\frac{1}{2\gamma}}(1+t)^{-\frac{3+2k}{4}\frac{1}{\gamma}}\sum_{l=0}^{k-1}(E_l(t))^{\frac{1}{2}}(1+t)^{-(\frac{3}{2}+\frac{k-l}{2})}\\
&+E_k(t)(1+t)^{-1}+(E_k(t))^{\frac{1}{2}}\sum_{l=0}^{k-l}(E_l(t))^{\frac{1}{2}}(1+t)^{-(1+\frac{k-l}{2})}.
\end{aligned} \tag{4.34}
$$

Similar to the estimates of $E_0(t)$ and $E_1(t)$, we have

$$
\begin{aligned}
\frac{d}{dt}(1+t)^{-1}E_k(t)&\lesssim((1+t)^{-1}E_k(t))^{1-\frac{1}{2\gamma}}(1+t)^{-\frac{5+2k}{4}\frac{1}{\gamma}}\\
&+((1+t)^{-1}E_k(t))^{1-\frac{1}{\gamma}}(1+t)^{-\frac{5+2k}{2}\frac{1}{\gamma}}\\
&+((1+t)^{-1}E_k(t))^{\frac{1}{2}}(1+t)^{-\frac{1}{2}}\\
&+((1+t)^{-1}E_k(t))^{\frac{1}{2}}\sum_{l=0}^{k-l}(E_l(t))^{\frac{1}{2}}(1+t)^{-(\frac{3}{2}+\frac{k-l}{2})}
\end{aligned} \tag{4.35}
$$

$$+ ((1+t)^{-1}E_k(t))^{\frac{1}{2}-\frac{1}{2\gamma}}(1+t)^{-\frac{5+2k}{4}\frac{1}{\gamma}}\sum_{l=0}^{k-1}(E_l(t))^{\frac{1}{2}}(1+t)^{-(2+\frac{k-l}{2})},$$

which together with Lemma 2.9, directly implies (1.14) with $2 \leq k \leq N$.

Therefore, combing the above results in Steps 2–4, we complete the proof of Theorem 1.1.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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