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Research article

An efficient numerical method based on QSC for multi-term variable-order time fractional mobile-immobile diffusion equation with Neumann boundary condition

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Abstract: In this work, we aimed at a kind of multi-term variable-order time fractional mobile-immobile diffusion (TF-MID) equation satisfying the Neumann boundary condition, with fractional orders $\alpha^m(t)$ for $m = 1, 2, \cdots, P$, and introduced a QSC-$L1^+$ scheme by applying the quadratic spline collocation (QSC) method along the spatial direction and using the $L1^+$ formula for the temporal direction. This new scheme was shown to be unconditionally stable and convergent with the accuracy $O(\tau^{\min\{3-\alpha^*-\alpha(0),\ 2\}} + \Delta x^2 + \Delta y^2)$, where Δx, Δy, and τ denoted the space-time mesh sizes. α^* was the maximum of $\alpha^m(t)$ over the time interval, and $\alpha(0)$ was the maximum of $\alpha^m(0)$ in all values of m. The QSC-$L1^+$ scheme, under certain appropriate conditions on $\alpha^m(t)$, is capable of attaining a second order convergence in time, even on a uniform space-time grid. Additionally, we also implemented a fast computation approach which leveraged the exponential-sum-approximation technique to increase the computational efficiency. A numerical example with different fractional orders was attached to confirm the theoretical findings.

Keywords: multi-term variable fractional order mobile-immobile equations; Neumann boundary condition; quadratic spline collocation; $L1^+$ method; numerical analysis; fast computation

1. Introduction

In recent years, the fractional partial differential equations (FPDEs) have been widely used to simulate various phenomena of anomalous diffusion, such as Brownian motion in fractal theory [1], and solute transport in porous media [2]. Samko and Ross [3] proposed an important kind of fractional operators, which are dependent on time and space. Recent studies showed that variable-order fractional derivatives can describe some complicated phenomena more precisely [4, 5], such as the viscoelasticity oscillators [6] and the motion of particles in special circumstances [7]. The multi-term variable-order time fractional mobile-immobile diffusion (TF-MID) equation offers a more precise and realistic depiction of solute diffusion in heterogeneous porous media compared to its single-term counterpart, as referenced in [8, 9].

Different kinds of numerical methods have been proposed for solving PDEs, such as the finite difference method [10–12] and finite element method [13–15]. The quadratic spline collocation (QSC) method stands out as an efficient discretization approach. The basis functions of the quadratic spline space are very smooth, and the smooth conditions at interpolation points are helpful to reduce the number of unknowns, which means the resulting algebraic equation has small scale. As a result, QSC has been widely adopted for solving both integer-order problems [16, 17] and fractional-order differential equations [18, 19].

Many efficient numerical schemes have been proposed to solve time fractional PDEs, including the classical $L1$ scheme [20, 21] and $L2\text{-}1_\sigma$ scheme [22]. She et al. [23] introduced a transformed $L1$ method to solve the multi-term time-fractional initial-boundary value problem. Yuan et al. [24] introduced linearized transformed $L1$ Galerkin finite element method for nonlinear time fractional Schrödinger equations. Chen et al. [25] studied a Grunwald-Letnikov scheme on uniform mesh for reaction sub-diffusion equations with weakly singular solutions and presented a sharp error estimate. Zhou et al. [26] proposed the nonuniform Alikhanov schemes on the nonuniform meshes for nonlinear time fractional parabolic equations. Ren et al. [27] established sharp $H1$ norm error estimates for the $L1$ formula and a fractional Crank-Nicolson scheme on nonuniform meshes for reaction sub-diffusion problems. However, for variable-order models, since the fractional derivative has a variable-dependent kernel, it is more difficult to construct suitable numerical approximations. Zheng and Wang [28] presented an $L1$ scheme for variable-order time-fractional diffusion equations. Du et al. [29] developed an $L2\text{-}1_\sigma$ formula for the variable-order time-fractional wave equation. Zhang et al. [30] considered an implicit numerical method to solve space-time variable-order fractional advection-diffusion equations. Liu et al. [31] proposed the regularity of the solution to a variable-order time-fractional diffusion equation, and constructed a fully discrete numerical scheme. Building on this foundation, we [32] expand the application of the $L1$ scheme by developing a first-order numerical scheme for the variable-order TF-MID with variable coefficients. Ji et al. [33] proposed the $L1^+$ scheme of constant-order FPDEs, which can be derived by performing the classical $L1$ scheme to the integral form of the FPDEs. The $L1^+$ scheme has almost the same computational cost as the classical $L1$ scheme, while improving the convergence order $(2 - \alpha)$ of the classical $L1$ scheme to second order, if the solution is sufficiently well-defined. In this paper, we will investigate the $L1^+$ formula for multi-term variable-order time fractional derivatives.

Furthermore, the fast implementation of numerical methods for time-fractional differential equations is another focus of researchers. Jiang et al. [34] introduced the sum-of-exponentials (SOE) technique, to greatly reduce the computational cost for the evaluation of the constant-order time fractional derivatives. Subsequently, Zhang et al. [35] employed the exponential sum approximation (ESA) method, adeptly handling the singular kernels of variable-order Caputo fractional derivatives. Building on this foundation, they developed a second-order fast evaluation method in [36].

In this paper, we combine the temporal $L1^+$ formula with the spatial QSC method to construct a QSC-$L1^+$ scheme, to solve two-dimensional multi-term variable-order TF-MID equations. The Neumann boundary conditions are incorporated into the framework of the energy method, which leads to a novel technique for the unconditionally stability and convergence of the QSC-$L1^+$ scheme. In addition, we conduct a comprehensive improvement and optimization of the ESA technique to fully meet the $L1^+$ formula. Such a fast evaluation demonstrates exceptional performance by significantly reducing the computational cost and memory requirements, particularly when combined with carefully selected parameters, which further highlights its notable advantages.

The structure of the paper is as follows. In Section 2, we introduce the multi-term variable-order TF-MID equations and propose the QSC-$L1^+$ scheme. Subsequently, we meticulously construct an energy method aimed at rigorously demonstrating the unconditional stability and convergence of the QSC-$L1^+$ scheme in Section 3. In Section 4, we harness the ESA technique to achieve fast computations along the temporal direction. Section 5 presents detailed results from numerical experiments, which not only substantiate our theoretical findings but also highlight the efficiency and practicality of the proposed scheme. In Section 6, we provide a concise summary of the entire work. It is worth noting that, we use C_i in this paper to denote constants which are independent of mesh sizes.

2. Equation and discretization

In this section, we consider a kind of two-dimensional multi-term variable-order TF-MID equation as follows, and the equation can be used to understand and simulate diffusion behavior of solutes in heterogeneous porous media [9],

$$u_t(x,y,t) + \sum_{m=1}^{P} {}_0^C D_t^{\alpha^m(t)} u(x,y,t) = \kappa \mathcal{L} u(x,y,t) + f(x,y,t), \tag{2.1}$$

defined in a rectangular spatial domain $\Omega = (x_L, x_R) \times (y_L, y_R)$, and the temporal interval is denoted by $[0, T]$. At the initial time, the solution satisfies

$$u(x,y,0) = u^0(x,y), \quad (x,y) \in \bar{\Omega} = \Omega \cup \partial\Omega, \tag{2.2}$$

and at the boundary $\partial\Omega$, the solution satisfies the Neumann condition

$$\frac{\partial u(x,y,t)}{\partial n} = \varphi(x,y,t), \quad (x,y,t) \in \partial\Omega \times (0,T]. \tag{2.3}$$

Here, the positive constant κ signifies the diffusion coefficient and f denotes a predefined smooth source function. Additionally, $\mathcal{L}$ stands as a spatial elliptic operator, which stands for:

$$\mathcal{L}u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

$\alpha^m(t)$ for $m = 1, 2, \cdots, P$ are the variable time fractional orders which satisfy

$$0 < \alpha_* \le \alpha^m(t) \le \alpha^* < 1, \; t \in [0,T], \; \lim_{t \to 0^+} (\alpha^m(t) - \alpha^m(0)) \ln t \text{ exists}. \tag{2.4}$$

For the solute transport problem in porous media, a small portion of solute transport follows the Fickian diffusion process, and it can be expressed as the term $u_t(x,y,t)$, while the majority of solute particle transport follows anomalous diffusion, which can be described by the variable-order Caputo fractional operator ${}_0^C D_t^{\alpha(t)} u(x,y,t)$. In order to distinguish anomalous diffusion with different speeds, we use multi-fractional orders ${}_0^C D_t^{\alpha^m(t)} u(x,y,t)$ in the modeling. The Caputo fractional operator ${}_0^C D_t^{\alpha^m(t)} u(x,y,t)$ is rigorously defined as

$$ {}_0^C D_t^{\alpha^m(t)} u(x,y,t) := \int_0^t \omega_{1-\alpha^m(t)}(t-s)\partial_s u(x,y,s)ds, $$

where the weight function $\omega_\beta(t)$ is specified by:

$$ \omega_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}. $$

To characterize the initial singularity of the solution, one can introduce the weighted Banach space $C_\mu^m((0,T];\mathcal{X})$, which incorporates with the norm $\|\cdot\|_{\mathcal{X}}$, where $m \geq 2$ and $0 \leq \mu < 1$, as defined in [37]. The space is defined as:

$$ C_\mu^m((0,T];\mathcal{X}) := \left\{ v \in C^1([0,T];\mathcal{X}) : \|v\|_{C_\mu^m((0,T];\mathcal{X})} < \infty \right\}, $$

with the norm given by:

$$ \|v\|_{C_\mu^m((0,T];\mathcal{X})} := \|v\|_{C^1([0,T];\mathcal{X})} + \sum_{l=2}^{m} \sup_{t\in(0,T]} t^{l-1-\mu} \left\| \frac{\partial^l v}{\partial t^l} \right\|_{\mathcal{X}}. $$

The eigenfunctions $\{\varphi_i\}_{i=1}^\infty$ of the Sturm-Liouville problem, which satisfy the equations

$$ -\mathcal{L}\varphi_i(x,y) = \lambda_i \varphi_i(x,y), \quad (x,y) \in \Omega; \quad \partial_n \varphi_i(x,y) = 0, \quad (x,y) \in \partial\Omega, $$

constitute an orthogonal basis in the $L^2(\Omega)$ space. The eigenvalues $\{\lambda_i\}_{i=1}^\infty$ are strictly positive and non-decreasing, approaching ∞ as i increases. By harnessing the theory of sectorial operators, we define the fractional Sobolev space

$$ \breve{H}^\gamma(\Omega) := \left\{ v \in L^2(\Omega) : |v|_{\breve{H}^\gamma}^2 := \sum_{i=1}^{\infty} \lambda_i^\gamma (v,\varphi_i)^2 < \infty \right\}, $$

equipped with the norm $\|v\|_{\breve{H}^\gamma} := \left(\|v\|_{L^2}^2 + |v|_{\breve{H}^\gamma}^2 \right)^{1/2}$. Moreover, $\breve{H}^\gamma(\Omega)$ is a subset of the fractional Sobolev space $H^\gamma(\Omega)$, distinguished by the characteristics outlined in [38]:

$$ \breve{H}^\gamma(\Omega) = \{ v \in H^\gamma(\Omega) : \mathcal{L}^s v(x,y) = 0 \}, $$

for all $(x,y) \in \partial\Omega$, where $s < \gamma/2$.

Lemma 2.1 ([8, 31]). *If condition* (2.4) *holds, suppose that* $u_0 \in \breve{H}^{\gamma+6}$ *for* $\gamma > 1/2$, $f \in H^1\left([0,T];\breve{H}^{s+4}\right) \cap H^2\left([0,T];\breve{H}^{s+2}\right) \cap H^3\left([0,T];\breve{H}^{s}\right)$ *for* $0 \leq s \leq \gamma$ *and* $\alpha \in C^2[0,T]$. *If* $\alpha^m(0) > 0$, *for* $m = 1,2,\cdots,P$, *we have* $u \in C^3\left((0,T];\breve{H}^\gamma(0,L)\right) \cap C^3_{1-\alpha^m(0)}\left((0,T];\breve{H}^\gamma(0,L)\right)$ *and*

$$ \|u\|_{C^1([0,T];\breve{H}^s(\Omega))} \leq C_0 \left(\left\|u^0\right\|_{\breve{H}^{s+2}(\Omega)} + \|f\|_{H^1(\breve{H}^{s+4})} + \|f\|_{H^2(\breve{H}^{s+2})} + \|f\|_{H^3(\breve{H}^{s})} \right), $$
$$ \|u\|_{C^3_{1-\alpha^m(0)}((0,T];\breve{H}^\gamma(0,L))} \leq C_1 \left(\left\|u^0\right\|_{\breve{H}^{\gamma+6}(0,L)} + \|f\|_{H^1(\breve{H}^{s+4})} + \|f\|_{H^2(\breve{H}^{s+2})} + \|f\|_{H^3(\breve{H}^{s})} \right). $$

Next, we will conduct a comprehensive exploration of the $L1^+$ scheme applied to the temporal domain, accompanied by the analysis of the QSC discretization in the spatial domain.

2.1. Temporal $L1^+$ scheme

Given a positive integer N, we divide the time interval $[0, T]$ uniformly as $0 = t_0 < t_1 < ... < t_N = T$. Therefore, we can conclude that $t_n = n\tau$, where τ is defined as $\tau = \frac{T}{N}$. For a given continuous function $v(t)$, we refer to its piecewise linear interpolation as $\Pi v(t)$. To quantify the discrepancy between the original function and its interpolation, we introduce the interpolation error $\theta_e v(t) = v(t) - \Pi v(t)$, which can be expressed as

$$\theta_e v(t) = \int_{t_{n-1}}^{t} (t-s)\partial_s^2 v(s)\,\mathrm{d}s - \frac{1}{\tau}\int_{t_{n-1}}^{t_n} (t-t_{n-1})(t_n - s)\,\partial_s^2 v(s)\,\mathrm{d}s, \quad t_{n-1} \le t \le t_n,\ 1 \le n \le N.$$

Based on Lemma 2.1, we have $\left\|\frac{\partial^2 v}{\partial t^2}\right\|_{\mathcal{X}} \le C_2 t^{-\alpha^m(0)}$. Therefore, we can verify that

$$|\theta_e v(t)| \le C_3\tau\left(t_n^{1-\alpha^m(0)} - t_{n-1}^{1-\alpha^m(0)}\right). \tag{2.5}$$

We represent v^n as the numerical approximation of $v(t)$ at the specific time instant $t = t_n$, and define $\mathfrak{T} = \{v^n,\ n = 0, 1, \cdots, N\}$ as a finite-dimensional function space that spans over the temporal grid. For convenience, we further define

$$\delta_t v^{n-\frac{1}{2}} = \frac{v^n - v^{n-1}}{\tau} \quad \text{and} \quad v^{n-\frac{1}{2}} = \frac{v^n + v^{n-1}}{2}.$$

Then, averaging the integral of ${}_0^C D_t^{\alpha^m(t)} v(t)$ over $[t_{n-1}, t_n]$, and approximating $\alpha^m(t)$ at the midpoint of this subinterval, we have

$$\frac{1}{\tau}\int_{t_{n-1}}^{t_n} {}_0^C D_t^{\alpha^m(t)} v(t)dt = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} {}_0^C D_t^{\tilde{\alpha}_n^m} v(t)dt + r_{1,n,m}, \tag{2.6}$$

where $\tilde{\alpha}_n^m := \alpha_{n-\frac{1}{2}}^m$, for $n = 1, 2, \cdots, N$ and $m = 1, 2, \cdots, P$. Based on the trapezoidal formula and Lemma 2.1, we can verify that

$$\begin{aligned}|r_{1,n,m}| &\le C_4\Big|\frac{1}{\tau}\int_{t_{n-1}}^{t_n}\int_0^t [\omega_{1-\alpha^m(t)}(t-s) - \omega_{1-\tilde{\alpha}_n^m}(t-s)]dsdt\Big| \\ &\le \frac{C_5}{\tau}\Big|\int_{t_{n-1}}^{t_n} \frac{\Gamma(2-\tilde{\alpha}_n^m)\left(t^{1-\alpha^m(t)} - t^{1-\tilde{\alpha}_n^m}\right)}{\Gamma(2-\tilde{\alpha}_n^m)\Gamma(2-\alpha^m(t))}dt\Big| + \frac{C_6}{\tau}\Big|\int_{t_{n-1}}^{t_n} \frac{t^{1-\tilde{\alpha}_n^m}[\Gamma(2-\tilde{\alpha}_n^m) - \Gamma(2-\alpha^m(t))]}{\Gamma(2-\tilde{\alpha}_n^m)\Gamma(2-\alpha^m(t))}dt\Big|,\end{aligned}$$

where C_4 is the upper bound for $v(t)$, and C_5, C_6 are positive constants related to C_4. According to the boundedness of $\Gamma(x)$ and applying Taylor's expansion for two numerators above, we have

$$|r_{1,n,m}| \le \frac{C_7}{\tau}\Big|\int_{t_{n-1}}^{t_n}\left[C_8(t-t_{n-\frac{1}{2}}) + C_9(t-t_{n-\frac{1}{2}})^2\right]dt\Big| + \frac{C_{10}}{\tau}\Big|\int_{t_{n-1}}^{t_n}\left[C_{11}(t-t_{n-\frac{1}{2}}) + C_{12}(t-t_{n-\frac{1}{2}})^2\right]dt\Big|,$$

where C_i for $i = 7, 8, \cdots, 12$ are the upper bounds of some functions. Since the integration of linear terms are equal to zero, we can obtain $\left|r_{1,n,m}\right| = O(\tau^2)$. Furthermore, the first term on the righthand side of (2.6) can be approximated by the $L1$ formula as

$$\frac{1}{\tau}\int_{t_{n-1}}^{t_n} {}_0^C D_t^{\tilde{\alpha}_n^m} v(t)dt = \frac{1}{\tau}\int_{t_{n-1}}^{t_n}\int_0^t \omega_{1-\tilde{\alpha}_n^m}(t-s)\partial_s \Pi v(s)dsdt + r_{2,n,m}. \tag{2.7}$$

According to the truncation error analysis in [33, 39], we have $r_{2,n,m} = O\Big(\tau^2 t_n^{-\tilde{\alpha}_n^m-\alpha^m(0)}\Big)$. Substituting Eq (2.7) into (2.6), we have the approximation for $n = 1, 2, \cdots, N$ that

$$\frac{1}{\tau}\int_{t_{n-1}}^{t_n} {}_0^C D_t^{\alpha^m(t)} v(t)dt = \sum_{k=1}^{n} a_{n-k+1}^{(n,m)}(v^k - v^{k-1}) + r_{1,n,m} + r_{2,n,m} := \bar{\boldsymbol{\delta}}_t^{\tilde{\alpha}_n^m} v^{n-\frac{1}{2}} + r_{1,n,m} + r_{2,n,m}, \tag{2.8}$$

where

$$a_{n-k+1}^{(n,m)} = \frac{1}{\tau^2}\int_{t_{n-1}}^{t_n}\int_{t_{k-1}}^{\min\{t,t_k\}} \omega_{1-\tilde{\alpha}_n^m}(t-s)dsdt, \quad k = 1, 2, \cdots, n, \ \ m = 1, 2, \cdots, P. \tag{2.9}$$

The discretization (2.8) for the variable-order Caputo fractional derivative ${}_0^C D_t^{\alpha^m(t)} v(t)$, with the coefficients (2.9), is called the $L1^+$ formula.

2.2. Spatial QSC method

For given positive integer M_x, we divide spatial domain $[x_L, x_R]$ uniformly as

$$\triangle_x := \{x_L = x_0 < x_1 < \ldots < x_{M_x} = x_R\},$$

with mesh size $\Delta x = \frac{x_R - x_L}{M_x}$. Then, we define the quadratic spline space with the variable x as

$$\mathcal{V}_x := \left\{ v \in C^1(x_L, x_R),\ v|_{[x_{i-1},x_i]} \in \mathbf{P}_2(\triangle_x), i = 1, 2, \ldots, M_x \right\},$$

where $\mathbf{P}_2(\cdot)$ represents the set of piecewise quadratic polynomials with the variable x. Similarly, we can define the quadratic spline space $\mathcal{V}_y$ for the variable y. Furthermore, we denote by $\triangle := \triangle_x \times \triangle_y$ the mesh partition of the spatial domain, and denote by $\mathcal{V} := \mathcal{V}_x \otimes \mathcal{V}_y$ the space of piecewise biquadratic polynomials.

Next, we consider the basis functions of the space $\mathcal{V}$. We let

$$\phi(x) = \frac{1}{2}\begin{cases} x^2, & 0 \le x \le 1, \\ -2(x-1)^2 + 2(x-1) + 1, & 1 \le x \le 2, \\ (3-x)^2, & 2 \le x \le 3, \\ 0, & \text{elsewhere}, \end{cases}$$

and define the quadratic B-splines

$$\phi_j(x) = \phi\left(\frac{x - x_L}{\Delta x} - j + 2\right), \quad j = 0, 1, \cdots, M_x + 1 \tag{2.10}$$

as the basis function of $\mathcal{V}_x$. Similarly, we get $\{\phi_j(y),\ j = 0, \cdots, M_y + 1\}$ as the basis functions for $\mathcal{V}_y$. Thus, the quadratic spline solution $u_h^n \in \mathcal{V}$ of Eq (2.1) is represented as

$$u_h^n(x, y) = \sum_{i=0}^{M_x+1}\sum_{j=0}^{M_y+1} c_{i,j}^n \phi_i(x)\phi_j(y), \quad n = 1, \cdots, N, \tag{2.11}$$

where $c_{i,j}^n$ are the coefficients to be solved.

In order to solve these unknowns, we choose the centers of the rectangular mesh $\triangle$ as the collocation points, denoted by

$$\xi = \{(\xi_i^x, \xi_j^y),\ i = 1, 2, \ldots, M_x,\ j = 1, 2, \cdots, M_y\},$$

where $\{\xi_i^x = \frac{1}{2}(x_{i-1} + x_i),\ i = 1, 2, \cdots, M_x\}$ and $\{\xi_j^y = \frac{1}{2}(y_{j-1} + y_j),\ j = 1, 2, \cdots, M_y\}$. We denote $\xi_0^x = x_L$, $\xi_{M_x+1}^x = x_R$ and $\xi_0^y = y_L$, $\xi_{M_y+1}^y = y_R$, and denote $\partial\xi = \{(x_L, \xi_j^y),\ j = 0, 1, \cdots, M_y + 1\} \cup \{(x_R, \xi_j^y),\ j = 0, 1, \cdots, M_y + 1\} \cup \{(\xi_i^x, y_L),\ i = 0, 1, \cdots, M_x + 1\} \cup \{(\xi_i^x, y_R),\ i = 0, 1, \cdots, M_x + 1\}$ as the boundary collocation points. Consequently, we select $\bar{\xi} = \xi \cup \partial\xi$ as the set encompassing both interior and boundary collocation points. For simplicity, we further define index sets: $\Lambda = \{(i, j),\ (\xi_i^x, \xi_j^y) \in \xi\}$, $\partial\Lambda = \{(i, j),\ (\xi_i^x, \xi_j^y) \in \partial\xi\}$, and $\bar{\Lambda} = \Lambda \cup \partial\Lambda$.

Taking the collocation points into expressions (2.11), we get

$$u_h^n\left(\xi_k^x, \xi_l^y\right) := \boldsymbol{\theta}_x\boldsymbol{\theta}_y c_{k,l}^n, \quad \frac{\partial^2}{\partial x^2}u_h^n\left(\xi_k^x, \xi_l^y\right) := \boldsymbol{\eta}_x\boldsymbol{\theta}_y c_{k,l}^n, \quad \frac{\partial^2}{\partial y^2}u_h^n\left(\xi_k^x, \xi_l^y\right) := \boldsymbol{\eta}_y\boldsymbol{\theta}_x c_{k,l}^n,$$

and the Neumann boundary condition

$$\begin{aligned} -\frac{\partial}{\partial x}u_h^n\left(\xi_0^x, \xi_l^y\right) &:= -\boldsymbol{\varsigma}_x\boldsymbol{\theta}_y c_{0,l}^n, \quad \frac{\partial}{\partial x}u_h^n\left(\xi_{M_x+1}^x, \xi_l^y\right) := \boldsymbol{\varsigma}_x\boldsymbol{\theta}_y c_{M_x+1,l}^n, \\ -\frac{\partial}{\partial y}u_h^n\left(\xi_k^x, \xi_0^y\right) &:= -\boldsymbol{\varsigma}_y\boldsymbol{\theta}_x c_{k,0}^n, \quad \frac{\partial}{\partial y}u_h^n\left(\xi_k^x, \xi_{M_y+1}^y\right) := \boldsymbol{\varsigma}_y\boldsymbol{\theta}_x c_{k,M_y+1}^n, \end{aligned}$$

where operators $\boldsymbol{\theta}_x$, $\boldsymbol{\varsigma}_x$, and $\boldsymbol{\eta}_x$ are defined as

$$\boldsymbol{\theta}_x c_{k,l}^n = \frac{1}{8}\begin{cases} 4\left(c_{0,l}^n + c_{1,l}^n\right), & k = 0, \\ c_{k-1,l}^n + 6c_{k,l}^n + c_{k+1,l}^n, & k = 1, 2, \cdots, M_x, \\ 4\left(c_{M_x,l}^n + c_{M_x+1,l}^n\right), & k = M_x + 1, \end{cases} \tag{2.12}$$

$$\boldsymbol{\varsigma}_x c_{k,l}^n = \frac{1}{\Delta x}\begin{cases} c_{1,l}^n - c_{0,l}^n, & k = 0, \\ c_{M_x+1,l}^n - c_{M_x,l}^n, & k = M_x + 1, \end{cases} \tag{2.13}$$

$$\boldsymbol{\eta}_x c_{k,l}^n = \frac{1}{\Delta x^2}\begin{cases} 0, & k = 0, M_x + 1, \\ (c_{k-1,l}^n - 2c_{k,l}^n + c_{k+1,l}^n), & k = 1, 2, \cdots, M_x. \end{cases} \tag{2.14}$$

We further define

$$\boldsymbol{\vartheta}_x c_{k,l}^n = \frac{1}{\Delta x}(c_{k,l}^n - c_{k-1,l}^n), \quad k = 1, 2, \cdots, M_x + 1,$$

which leads to

$$\boldsymbol{\eta}_x c_{k,l}^n = \frac{1}{\Delta x}(\boldsymbol{\vartheta}_x c_{k+1,l}^n - \boldsymbol{\vartheta}_x c_{k,l}^n), \quad k = 1, 2, \cdots, M_x.$$

In addition, the operators $\boldsymbol{\theta}_y$, $\boldsymbol{\eta}_y$, and $\boldsymbol{\vartheta}_y$ are defined along the y direction similarly. We need to notice that $\boldsymbol{\varsigma}_x$ and $\boldsymbol{\varsigma}_y$ are discretizations for the Neumann boundary conditions, and their definitions are compatible with $\boldsymbol{\vartheta}_x$ and $\boldsymbol{\vartheta}_y$, respectively. In this paper, we keep both definitions without confusion. Next, we will delve into the full discretization scheme, which is built upon the spatial and temporal discretizations, to construct a robust numerical method for Eqs (2.1)–(2.3).

2.3. QSC-L1$^+$ scheme

We consider Eq (2.1) on the time subinterval $[t_{n-1}, t_n]$, and take the integral average to obtain

$$\frac{1}{\tau}\int_{t_{n-1}}^{t_n} u_t(x,y,t)dt + \frac{1}{\tau}\int_{t_{n-1}}^{t_n}\sum_{m=1}^{P} {}_0^C D_t^{\alpha^m(t)} u(x,y,t)dt = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} \kappa\mathcal{L}u(x,y,t)dt + \frac{1}{\tau}\int_{t_{n-1}}^{t_n} f(x,y,t)dt. \quad (2.15)$$

Through calculations, we can validate the first term of Eq (2.15),

$$\frac{1}{\tau}\int_{t_{n-1}}^{t_n} u_t(x,y,t)dt = \frac{u^n(x,y) - u^{n-1}(x,y)}{\tau} = \boldsymbol{\delta}_t u^{n-\frac{1}{2}}(x,y). \quad (2.16)$$

Regarding the first term at the righthand side of Eq (2.15),

$$\frac{1}{\tau}\int_{t_{n-1}}^{t_n} \kappa\mathcal{L}u(x,y,t)dt = \kappa\mathcal{L}u^{n-\frac{1}{2}}(x,y) + r_{3,n},$$

where

$$r_{3,n} = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} \kappa\mathcal{L}u\,(x,y,t)\,dt - \kappa\mathcal{L}u^{n-\frac{1}{2}}(x,y) = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} \kappa\mathcal{L}\theta_e u\,(x,y,t)\,dt,$$

and based on estimations (2.5), it satisfies

$$|r_{3,n}| \le \frac{\kappa}{\tau}\int_{t_{n-1}}^{t_n} |\mathcal{L}\theta_e u\,(x,y,t)\,|dt \le C_{13}\tau\left(t_n^{1-\alpha^m(0)} - t_{n-1}^{1-\alpha^m(0)}\right).$$

Similarly, for the last term of Eq (2.15), we have

$$\frac{1}{\tau}\int_{t_{n-1}}^{t_n} f(x,y,t)dt = f^{n-\frac{1}{2}}(x,y) + r_{4,n}, \quad (2.17)$$

where $r_{4,n}$ satisfies

$$|r_{4,n}| \le \frac{C_{14}}{2\tau}\int_{t_{n-1}}^{t_n} (t-t_n)\,(t-t_{n-1})\,dt = O(\tau^2),$$

with C_{14} as an upper bound for $f_{tt}(x,y,t)$.

Utilizing Eqs (2.16) and (2.17), in conjunction with the discretization method outlined in (2.8), Eq (2.15) can be reformulated as

$$\boldsymbol{\delta}_t u^{n-\frac{1}{2}}(x,y) + \sum_{m=1}^{P} \bar{\boldsymbol{\delta}}_t^{\tilde{\alpha}_n^m} u^{n-\frac{1}{2}}(x,y) = \kappa\mathcal{L}u^{n-\frac{1}{2}}(x,y) + f^{n-\frac{1}{2}}(x,y) + R^n, \quad (2.18)$$

with the truncation errors

$$R^n = \sum_{m=1}^{P}\left(r_{1,n,m} + r_{2,n,m}\right) + r_{3,n} + r_{4,n} = O\Big(\tau^2\sum_{m=1}^{P} t_n^{-\tilde{\alpha}_n^m - \alpha^m(0)} + \tau\big(t_n^{1-\alpha^m(0)} - t_{n-1}^{1-\alpha^m(0)}\big) + \tau^2\Big). \quad (2.19)$$

Subsequently, we will proceed to approximate the solution of Eq (2.1) with the QSC method. Specifically, we substitute $u_h^n(x,y)$ of the form given by (2.11) into Eq (2.18), neglect the truncation errors, and obtain

$$\begin{aligned}&\sum_{i=0}^{M_x+1}\sum_{j=0}^{M_y+1}\boldsymbol{\delta}_t c_{i,j}^{n-\frac{1}{2}}\phi_i(x)\phi_j(y)+\sum_{i=0}^{M_x+1}\sum_{j=0}^{M_y+1}\sum_{m=1}^{P}\bar{\boldsymbol{\delta}}_t^{\tilde{\alpha}_n^m}c_{i,j}^{n-\frac{1}{2}}\phi_i(x)\phi_j(y)\\&=\kappa\sum_{i=0}^{M_x+1}\sum_{j=0}^{M_y+1}c_{i,j}^{n-\frac{1}{2}}\left[\phi_i''(x)\phi_j(y)+\phi_i(x)\phi_j''(y)\right]+f^{n-\frac{1}{2}}(x,y),\quad (x,y)\in\Omega,\ 1\le n\le N.\end{aligned}\tag{2.20}$$

Taking the collocation points (ξ_i^x,ξ_j^y) into Eq (2.20), we get the QSC-$L1^+$ scheme,

$$\boldsymbol{\delta}_t\boldsymbol{\theta}_x\boldsymbol{\theta}_y c_{i,j}^{n-\frac{1}{2}}+\sum_{m=1}^{P}\bar{\boldsymbol{\delta}}_t^{\tilde{\alpha}_n^m}\boldsymbol{\theta}_x\boldsymbol{\theta}_y c_{i,j}^{n}=\kappa(\boldsymbol{\eta}_x\boldsymbol{\theta}_y+\boldsymbol{\eta}_y\boldsymbol{\theta}_x)c_{i,j}^{n-\frac{1}{2}}+f_{i,j}^{n-\frac{1}{2}},\quad (i,j)\in\Lambda,\quad 1\le n\le N.\tag{2.21}$$

For the initial condition (2.2), we have

$$\boldsymbol{\theta}_x\boldsymbol{\theta}_y c_{i,j}^0=u_{i,j}^0,\quad (i,j)\in\bar{\Lambda},\tag{2.22}$$

and on the boundary collocation points, we have

$$-\boldsymbol{\varsigma}_x\boldsymbol{\theta}_y c_{0,j}^{n-\frac{1}{2}}=\varphi_{0,j}^{n-\frac{1}{2}},\quad \boldsymbol{\varsigma}_x\boldsymbol{\theta}_y c_{M_x+1,j}^{n-\frac{1}{2}}=\varphi_{M_x+1,j}^{n-\frac{1}{2}},\quad -\boldsymbol{\varsigma}_y\boldsymbol{\theta}_x c_{i,0}^{n-\frac{1}{2}}=\varphi_{i,0}^{n-\frac{1}{2}},\quad \boldsymbol{\varsigma}_y\boldsymbol{\theta}_x c_{i,M_y+1}^{n-\frac{1}{2}}=\varphi_{i,M_y+1}^{n-\frac{1}{2}}.\tag{2.23}$$

Remark 2.1. *If the equation includes additional convection terms, such as u_x+u_y, the corresponding QSC-$L1^+$ scheme* (2.21) *will also be appended an additional term $(\boldsymbol{\vartheta}_x\boldsymbol{\theta}_y+\boldsymbol{\theta}_x\boldsymbol{\vartheta}_y)c_{i,j}^{n-\frac{1}{2}}$, which leads to a sparse matrix-vector multiplication. Therefore, the QSC-$L1^+$ scheme can be applied for TF-MID equations with convection terms, with additional little computational cost.*

Next, we will delve into a comprehensive analysis of the stability and convergence properties of the proposed numerical scheme.

3. Numerical analysis of the QSC-$L1^+$ method

Before conducting the numerical analysis, it is imperative to establish some fundamental definitions concerning inner products and norms. Specifically, we define $\mathcal{M}_h=\{\boldsymbol{w},\ \boldsymbol{w}=\{w_{i,j},\ (i,j)\in\Lambda\}\}$ as the spatial grid function space and further introduce $\mathring{\mathcal{M}}_h=\{\boldsymbol{w}\in\mathcal{M}_h,\ \boldsymbol{\varsigma}_x w_{i,j}=\boldsymbol{\varsigma}_y w_{i,j}=0$ for $(i,j)\in\partial\Lambda\}$. For two functions $w,v\in\mathring{\mathcal{M}}_h$, we proceed to define the inner product,

$$\langle\boldsymbol{w},\boldsymbol{v}\rangle:=\Delta x\Delta y\sum_{i=1}^{M_x}\sum_{j=1}^{M_y}w_{i,j}v_{i,j},$$

$$\langle\boldsymbol{\vartheta}_x\boldsymbol{w},\boldsymbol{\vartheta}_x\boldsymbol{v}\rangle_x:=\Delta x\Delta y\sum_{i=1}^{M_x+1}\sum_{j=1}^{M_y}\left(\boldsymbol{\vartheta}_x w_{i,j}\right)\left(\boldsymbol{\vartheta}_x v_{i,j}\right),\quad \langle\boldsymbol{\vartheta}_y\boldsymbol{w},\boldsymbol{\vartheta}_y\boldsymbol{v}\rangle_y:=\Delta x\Delta y\sum_{i=1}^{M_x}\sum_{j=1}^{M_y+1}\left(\boldsymbol{\vartheta}_y w_{i,j}\right)\left(\boldsymbol{\vartheta}_y v_{i,j}\right),$$

which induce the associated discrete norms

$$\|\boldsymbol{w}\| := \sqrt{\langle \boldsymbol{w}, \boldsymbol{w}\rangle}, \quad |\boldsymbol{w}|_{1x} := \sqrt{\langle \boldsymbol{\vartheta}_x \boldsymbol{w}, \boldsymbol{\vartheta}_x \boldsymbol{w}\rangle_x}\,, \quad |\boldsymbol{w}|_{1y} := \sqrt{\langle \boldsymbol{\vartheta}_y \boldsymbol{w}, \boldsymbol{\vartheta}_y \boldsymbol{w}\rangle_y}\,.$$

We can reformulate scheme (2.21) as

$$\begin{aligned}&\Big(1+\tau\sum_{m=1}^{P} a_1^{(n,m)}\Big)\Big(\boldsymbol{\theta}_x\boldsymbol{\theta}_y c_{i,j}^n - \boldsymbol{\theta}_x\boldsymbol{\theta}_y c_{i,j}^{n-1}\Big) + \tau\sum_{k=1}^{n-1}\sum_{m=1}^{P} a_{n-k+1}^{(n,m)}\Big(\boldsymbol{\theta}_x\boldsymbol{\theta}_y c_{i,j}^k - \boldsymbol{\theta}_x\boldsymbol{\theta}_y c_{i,j}^{k-1}\Big)\\ &= \tau\kappa(\boldsymbol{\eta}_x\boldsymbol{\theta}_y + \boldsymbol{\eta}_y\boldsymbol{\theta}_x)c_{i,j}^{n-\frac{1}{2}} + \tau f_{i,j}^{n-\frac{1}{2}}.\end{aligned} \tag{3.1}$$

To facilitate clarity and consistency in our notation, we uniformly define the coefficients for Eq (3.1) as follows:

$$b_1^{(n)} = 1+\tau\sum_{m=1}^{P} a_1^{(n,m)}, \quad b_{n-k+1}^{(n)} = \tau\sum_{m=1}^{P} a_{n-k+1}^{(n,m)}, \quad 1\le k\le n-1.$$

With these new coefficients, the QSC-$L1^+$ method (2.21) can be rewritten in a more compact form as

$$\sum_{k=1}^{n} b_{n-k+1}^{(n)}\Big(\boldsymbol{\theta}_x\boldsymbol{\theta}_y c_{i,j}^k - \boldsymbol{\theta}_x\boldsymbol{\theta}_y c_{i,j}^{k-1}\Big) = \tau\kappa\Big(\boldsymbol{\eta}_x\boldsymbol{\theta}_y + \boldsymbol{\eta}_y\boldsymbol{\theta}_x\Big)c_{i,j}^{n-\frac{1}{2}} + \tau f_{i,j}^{n-\frac{1}{2}}, \quad (i,j)\in\Lambda,\ 1\le n\le N. \tag{3.2}$$

We notice that the orders of magnitude of the coefficients can be estimated as $b_1^{(n)} = O(1)$, $b_2^{(n)} = O(\tau^{1-\tilde{\alpha}_n^m})$. Considering the coefficient properties outlined in [33, 39], the newly defined set of coefficients $\{b_{n-k+1}^{(n)}, k = 1, 2, \cdots, n\}$ fulfills the properties stated in the following lemma.

Lemma 3.1. *For sufficiently small values of τ, the coefficients $b_k^{(n)}$ are monotonically decreasing for $k = 1, 2, \cdots, n$, i.e.,*

$$b_1^{(n)} > b_2^{(n)} > \cdots > b_n^{(n)} > 0,$$

where n is a fixed time index.

3.1. Basic preparations

Before the numerical analysis, we need to provide some basic lemmas which are necessary in the stability and convergence analysis. Here, we first introduce some lemmas on the coefficients of the QSC-$L1^+$ method.

Lemma 3.2. *We assume that the fractional order $\alpha^m(t)$ satisfies $(\alpha^m(t))' \le 0$, for $0 \le t \le T$ and $m = 1, 2, \cdots, P$, then the coefficients $b_{n-k}^{(n)}$ in the QSC-$L1^+$ scheme (3.2) fulfill the estimation*

$$b_{n-k}^{(n)} \le (1 + C_{15}\tau)\, b_{n-k}^{(n-1)}, \quad 1\le k\le n-1, \quad 2\le n\le N.$$

Lemma 3.3. *The coefficients of the QSC-$L1^+$ scheme have a positive lower bound, i.e.,*

$$\sum_{m=1}^{P} a_n^{(n,m)} \ge \sum_{m=1}^{P} \frac{T^{-\tilde{\alpha}_n^m}}{\Gamma(1-\tilde{\alpha}_n^m)} \ge C_{16},$$

where $n \ge 2$.

Lemma 3.4. *A special summation of the coefficients of the QSC-L*1^+ *scheme is bounded, i.e.,*

$$\sum_{k=1}^{n-1} b_k^{(k)} \leq C_{17}.$$

The proof of above lemmas can be referred to [40]. Furthermore, we also require the subsequent lemmas concerning the operators that have been previously defined.

Lemma 3.5 ([22])**.** *If the coefficients* $b_k^{(n)}$ *are monotonically decreasing, for* $k = 1, 2, \cdots, n$*, then we have the estimate,*

$$\sum_{k=1}^{n} b_{n-k+1}^{(n)} \left\langle \boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^k - \boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^{k-1}, \boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^n \right\rangle \geq \frac{1}{2}\left[\sum_{k=1}^{n} b_{n-k+1}^{(n)} \left(\left\|\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^k\right\|^2 - \left\|\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^{k-1}\right\|^2\right)\right],$$

where $\boldsymbol{c}^k = \{c_{i,j}^k,\ (i,j) \in \bar{\Lambda}\}$ *are the coefficients in the quadratic spline approximation* u_h^k*.*

Lemma 3.6. *For function* $\boldsymbol{w} \in \mathring{\mathcal{M}}_h$*, we have*

$$\frac{1}{2}\|\boldsymbol{w}\|^2 \leq \langle \boldsymbol{\theta}_x\boldsymbol{w}, \boldsymbol{w}\rangle \leq \|\boldsymbol{w}\|^2, \quad \frac{1}{2}\|\boldsymbol{w}\|^2 \leq \langle \boldsymbol{\theta}_y\boldsymbol{w}, \boldsymbol{w}\rangle \leq \|\boldsymbol{w}\|^2.$$

Proof. We prove the first estimation for simplicity. According to reference [40], the operator $\boldsymbol{\theta}_x$ can be decomposed as $\boldsymbol{\theta}_x = \boldsymbol{\zeta}_x^2$, where $\boldsymbol{\zeta}_x$ is also a spatial operator. Similarly, $\boldsymbol{\theta}_y = \boldsymbol{\zeta}_y^2$. Since $\left\|\boldsymbol{\zeta}_x\boldsymbol{w}\right\|^2 = \langle \boldsymbol{\zeta}_x\boldsymbol{w}, \boldsymbol{\zeta}_x\boldsymbol{w}\rangle = \langle \boldsymbol{\theta}_x\boldsymbol{w}, \boldsymbol{w}\rangle$, we can get from the definition of $\boldsymbol{\theta}_x$ in (2.12) that

$$\langle \boldsymbol{\theta}_x\boldsymbol{w}, \boldsymbol{w}\rangle = \frac{\Delta x\Delta y}{8}\sum_{j=1}^{M_y}\left[(w_{0,j})(w_{1,j}) + 2\sum_{i=1}^{M_x-1}(w_{i,j})(w_{i+1,j}) + 6\sum_{i=1}^{M_x}(w_{i,j})^2 + (w_{M_x+1,j})(w_{M_x,j})\right]. \tag{3.3}$$

According to the homogeneous Neumann boundary conditions, we get $w_{0,j} = w_{1,j}$ and $w_{M_x+1,j} = w_{M_x,j}$. Then, we have

$$\langle \boldsymbol{\theta}_x\boldsymbol{w}, \boldsymbol{w}\rangle = \frac{\Delta x\Delta y}{8}\sum_{j=1}^{M_y}\left[(w_{1,j})^2 + 2\sum_{i=1}^{M_x-1}(w_{i,j})(w_{i+1,j}) + 6\sum_{i=1}^{M_x}(w_{i,j})^2 + (w_{M_xj})^2\right]. \tag{3.4}$$

We utilize the inequality $2ab \leq a^2 + b^2$ in equality (3.4) to get

$$\langle \boldsymbol{\theta}_x\boldsymbol{w}, \boldsymbol{w}\rangle \leq \Delta x\Delta y\sum_{j=1}^{M_y}\sum_{i=1}^{M_x}(w_{i,j})^2 = \|\boldsymbol{w}\|^2.$$

Then, using the inequality $2ab \geq -a^2 - b^2$ in equality (3.4), we can get

$$\langle \boldsymbol{\theta}_x\boldsymbol{w}, \boldsymbol{w}\rangle \geq \frac{\Delta x\Delta y}{8}\sum_{j=1}^{M_y}\left[6(w_{1,j})^2 + 4\sum_{i=2}^{M_x-1}(w_{i,j})^2 + 6(w_{M_x,j})^2\right] \geq \frac{1}{2}\|\boldsymbol{w}\|^2.$$

The second estimation can be obtained in the similar way.

Lemma 3.7. *For any* $\boldsymbol{v}^n \in \mathring{\mathcal{M}}_h$, $n = 1, 2, \cdots, N$, *we have*

$$\left\langle (\boldsymbol{\eta}_x\boldsymbol{\theta}_y + \boldsymbol{\eta}_y\boldsymbol{\theta}_x)\boldsymbol{v}^{n-\frac{1}{2}}, \boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{v}^n \right\rangle \leq -\frac{1}{4}\left(\left|\boldsymbol{\zeta}_x\boldsymbol{\theta}_y\boldsymbol{v}^n\right|_{1x}^2 + \left|\boldsymbol{\zeta}_y\boldsymbol{\theta}_x\boldsymbol{v}^n\right|_{1y}^2 \right) + \frac{1}{4}\left(\left|\boldsymbol{\zeta}_x\boldsymbol{\theta}_y\boldsymbol{v}^{n-1}\right|_{1x}^2 + \left|\boldsymbol{\zeta}_y\boldsymbol{\theta}_x\boldsymbol{v}^{n-1}\right|_{1y}^2 \right).$$

Proof. Recalling the notation $\boldsymbol{v}^{n-\frac{1}{2}} = \frac{1}{2}\left(\boldsymbol{v}^n + \boldsymbol{v}^{n-1}\right)$, we have

$$\begin{aligned} \left\langle (\boldsymbol{\eta}_x\boldsymbol{\theta}_y + \boldsymbol{\eta}_y\boldsymbol{\theta}_x)\boldsymbol{v}^{n-\frac{1}{2}}, \boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{v}^n \right\rangle =& \frac{1}{2}\left\langle \boldsymbol{\eta}_x\boldsymbol{\theta}_y\boldsymbol{v}^{n-1}, \boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{v}^n \right\rangle + \frac{1}{2}\left\langle \boldsymbol{\eta}_x\boldsymbol{\theta}_y\boldsymbol{v}^{n}, \boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{v}^n \right\rangle + \frac{1}{2}\left\langle \boldsymbol{\eta}_y\boldsymbol{\theta}_x\boldsymbol{v}^{n-1}, \boldsymbol{\theta}_y\boldsymbol{\theta}_x\boldsymbol{v}^n \right\rangle \\ &+ \frac{1}{2}\left\langle \boldsymbol{\eta}_y\boldsymbol{\theta}_x\boldsymbol{v}^{n}, \boldsymbol{\theta}_y\boldsymbol{\theta}_x\boldsymbol{v}^n \right\rangle := \frac{1}{2}\sum_{i=1}^{4} P_i. \end{aligned} \tag{3.5}$$

We first present the estimation for P_1,

$$P_1 = \Delta x\Delta y \sum_{i=1}^{M_x}\sum_{j=1}^{M_y} \left(\boldsymbol{\eta}_x\boldsymbol{\theta}_y v_{i,j}^{n-1}\right)\left(\boldsymbol{\theta}_x\boldsymbol{\theta}_y v_{i,j}^{n}\right) = \Delta y \sum_{i=1}^{M_x}\sum_{j=1}^{M_y} \left(\boldsymbol{\vartheta}_x\boldsymbol{\theta}_y v_{i+1,j}^{n-1} - \boldsymbol{\vartheta}_x\boldsymbol{\theta}_y v_{i,j}^{n-1}\right)\left(\boldsymbol{\theta}_x\boldsymbol{\theta}_y v_{i,j}^{n}\right).$$

Applying the homogeneous Neumann boundary conditions $\boldsymbol{\varsigma}_x\boldsymbol{\theta}_y v_{0,j}^{n-1} = 0$ and $\boldsymbol{\varsigma}_x\boldsymbol{\theta}_y v_{M_x+1,j}^{n-1} = 0$, for $j = 1, 2, \cdots, M_y$, we rearrange the terms in P_1 and obtain

$$P_1 = -\Delta x\Delta y \sum_{i=1}^{M_x+1}\sum_{j=1}^{M_y} \left(\boldsymbol{\vartheta}_x\boldsymbol{\theta}_y v_{i,j}^{n-1}\right)\left(\boldsymbol{\vartheta}_x\boldsymbol{\theta}_x\boldsymbol{\theta}_y v_{i,j}^{n}\right) = -\left\langle \boldsymbol{\vartheta}_x\boldsymbol{\theta}_y\boldsymbol{v}^{n-1}, \boldsymbol{\vartheta}_x\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{v}^n \right\rangle.$$

Likewise, we have

$$P_2 = -\left\langle \boldsymbol{\vartheta}_x\boldsymbol{\theta}_y\boldsymbol{v}^{n}, \boldsymbol{\vartheta}_x\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{v}^n \right\rangle = -\left|\boldsymbol{\zeta}_x\boldsymbol{\theta}_y\boldsymbol{v}^n\right|_{1x}^2.$$

Furthermore, we use the Cauchy-Schwarz inequality and $-2ab \leq a^2 + b^2$ to obtain

$$P_1 + P_2 \leq -\left|\boldsymbol{\zeta}_x\boldsymbol{\theta}_y\boldsymbol{v}^n\right|_{1x}^2 + \frac{1}{2}\left(\left|\boldsymbol{\zeta}_x\boldsymbol{\theta}_y\boldsymbol{v}^n\right|_{1x}^2 + \left|\boldsymbol{\zeta}_x\boldsymbol{\theta}_y\boldsymbol{v}^{n-1}\right|_{1x}^2 \right) = -\frac{1}{2}\left(\left|\boldsymbol{\zeta}_x\boldsymbol{\theta}_y\boldsymbol{v}^n\right|_{1x}^2 - \left|\boldsymbol{\zeta}_x\boldsymbol{\theta}_y\boldsymbol{v}^{n-1}\right|_{1x}^2 \right).$$

The terms P_3 and P_4 have similar results. Thus, we can complete the proof.

Lemma 3.8 ([41]). *We suppose* v^n, $w^n \in \mathfrak{T}$ *satisfy the inequality* $v^n \leq (1 + \tau C_{18})v^{n-1} + \tau w^{n-1}$, *with* $n = 1, 2, \ldots, N$, *then we can obtain*

$$v^n \leq e^{C_{18}n\tau}\left[v^0 + \tau\sum_{l=0}^{n-1} w^l \right],$$

where C_{18} *is a positive constant.*

Given the lemmas presented above, we will focus on the stability of the QSC-$L1^+$ scheme (2.21)–(2.23), or its equivalent form (3.2).

3.2. Stability analysis

Theorem 3.1. *We denote by* $\boldsymbol{c}^n = \{c^n_{i,j},\ (i,j) \in \Lambda, 0 \le n \le N\}$ *the coefficients of the approximation* (2.11) *solved by the QSC-L1*$^+$ *method. If the fractional orders* $\alpha^m(t)$ *are monotonically decreasing functions, then the following estimate holds:*

$$\begin{aligned}&\left\|\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^n\right\|^2 + \frac{\tau\kappa}{4}\left(\left|\boldsymbol{\zeta}_x\boldsymbol{\theta}_y\boldsymbol{c}^n\right|^2_{1x} + \left|\boldsymbol{\zeta}_y\boldsymbol{\theta}_x\boldsymbol{c}^n\right|^2_{1y}\right)\\ &\le C_{19}\left[\left\|\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^0\right\|^2 + \frac{\tau\kappa}{2}\left(\left|\boldsymbol{\zeta}_x\boldsymbol{\theta}_y\boldsymbol{c}^0\right|^2_{1x} + \left|\boldsymbol{\zeta}_y\boldsymbol{\theta}_x\boldsymbol{c}^0\right|^2_{1y}\right)\right] + C_{20}\tau\sum_{k=1}^{n}\left\|\boldsymbol{f}^{k-\frac{1}{2}}\right\|^2.\end{aligned}$$

Proof. We multiply Eq (3.2) by $2\Delta x\Delta y\boldsymbol{\theta}_x\boldsymbol{\theta}_y c^n_{i,j}$. Taking summation for the indices i from 1 to M_x and for j from 1 to M_y yield

$$\sum_{k=1}^{n} b^{(n)}_{n-k+1}\left\langle \boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^k - \boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^{k-1}, 2\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^n\right\rangle = \tau\kappa\left\langle(\boldsymbol{\eta}_x\boldsymbol{\theta}_y + \boldsymbol{\eta}_y\boldsymbol{\theta}_x)\boldsymbol{c}^{n-\frac{1}{2}}, 2\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^n\right\rangle + \tau\left\langle \boldsymbol{f}^{n-\frac{1}{2}}, 2\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^n\right\rangle.$$

According to Lemmas 3.5 and 3.7, we achieve the following estimation

$$\begin{aligned}&\sum_{k=1}^{n} b^{(n)}_{n-k+1}\left\|\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^k\right\|^2 + \frac{\tau\kappa}{2}\left(\left|\boldsymbol{\zeta}_x\boldsymbol{\theta}_y\boldsymbol{c}^n\right|^2_{1x} + \left|\boldsymbol{\zeta}_y\boldsymbol{\theta}_x\boldsymbol{c}^n\right|^2_{1y}\right)\\ &\le \sum_{k=1}^{n-1} b^{(n)}_{n-k}\left\|\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^k\right\|^2 + b^{(n)}_n\left\|\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^0\right\|^2 + \frac{\tau\kappa}{2}\left(\left|\boldsymbol{\zeta}_x\boldsymbol{\theta}_y\boldsymbol{c}^{n-1}\right|^2_{1x} + \left|\boldsymbol{\zeta}_y\boldsymbol{\theta}_x\boldsymbol{c}^{n-1}\right|^2_{1y}\right) + 2\tau\left\langle \boldsymbol{f}^{n-\frac{1}{2}}, \boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^n\right\rangle.\end{aligned} \tag{3.6}$$

Then, based on Lemma 3.2, we obtain

$$\sum_{k=1}^{n-1} b^{(n)}_{n-k+1}\left\|\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^k\right\|^2 \le (1 + C_{15}\tau)\sum_{k=1}^{n-1} b^{(n-1)}_{n-k}\left\|\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^k\right\|^2. \tag{3.7}$$

Denote

$$G^0 = \frac{\tau\kappa}{2}\left(\left|\boldsymbol{\zeta}_x\boldsymbol{\theta}_y\boldsymbol{c}^0\right|^2_{1x} + \left|\boldsymbol{\zeta}_y\boldsymbol{\theta}_x\boldsymbol{c}^0\right|^2_{1y}\right),\ G^n = \sum_{k=1}^{n} b^{(n)}_{n-k+1}\left\|\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^k\right\|^2 + \frac{\tau\kappa}{2}\left(\left|\boldsymbol{\zeta}_x\boldsymbol{\theta}_y\boldsymbol{c}^n\right|^2_{1x} + \left|\boldsymbol{\zeta}_y\boldsymbol{\theta}_x\boldsymbol{c}^n\right|^2_{1y}\right),$$

where $1 \le n \le N$. With inequality (3.7), the inequality (3.6) can be simplified as

$$G^n \le (1 + C_{15}\tau)\,G^{n-1} + b^{(n)}_n\left\|\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^0\right\|^2 + 2\tau\left\langle \boldsymbol{f}^{n-\frac{1}{2}}, \boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^n\right\rangle.$$

We utilize Lemma 3.8 to derive that

$$G^n \le e^{C_{15}n\tau}\left[G^0 + \sum_{k=1}^{n-1} b^{(k)}_k\left\|\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^0\right\|^2 + 2\tau\sum_{k=1}^{n}\left\langle \boldsymbol{f}^{k-\frac{1}{2}}, \boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^k\right\rangle\right], \quad 1 \le n \le N. \tag{3.8}$$

According to the definition of $\left\{b^{(n)}_{n-k+1},\ k = 1, 2, \cdots, n\right\}$, we have by Lemma 3.3 that $b^{(n)}_1 = 1 + \tau a^{(n)}_1 > 1 + C_{16}\tau$, and $b^{(n)}_{n-k+1} > C_{16}\tau$, for $k = 1, 2, \cdots, n-1$. Then, G^n has the lower bound

$$G^n \ge (1 + C_{16}\tau)\left\|\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^n\right\|^2 + C_{16}\tau\sum_{k=1}^{n-1}\left\|\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^k\right\|^2 + \frac{\tau\kappa}{2}\left(\left|\boldsymbol{\zeta}_x\boldsymbol{\theta}_y\boldsymbol{c}^n\right|^2_{1x} + \left|\boldsymbol{\zeta}_y\boldsymbol{\theta}_x\boldsymbol{c}^n\right|^2_{1y}\right). \tag{3.9}$$

We combine estimates (3.8) and (3.9) to conclude that for $1 \le n \le N$,

$$\begin{aligned}&(1+C_{16}\tau)\left\|\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^n\right\|^2+C_{16}\tau\sum_{k=1}^{n-1}\left\|\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^k\right\|^2+\frac{\tau\kappa}{2}\left(\left|\boldsymbol{\zeta}_x\boldsymbol{\theta}_y\boldsymbol{c}^n\right|_{1x}^2+\left|\boldsymbol{\zeta}_y\boldsymbol{\theta}_x\boldsymbol{c}^n\right|_{1y}^2\right)\\&\le e^{C_{15}T}\left[\frac{\tau\kappa}{2}\left(\left|\boldsymbol{\zeta}_x\boldsymbol{\theta}_y\boldsymbol{c}^0\right|_{1x}^2+\left|\boldsymbol{\zeta}_y\boldsymbol{\theta}_x\boldsymbol{c}^0\right|_{y}^2\right)+\sum_{k=1}^{n-1}b_k^{(k)}\left\|\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^0\right\|^2\right]+2e^{C_{15}T}\tau\sum_{k=1}^{n}\left\langle\boldsymbol{f}^{k-\frac{1}{2}},\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^k\right\rangle.\end{aligned}\tag{3.10}$$

Next, we will analyze the terms in (3.10) individually. First, based on Lemma 3.6, we obtain the estimates

$$\begin{aligned}\frac{\tau\kappa}{2}\left(\left|\boldsymbol{\zeta}_x\boldsymbol{\theta}_y\boldsymbol{c}^n\right|_{1x}^2+\left|\boldsymbol{\zeta}_y\boldsymbol{\theta}_x\boldsymbol{c}^n\right|_{1y}^2\right)&\ge\frac{\tau\kappa}{4}\left(\left|\boldsymbol{\theta}_y\boldsymbol{c}^n\right|_{1x}^2+\left|\boldsymbol{\theta}_x\boldsymbol{c}^n\right|_{1y}^2\right),\\\frac{\tau\kappa}{2}\left(\left|\boldsymbol{\zeta}_x\boldsymbol{\theta}_y\boldsymbol{c}^0\right|_{1x}^2+\left|\boldsymbol{\zeta}_y\boldsymbol{\theta}_x\boldsymbol{c}^0\right|_{1y}^2\right)&\le\frac{\tau\kappa}{2}\left(\left|\boldsymbol{\theta}_y\boldsymbol{c}^0\right|_{1x}^2+\left|\boldsymbol{\theta}_x\boldsymbol{c}^0\right|_{1y}^2\right).\end{aligned}\tag{3.11}$$

By the inequality $2ab \le 2\varepsilon a^2 + (1/2\varepsilon)b^2$,

$$2e^{C_{15}T}\tau\sum_{k=1}^{n}\left\langle\boldsymbol{f}^{k-\frac{1}{2}},\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^k\right\rangle\le C_{16}\tau\sum_{k=1}^{n}\left\|\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^k\right\|^2+\frac{\tau e^{2C_{15}T}}{C_{16}}\sum_{k=1}^{n}\left\|\boldsymbol{f}^{k-\frac{1}{2}}\right\|^2.\tag{3.12}$$

Taking the estimates (3.11) – (3.12) into (3.10), we can derive the conclusion of the theorem.

3.3. Convergence analysis

Recall that the quadratic spline interpolation $\mathcal{I}w(x,y)$ of the function $w(x,y)$ satisfies

$$(\mathcal{I}w)\left(\xi_i^x,\xi_j^y\right)=w\left(\xi_i^x,\xi_j^y\right),\quad (i,j)\in\Lambda,\tag{3.13}$$

and satisfies

$$(\mathcal{I}w)_x\left(x_L,\xi_j^y\right)=w_x\left(x_L,\xi_j^y\right),\quad (\mathcal{I}w)_x\left(x_R,\xi_j^y\right)=w_x\left(x_R,\xi_j^y\right),\quad j=0,1,\cdots,M_y+1,$$

on the left and right boundaries, as well as

$$(\mathcal{I}w)_y\left(\xi_i^x,y_L\right)=w_y\left(\xi_i^x,y_L\right),\quad (\mathcal{I}w)_y\left(\xi_i^x,y_R\right)=w_y\left(\xi_i^x,y_R\right),\quad i=0,1,\cdots,M_x+1,$$

on the other two boundaries. For $w(x,y)\in C^4(\bar\Omega)$, we denote a norm $\|w\|_c=\max_{(i,j)\in\Lambda}|w(\xi_i^x,\xi_j^y)|$, and the interpolation error satisfies [17,42]

$$\|(\mathcal{I}w-w)_{xx}\|_c=O(\Delta x^2),\quad \|(\mathcal{I}w-w)_{yy}\|_c=O(\Delta y^2).\tag{3.14}$$

We denote $\boldsymbol{u}^n=\{u^n(\xi_i^x,\xi_j^y),\ (i,j)\in\bar\Lambda\}$ as the true solution, and $\boldsymbol{u}_h^n=\{u_h^n(\xi_i^x,\xi_j^y),\ (i,j)\in\bar\Lambda\}$ as the numerical solution by the the QSC-$L1^+$ scheme. Then, the convergence of the QSC-$L1^+$ scheme can be derived as follows.

Theorem 3.2. *If the fractional orders $\alpha^m(t)$ satisfy $0<\alpha_*\le\alpha^m(t)\le\alpha^*<1$, then the numerical solution of the QSC-$L1^+$ scheme converges and satisfies*

$$\left\|\boldsymbol{u}^n-\boldsymbol{u}_h^n\right\|\le C_{21}(\tau^{\min\{3-\alpha^*-\alpha(0),\ 2\}}+\Delta x^2+\Delta y^2).$$

Proof. With Eq (2.18), for any fixed n, we can construct the following difference equation on the interpolation $\mathcal{I}u^n(x,y)$:

$$\boldsymbol{\delta}_t\mathcal{I}u^{n-\frac{1}{2}}(x,y)+\sum_{m=1}^{P}\bar{\boldsymbol{\delta}}_t^{\tilde{\alpha}_n^m}\mathcal{I}u^{n-\frac{1}{2}}(x,y)=\kappa\left[\mathcal{I}u_{xx}^{n-\frac{1}{2}}(x,y)+\mathcal{I}u_{yy}^{n-\frac{1}{2}}(x,y)\right]+f^{n-\frac{1}{2}}(x,y)+g^{n-\frac{1}{2}}(x,y), \quad (3.15)$$

where

$$\begin{aligned} g^{n-\frac{1}{2}}(x,y)=&\boldsymbol{\delta}_t(\mathcal{I}u-u)^{n-\frac{1}{2}}(x,y)+\sum_{m=1}^{P}\bar{\boldsymbol{\delta}}_t^{\tilde{\alpha}_n^m}(\mathcal{I}u-u)^{n-\frac{1}{2}}(x,y)\\ &-\kappa\left[(\mathcal{I}u-u)_{xx}^{n-\frac{1}{2}}(x,y)+(\mathcal{I}u-u)_{yy}^{n-\frac{1}{2}}(x,y)\right]+R^n, \end{aligned} \quad (3.16)$$

with the definition of R^n in (2.19). We take the collocation points (ξ_i^x,ξ_j^y) for $(i,j)\in\Lambda$ into (3.15)–(3.16), and obtain

$$\begin{aligned} &\sum_{k=1}^{n}b_{n-k+1}^{(n)}\left[\mathcal{I}u^k(\xi_i^x,\xi_j^y)-\mathcal{I}u^{k-1}(\xi_i^x,\xi_j^y)\right]\\ &=\tau\kappa\left[(\mathcal{I}u)_{xx}^{n-\frac{1}{2}}(\xi_i^x,\xi_j^y)+(\mathcal{I}u)_{yy}^{n-\frac{1}{2}}(\xi_i^x,\xi_j^y)\right]+\tau f^{n-\frac{1}{2}}(\xi_i^x,\xi_j^y)+\tau g^{n-\frac{1}{2}}(\xi_i^x,\xi_j^y). \end{aligned} \quad (3.17)$$

Based on (2.19) and properties (3.14), $g^{n-\frac{1}{2}}$ can be estimated by

$$\|\boldsymbol{g}^{n-\frac{1}{2}}\|_c\le C_{22}\Big(\Delta x^2+\Delta y^2+\tau^2\sum_{m=1}^{P}t_n^{-\tilde{\alpha}_n^m-\alpha^m(0)}+\tau(t_n^{1-\alpha^m(0)}-t_{n-1}^{1-\alpha^m(0)})+\tau^2\Big). \quad (3.18)$$

Since $\mathcal{I}u^n(x,y)\in\mathcal{V}$, we assume $\mathcal{I}u^n(x,y)$ can be written as

$$\mathcal{I}u^n(x,y)=\sum_{i=0}^{M_x+1}\sum_{j=0}^{M_y+1}d_{i,j}^n\phi_i(x)\phi_j(y),$$

where $d_{i,j}^n$ are degrees of freedom (DOFs) of $\mathcal{I}u^n(x,y)$. Then, Eq (3.17) can be rewritten as

$$\sum_{k=1}^{n}b_{n-k+1}^{(n)}\Big(\boldsymbol{\theta}_x\boldsymbol{\theta}_y d_{i,j}^k-\boldsymbol{\theta}_x\boldsymbol{\theta}_y d_{i,j}^{k-1}\Big)=\tau\kappa\Big(\boldsymbol{\eta}_x\boldsymbol{\theta}_y+\boldsymbol{\eta}_y\boldsymbol{\theta}_x\Big)d_{i,j}^{n-\frac{1}{2}}+\tau f_{i,j}^{n-\frac{1}{2}}+\tau g_{i,j}^{n-\frac{1}{2}}, \quad (i,j)\in\Lambda, \quad (3.19)$$

where $g_{i,j}^{n-\frac{1}{2}}=g^{n-\frac{1}{2}}(\xi_i^x,\xi_j^y)$, and the boundary conditions for $(i,j)\in\partial\Lambda$, $1\le n\le N$,

$$-\boldsymbol{\varsigma}_x\boldsymbol{\theta}_y d_{0,j}^{n-\frac{1}{2}}=\varphi_{0,j}^{n-\frac{1}{2}},\quad \boldsymbol{\varsigma}_x\boldsymbol{\theta}_y d_{M_x+1,j}^{n-\frac{1}{2}}=\varphi_{M_x+1,j}^{n-\frac{1}{2}},\quad -\boldsymbol{\varsigma}_y\boldsymbol{\theta}_y d_{i,0}^{n-\frac{1}{2}}=\varphi_{i,0}^{n-\frac{1}{2}},\quad \boldsymbol{\varsigma}_x\boldsymbol{\theta}_y d_{i,M_y+1}^{n-\frac{1}{2}}=\varphi_{i,M_y+1}^{n-\frac{1}{2}}. \quad (3.20)$$

Denote $\boldsymbol{e}^n=\boldsymbol{d}^n-\boldsymbol{c}^n$, and we substitute (3.2) and (2.23) from (3.19) and (3.20) to obtain

$$\sum_{k=1}^{n}b_{n-k+1}^{(n)}\left(\boldsymbol{\theta}_x\boldsymbol{\theta}_y e_{i,j}^k-\boldsymbol{\theta}_x\boldsymbol{\theta}_y e_{i,j}^{k-1}\right)=\tau\kappa\left(\boldsymbol{\eta}_x\boldsymbol{\theta}_y+\boldsymbol{\eta}_y\boldsymbol{\theta}_x\right)e_{i,j}^{n-\frac{1}{2}}+\tau g_{i,j}^{n-\frac{1}{2}}, \quad (i,j)\in\Lambda,\ 1\le n\le N,$$

and for $(i,j)\in\partial\Lambda$, $\quad 1\le n\le N$ such that

$$-\boldsymbol{\varsigma}_x\boldsymbol{\theta}_y e_{0,j}^{n-\frac{1}{2}}=0,\quad \boldsymbol{\varsigma}_x\boldsymbol{\theta}_y e_{M_x+1,j}^{n-\frac{1}{2}}=0,\quad -\boldsymbol{\varsigma}_y\boldsymbol{\theta}_y e_{i,0}^{n-\frac{1}{2}}=0,\quad \boldsymbol{\varsigma}_x\boldsymbol{\theta}_y e_{i,M_y+1}^{n-\frac{1}{2}}=0.$$

Applying estimation (3.10) in the proof of Theorem 3.1, together with $\boldsymbol{e}^0 = 0$, we can get

$$\left\|\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{e}^n\right\|^2 \leq 2e^{C_{15}T}\tau\sum_{k=1}^{n}\left\langle \boldsymbol{g}^{k-\frac{1}{2}}, \boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{c}^k\right\rangle. \tag{3.21}$$

Defining $E^n = \max_{1\leq l\leq n}\left\|\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{e}^l\right\|$, for $n = 1, 2, \cdots, N$, we have from (3.21) that

$$\left\|\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{e}^l\right\|^2 \leq 2\tau e^{C_{15}T}E^l\sum_{k=1}^{l}\left\|\boldsymbol{g}^{k-\frac{1}{2}}\right\| \leq 2\tau e^{C_{15}T}E^n\sum_{k=1}^{n}\left\|\boldsymbol{g}^{k-\frac{1}{2}}\right\|.$$

Taking the maximum on the left-hand side for $l = 1, 2, \cdots, n$, we have

$$(E^n)^2 \leq 2\tau e^{C_{15}T}E^n\sum_{k=1}^{n}\left\|\boldsymbol{g}^{k-\frac{1}{2}}\right\|,$$

which leads to

$$\left\|\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{e}^n\right\| \leq E^n \leq 2\tau e^{C_{15}T}\sum_{k=1}^{n}\left\|\boldsymbol{g}^{k-\frac{1}{2}}\right\| \leq C_{23}\tau\sum_{k=1}^{n}\left\|\boldsymbol{g}^{k-\frac{1}{2}}\right\|_c.$$

Based on the estimation (3.18), we see

$$C_{23}\tau\sum_{k=1}^{n}\left\|\boldsymbol{g}^{k-\frac{1}{2}}\right\|_c \leq C_{24}\left[T\left(\tau^2+\Delta x^2+\Delta y^2\right)+\tau\sum_{m=1}^{P}\sum_{k=1}^{n}\left(\tau^2 t_k^{-\tilde{\alpha}_k^m-\alpha^m(0)}\right)+\tau^2 t_n^{1-\alpha^m(0)}\right]. \tag{3.22}$$

Next, we give further discussions on the values of t_n.

(I) If $t_n \leq 1$, we have $t_n^{1-\alpha^m(0)} \leq t_n^{1-\alpha(0)}$, and $t_k^{-\tilde{\alpha}_k^m-\alpha^m(0)} \leq t_k^{-\alpha^*-\alpha(0)}$ for $k \leq n$, where $\tilde{\alpha}_n = \max_{1\leq m\leq P}\tilde{\alpha}_n^m$, and $\alpha(0) = \max_{1\leq m\leq P}\alpha^m(0)$. Therefore, we have

$$\tau\sum_{k=1}^{n}\left(\tau^2 t_k^{-\tilde{\alpha}_k^m-\alpha^m(0)}\right) \leq \tau^{3-\alpha^*-\alpha(0)}+\tau^2\int_{t_1}^{t_n}t^{-\alpha^*-\alpha(0)}dt = \frac{\tau^2 t_n^{1-\alpha^*-\alpha(0)}}{1-\alpha^*-\alpha(0)}-\frac{\alpha^*+\alpha(0)}{1-\alpha^*-\alpha(0)}\tau^{3-\alpha^*-\alpha(0)}. \tag{3.23}$$

(II) If $t_n > 1$, we have $t_n^{1-\alpha^m(0)} \leq t_n^{1-\bar{\alpha}(0)}$ for $\bar{\alpha}(0) = \min_{1\leq m\leq P}\alpha^m(0)$. Meanwhile, we define an integer $\bar{k} = \lfloor 1/\tau \rfloor$, which leads to $t_k \leq 1$ for $k \leq \bar{k}$, and we can refer to the case (I) above for the summation. For $k \geq \bar{k}+1$, we have $t_k > 1$ and $t_k^{-\tilde{\alpha}_k^m-\alpha^m(0)} \leq t_k^{-\alpha_*-\bar{\alpha}(0)}$, then we have

$$\tau\sum_{k=1}^{n}\left(\tau^2 t_k^{-\tilde{\alpha}_k^m-\alpha^m(0)}\right) = \frac{t_{\bar{k}}^{1-\alpha^*-\alpha(0)}}{1-\alpha^*-\alpha(0)}\tau^2-\frac{\alpha^*+\alpha(0)}{1-\alpha^*-\alpha(0)}\tau^{3-\alpha^*-\alpha(0)}+\frac{t_n^{1-\alpha_*-\bar{\alpha}(0)}-t_{\bar{k}}^{1-\alpha_*-\bar{\alpha}(0)}}{1-\alpha_*-\alpha(0)}\tau^2. \tag{3.24}$$

Thus, we can substitute (3.23) and (3.24) into (3.22) to obtain

$$\left\|\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{e}^n\right\| \leq C_{21}\left(\tau^{\min\{3-\alpha^*-\alpha(0),\,2\}}+\Delta x^2+\Delta y^2\right). \tag{3.25}$$

Since $(\mathcal{I}u-u)^n\left(\xi_i^x, \xi_j^y\right) = 0$, we can get

$$\left\|\boldsymbol{u}^n-\boldsymbol{u}_h^n\right\| = \|\mathcal{I}\boldsymbol{u}^n-\boldsymbol{u}_h^n\| = \|\boldsymbol{\theta}_x\boldsymbol{\theta}_y\boldsymbol{e}^n\|.$$

With the estimate provided in (3.25), we can obtain the convergence result.

Remark 3.1. *If the fractional orders $\alpha^m(t)$ satisfy $\alpha^* + \alpha(0) < 1$, one can obtain*

$$\left\|\boldsymbol{u}^n - \boldsymbol{u}_h^n\right\| \leq C_{25}\left(\tau^2 + \Delta x^2 + \Delta y^2\right).$$

Remark 3.2. *The convergence of the numerical solution always depends on the well-posedness of the solution. If the fractional orders do not satisfy the restriction* (2.4)*, the well-posedness of the solution in Lemma 2.1 will not hold anymore. Under some weak regularity of the solution, such as*

$$\|u(X,t)\|_2 \leq C, \quad \|u'(X,t)\|_2 + t\|u''(X,t)\|_1 + t^2\|u'''(X,t)\|_1 \leq Ct^{\sigma-1},$$

for a positive parameter σ, then it can be obtained similarly that

$$\left\|\boldsymbol{u}^n - \boldsymbol{u}_h^n\right\| \leq C\left(\tau^{\min\{\sigma,2\}} + \Delta x^2 + \Delta y^2\right).$$

Remark 3.3. *If the solution satisfies $u \in C^2[0,T]$, the QSC-$L1^+$ scheme can achieve second temporal convergence order, which fits well with the example confirmation in Ref. [33].*

4. Acceleration techniques

The Caputo time fractional operator is a nonlocal operator, and its discretization typically involves the numerical approximations at all the historical time points, which can be prohibitively expensive, especially for complicated systems and long-term simulations. To mitigate this computational burden, we utilize the ESA technique introduced in [35] to reduce the computational cost during the implementation of the $L1^+$ formula. The primary task is to efficiently approximate the function $t^{-\tilde{\alpha}_n^m}$ in the integration (2.7) over the subinterval $[\tau, T]$, while the implementation on the subinterval $[0,\tau]$ does not require acceleration.

It is reported in [35] that for a small tolerance $\varepsilon > 0$, one can define

$$h = \frac{2\pi}{\log 3 + \alpha^* \log(\cos 1)^{-1} + \log \varepsilon^{-1}}, \quad \underline{N} = \left\lceil \frac{1}{h}\frac{1}{\alpha_*}\Big(\log\varepsilon + \log\Gamma(1+\alpha^*)\Big)\right\rceil,$$
$$\overline{N} = \left\lfloor \frac{1}{h}\Big(\log\frac{T}{\tau} + \log\log\varepsilon^{-1} + \log\alpha_* + 2^{-1}\Big)\right\rfloor.$$

For $t \in [t_{n-1}, t_n]$, $s \in [0, t_{n-2}]$, such that $t - s \geq \tau$, and the exponential function in the definition of the variable order Caputo time fractional derivative can be approximated by

$$\Big(\frac{t-s}{T}\Big)^{-\tilde{\alpha}_n^m} \approx \sum_{r=\underline{N}+1}^{\overline{N}} \varrho^{(n,r)} e^{\frac{-\chi^{(r)}(t-s)}{T}},$$

where

$$\chi^{(r)} = e^{rh}, \quad \varrho^{(n,r)} = \frac{h e^{\tilde{\alpha}_n^m rh}}{\Gamma(\tilde{\alpha}_n^m)}.$$

Now for any $v \in \mathfrak{T}$ and $1 \leq m \leq P$, the $L1^+$ operator $\bar{\boldsymbol{\delta}}_t^{\tilde{\alpha}_n^m} v(t_{n-\frac{1}{2}})$ with $n \geq 2$ defined in (2.8) can be decomposed as

$$\begin{aligned}\bar{\boldsymbol{\delta}}_t^{\tilde{\alpha}_n^m} v\left(t_{n-\frac{1}{2}}\right) &= \frac{1}{\tau}\int_{t_{n-1}}^{t_n}\int_0^{t_{n-2}} \partial_t \Pi v(s)\, \omega_{1-\tilde{\alpha}_n^m}(t-s)dsdt + \frac{1}{\tau}\int_{t_{n-1}}^{t_n}\int_{t_{n-2}}^{t} \partial_t \Pi v(s)\, \omega_{1-\tilde{\alpha}_n^m}(t-s)dsdt \\ &:= I_{t_0}^{t_{n-2}}(t_{n-\frac{1}{2}}) + \sum_{k=n-1}^{n} a_{n-k+1}^{(n,m)}(v^k - v^{k-1}).\end{aligned} \tag{4.1}$$

The first term in (4.1) represents the historical summation, which can be written as

$$
\begin{aligned}
I_{t_0}^{t_{n-2}}\big(t_{n-\frac{1}{2}}\big) &= \frac{T^{-\tilde{\alpha}_n^m}}{\tau\Gamma(1-\tilde{\alpha}_n^m)}\int_{t_{n-1}}^{t_n}\int_0^{t_{n-2}} \partial_t \Pi v(s)\Big(\frac{t-s}{T}\Big)^{-\tilde{\alpha}_n^m} ds dt \\
&\approx \frac{T^{-\tilde{\alpha}_n^m}}{\tau\Gamma(1-\tilde{\alpha}_n^m)}\int_{t_{n-1}}^{t_n}\int_0^{t_{n-2}} \partial_t \Pi v(s) \sum_{r=\underline{N}+1}^{\overline{N}} \varrho^{(n,r)} e^{-\chi^{(r)}\frac{t-s}{T}} ds dt \\
&:= \frac{T^{-\tilde{\alpha}_n^m}}{\tau\Gamma(1-\tilde{\alpha}_n^m)} \sum_{r=\underline{N}+1}^{\overline{N}} \varrho^{(n,r)} b^{(n,r)} V^{(n,r)},
\end{aligned}
\tag{4.2}
$$

where

$$
b^{(n,r)} = \int_{t_{n-1}}^{t_n} e^{-\chi^{(r)}\frac{t-t_{n-2}}{T}} dt, \quad V^{(n,r)} = \int_0^{t_{n-2}} \partial_t \Pi v(s) e^{-\chi^{(r)}\frac{t_{n-2}-s}{T}} ds, \quad r = \underline{N}+1, \cdots, \overline{N}.
$$

We can see that $V^{(2,r)} = 0$, and $V^{(n,r)}$ can be expressed as obtained recursively by

$$
\begin{aligned}
V^{(n,r)} &= \int_0^{t_{n-3}} \partial_t \Pi v(s) e^{-\chi^{(r)}\frac{t_{n-2}-s}{T}} ds + \int_{t_{n-3}}^{t_{n-2}} \partial_t \Pi v(s) e^{-\chi^{(r)}\frac{t_{n-2}-s}{T}} ds \\
&= e^{-\chi^{(r)}\frac{\tau}{T}} V^{(n-1,r)} + \frac{T}{\chi^{(r)}\tau}\big(1 - e^{-\chi^{(r)}\frac{\tau}{T}}\big)(v^{n-2} - v^{n-3}).
\end{aligned}
$$

This means that $V^{(n,r)}$ can be recursively obtained with the value $V^{(n-1,r)}$ at the previous time instant. Taking expressions (4.2) into (4.1), we can get the fast evaluation method of the $L1^+$ formula for the variable order fractional operator,

$$
\tilde{\delta}_t^{\tilde{\alpha}_n^m} v(t_{n-\frac{1}{2}}) := \sum_{k=n-1}^{n} a_{n-k+1}^{(n,m)}(v^k - v^{k-1}) + \frac{T^{-\tilde{\alpha}_n^m}}{\tau\Gamma(1-\tilde{\alpha}_n^m)} \sum_{r=\underline{N}+1}^{\overline{N}} \varrho^{(n,r)} b^{(n,r)} V^{(n,r)}.
\tag{4.3}
$$

Using the fast evaluation (4.3), we can get an accelerated scheme, named the QSC-F$L1^+$ scheme. Actually, when $n = 1, 2$, we still employ (2.21) to compute Eq (2.1). For $n \geq 3$, we take the following QSC-F$L1^+$ scheme,

$$
\boldsymbol{\delta}_t \boldsymbol{\theta}_x \boldsymbol{\theta}_y c_{i,j}^{n-\frac{1}{2}} + \sum_{m=1}^{P} \tilde{\boldsymbol{\delta}}_t^{\tilde{\alpha}_n^m} \boldsymbol{\theta}_x \boldsymbol{\theta}_y c_{i,j}^n = \kappa(\boldsymbol{\eta}_x \boldsymbol{\theta}_y + \boldsymbol{\eta}_y \boldsymbol{\theta}_x) c_{i,j}^{n-\frac{1}{2}} + f_{i,j}^{n-\frac{1}{2}}.
\tag{4.4}
$$

Compared with the computational cost $O(M_x M_y N^2)$ for the QSC-$L1^+$ scheme (2.21), the cost for the QSC-F$L1^+$ scheme (4.4) is reduced to $O(M_x M_y N \log^2 N)$. Moreover, the memory requirement is also significantly decreased from $O(M_x M_y N)$ to $O(M_x M_y \log^2 N)$.

5. Numerical experiments

Example 5.1. We limit Eq (2.1) within the spatial domain $\Omega = (0,1) \times (0,1)$ and over the temporal interval $[0, T] = [0, 1]$, and consider various combinations of the subsequent variable fractional time orders,

$$
\begin{aligned}
&\alpha^1(t) = 0.3 + 0.5t, \qquad \alpha^2(t) = 0.9 - 0.5t - \frac{1}{4\pi}\left[\sin\left(2\pi(1-t)\right)\right], \\
&\alpha^3(t) = 0.45 - 0.3t, \qquad \alpha^4(t) = \left|3(t-0.5)^2 - 0.2\right| + 0.3.
\end{aligned}
$$

We take $\kappa = 1$, $u^0(x, y) = \cos(\pi x)\cos(\pi y)$, and impose the homogeneous Neumann boundary conditions and a homogeneous source function.

Since the true solution is unknown, the numerical solution on a finer space-time mesh is chosen as the reference solution for comparison, then the error can be measured by

$$Err^2 := \Delta x \Delta y \sum_{i=0}^{M_x+1} \sum_{j=0}^{M_y+1} \left| u_h^n(\xi_i^x, \xi_j^y) - u_h^{2n}(\xi_{2i}^x, \xi_{2j}^y) \right|^2 .$$

Now, we compute the convergence orders of the new proposed scheme. We first choose $M_x = M_y = 2^{12}$, which is small enough, and change the value of N from 2^9 to 2^{12}. The observed errors near the initial time point and the corresponding orders of convergence in time are shown in Table 1. It can be seen that the temporal convergence orders fit the theoretical order $O(\tau^{\min\{3-\alpha^*-\alpha(0),\, 2\}})$ very well. The errors at the final point and corresponding orders are shown in Table 2. It seems that the initial weak singularity almost does not effect the convergence orders at the instances far away. Then, we choose $N = 2^{10}$ such that temporal mesh size is small enough, and change the value of M_x and M_y from 2^6 to 2^9. The observed errors and the spatial orders are shown in Table 3. We can see that the numerical results are consistent with the theoretical results.

Besides, in order to thoroughly investigate the effectiveness and efficiency of the ESA technique, we compare the observed errors and the running time for both the QSC-$L1^+$ scheme and the QSC-F$L1^+$ scheme in Table 4. We can clearly observe that the QSC-F$L1^+$ scheme requires significantly less running time compared to the QSC-$L1^+$ scheme, while achieving comparable levels of errors with the same discretization parameters. This finding exhibits the superiority of the ESA technique in terms of computational efficiency.

Table 1. Errors and temporal convergence orders of QSC-$L1^+$ near the initial time point.

	$(\alpha^1(t), \alpha^3(t))$		$(\alpha^1(t), \alpha^4(t))$		$(\alpha^2(t), \alpha^4(t))$	
N	Err	$order$	Err	$order$	Err	$order$
2^9	4.00e-05	—	5.93e-05	—	5.12e-05	—
2^{10}	7.05e-06	2.50	1.84e-05	1.68	1.98e-05	1.37
2^{11}	1.26e-06	2.48	6.40e-06	1.52	8.31e-06	1.25
2^{12}	2.30e-07	2.45	2.42e-06	1.40	3.65e-06	1.19
		≈ 2.00		≈ 1.30		≈ 1.20

Table 2. Errors and temporal convergence orders of QSC-$L1^+$ at final point.

	$(\alpha^1(t), \alpha^3(t))$		$(\alpha^1(t), \alpha^4(t))$		$(\alpha^2(t), \alpha^4(t))$	
N	Err	$order$	Err	$order$	Err	$order$
2^7	1.11e-06	—	2.47e-06	—	2.80e-06	—
2^8	2.76e-07	1.99	3.69e-07	2.74	5.37e-07	2.38
2^9	6.89e-08	2.00	8.88e-08	2.05	1.31e-07	2.03
2^{10}	1.72e-08	2.00	2.15e-08	2.04	3.24e-08	2.01
		≈ 2.00		≈ 2.00		≈ 2.00

Table 3. Errors and spatial convergence orders of the QSC-$L1^+$ scheme.

	$(\alpha^1(t), \alpha^3(t))$		$(\alpha^1(t), \alpha^4(t))$		$(\alpha^2(t), \alpha^4(t))$	
$M_x = M_y$	Err	$order$	Err	$order$	Err	$order$
2^6	4.58e-06	—	4.06e-06	—	4.87e-06	—
2^7	1.15e-06	1.99	1.02e-06	2.00	1.22e-06	1.99
2^8	2.87e-07	2.00	2.54e-07	2.00	3.05e-07	2.00
2^9	7.16e-08	2.00	6.35e-08	1.99	7.61e-08	2.00
		≈ 2.00		≈ 2.00		≈ 2.00

Table 4. Observing errors and computational cost of the numerical schemes.

			QSC-$L1^+$		QSC-F$L1^+$	
	$M_x = M_y$	N	Err	$Time\ (Sec.)$	Err	$Time\ (Sec.)$
$(\alpha^1(t), \alpha^2(t))$	2^6	2^9	4.80e-06	2.2	5.03e-06	2.6
	2^7	2^{10}	1.20e-06	36.9	1.18e-06	33.3
	2^8	2^{11}	3.00e-07	446.8	2.79e-07	274.1
	2^9	2^{12}	7.51e-08	11217.5	5.22e-08	4178.5
$(\alpha^1(t), \alpha^4(t))$	2^6	2^9	4.44e-06	2.1	4.44e-06	2.8
	2^7	2^{10}	1.34e-06	37.0	1.34e-06	36.1
	2^8	2^{11}	2.88e-07	438.8	2.88e-07	300.0
	2^9	2^{12}	8.29e-08	11079.5	8.29e-08	4533.4
$(\alpha^2(t), \alpha^4(t))$	2^6	2^9	4.29e-06	2.1	4.52e-06	2.6
	2^7	2^{10}	1.07e-06	36.7	1.05e-06	33.1
	2^8	2^{11}	2.69e-07	447.1	2.63e-07	280.6
	2^9	2^{12}	6.72e-08	11173.7	1.74e-08	4068.3

Example 5.2. We extend Eq (2.1) into three-dimensional spatial domain $\Omega = (0,1)\times(0,1)\times(0,1)$, and still on the temporal interval $[0,T] = [0,1]$, with the same variable fractional orders as Example 5.1. We choose the proper source function, such that the true solution is $u(x,y,z,t) = (1+t^3)\cos\pi x\cos\pi y\cos\pi z$, which satisfies the homogeneous Neumann boundary conditions and the initial value $u^0(x,y) = \cos\pi x\cos\pi y\cos\pi z$. For such a problem, there is no singularity at the initial point, and the theoretical convergence order is $O(\tau^2 + \Delta x^2 + \Delta y^2)$. To verify such a result, we first choose $M_x = M_y = M_z = 2^6$, and change the value of N from 2^2 to 2^5. The observed errors at the final point and the corresponding temporal convergence orders are shown in Table 5. Then, we choose $N = 2^8$, and change the value of M_x, M_y, and M_z from 2^2 to 2^5. The observed errors and the spatial convergence orders are shown in Table 6. We can see that the numerical results are consistent with the theoretical results.

Table 5. Errors and temporal convergence orders of QSC-$L1^+$ for Example 5.2.

	$(\alpha^1(t), \alpha^3(t))$		$(\alpha^1(t), \alpha^4(t))$		$(\alpha^2(t), \alpha^4(t))$	
N	Err	$order$	Err	$order$	Err	$order$
2^2	1.62e-02	—	1.61e-02	—	1.62e-02	—
2^3	4.08e-03	1.99	4.07e-03	1.98	4.09e-03	1.99
2^4	1.07e-03	1.93	1.06e-03	1.94	1.07e-03	1.93
2^5	3.15e-04	1.80	3.13e-04	1.76	3.15e-04	1.76
		≈ 2.00		≈ 2.00		≈ 2.00

Table 6. Errors and spatial convergence orders of the QSC-$L1^+$ scheme for Example 5.2.

	$(\alpha^1(t), \alpha^3(t))$		$(\alpha^1(t), \alpha^4(t))$		$(\alpha^2(t), \alpha^4(t))$	
$M_x = M_y = M_z$	Err	$order$	Err	$order$	Err	$order$
2^2	1.51e-02	—	1.51e-02	—	1.51e-02	—
2^3	3.97e-03	1.92	3.95e-03	1.93	3.96e-03	1.93
2^4	1.01e-03	1.97	1.00e-03	1.98	1.01e-03	1.97
2^5	2.55e-04	1.99	2.54e-04	1.98	2.55e-04	1.99
		≈ 2.00		≈ 2.00		≈ 2.00

6. Concluding remark

In this paper, we have proposed a novel QSC-$L1^+$ scheme specifically designed for solving the multi-term variable-order TF-MID equation with Neumann boundary conditions, which is often used to model solute transport in porous media. The new scheme has been proved to be unconditionally stable and convergent with order $O(\tau^{\min\{3-\alpha^*-\alpha(0),\,2\}} + \Delta x^2 + \Delta y^2)$, with some proper assumptions on $\alpha^m(t)$ and without any restrictions on the solution of the original model. The numerical experiments have demonstrated that the results obtained using the proposed QSC-$L1^+$ scheme align well with the theoretical analysis, even when the variable-order function $\alpha^m(t)$ is not monotonically decreasing or even not smooth. Furthermore, to enhance computational efficiency, we incorporate a fast evaluation method based on the ESA technique into the QSC-$L1^+$ scheme, resulting in the QSC-F$L1^+$ scheme. This refined approach significantly reduces the computation cost, making it more practical for large-scale and time-consuming simulations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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