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ERA, 33(2): 613–641.
DOI: 10.3934/era.2025029
Received: 04 December 2024
Revised: 20 January 2025
Accepted: 21 January 2025
Published: 10 February 2025

Research article

Heterogeneous anti-synchronization of stochastic complex dynamical networks involving uncertain dynamics: an approach of the space-time discretizations

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Abstract: The modeling of discrete space-time stochastic heterogeneous complex networks with unknown factors was achieved through the utilization of differencing techniques with respect to the time and space variables of the nodes' states. Via the space-time discrete Lyapunov-Krasovskii functional and the approach of linear matrix inequality, this paper derived the mean-squared asymptotic anti-synchronization of the aforementioned discrete networks. This was achieved by defining an updated law for the hitherto unknown parameters and incorporating boundary values within the feedback controller. The theoretical and experimental findings indicated that the feedback controller at the boundary represented a more effective and cost-effective control technique for the networks. Furthermore, an adaptive rule has been designed to identify the uncertainties that occur in the networks with a high degree of accuracy. In particular, this rule enabled the response networks to identify unknown information in the drive networks with a high level of precision by incorporating an adaptive updating mechanism. Finally, a numerical example was provided to elucidate the viability of the ongoing investigation.

Keywords: heterogeneous networks; complex networks; inverse synchronization; spatial discretization; parameter recognition

1. Introduction

Complex dynamical networks (CDNs) refer to systems composed of interconnected nodes or entities, where the relationships between nodes are not trivial and often exhibit nontrivial patterns or structures, e.g., transport networks and 5G communication networks, social networks and Covid-19 epidemic networks. The study of complex networks has become a significant interdisciplinary field, drawing insights and methodologies from mathematics, computer science, physics, biology, sociology, and other disciplines [1–3]. Complex networks have found applications in diverse fields, including biology (e.g., gene regulatory networks, protein-protein interaction networks), social sciences (e.g., social networks, collaboration networks), transportation (road networks, air traffic networks), and information systems (e.g., the World Wide Web, citation networks); see papers [4–6]. Researchers in this field often use tools from graph theory, statistical physics, and machine learning to analyze and model the intricate structures and behaviors of complex systems (see refs. [7–9]).

A system composed of nonidentical networks may refer to a heterogeneous network. It consists of different types of nodes or entities with diverse characteristics, and the connections between them can vary in nature. Examples include multiplex networks, where nodes can have different types of relationships in different layers, and interdependent networks, where the failure of nodes in one network can affect nodes in another. The synchronization of nonidentical networks is a complex and interdisciplinary topic that spans various fields, including physics, engineering, and network science. In the context of nonidentical networks, achieving synchronization implies that nodes with different dynamics or characteristics become correlated or coordinated in their behavior. A common approach is to consider a drive network influencing the behavior of a response network. The drive network might have more influence or a different role compared to the response network, and achieving synchronization involves aligning the dynamics of the two networks. For the synchronization issues of nonidentical CDNs, the readers could refer to the literatures [10–14].

Anti-synchronization of networks involves achieving a specific type of coordination or correlation between nodes in a network. In synchronization, nodes in a network tend to evolve in a coordinated manner. In contrast, anti-synchronization is a state where the nodes exhibit an antiphase relationship, meaning that their dynamics are such that when one node is in a certain state, the other is in the opposite state; see refs. [15–20]. Adaptive anti-synchronization involves incorporating mechanisms that allow the network to adapt to changes in its internal or external conditions. These adaptive mechanisms can include adjusting parameters, coupling strengths, or introducing control signals based on feedback from the system or external measurements. Adaptive anti-synchronization has potential applications in various fields, including secure communication systems, networked control systems, and complex systems where maintaining a specific antiphase relationship between components is important, see refs. [18–20].

Stochastic network plays a significant role in the evolution of the network. These uncertainties can arise from various sources, including random events, probabilistic interactions, or dynamic processes. Analyzing and modeling stochastic networks is essential for understanding their behavior and making informed decisions in scenarios where uncertainty is inherent; see refs. [21–26]. Uncertain networks refer to network structures where there is uncertainty or variability in the connections between nodes, the strengths of these connections, or other properties of the network. This uncertainty can arise from various sources, including incomplete information, dynamic changes in network topology, or random fluctuations. Understanding and modeling uncertain networks are crucial for various applications, and researchers often employ different approaches to address the challenges associated with uncertainty. Researchers have developed various approaches and methodologies to address these challenges; please see publications [27–32].

Discrete space or discrete time networks, as opposed to continuous space or continuous time networks, have distinct advantages in various applications. For example, discrete time systems are often easier to synchronize and coordinate. Events or updates can be synchronized at specific time intervals, simplifying the management of network-wide activities. Discrete time models are often simpler to develop and analyze. In signal processing applications, discrete time signals are often preferred for digital signal processing. Discrete time allows for the application of powerful digital filtering techniques and efficient algorithms [33]. On the other hand, discretizing space simplifies computations, making it more computationally efficient, especially for systems with a large number of nodes or complex interactions. Nodes in the network are located at specific points in space, simplifying the description of their positions and connections. In certain applications, discretizing space allows for more efficient utilization of resources [34]. To the authors' knowledge, despite numerous reports having focused on the research of discrete-time complex networks [9, 15, 35–37] or continuous-time complex networks with reaction diffusions [5,38–42], the studies of the discrete networks in both time and space are very few; please see ref. [43]. Specifically, almost no paper involves the adaptive anti-synchronization of time-space discrete drive-response nonidentical CDNs.

Based on the above discussion, the primary purpose of this paper is to study the adaptive anti-synchronization problems of space-time discrete heterogeneous stochastic multilayer CDNs with multiple delays. To achieve this, a space-time discrete Lyapunov-Krasovskii functional and the linear matrix inequality (LMI) method are used to design an adaptive updated law for unknown parameter and feedback controller that incorporates boundary information. In particular, the adaptive parameter in the response networks can achieve the accuracy identification to the unknown information in the drive networks. This document presents key contributions that are distinct from existing works. These contributions are briefly summarized.

1) This paper differs from the continuous time networks in [6–8, 10–14, 44–46] or reaction-diffusion networks in [5, 38–42]; a CDNs in time and space is modeled by a discrete way.
2) In contrast to the feedback controller in the networks presented in previous reports [9, 15, 35–37], a feedback controller in the boundary is built to discrete space-time CDNs, which is a more effective and low-cost control technique for the networks.
3) In contrast to previous work on the incomplete synchronization to heterogeneous CDNs, e.g., quasi-synchronization [10–14], the present paper addresses the problem of anti-synchronization to heterogeneous discrete space-time CDNs, which belongs to the category of the complete synchronization.
4) In order to fill the gap that no report had been published for the adaptive control of discrete space-time CDNs until now, this paper considers the uncertainty in the model of CDNs and designs an adaptive updated rule. The adaptive parameter in the response networks can achieve the accuracy identification to the uncertain factor.

This article has the following structure. In Section 2, the drive-response discrete space-time heterogeneous CDNs with multiple delays have been established. The adaptive updated law of the unknown parameter and adaptive anti-synchronization of the drive-response heterogeneous CDNs are discussed in Section 3. In Section 4, the theoretical results are confirmed by a numerical example. A brief summary is given in Section 5.

Notations: Denote $\mathbb{R}_0 = [0, +\infty)$, $\mathbb{Z}_0 = \{0, 1, 2, \ldots\}$; $\mathbb{Z}_+ = \mathbb{Z}_0 \setminus \{0\}$. We use symbol $\otimes$ to denote the Kronecker product. Define $\|A\|_\infty = \max_{1\le i\le m, 1\le j\le n} |a_{ij}|$ for $A = (a_{ij})_{m\times n} \in \mathbb{R}^{m\times n}$. For any sets I, J, I_J stands for set $I \cap J$. I represents some suitable dimensional identical matrix. Also, denote

$$\Delta z_k^{[\cdot]} = z_{k+1}^{[\cdot]} - z_k^{[\cdot]}, \quad \Delta_\hbar z_k^{[\delta]} = \frac{z_k^{[\delta+1]} - z_k^{[\delta]}}{\hbar}, \quad \Delta_\hbar^2 z_k^{[\delta]} = \Delta_\hbar(\Delta_\hbar z_k^{[\delta]}),$$

where $z = z_k^{[\delta]} : \mathbb{Z}^2 \to \mathbb{R}^M$, $(\delta, k) \in \mathbb{Z}^2$, $\hbar > 0$.

2. Digital scheme of space-time CDNs

In the following model, $\mathbf{F}, \sigma \in \mathbb{R}^M$ are the activation and noise functions in sequence; $\boldsymbol{\Gamma}_l \in \mathbb{R}^{M\times M}$ and $G_l = (G_{ij,l})_{m\times m}$ represent the inner and outer linking matrix in sequence $l = 1, 2, \ldots, p$, $i, j = 1, 2, \ldots, m$; $w \in \mathbb{R}$ denotes the Brownian motion on certain probability space. This paper discusses the general CDNs with reaction diffusions, unknown dynamics, and multiple delays, which are constructed by

$$\begin{aligned} \mathrm{d}\mathbf{w}_i(x,t) = \Big[A\frac{\partial^2 \mathbf{w}_i(x,t)}{\partial x^2} + C\mathbf{w}_i(x,t) + \boldsymbol{\gamma}(t)\mathbf{F}(\mathbf{w}_i(x,t)) \\ + \sum_{l=1}^{p}\sum_{j=1}^{m} c_l G_{ij,l}\boldsymbol{\Gamma}_l \mathbf{w}_j(x, t-\tau_l)\Big]\mathrm{d}t + \sum_{l=1}^{p}\hat{\sigma}(\mathbf{w}_i(x, t-\tau_l))\mathrm{d}w(t) + \zeta, \end{aligned} \tag{2.1}$$

where $(x, t) \in (0, L) \times \mathbb{R}_0$ for some $L > 0$, or in a discrete form

$$\begin{aligned} \mathbf{w}_{i,k+1}^{[\delta]} = e^{Ch}\mathbf{w}_{i,k}^{[\delta]} + (e^{Ch} - I)C^{-1}\Big[A\Delta_{\hbar}^2\mathbf{w}_{i,k}^{[\delta-1]} + \boldsymbol{\gamma}_k\mathbf{F}(\mathbf{w}_{i,k}^{[\delta]}) \\ + \sum_{l=1}^{p}\sum_{j=1}^{m} c_l G_{ij,l}\boldsymbol{\Gamma}_l\mathbf{w}_{j,k-\tau_l}^{[\delta]} + \sum_{l=1}^{p}\sigma(\mathbf{w}_{i,k-\tau_l}^{[\delta]})\Delta_h w_k + \zeta\Big], \end{aligned} \tag{2.2}$$

where $(\delta, k) \in (0, \mathcal{L})_{\mathbb{Z}} \times \mathbb{Z}_0$, $\mathcal{L} \geq 2$ is some integer; $\mathbf{w}_i = (\mathbf{w}_{i1}, \mathbf{w}_{i2}, \ldots, \mathbf{w}_{iM})^T \in \mathbb{R}^M$ stands for the state of the ith transportation node; $\boldsymbol{\gamma}_k$ is adaptive; $\mathbf{w}_{i,k}^{[\delta]}$ and w_k are the simplified forms of $\mathbf{w}_i(\delta\hbar, kh)$ and $w(kh)$ in sequence $i = 1, 2, \ldots, m$; $\Delta_h w_k = \frac{w_{k+1} - w_k}{\sqrt{h}}$; $\hbar$ and h denote the space and time steps' length, respectively; $A \in \mathbb{R}^{M\times M}$, C is nonsingular M-order square matrix; $c_l \geq 0$, $\hat{\sigma}(\cdot) = \sqrt{h}\sigma(\cdot)$, $\tau_l \in [1, +\infty)_{\mathbb{Z}}$ is the time delay, $l = 1, 2, \ldots, p$; $\zeta \in \mathbb{R}^M$ is a corrective vector. Furthermore, CDNs (2.2) possesses the following initial and controlling boundary values:

$$\begin{cases} \mathbf{w}_{i,s}^{[\delta]} = \phi_{i,s}^{[\delta]}, \quad \forall(\delta, s) \in [0, \mathcal{L}]_{\mathbb{Z}} \times [-\tau_\infty, 0]_{\mathbb{Z}}, \\ \Delta_{\hbar}\mathbf{w}_{i,k}^{[\delta]}\Big|_{\delta=0} = 0, \quad \Delta_{\hbar}\mathbf{w}_{i,k}^{[\delta]}\Big|_{\delta=\mathcal{L}-1} = \xi_{i,k}, \quad \forall k \in \mathbb{Z}_0, \end{cases}$$

where $\tau_\infty := \max_{\delta=1,2,\ldots,p}\tau_l < \infty$, ξ_i denotes the boundary input, $i = 1, 2, \ldots, m$.

Remark 2.1. *CDNs (2.2) is the discrete form of CDNs (2.1) by using the exponential Euler difference (see [33]) for variable t and the central finite difference (see [47, 48]) for variable x.*

Remark 2.2. *It can be seen from the boundary control described above that, in contrast to the feedback controls (FCs) that are designed within the networks, the boundary control described in this study is only active at the boundary point $\mathcal{L} - 1$. This control method is a more effective and low-cost technique than the FCs described in the paper [48], which occurs for all values of $\delta \in [1, \mathcal{L} - 1]_{\mathbb{Z}}$.*

Let the matrix $G_\cdot$ and the activation and noise functions $\mathbf{F}, \sigma$ meet the following assumptions.

(c_1) $G_l = (G_{ij,l})_{m\times m}$ is irreducible and $G_{ij,l} = G_{ji,l} \geq 0$ for $i \neq j$, $G_{ii,l} = -\sum_{j=1, j\neq i}^{m} G_{ij,l}$, $i, j = 1, 2, \ldots, m$, $l = 1, 2, \ldots, p$.

(c_2) $\mathbf{F}(\cdot)$ and $\sigma(\cdot)$ are odd, and there exist positive constant $\mathbf{F}_0$, and M-order matrices $L_f, L_\sigma \geq 0$ such that

$$\|\mathbf{F}(z)\|_\infty \leq \mathbf{F}_0,$$
$$[\mathbf{F}(z) - \mathbf{F}(y)]^T[\mathbf{F}(z) - \mathbf{F}(y)] \leq (z-y)^T L_f (z-y),$$
$$[\sigma(z) - \sigma(y)]^T[\sigma(z) - \sigma(y)] \leq (z-y)^T L_\sigma (z-y),$$

where $z, y \in \mathbb{R}^M$.

Let us consider the drive CDNs below:

$$\mathbf{z}_{k+1}^{[\delta]} = e^{Ch}\mathbf{z}_k^{[\delta]} + (e^{Ch} - I)C^{-1}\Big[A\Delta_\hbar^2\mathbf{z}_k^{[\delta-1]} + \gamma\mathbf{F}(\mathbf{z}_k^{[\delta]}) + \sum_{l=1}^{p}\sum_{j=1}^{m} c_l G_{ij,l}\mathbf{\Gamma}_l\mathbf{z}_{k-\tau_l}^{[\delta]} + \sum_{l=1}^{p}\sigma(\mathbf{z}_{k-\tau_l}^{[\delta]})\Delta_h w_k - \zeta\Big], \tag{2.3}$$

where $(\delta, k) \in (0, \mathcal{L})_{\mathbb{Z}} \times \mathbb{Z}_0$; γ is an unknown parameter. The initial and boundary values are described by

$$\begin{cases} \mathbf{z}_s^{[\delta]} = \psi_s^{[\delta]}, \quad \forall(\delta, s) \in [0, \mathcal{L}]_{\mathbb{Z}} \times [-\tau_\infty, 0]_{\mathbb{Z}}, \\ \Delta_\hbar \mathbf{z}_k^{[\delta]}\Big|_{\delta=0} = 0 = \Delta_\hbar \mathbf{z}_k^{[\delta]}\Big|_{\delta=\mathcal{L}-1}, \quad \forall k \in \mathbb{Z}_0. \end{cases}$$

CDNs (2.2) can be regarded as the response networks of the drive networks (2.3). Let the anti-synchronous error between the drive-response networks (2.3) and (2.2) be represented by $\mathbf{e}_i = \mathbf{w}_i + \mathbf{z}$ for $i = 1, 2, \ldots, m$. From CDNs (2.3) and (2.2), it follows

$$\begin{aligned} \mathbf{e}_{i,k+1}^{[\delta]} = {} & e^{Ch}\mathbf{e}_{i,k}^{[\delta]} + (e^{Ch} - I)C^{-1}\Big[A\Delta_\hbar^2\mathbf{e}_{i,k}^{[\delta-1]} + \gamma\tilde{\mathbf{F}}(\mathbf{e}_{i,k}^{[\delta]}) \\ & +(\boldsymbol{\gamma}_k - \gamma)\mathbf{F}(\mathbf{w}_{i,k}^{[\delta]}) + \sum_{l=1}^{p}\sum_{j=1}^{m} c_l G_{ij,l}\mathbf{\Gamma}_l\mathbf{e}_{j,k-\tau_l}^{[\delta]} + \sum_{l=1}^{p}\tilde{\sigma}(\mathbf{e}_{i,k-\tau_l}^{[\delta]})\Delta_h w_k\Big], \end{aligned}$$

where $i = 1, 2, \ldots, m$, or in a compact form

$$\begin{aligned} \bar{\mathbf{e}}_{k+1}^{[\delta]} = {} & (e^{Ch})_{\otimes m}\bar{\mathbf{e}}_k^{[\delta]} + (C_*)_{\otimes m}\Big[(A)_{\otimes m}\Delta_\hbar^2\bar{\mathbf{e}}_k^{[\delta-1]} + \gamma\tilde{\mathbf{F}}_\otimes(\bar{\mathbf{e}}_k^{[\delta]}) \\ & +(\boldsymbol{\gamma}_k - \gamma)\mathbf{F}_\otimes(\mathbf{w}_k^{[\delta]}) + \sum_{l=1}^{p} c_l(\mathbf{\Gamma}_l)_{\otimes G_l}\bar{\mathbf{e}}_{k-\tau_l}^{[\delta]} + \sum_{l=1}^{p}\tilde{\sigma}_\otimes^w(\bar{\mathbf{e}}_{k-\tau_l}^{[\delta]})\Big], \end{aligned} \tag{2.4}$$

where $(\delta, k) \in (0, \mathcal{L})_{\mathbb{Z}} \times \mathbb{Z}_0$, $\bar{\mathbf{e}} = (\mathbf{e}_1^T, \ldots, \mathbf{e}_m^T)^T$, $C_* = (e^{Ch} - I)C^{-1}$,

$$\tilde{\mathbf{F}}(\mathbf{e}_i) = \mathbf{F}(\mathbf{w}_i) + \mathbf{F}(\mathbf{z}), \quad \tilde{\sigma}(\mathbf{e}_i) = \sigma(\mathbf{w}_i) + \sigma(\mathbf{z}), \quad i = 1, 2, \ldots, m,$$

$$\tilde{\mathbf{F}}_\otimes(\bar{\mathbf{e}}) = (\tilde{\mathbf{F}}(\mathbf{e}_1), \ldots, \tilde{\mathbf{F}}(\mathbf{e}_m))^T, \quad \mathbf{F}_\otimes(\mathbf{w}) = (\mathbf{F}(\mathbf{w}_1), \ldots, \mathbf{F}(\mathbf{w}_m))^T,$$
$$\tilde{\sigma}_\otimes(\bar{\mathbf{e}}) = (\tilde{\sigma}(\mathbf{e}_1), \ldots, \tilde{\sigma}(\mathbf{e}_m))^T, \quad \tilde{\sigma}_\otimes^w(\bar{\mathbf{e}}) = (\tilde{\sigma}(\mathbf{e}_1), \ldots, \tilde{\sigma}(\mathbf{e}_m))^T\Delta_h w.$$

In the model above, $(A)_{\otimes m} := I_m \otimes A$ and $(A)_{\otimes B} := B \otimes A$ for suitable dimensional matrices A, B and the m-order identical matrix I_m.

According to the initial and boundary values in CDNs (2.2) and (2.3), the error system (2.4) possesses the initial and controlling boundary conditions

$$\begin{cases} \bar{\mathbf{e}}_s^{[\delta]} = \varphi_s^{[\delta]}, \quad \forall(\delta, s) \in [0, \mathcal{L}]_{\mathbb{Z}} \times [-\tau_\infty, 0]_{\mathbb{Z}}, \\ \Delta_\hbar \bar{\mathbf{e}}_k^{[\delta]}\Big|_{\delta=0} = 0, \quad \Delta_\hbar \bar{\mathbf{e}}_k^{[\delta]}\Big|_{\delta=\mathcal{L}-1} = \bar{\xi}_k, \quad \forall k \in \mathbb{Z}_0, \end{cases} \tag{2.5}$$

where $\varphi = (\phi_1 + \psi, \ldots, \phi_m + \psi)^T$, $\bar{\xi} = (\xi_1, \ldots, \xi_m)^T$.

The framework of the drive-response-error CDNs (2.2)–(2.4) is figured as that in Figure 1.

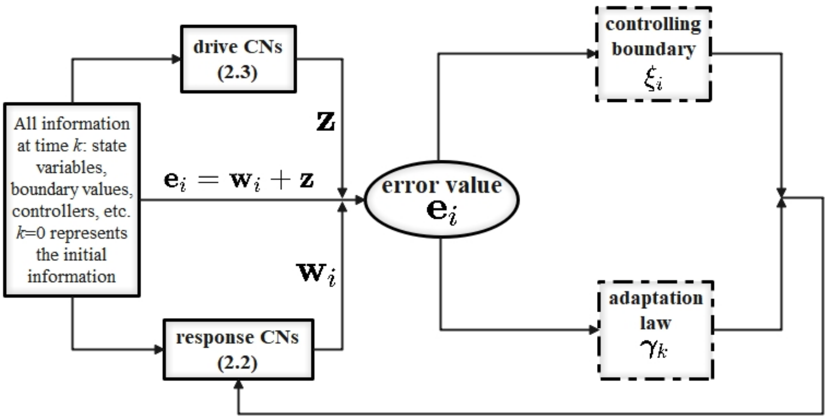

Figure 1. Framework of the drive-response-error boundary controlling CDNs.

Remark 2.3. *In [43], Zhang and Li discussed a class of discrete space-time CDNs and the issue of switching clusters' synchronization had been addressed perfectly in this article. Compared with the work above, this paper possesses some merits below. First, CDNs (2.2) of this paper is involved with the stochastic disturbances. Second, the drive-response networks (2.2) and (2.3) are heterogeneous, but the CDNs in [43] were homogeneous. Finally, it exists the uncertainty in CDNs (2.2), which extensively increases the researching difficulty and the applicable categories of the networks.*

Remark 2.4. *Literatures [44–46] used adaptive methods to identify and estimate unknown parameters of continuous time transportation networks without spatial variables. However, identifying uncertain parameters in spatiotemporal discrete heterogeneous networks presents two challenges that must be overcome. In the first, the introduction of spatial diffusion will complicate the research of this topic. Additionally, based on the applicant's recent research experience in [43], the calculation of the time discrete Lyapunov function is more complex than that of the continuous time Lyapunov function. You can see this point in* Eq (3.6)*, where the term* $\pounds_k$ *exists, but remember that there is no* $\pounds_k$ *in the continuous case.*

Definition 2.1 ([49])**.** *The drive-response networks (2.3) and (2.2) are said to be mean squared asymptotic anti-synchronization (MSAAS) via boundary control* $\bar{\xi}_k$ *in (2.5) if*

$$\lim_{k\to\infty} \sum_{i=1}^{m} \sum_{\delta=0}^{\mathcal{L}} \mathbb{E}\|\mathbf{e}_{i,k}^{[\delta]}\|^2 = \lim_{k\to\infty} \sum_{i=1}^{m} \sum_{\delta=0}^{\mathcal{L}} \mathbb{E}\left\|\mathbf{w}_{i,k}^{[\delta]} + \mathbf{z}_k^{[\delta]}\right\|^2 = 0, \quad \forall k \in \mathbb{Z}_0.$$

Remark 2.5. *In the existing publications, papers [9,15,35–37] investigated single discrete-time CDNs and reports [5, 38–42] considered the continuous-time CDNs with reaction diffusions. The drive-response CDNs (2.3) and (2.2) are distinct with them by discussing both time and space discretizations in the model of CDNs. Although the time-space discrete CDNs has been considered in our work in literature [43], it has some clear distinctions. The biggest difference is that the drive-response CDNs in literature [43] are identical, but drive-response CDNs (2.3) and (2.2) are nonidentical. Besides, this paper considers MSAAS with adaptive updated rule, which is different from the synchronization in literature [43].*

Throughout this article, suppose that

$$\sum_{i=1}^{m}\sum_{\delta=1}^{\mathcal{L}-1}\mathbb{E}\|\phi_{i,s}^{[\delta]}\|^2<\infty,\quad \sum_{\delta=1}^{\mathcal{L}-1}\mathbb{E}\|\psi_{s}^{[\delta]}\|^2<\infty,$$

where $s\in[-\tau_\infty,0]_{\mathbb{Z}}$. By right of initial values (2.5), it has

$$\sum_{\delta=1}^{\mathcal{L}-1}\mathbb{E}\|\bar{\mathbf{e}}_{s}^{[\delta]}\|^2<\infty,\quad \forall s\in[-\tau_\infty,0]_{\mathbb{Z}}. \tag{2.6}$$

In view of Corollary 3.1 of [50], Subsection 2.3, and Inequalities 3.2.1 and 3.2.3 of [51], one has the following lemma.

Lemma 2.1 ([50, 51])**.** *The inequalities below are valid for m-dimensional vector* $\mathbf{z}$ *and m-order positive definitive matrix G.*

(i) $\sum_{\delta=0}^{\mathcal{L}-1}\Delta_\hbar\mathbf{z}^{[\delta]T}G\Delta_\hbar\mathbf{z}^{[\delta]}\le\frac{\mu_{\mathcal{L}}}{\hbar^2}\sum_{\delta=0}^{\mathcal{L}}\mathbf{z}^{[\delta]T}G\mathbf{z}^{[\delta]}$ *when* $\mathbf{z}_0=0$, *where* $\mu_{\mathcal{L}}=4\cos^2\frac{\pi}{2\mathcal{L}+1}$.

(ii) $\sum_{\delta=1}^{\mathcal{L}-1}\Delta_\hbar\mathbf{z}^{[\delta]T}G\Delta_\hbar\mathbf{z}^{[\delta]}\ge\frac{\chi_{\mathcal{L}}}{\hbar^2}\sum_{\delta=1}^{\mathcal{L}}\mathbf{z}^{[\delta]T}G\mathbf{z}^{[\delta]}-\frac{1}{\hbar^2}(\mathbf{z}_1+\mathbf{z}_{\mathcal{L}})^TG(\mathbf{z}_1+\mathbf{z}_{\mathcal{L}})$, *where* $\chi_{\mathcal{L}}=4\sin^2\frac{\pi}{2\mathcal{L}}$.

(iii) $\sum_{\delta=1}^{\mathcal{L}-1}\Delta_\hbar\mathbf{z}^{[\delta]T}G\Delta_\hbar\mathbf{z}^{[\delta]}+\frac{1}{\hbar^2}[\mathbf{z}_1+(-1)^{\mathcal{L}}\mathbf{z}_{\mathcal{L}}]^TG[\mathbf{z}_1+(-1)^{\mathcal{L}}\mathbf{z}_{\mathcal{L}}]\le\frac{4-\chi_{\mathcal{L}}}{\hbar^2}\sum_{\delta=1}^{\mathcal{L}}\mathbf{z}^{[\delta]T}G\mathbf{z}^{[\delta]}$.

3. Adaptive anti-synchronization with parameter identification

This section constructs the space-time discrete stochastic Lyapunov-Krasovskii functional to the error CDNs (2.4). Subsequently, an adaptive update law of uncertain parameter γ is designed. Further, boundary input and controlling gain are designed to perform mean squared asymptotic anti-synchronization of the drive-response networks (2.3) and (2.2), i.e., some asymptotical stability criterions in mean squared sense are established to the error CDNs (2.4). Meanwhile, it achieves the identification of the uncertain parameter.

In accordance with the error CDNs (2.4), a space-time discrete Lyapunov-Krasovskii functional candidate is constructed by

$$V_k=V_{0,k}+V_{1,k}+V_{2,k},$$

where

$$V_{0,k} = \frac{1}{\alpha}(\gamma_k - \gamma)^2, \quad V_{1,k} = \sum_{\delta=1}^{\mathcal{L}-1} \bar{\mathbf{e}}_k^{[\delta]T}(P)_{\otimes m}\bar{\mathbf{e}}_k^{[\delta]},$$

$$V_{2,k} = \sum_{\delta=1}^{\mathcal{L}-1}\sum_{l=1}^{p}\sum_{q=k-\tau_l}^{k-1}\Big[d_2 p\bar{\mathbf{e}}_q^{[\delta]T}(L_\sigma)_{\otimes m}\bar{\mathbf{e}}_q^{[\delta]} + d_3 pc_l^2\bar{\mathbf{e}}_q^{[\delta]T}(\mathbf{\Gamma}_l^T)_{\otimes G_l^T}(\mathbf{\Gamma}_l)_{\otimes G_l}\bar{\mathbf{e}}_q^{[\delta]}\Big],$$

where $P > 0$ is an M-order matrix, and d_2, d_3, α are some positive constants.

The chief research scheme of this paper can be summarized in the flowchart described by Figure 2.

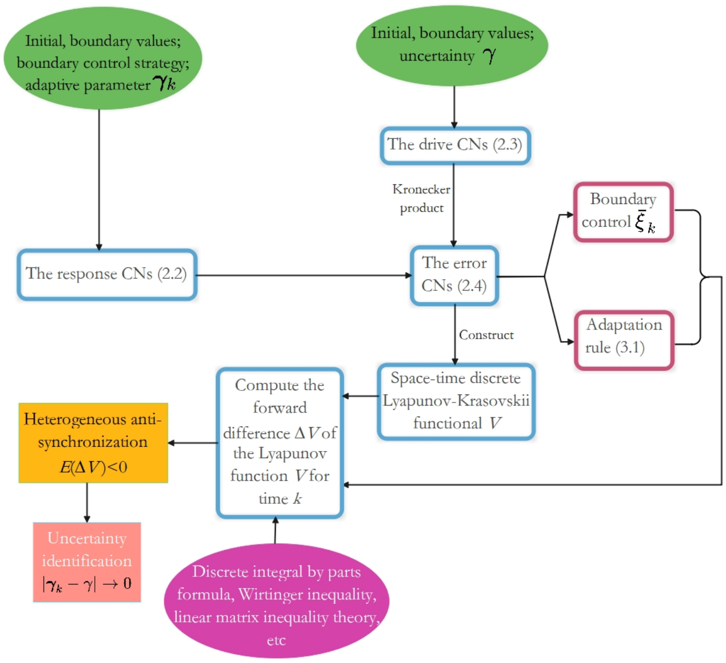

Figure 2. The flowchart of the chief research scheme.

3.1. Adaptive updated law

Set $\eta_k^{[\delta]} = \bar{\mathbf{e}}_{k+1}^{[\delta]} - \bar{\mathbf{e}}_k^{[\delta]}$, $\forall(\delta, k) \in (0, \mathcal{L})_{\mathbb{Z}} \times \mathbb{Z}_0$. We design the adaptive updated law for unknown parameter γ_k as below:

$$\gamma_{k+1} = \gamma_k - \Upsilon_k, \quad \Upsilon_k = \alpha\sum_{\delta=1}^{\mathcal{L}-1}[\bar{\mathbf{e}}_k^{[\delta]T}(PC_*)_{\otimes m} + \eta_k^{[\delta]T}(QC_*)_{\otimes m}]\mathbf{F}_\otimes(\mathbf{w}_k^{[\delta]}), \tag{3.1}$$

where $k \in \mathbb{Z}_0$. Hereby, Q is a positive definitive M-order matrix, which will be calculated by some LMIs later. Remarkably, it follows from (c_2) that

$$(\mathbf{u}^T H\mathbf{v})^2 \leq M^3\|H\|_\infty^2\|\mathbf{v}\|_\infty^2\|\mathbf{u}\|^2, \quad \|\mathbf{F}_\otimes(\mathbf{w})\|_\infty \leq \mathbf{F}_0, \tag{3.2}$$

where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^M, \mathbf{w} \in \mathbb{R}^{mM}, H \in \mathbb{R}^{M\times M}$. So, it implies

$$\mathbb{E}\Upsilon_k^2 \leq 2\alpha^2\mathcal{L}\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\Big\{\Big[\bar{\mathbf{e}}_k^{[\delta]T}(PC_*)_{\otimes m}\mathbf{F}_\otimes(\mathbf{w}_k^{[\delta]})\Big]^2 + \Big[\eta_k^{[\delta]T}(QC_*)_{\otimes m}\mathbf{F}_\otimes(\mathbf{w}_k^{[\delta]})\Big]^2\Big\}$$

$$\leq 2\alpha^2(mM)^3\mathcal{L}\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\left[\|PC_*\|_\infty^2\|\mathbf{F}_\otimes(\mathbf{w}_k^{[\delta]})\|_\infty^2\bar{\mathbf{e}}_k^{[\delta]T}\bar{\mathbf{e}}_k^{[\delta]} + \|QC_*\|_\infty^2\|\mathbf{F}_\otimes(\mathbf{w}_k^{[\delta]})\|_\infty^2\eta_k^{[\delta]T}\eta_k^{[\delta]}\right]$$

$$\leq 2\alpha^2(mM)^3\mathbf{F}_0^2\mathcal{L}\max\{\|PC_*\|_\infty^2, \|QC_*\|_\infty^2\}\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\left[\bar{\mathbf{e}}_k^{[\delta]T}\bar{\mathbf{e}}_k^{[\delta]} + \eta_k^{[\delta]T}\eta_k^{[\delta]}\right], \tag{3.3}$$

for all $k \in \mathbb{Z}_0$. Suppose that

(c_3) Constants α, α_* satisfy $0 < 2\alpha(mM)^3\mathbf{F}_0^2\mathcal{L}\max\{\|PC_*\|_\infty^2, \|QC_*\|_\infty^2\} < \alpha_*$.

According to (3.3) and assumption (c_3), it yields

$$\begin{aligned}
\mathbb{E}[\Delta V_{0,k}] &= \frac{1}{\alpha}\mathbb{E}(\boldsymbol{\gamma}_{k+1} - \gamma)^2 - \frac{1}{\alpha}\mathbb{E}(\boldsymbol{\gamma}_k - \gamma)^2 \\
&= \frac{1}{\alpha}\mathbb{E}(\boldsymbol{\gamma}_{k+1} - \boldsymbol{\gamma}_k)(\boldsymbol{\gamma}_{k+1} + \boldsymbol{\gamma}_k - 2\gamma) \\
&= \frac{1}{\alpha}\mathbb{E}\Upsilon_k^2 - \frac{2}{\alpha}\mathbb{E}[(\boldsymbol{\gamma}_k - \gamma)\Upsilon_k] \\
&\leq \alpha_*\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\left[\bar{\mathbf{e}}_k^{[\delta]T}\bar{\mathbf{e}}_k^{[\delta]} + \eta_k^{[\delta]T}\eta_k^{[\delta]}\right] - 2\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}(\boldsymbol{\gamma}_k - \gamma)[\bar{\mathbf{e}}_k^{[\delta]T}(PC_*)_{\otimes m} \\
&\quad + \eta_k^{[\delta]T}(QC_*)_{\otimes m}]\mathbf{F}_\otimes(\mathbf{w}_k^{[\delta]}), \quad \forall k \in \mathbb{Z}_0.
\end{aligned} \tag{3.4}$$

Remark 3.1. *In the designation of adaptive updated law (3.1), it is dependent of the value of* $\eta_k^{[\delta]} = \bar{\mathbf{e}}_{k+1}^{[\delta]} - \bar{\mathbf{e}}_k^{[\delta]}$*, namely, the value of* $\bar{\mathbf{e}}$ *at time* $k+1$*. If the current time is k, by the iterative relationship (3.1), the states' values of the response network (2.2) at time* $k+1$ *can be determined by the whole relative values at time k or before the time k, and so are the states' values of the response network (2.3).*

Remark 3.2. *In the construction of adaptation rules for continuous CDNs ([44–46]), it was observed that all of them were not involved with the term* $\eta_k^{[\delta]}$*. In other words, it was ascertained that the second term on the righthand side of* Υ_k *in (3.1) does not exist. Why? The rationale will be expounded in the ensuing discourse on heterogeneous anti-synchronization. The calculations of inequalities (3.3) and (3.4) demonstrate that the presence of term* $\eta_k^{[\delta]}$ *greatly increases the complexity and difficulty of the current research.*

Remark 3.3. *In the studies of nonidentical CDNs [10–14], nobody designs the adaptive updated rule such as* Eq (3.1)*, so the concept of quasi-synchronization was proposed and researched. In this article, an adaptive parameter with updated rule* Eq (3.1) *has been designed to perform the complete MSAAS for the drive-response CDNs (2.3) and (2.2).*

3.2. Heterogeneous anti-synchronization with uncertain recognition

The boundary controlling input is designed by

$$\bar{\xi}_k = -\sum_{\delta=1}^{\mathcal{L}-1}(\Pi)_{\otimes m}\bar{\mathbf{e}}_k^{[\delta]}, \quad \forall k \in \mathbb{Z}_0, \tag{3.5}$$

where $\Pi \in \mathbb{R}^{M\times M}$ represents the controlling gain.

Theorem 3.1. *Let assumptions* (c_1)–(c_3) *hold,* $\det(A) \neq 0$, $\det(C) \neq 0$, $\varepsilon \in (0,1)$*, and the updated law for adaptive parameter* $\boldsymbol{\gamma}$ *and the boundary controlling input be designed as those in (3.1) and (3.5), respectively. Suppose further that*

(c_4) $\mathbb{E}\boldsymbol{\gamma}_0^2 < \infty$ *and* $|\gamma| \leq \gamma_*$*, where* γ_* *is a known constant.*

(c_5) *There exist matrices* $P > 0$, $Q > 0$, $R > 0$, $\Phi > 0$, $\mho > 0$ *and positive numbers* α_*, d_1, d_2, d_3 *so that*

$$-2PC_*A + \frac{\mathcal{L}\hbar^2}{\chi_{\mathcal{L}}}\mho + \frac{\mu_{\mathcal{L}-1}}{\hbar^2}R < 0,$$

$$\Omega := \begin{bmatrix} \Omega_{11} & \Omega_{12} & 0 & (QC_*A)_{\otimes m} & 0 & 0 & (QC_*)_{\otimes m} \\ * & \Omega_{22} & -\frac{1}{\hbar}(\Phi)_{\otimes m} & 0 & 0 & 0 & (PC_*)_{\otimes m} \\ * & * & -(\mho)_{\otimes m} & 0 & 0 & 0 & 0 \\ * & * & * & -(R)_{\otimes m} & 0 & 0 & 0 \\ * & * & * & * & \Omega_{55} & 0 & 0 \\ * & * & * & * & * & \Omega_{66} & 0 \\ * & * & * & * & * & * & -(d_3 I)_{\otimes m} \end{bmatrix} < 0,$$

where

$$\Omega_{11} = (\alpha_* I)_{\otimes m} - (2 - \varepsilon - \varepsilon\gamma_*^2)(Q)_{\otimes m} + (P)_{\otimes m}, \quad \Omega_{12} = [Q(e^{Ch} - I)]_{\otimes m},$$

$$\Omega_{22} = (\alpha_* I)_{\otimes m} - \frac{2}{\hbar}(\Phi)_{\otimes m} + \varepsilon\gamma_*^2(P)_{\otimes m} + d_1(L_f)_{\otimes m}$$

$$+2\Big[P(e^{Ch} - I)\Big]_{\otimes m} + d_2 p^2(L_\sigma)_{\otimes m} + d_3 p \sum_{l=1}^{p} c_l^2(\boldsymbol{\Gamma}_l^T)_{\otimes G_l^T}(\boldsymbol{\Gamma}_l)_{\otimes G_l},$$

$$\Omega_{55} = -(d_1 I)_{\otimes m} + \frac{1}{\varepsilon}(C_*^T P C_*)_{\otimes m} + \frac{1}{\varepsilon}(C_*^T Q C_*)_{\otimes m},$$

$$\Omega_{66} = -(d_2 I)_{\otimes m} + \frac{1}{\varepsilon}(C_*^T Q C_*)_{\otimes m}.$$

Then, the response CDNs (2.2) asymptotically inversely synchronizes the drive CDNs (2.3) in the mean squared sense, and $\mathbb{E}|\boldsymbol{\gamma}_k - \gamma|^2 \to 0$ *as* $k \to \infty$*. Moreover, the controlling gain* $\Pi = [PC_*A]^{-1}\Phi$.

Proof. Taking α fulfills assumption (c_3). Based on the Lyapunov-Krasovskii functional as before and the error networks (2.4), it attains

$$\begin{aligned} \mathbb{E}[\Delta V_{1,k}] &= \mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1} \bar{\mathbf{e}}_{k+1}^{[\delta]T}(P)_{\otimes m}\bar{\mathbf{e}}_{k+1}^{[\delta]} - \mathbb{E}[V_{1,k}] \\ &= \mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1} (\eta_k^{[\delta]T} + \bar{\mathbf{e}}_k^{[\delta]T})(P)_{\otimes m}(\eta_k^{[\delta]} + \bar{\mathbf{e}}_k^{[\delta]}) - \mathbb{E}[V_{1,k}] \end{aligned}$$

$$= \underbrace{\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\eta_k^{[\delta]T}(P)_{\otimes m}\eta_k^{[\delta]}}_{\pounds_k} + \underbrace{2\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\bar{\mathbf{e}}_k^{[\delta]T}(P)_{\otimes m}\eta_k^{[\delta]}}_{\Lambda_{1,k}}, \tag{3.6}$$

where

$$\begin{aligned}
\Lambda_{1,k} &= 2\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\bar{\mathbf{e}}_k^{[\delta]T}(P)_{\otimes m}\eta_k^{[\delta]} \\
&= 2\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\bar{\mathbf{e}}_k^{[\delta]T}\Big[P(e^{Ch}-I)\Big]_{\otimes m}\bar{\mathbf{e}}_k^{[\delta]} \\
&\quad \underbrace{+ 2\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\bar{\mathbf{e}}_k^{[\delta]T}(PC_*A)_{\otimes m}\Delta_\hbar^2\bar{\mathbf{e}}_k^{[\delta-1]}}_{\Lambda_{2,k}} \\
&\quad \underbrace{+ 2\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\gamma\bar{\mathbf{e}}_k^{[\delta]T}(PC_*)_{\otimes m}\tilde{\mathbf{F}}_\otimes(\bar{\mathbf{e}}_k^{[\delta]})}_{\Theta_{1,k}} \\
&\quad +2\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\bar{\mathbf{e}}_k^{[\delta]T}(PC_*)_{\otimes m}\sum_{l=1}^{p}c_l(\mathbf{\Gamma}_l)_{\otimes G_l}\bar{\mathbf{e}}_{k-\tau_l}^{[\delta]} \\
&\quad +2\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\bar{\mathbf{e}}_k^{[\delta]T}(\boldsymbol{\gamma}_k-\gamma)(PC_*)_{\otimes m}\mathbf{F}_\otimes(\mathbf{w}_k^{[\delta]}), \quad \forall k\in\mathbb{Z}_0.
\end{aligned} \tag{3.7}$$

Via the discrete formula of integration by parts and boundary values in (2.5), it computes

$$\begin{aligned}
\Lambda_{2,k} &= 2\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\bar{\mathbf{e}}_k^{[\delta]T}(PC_*A)_{\otimes m}\Delta_\hbar^2\bar{\mathbf{e}}_k^{[\delta-1]} \\
&= \frac{2}{\hbar}\mathbb{E}\bar{\mathbf{e}}_k^{[\delta]T}(PC_*A)_{\otimes m}\Delta_\hbar\bar{\mathbf{e}}_k^{[\delta-1]}\Big|_1^{\mathcal{L}} - 2\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\Delta_\hbar\bar{\mathbf{e}}_k^{[\delta]T}(PC_*A)_{\otimes m}\Delta_\hbar\bar{\mathbf{e}}_k^{[\delta]} \\
&= \underbrace{\frac{2}{\hbar}\mathbb{E}\bar{\mathbf{e}}_k^{[\mathcal{L}]T}(PC_*A)_{\otimes m}\bar{\xi}_k}_{\aleph_k} - 2\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\Delta_\hbar\bar{\mathbf{e}}_k^{[\delta]T}(PC_*A)_{\otimes m}\Delta_\hbar\bar{\mathbf{e}}_k^{[\delta]}, \quad \forall k\in\mathbb{Z}_0.
\end{aligned} \tag{3.8}$$

Define $\hat{\bar{\mathbf{e}}}_k^{[\delta]} = \bar{\mathbf{e}}_k^{[\mathcal{L}]} - \bar{\mathbf{e}}_k^{[\delta]}$, $\forall(\delta,k)\in[0,\mathcal{L}]_{\mathbb{Z}}\times\mathbb{Z}_0$. It should be noted that

$$\hat{\bar{\mathbf{e}}}_k^{[\mathcal{L}]} = 0, \quad \hat{\bar{\mathbf{e}}}_k^{[1]} = \bar{\mathbf{e}}_k^{[\mathcal{L}]} - \bar{\mathbf{e}}_k^{[1]}, \quad \forall k\in\mathbb{Z}_0.$$

Together with the discrete Wirtinger's inequalities in item (*ii*) of Lemma 2.1, it results in

$$\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\Delta_\hbar\bar{\mathbf{e}}_k^{[\delta]T}(\mho)_{\otimes m}\Delta_\hbar\bar{\mathbf{e}}_k^{[\delta]} = \mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\Delta_\hbar\hat{\bar{\mathbf{e}}}_k^{[\delta]T}(\mho)_{\otimes m}\Delta_\hbar\hat{\bar{\mathbf{e}}}_k^{[\delta]}$$

$$\geq \frac{\mathcal{X}\mathcal{L}}{\hbar^2}\mathbb{E}\sum_{\delta=1}^{\mathcal{L}}\hat{\mathbf{e}}_k^{[\delta]T}(\mho)_{\otimes m}\hat{\mathbf{e}}_k^{[\delta]} - \frac{1}{\hbar^2}\mathbb{E}(\hat{\mathbf{e}}_k^{[\mathcal{L}]T}+\hat{\mathbf{e}}_k^{[1]T})(\mho)_{\otimes m}(\hat{\mathbf{e}}_k^{[\mathcal{L}]}+\hat{\mathbf{e}}_k^{[1]})$$
$$= \frac{\mathcal{X}\mathcal{L}}{\hbar^2}\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\hat{\mathbf{e}}_k^{[\delta]T}(\mho)_{\otimes m}\hat{\mathbf{e}}_k^{[\delta]} - \frac{1}{\hbar^2}\mathbb{E}(\bar{\mathbf{e}}_k^{[\mathcal{L}]T}-\bar{\mathbf{e}}_k^{[1]T})(\mho)_{\otimes m}(\bar{\mathbf{e}}_k^{[\mathcal{L}]}-\bar{\mathbf{e}}_k^{[1]})$$
$$= \frac{\mathcal{X}\mathcal{L}}{\hbar^2}\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\hat{\mathbf{e}}_k^{[\delta]T}(\mho)_{\otimes m}\hat{\mathbf{e}}_k^{[\delta]} - \mathbb{E}\left(\sum_{\delta=1}^{\mathcal{L}-1}\Delta_\hbar\bar{\mathbf{e}}_k^{[\delta]T}\right)(\mho)_{\otimes m}\left(\sum_{\delta=1}^{\mathcal{L}-1}\Delta_\hbar\bar{\mathbf{e}}_k^{[\delta]}\right)$$
$$\geq \frac{\mathcal{X}\mathcal{L}}{\hbar^2}\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\hat{\mathbf{e}}_k^{[\delta]T}(\mho)_{\otimes m}\hat{\mathbf{e}}_k^{[\delta]} - (\mathcal{L}-1)\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\Delta_\hbar\bar{\mathbf{e}}_k^{[\delta]T}(\mho)_{\otimes m}\Delta_\hbar\bar{\mathbf{e}}_k^{[\delta]},$$

which indicates

$$\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\hat{\mathbf{e}}_k^{[\delta]T}(\mho)_{\otimes m}\hat{\mathbf{e}}_k^{[\delta]} \leq \frac{\mathcal{L}\hbar^2}{\mathcal{X}\mathcal{L}}\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\Delta_\hbar\bar{\mathbf{e}}_k^{[\delta]T}(\mho)_{\otimes m}\Delta_\hbar\bar{\mathbf{e}}_k^{[\delta]}, \quad \forall k\in\mathbb{Z}_0. \tag{3.9}$$

According to the error CDNs (2.4), it gets

$$\begin{aligned}\eta_k^{[\delta]} = {}& (e^{Ch}-I)_{\otimes m}\bar{\mathbf{e}}_k^{[\delta]} + (C_*)_{\otimes m}\Big[(A)_{\otimes m}\Delta_\hbar^2\bar{\mathbf{e}}_k^{[\delta-1]} \\ & +\gamma\tilde{\mathbf{F}}_\otimes(\bar{\mathbf{e}}_k^{[\delta]}) + (\boldsymbol{\gamma}_k-\gamma)\mathbf{F}_\otimes(\mathbf{w}_k^{[\delta]}) + \sum_{l=1}^{p}c_l(\boldsymbol{\Gamma}_l)_{\otimes G_l}\bar{\mathbf{e}}_{k-\tau_l}^{[\delta]} + \sum_{l=1}^{p}\tilde{\sigma}_\otimes^w(\bar{\mathbf{e}}_{k-\tau_l}^{[\delta]})\Big],\end{aligned} \tag{3.10}$$

where $(\delta,k)\in(0,\mathcal{L})\times\mathbb{Z}_0$. It induces from (3.10) that

$$\begin{aligned}2\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\eta_k^{[\delta]T}(Q)_{\otimes m}\eta_k^{[\delta]} = {}& 2\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\eta_k^{[\delta]T}[Q(e^{Ch}-I)]_{\otimes m}\bar{\mathbf{e}}_k^{[\delta]} \\ & +2\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\eta_k^{[\delta]T}(QC_*A)_{\otimes m}\Delta_\hbar^2\bar{\mathbf{e}}_k^{[\delta-1]} \\ & +\underbrace{2\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\gamma\eta_k^{[\delta]T}(QC_*)_{\otimes m}\tilde{\mathbf{F}}_\otimes(\bar{\mathbf{e}}_k^{[\delta]})}_{\Theta_{2,k}} \\ & +2\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\eta_k^{[\delta]T}(QC_*)_{\otimes m}\sum_{l=1}^{p}c_l(\boldsymbol{\Gamma}_l)_{\otimes G_l}\bar{\mathbf{e}}_{k-\tau_l}^{[\delta]} \\ & +\underbrace{2\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}(\boldsymbol{\gamma}_k-\gamma)\eta_k^{[\delta]T}(QC_*)_{\otimes m}\mathbf{F}_\otimes(\mathbf{w}_k^{[\delta]})}_{\P_k} \\ & +\underbrace{2\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\eta_k^{[\delta]T}(QC_*)_{\otimes m}\sum_{l=1}^{p}\tilde{\sigma}_\otimes^w(\bar{\mathbf{e}}_{k-\tau_l}^{[\delta]})}_{\Lambda_{4,k}}, \quad \forall k\in\mathbb{Z}_0.\end{aligned} \tag{3.11}$$

According to the elementary inequality in Lemma 1.13 of [52], i.e., $2X^TY \le \varepsilon X^TX + \frac{1}{\varepsilon}Y^TY$ for any $X, Y \in \mathbb{R}^{mM}$ and $\varepsilon > 0$, it generates

$$\begin{aligned}\Theta_{1,k} &= 2\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\gamma\bar{\mathbf{e}}_k^{[\delta]T}(PC_*)_{\otimes m}\tilde{\mathbf{F}}_{\otimes}(\bar{\mathbf{e}}_k^{[\delta]})\\ &\le \varepsilon\gamma_*^2\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\bar{\mathbf{e}}_k^{[\delta]T}(P)_{\otimes m}\bar{\mathbf{e}}_k^{[\delta]} + \frac{1}{\varepsilon}\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\tilde{\mathbf{F}}_{\otimes}^T(\bar{\mathbf{e}}_k^{[\delta]})(C_*^TPC_*)_{\otimes m}\tilde{\mathbf{F}}_{\otimes}(\bar{\mathbf{e}}_k^{[\delta]}),\end{aligned} \tag{3.12}$$

and, similarly,

$$\begin{aligned}\Theta_{2,k} &= 2\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\gamma\eta_k^{[\delta]T}(QC_*)_{\otimes m}\tilde{\mathbf{F}}_{\otimes}(\bar{\mathbf{e}}_k^{[\delta]})\\ &\le \varepsilon\gamma_*^2\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\eta_k^{[\delta]T}(Q)_{\otimes m}\eta_k^{[\delta]} + \frac{1}{\varepsilon}\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\tilde{\mathbf{F}}_{\otimes}^T(\bar{\mathbf{e}}_k^{[\delta]})(C_*^TQC_*)_{\otimes m}\tilde{\mathbf{F}}_{\otimes}(\bar{\mathbf{e}}_k^{[\delta]}), \quad \forall k \in \mathbb{Z}_0.\end{aligned} \tag{3.13}$$

From (3.11), we have

$$\begin{aligned}\Lambda_{4,k} &= 2\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\eta_k^{[\delta]T}(QC_*)_{\otimes m}\sum_{l=1}^{p}\tilde{\sigma}_{\otimes}^{w}(\bar{\mathbf{e}}_{k-\tau_l}^{[\delta]})\\ &\le \varepsilon\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\eta_k^{[\delta]T}(Q)_{\otimes m}\eta_k^{[\delta]} + \frac{1}{\varepsilon}\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\sum_{l=1}^{p}[\tilde{\sigma}_{\otimes}^{w}(\bar{\mathbf{e}}_{k-\tau_l}^{[\delta]})]^T(C_*^TQC_*)_{\otimes m}\sum_{l=1}^{p}\tilde{\sigma}_{\otimes}^{w}(\bar{\mathbf{e}}_{k-\tau_l}^{[\delta]})\\ &= \varepsilon\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\eta_k^{[\delta]T}(Q)_{\otimes m}\eta_k^{[\delta]} + \frac{1}{\varepsilon}\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\sum_{l=1}^{p}\tilde{\sigma}_{\otimes}^{T}(\bar{\mathbf{e}}_{k-\tau_l}^{[\delta]})(C_*^TQC_*)_{\otimes m}\sum_{l=1}^{p}\tilde{\sigma}_{\otimes}(\bar{\mathbf{e}}_{k-\tau_l}^{[\delta]}),\end{aligned} \tag{3.14}$$

where $k \in \mathbb{Z}_0$. In (3.14), note that $\Delta_h w_k$ is independent of $\bar{\mathbf{e}}_s^{[\cdot]}$ for all $s \le k$, and we use the fact of $\mathbb{E}(\Delta_h w_\cdot)^2 = 1$.

In terms of assumption (c_2), it deduces

$$d_1\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\tilde{\mathbf{F}}_{\otimes}^T(\bar{\mathbf{e}}_k^{[\delta]})\tilde{\mathbf{F}}_{\otimes}(\bar{\mathbf{e}}_k^{[\delta]}) \le \mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\bar{\mathbf{e}}_k^{[\delta]T}d_1(L_f)_{\otimes m}\bar{\mathbf{e}}_k^{[\delta]}, \tag{3.15}$$

$$d_2\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\sum_{l=1}^{p}\tilde{\sigma}_{\otimes}^T(\bar{\mathbf{e}}_{k-\tau_l}^{[\delta]})\sum_{l=1}^{p}\tilde{\sigma}_{\otimes}(\bar{\mathbf{e}}_{k-\tau_l}^{[\delta]}) \le d_2p\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\sum_{l=1}^{p}\bar{\mathbf{e}}_{k-\tau_l}^{[\delta]T}(L_\sigma)_{\otimes m}\bar{\mathbf{e}}_{k-\tau_l}^{[\delta]}, \tag{3.16}$$

$$\begin{aligned}&d_3\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\sum_{l=1}^{p}\left[c_l(\mathbf{\Gamma}_l)_{\otimes G_l}\bar{\mathbf{e}}_{k-\tau_l}^{[\delta]}\right]^T\sum_{l=1}^{p}c_l(\mathbf{\Gamma}_l)_{\otimes G_l}\bar{\mathbf{e}}_{k-\tau_l}^{[\delta]}\\ &\le d_3p\mathbb{E}\sum_{\delta=1}^{\mathcal{L}-1}\sum_{l=1}^{p}\bar{\mathbf{e}}_{k-\tau_l}^{[\delta]T}c_l^2(\mathbf{\Gamma}_l^T)_{\otimes G_l^T}(\mathbf{\Gamma}_l)_{\otimes G_l}\bar{\mathbf{e}}_{k-\tau_l}^{[\delta]},\end{aligned} \tag{3.17}$$

where $k \in \mathbb{Z}_0$. Based on the definition of $V_{2,k}$, we have

$$\begin{aligned}
\mathbb{E}[\Delta V_{2,k}] &= \mathbb{E}V_{2,k+1} - \mathbb{E}V_{2,k} \\
&= d_2 p^2 \mathbb{E} \sum_{\delta=1}^{\mathcal{L}-1} \bar{\mathbf{e}}_k^{[\delta]T} (L_\sigma)_{\otimes m} \bar{\mathbf{e}}_k^{[\delta]} \\
&\quad + d_3 p \mathbb{E} \sum_{\delta=1}^{\mathcal{L}-1} \bar{\mathbf{e}}_k^{[\delta]T} \sum_{l=1}^{p} c_l^2 (\mathbf{\Gamma}_l^T)_{\otimes G_l^T} (\mathbf{\Gamma}_l)_{\otimes G_l} \bar{\mathbf{e}}_k^{[\delta]} \\
&\quad - d_2 p \mathbb{E} \sum_{\delta=1}^{\mathcal{L}-1} \sum_{l=1}^{p} \bar{\mathbf{e}}_{k-\tau_l}^{[\delta]T} (L_\sigma)_{\otimes m} \bar{\mathbf{e}}_{k-\tau_l}^{[\delta]} \\
&\quad - d_3 p \mathbb{E} \sum_{\delta=1}^{\mathcal{L}-1} \sum_{l=1}^{p} \bar{\mathbf{e}}_{k-\tau_l}^{[\delta]T} c_l^2 (\mathbf{\Gamma}_l^T)_{\otimes G_l^T} (\mathbf{\Gamma}_l)_{\otimes G_l} \bar{\mathbf{e}}_{k-\tau_l}^{[\delta]},
\end{aligned} \tag{3.18}$$

where $k \in \mathbb{Z}_0$.

From Lemma 2.1, we have

$$\begin{aligned}
\mathbb{E} \sum_{\delta=1}^{\mathcal{L}-1} \Delta_\hbar^2 \bar{\mathbf{e}}_k^{[l-1]T} (R)_{\otimes m} \Delta_\hbar^2 \bar{\mathbf{e}}_k^{[\delta-1]} &\leq \mathbb{E} \sum_{\delta=0}^{\mathcal{L}-2} \Delta_\hbar^2 \bar{\mathbf{e}}_k^{[\delta]T} (R)_{\otimes m} \Delta_\hbar^2 \bar{\mathbf{e}}_k^{[\delta]} \\
&\leq \frac{\mu_{\mathcal{L}-1}}{\hbar^2} \mathbb{E} \sum_{\delta=1}^{\mathcal{L}-1} \Delta_\hbar \bar{\mathbf{e}}_k^{[\delta]T} (R)_{\otimes m} \Delta_\hbar \bar{\mathbf{e}}_k^{[\delta]}, \quad \forall k \in \mathbb{Z}_0.
\end{aligned} \tag{3.19}$$

In view of (3.8) and by boundary input (3.5), it has

$$\begin{aligned}
\aleph_k &= \frac{2}{\hbar} \mathbb{E} \bar{\mathbf{e}}_k^{[\mathcal{L}]T} (PC_* A)_{\otimes m} \bar{\xi}_k \\
&= -\frac{2}{\hbar} \mathbb{E} \sum_{\delta=1}^{\mathcal{L}-1} \bar{\mathbf{e}}_k^{[\mathcal{L}]T} (PC_* A \Pi)_{\otimes m} \bar{\mathbf{e}}_k^{[\delta]} \\
&= -\frac{2}{\hbar} \mathbb{E} \sum_{\delta=1}^{\mathcal{L}-1} \hat{\mathbf{e}}_k^{[\delta]T} (\Phi)_{\otimes m} \bar{\mathbf{e}}_k^{[\delta]} - \frac{2}{\hbar} \mathbb{E} \sum_{\delta=1}^{\mathcal{L}-1} \bar{\mathbf{e}}_k^{[\delta]T} (\Phi)_{\otimes m} \bar{\mathbf{e}}_k^{[\delta]}, \quad \forall k \in \mathbb{Z}_0.
\end{aligned} \tag{3.20}$$

Set

$$\tilde{\Omega} = \begin{bmatrix} \Omega & 0 \\ 0 & -2PC_* A + \frac{\mathcal{L}\hbar^2}{\chi_{\mathcal{L}}} \mho + \frac{\mu_{\mathcal{L}-1}}{\hbar^2} R \end{bmatrix}.$$

On the grounds of (3.4), (3.6)–(3.20), and assumption (c_5), it induces $\tilde{\Omega} < 0$ and

$$\mathbb{E}[\Delta V_k] \leq \mathbb{E} \sum_{\delta=1}^{\mathcal{L}-1} \mathcal{H}_k^{[\delta]T} \tilde{\Omega} \mathcal{H}_k^{[\delta]}, \quad \forall k \in \mathbb{Z}_0,$$

where

$$\mathcal{H}_k^{[\delta]} = \left[\eta_k^{[\delta]}, \bar{\mathbf{e}}_k^{[\delta]}, \hat{\mathbf{e}}_k^{[\delta]}, \Delta_\hbar^2 \bar{\mathbf{e}}_k^{[\delta-1]}, \tilde{\mathbf{F}}_\otimes(\bar{\mathbf{e}}_k^{[\delta]}), \sum_{l=1}^{p} \tilde{\sigma}_\otimes(\bar{\mathbf{e}}_{k-\tau_l}^{[\delta]}), \sum_{l=1}^{p} c_l (\mathbf{\Gamma}_l)_{\otimes G_l} \bar{\mathbf{e}}_{k-\tau_l}^{[\delta]}, \Delta_\hbar \bar{\mathbf{e}}_k^{[\delta]} \right]^T$$

for any $(\delta, k) \in (0, \mathcal{L})_{\mathbb{Z}} \times \mathbb{Z}_0$. Obviously,

$$\sum_{j=0}^{k-1} \mathbb{E}[\Delta V_j] \leq \mathbb{E} \sum_{j=0}^{k-1} \sum_{\delta=1}^{\mathcal{L}-1} \mathcal{H}_j^{[\delta]T} \Omega \mathcal{H}_j^{[\delta]} \leq \mathbb{E} \sum_{j=0}^{k-1} \sum_{\delta=1}^{\mathcal{L}-1} \bar{\mathbf{e}}_j^{[\delta]T} \Omega \bar{\mathbf{e}}_j^{[\delta]} \leq \lambda_\Omega^+ \mathbb{E} \sum_{j=0}^{k-1} \sum_{\delta=1}^{\mathcal{L}-1} \bar{\mathbf{e}}_j^{[\delta]T} \bar{\mathbf{e}}_j^{[\delta]}$$

$$\Rightarrow \mathbb{E} \sum_{j=0}^{k-1} \sum_{\delta=1}^{\mathcal{L}-1} \bar{\mathbf{e}}_j^{[\delta]T} \bar{\mathbf{e}}_j^{[\delta]} \leq \frac{1}{\lambda_\Omega^+} \sum_{j=0}^{k-1} \mathbb{E}[\Delta V_j] \leq -\frac{1}{\lambda_\Omega^+} \mathbb{E} V_0 < \infty,$$

which implies

$$\mathbb{E} \sum_{\delta=1}^{\mathcal{L}-1} \bar{\mathbf{e}}_k^{[\delta]T} \bar{\mathbf{e}}_k^{[\delta]} \to 0,$$

as $k \to \infty$. Hereby, λ_Ω^+ denotes the maximum eigenvalue of Ω. According to boundary information in (3.5), it exists constant $M_1 > 0$ so that

$$\|\bar{\mathbf{e}}_k^{[\mathcal{L}]}\|^2 \leq M_1 \sum_{\delta=1}^{\mathcal{L}-1} \|\bar{\mathbf{e}}_k^{[\delta]}\|^2, \quad \forall k \in \mathbb{Z}_0. \tag{3.21}$$

Additionally,

$$\|\bar{\mathbf{e}}_k^{[0]}\|^2 = \|\bar{\mathbf{e}}_k^{[1]}\|^2 \leq \sum_{\delta=1}^{\mathcal{L}-1} \|\bar{\mathbf{e}}_k^{[\delta]}\|^2, \quad \forall k \in \mathbb{Z}_0. \tag{3.22}$$

Together with (3.21) and (3.22), it causes

$$\sum_{\delta=0}^{\mathcal{L}} \|\bar{\mathbf{e}}_k^{[\delta]}\|^2 \to 0, \text{ as } k \to \infty.$$

Videlicet, the error CDNs (2.4) achieves global asymptotic stability in the mean squared sense. To wit, the response CDNs (2.2) asymptotically inversely synchronizes the drive CDNs (2.3). Based on the error CDNs (2.4) and the same discussions in [52, Theorem 4.20], it directly gains $\boldsymbol{\gamma}_k \to \gamma$, as $k \to \infty$. This achieves the proof. □

3.3. Homogeneous anti-synchronization

If $\boldsymbol{\gamma}_k \equiv \gamma$ for all $k \in \mathbb{Z}_0$, then the error CDNs (2.4) is transformed into

$$\begin{aligned} \bar{\mathbf{e}}_{k+1}^{[\delta]} = (e^{Ch})_{\otimes m} \bar{\mathbf{e}}_k^{[\delta]} + [(e^{Ch} - I)C^{-1}]_{\otimes m} \Big[(A)_{\otimes m} \Delta_{\hbar}^2 \bar{\mathbf{e}}_k^{[\delta-1]} \\ + \gamma \tilde{\mathbf{F}}_{\otimes}(\bar{\mathbf{e}}_k^{[\delta]}) + \sum_{l=1}^{p} c_l (\boldsymbol{\Gamma}_l)_{\otimes G_l} \bar{\mathbf{e}}_{k-\tau_l}^{[\delta]} + \sum_{l=1}^{p} \tilde{\sigma}_{\otimes}^{w}(\bar{\mathbf{e}}_{k-\tau_l}^{[\delta]}) \Big], \end{aligned} \tag{3.23}$$

where $(\delta, k) \in (0, \mathcal{L})_{\mathbb{Z}} \times \mathbb{Z}_0$.

By a similar proof to Theorem 3.1, we can gain the corollary below.

Corollary 3.1. *Let assumptions* (c_1)–(c_3) *hold,* $\det(A) \neq 0$, $\det(C) \neq 0$, $\varepsilon \in (0, 1)$*, and the boundary input be designed as that in (3.5). Assume that*

(c_6) *$\boldsymbol{\gamma}_k \equiv \gamma$ for all $k \in \mathbb{Z}_0$.*

(c_7) *Assumption (c_5) is valid with $\alpha_* = 0$.*

Furthermore, the controlling gain is designed as before. Then, the response CDNs (2.2) asymptotically anti-synchronizes the drive CDNs (2.3) in the mean squared sense.

According to Theorem 3.1, a realizable algorithm for adaptive anti-synchronization of the drive-response CDNs (2.2) and (2.3) is designed as follows.

Algorithm 1 Adaptive anti-synchronization of the drive-response CDNs (2.2) and (2.3)

(a_1) Initialing the values of the coefficients to the drive-response CDNs (2.2) and (2.3) and the value of parameter $\varepsilon \in (0, 1)$.

(a_2) Computing LMIs in Theorem 3.1. If it is unviable, modify the values of in step (a_1); otherwise, switch to next step.

(a_3) Receiving the values of α_* and m-order matrices in assumption (c_5). Calculating the controlling gain defined in (c_5) and determining the boundary input in (3.5).

(a_4) Take α to satisfy $0 < 2\alpha(mM)^3\mathbf{F}_0^2\mathcal{L}\max\{\|PC_*\|_\infty^2, \|QC_*\|_\infty^2\} < \alpha_*$ and calculate the updated law (3.1) of the unknown parameter γ.

(a_5) Write the iterative program based on the drive-response CDNs (2.2) and (2.3) and plot the response trajectories.

Remark 3.4. *Discrete-time CDNs have been the focus of the authors' previous works [9, 15, 35–37]. This article deals with discrete-time CDNs with discrete spatial diffusion. The associated calculations, such as the Lyapunov-Krasovskii functional, are more complex than the case of a single time-variable. In this paper, we present a boundary feedback controller for discrete-time and spatial CDNs, whereas previous reports [9, 15, 35–37] have used a traditional feedback controller for CDNs. Previous studies [9, 15, 35–37] have focused on homogeneous CDNs, whereas this article examines heterogeneous CDNs.*

Remark 3.5. *Previous publications [10–14] have examined continuous-time heterogeneous networks. In contrast, this article addresses the following aspects: 1) The paper considers the issue of discrete space transportation networks, taking into account their heterogeneity. 2) It achieves mean squared asymptotic anti-synchronization for the networks, which differs significantly from the quasi-synchronization discussed in previous publications [10–14]. 3) The response networks in these publications [10–14] have a predetermined heterogeneous coefficient. We have designed an adaptive parameter in the response networks that tends to drive-response CDNs homogeneously by continuously adjusting it according to the updated law. At this moment, the adaptive parameter in the response CDNs can perfectly identify the unknown parameter in the drive CDNs according to the updated law.*

4. Experiments

Supposing the drive CDNs are expressed by

$$\mathbf{z}_{k+1}^{[\delta]} = e^{Ch}\mathbf{z}_k^{[\delta]} + (e^{Ch}-I)C^{-1}\Bigg\{0.1\Delta_\hbar^2\mathbf{z}_k^{[\delta-1]} + \gamma\begin{bmatrix}0.3 & 0\\ 0 & 0.1\end{bmatrix}\begin{bmatrix}\sin(\mathbf{z}_{1,k}^{[\delta]})\\ \sin(\mathbf{z}_{2,k}^{[\delta]})\end{bmatrix}$$
$$+\begin{bmatrix}0.1 & 0.1\\ 0.5 & 0.5\end{bmatrix}\begin{bmatrix}\frac{\mathbf{z}_{1,k-1}^{[\delta]}}{1+|\mathbf{z}_{2,k-1}^{[\delta]}|}\\ \frac{\mathbf{z}_{2,k-2}^{[\delta]}}{1+|\mathbf{z}_{1,k-2}^{[\delta]}|}\end{bmatrix}\Delta_h w_k - \begin{bmatrix}5.2027\\ 5.6240\end{bmatrix}\Bigg\}, \quad \forall(\delta,k)\in(0,6)_{\mathbb{Z}}\times\mathbb{Z}_0, \tag{4.1}$$

where $\mathbf{z} = \begin{bmatrix}\mathbf{z}_1\\ \mathbf{z}_2\end{bmatrix}$, $C = -\begin{bmatrix}8 & 0\\ 8 & 24\end{bmatrix}$, $h = 0.01$, $\hbar = 0.25$, $\gamma = 0.1$. The boundary values of the drive CDNs (4.1) are given by $\Delta_\hbar\mathbf{z}_k^{[0]} = \Delta_\hbar\mathbf{z}_k^{[5]} = 0$, $\forall k\in\mathbb{Z}_0$. The response CDNs of the drive CDNs (4.1) are described as

$$\begin{cases}\mathbf{w}_{1,k+1}^{[\delta]} = e^{Ch}\mathbf{w}_{1,k}^{[\delta]} + (e^{Ch}-I)C^{-1}\Bigg\{0.1\Delta_\hbar^2\mathbf{w}_{1,k}^{[\delta-1]} + \tilde{\gamma}_k\begin{bmatrix}0.3 & 0\\ 0 & 0.1\end{bmatrix}\begin{bmatrix}\sin(\mathbf{w}_{11,k}^{[\delta]})\\ \sin(\mathbf{w}_{12,k}^{[\delta]})\end{bmatrix}\\ \quad -0.5\mathbf{\Gamma}_1\mathbf{w}_{1,k-1}^{[\delta]} + 0.5\mathbf{\Gamma}_1\mathbf{w}_{2,k-1}^{[\delta]} - 0.8\mathbf{\Gamma}_2\mathbf{w}_{1,k-2}^{[\delta]} + 0.8\mathbf{\Gamma}_2\mathbf{w}_{2,k-2}^{[\delta]}\\ \quad +\begin{bmatrix}0.1 & 0.1\\ 0.5 & 0.5\end{bmatrix}\begin{bmatrix}\frac{\mathbf{w}_{11,k-1}^{[\delta]}}{1+|\mathbf{w}_{12,k-1}^{[\delta]}|}\\ \frac{\mathbf{w}_{12,k-2}^{[\delta]}}{1+|\mathbf{w}_{11,k-2}^{[\delta]}|}\end{bmatrix}\Delta_h w_k + \begin{bmatrix}5.2027\\ 5.6240\end{bmatrix}\Bigg\},\\ \mathbf{w}_{2,k+1}^{[\delta]} = e^{Ch}\mathbf{w}_{2,k}^{[\delta]} + (e^{Ch}-I)C^{-1}\Bigg\{0.1\Delta_\hbar^2\mathbf{w}_{2,k}^{[\delta-1]} + \tilde{\gamma}_k\begin{bmatrix}0.3 & 0\\ 0 & 0.1\end{bmatrix}\begin{bmatrix}\sin(\mathbf{w}_{21,k}^{[\delta]})\\ \sin(\mathbf{w}_{22,k}^{[\delta]})\end{bmatrix}\\ \quad -0.5\mathbf{\Gamma}_1\mathbf{w}_{1,k-1}^{[\delta]} + 0.5\mathbf{\Gamma}_1\mathbf{w}_{2,k-1}^{[\delta]} - 0.8\mathbf{\Gamma}_2\mathbf{w}_{1,k-2}^{[\delta]} + 0.8\mathbf{\Gamma}_2\mathbf{w}_{2,k-2}^{[\delta]}\\ \quad +\begin{bmatrix}0.1 & 0.1\\ 0.5 & 0.5\end{bmatrix}\begin{bmatrix}\frac{\mathbf{w}_{21,k-1}^{[\delta]}}{1+|\mathbf{w}_{22,k-1}^{[\delta]}|}\\ \frac{\mathbf{w}_{22,k-2}^{[\delta]}}{1+|\mathbf{w}_{21,k-2}^{[\delta]}|}\end{bmatrix}\Delta_h w_k + \begin{bmatrix}5.2027\\ 5.6240\end{bmatrix}\Bigg\},\end{cases} \tag{4.2}$$

where $(\delta,k)\in(0,6)_{\mathbb{Z}}\times\mathbb{Z}_0$, $\tilde{\gamma}_0 = 0$,

$$\mathbf{w}_1 = \begin{bmatrix}\mathbf{w}_{11}\\ \mathbf{w}_{12}\end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix}\mathbf{w}_{21}\\ \mathbf{w}_{22}\end{bmatrix}, \quad \mathbf{\Gamma}_1 = \begin{bmatrix}0 & -0.2\\ 0.7 & 0\end{bmatrix}, \quad \mathbf{\Gamma}_2 = \begin{bmatrix}0 & 0.5\\ -0.3 & 0\end{bmatrix}.$$

The boundary values of the response CDNs (4.2) are given by $\Delta_\delta\mathbf{w}_{1,k}^{[0]} = \Delta_\delta\mathbf{w}_{2,k}^{[0]} = 0$, $\Delta_\delta\mathbf{w}_{1,k}^{[5]} = \xi_{1,k}$, $\Delta_\delta\mathbf{w}_{2,k}^{[5]} = \xi_{2,k}$, $\forall k\in\mathbb{Z}_0$.

According to the information above and Algorithm 1, a flow diagram of achieving the adaptive anti-synchronization of the drive-response CDNs (4.1) and (4.2) is drawn in Figure 3.

Obviously, (c_1)–(c_2) in Theorem 3.1 hold. Taking $\varepsilon = 0.5$, $\gamma_* = 0.2$ and computing the LMIs in Theorem 3.1, the LMIs in (c_5) are valid for $\alpha_* = 1.5763$, $d_1 = 29.0039$, $d_2 = 9.1156$, $d_3 = 10.6818$,

$$P = \begin{bmatrix}1229.1121 & 33.1733\\ 33.1733 & 603.1577\end{bmatrix}, \quad Q = \begin{bmatrix}1149.4085 & 38.0849\\ 38.0849 & 655.5776\end{bmatrix},$$

$$R = \begin{bmatrix}0.0204 & 0.0006\\ 0.0006 & 0.0103\end{bmatrix}, \quad \Phi = \begin{bmatrix}0.3385 & 0.0492\\ 0.0492 & 0.2912\end{bmatrix}, \quad \mho = \begin{bmatrix}0.6374 & -0.0027\\ -0.0027 & 0.2390\end{bmatrix}.$$

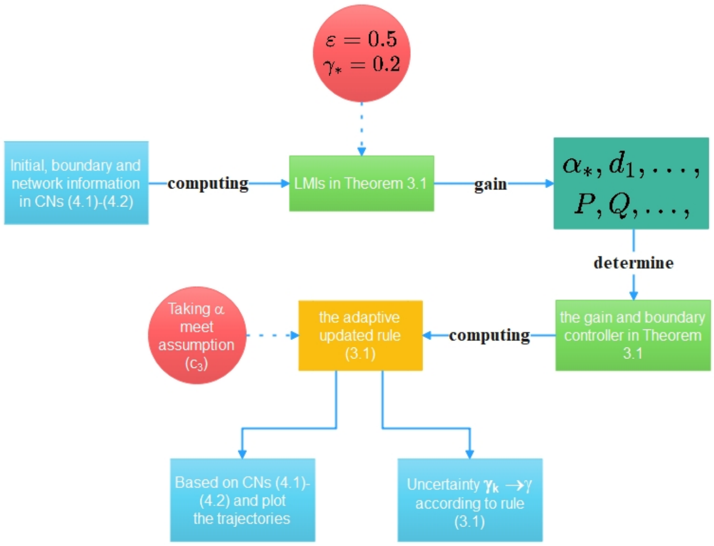

Figure 3. Flow diagram of achieving the adaptive anti-synchronization.

Further, taking α in (c_3) small enough, it is easy to ensure that (c_3) is satisfied.

So, the gain and controller are determined as

$$\Pi = \begin{bmatrix} 0.2847 & 0.0281 \\ 0.0863 & 0.5425 \end{bmatrix},$$

$$\begin{bmatrix} \xi_{1,k} \\ \xi_{2,k} \end{bmatrix} = -\sum_{\delta=1}^{5} \begin{bmatrix} 0.2847 & 0.0281 & 0 & 0 \\ 0.0863 & 0.5425 & 0 & 0 \\ 0 & 0 & 0.2847 & 0.0281 \\ 0 & 0 & 0.0863 & 0.5425 \end{bmatrix} \begin{bmatrix} \mathbf{e}_{11,k}^{[\delta]} \\ \mathbf{e}_{12,k}^{[\delta]} \\ \mathbf{e}_{21,k}^{[\delta]} \\ \mathbf{e}_{22,k}^{[\delta]} \end{bmatrix},$$

where $\mathbf{e}_{11,k}^{[\delta]} = \mathbf{w}_{11,k}^{[\delta]} + \mathbf{z}_{1,k}^{[\delta]}$, $\mathbf{e}_{12,k}^{[\delta]} = \mathbf{w}_{12,k}^{[\delta]} + \mathbf{z}_{2,k}^{[\delta]}$, $\mathbf{e}_{21,k}^{[\delta]} = \mathbf{w}_{21,k}^{[\delta]} + \mathbf{z}_{1,k}^{[\delta]}$, $\mathbf{e}_{22,k}^{[\delta]} = \mathbf{w}_{22,k}^{[\delta]} + \mathbf{z}_{2,k}^{[\delta]}$, $(\delta, k) \in (0, 6)_{\mathbb{Z}} \times \mathbb{Z}_0$. The update law for the adaptive parameter $\tilde{\gamma}_k$ is outlined as

$$\begin{aligned} \tilde{\gamma}_{k+1} = \tilde{\gamma}_k - 0.01 \sum_{\delta=1}^{5} & \begin{bmatrix} \mathbf{e}_{11,k}^{[\delta]} \\ \mathbf{e}_{12,k}^{[\delta]} \\ \mathbf{e}_{21,k}^{[\delta]} \\ \mathbf{e}_{22,k}^{[\delta]} \end{bmatrix}^T \begin{bmatrix} 11.8004 & 0.2949 & 0 & 0 \\ 0.1017 & 5.3624 & 0 & 0 \\ 0 & 0 & 11.8004 & 0.2949 \\ 0 & 0 & 0.1017 & 5.3624 \end{bmatrix} \begin{bmatrix} \sin(\mathbf{w}_{11,k}^{[\delta]}) \\ \sin(\mathbf{w}_{12,k}^{[\delta]}) \\ \sin(\mathbf{w}_{21,k}^{[\delta]}) \\ \sin(\mathbf{w}_{22,k}^{[\delta]}) \end{bmatrix} \\ -0.01 \sum_{\delta=1}^{5} & \begin{bmatrix} \mathbf{e}_{11,k+1}^{[\delta]} - \mathbf{e}_{11,k}^{[\delta]} \\ \mathbf{e}_{12,k+1}^{[\delta]} - \mathbf{e}_{12,k}^{[\delta]} \\ \mathbf{e}_{21,k+1}^{[\delta]} - \mathbf{e}_{21,k}^{[\delta]} \\ \mathbf{e}_{22,k+1}^{[\delta]} - \mathbf{e}_{22,k}^{[\delta]} \end{bmatrix}^T \begin{bmatrix} 11.0326 & 0.3386 & 0 & 0 \\ 0.1300 & 5.8284 & 0 & 0 \\ 0 & 0 & 11.0326 & 0.3386 \\ 0 & 0 & 0.1300 & 5.8284 \end{bmatrix} \begin{bmatrix} \sin(\mathbf{w}_{11,k}^{[\delta]}) \\ \sin(\mathbf{w}_{12,k}^{[\delta]}) \\ \sin(\mathbf{w}_{21,k}^{[\delta]}) \\ \sin(\mathbf{w}_{22,k}^{[\delta]}) \end{bmatrix}, \end{aligned} \tag{4.3}$$

where the initial value is $\tilde{\gamma}_0 = 0$, $k \in \mathbb{Z}_+$. Apparently, (c_4) holds. In sum, all assumptions in Theorem 3.1 are valid. By Theorem 3.1, the response CDNs (4.2) asymptotically inversely synchronizes the drive CDNs (4.1); please see Figures 4–19.

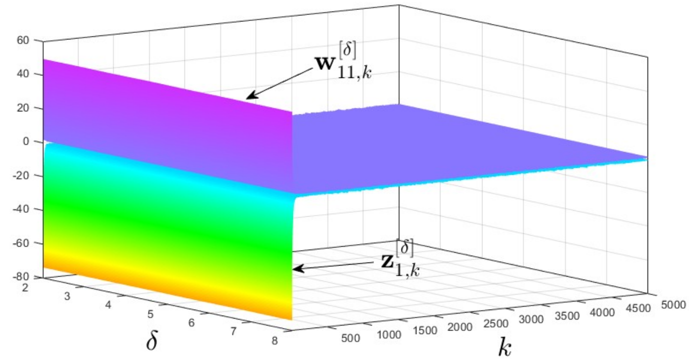

Figure 4. 3D anti-synchronization of states $\mathbf{w}_{11}$ and $\mathbf{z}_1$.

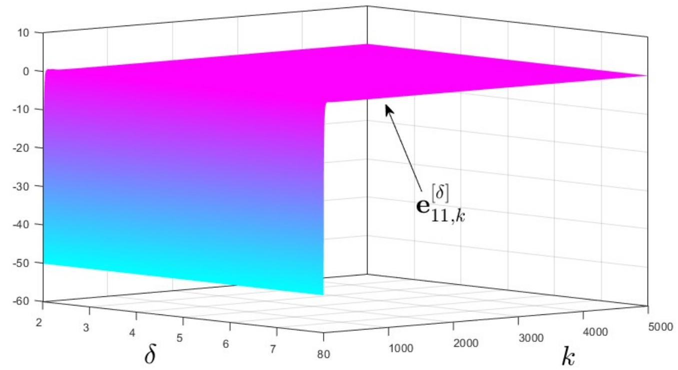

Figure 5. The error of states $\mathbf{w}_{11}$ and $\mathbf{z}_1$ in 3D space.

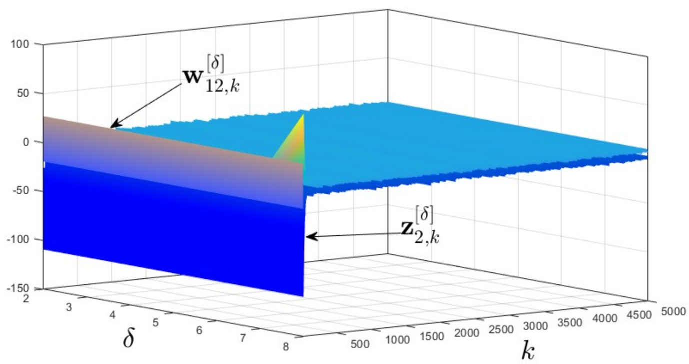

Figure 6. 3D anti-synchronization of states $\mathbf{w}_{12}$ and $\mathbf{z}_2$.

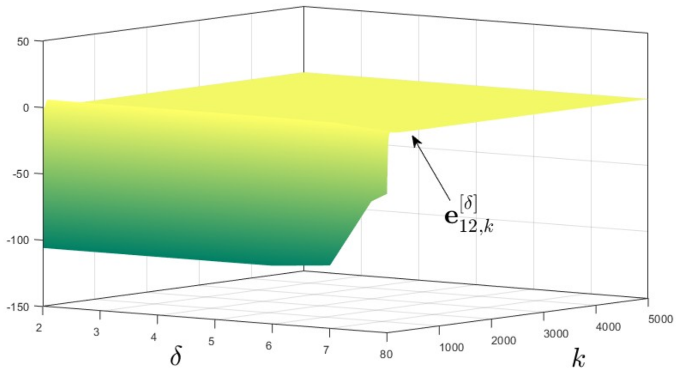

Figure 7. The error of states $\mathbf{w}_{12}$ and $\mathbf{z}_2$ in 3D space.

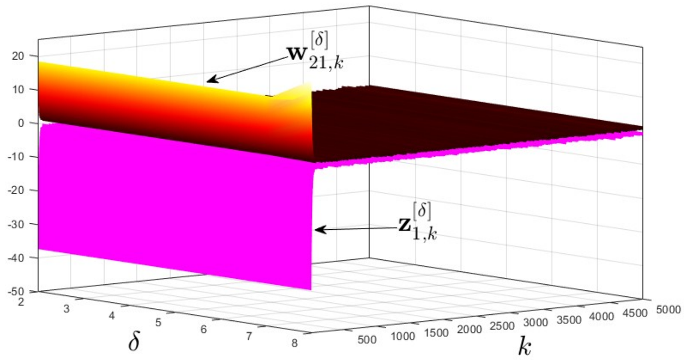

Figure 8. 3D anti-synchronization of states $\mathbf{w}_{21}$ and $\mathbf{z}_1$.

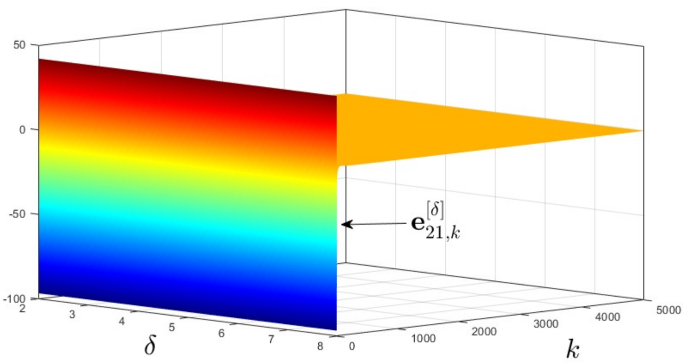

Figure 9. The error of states $\mathbf{w}_{21}$ and $\mathbf{z}_1$ in 3D space.

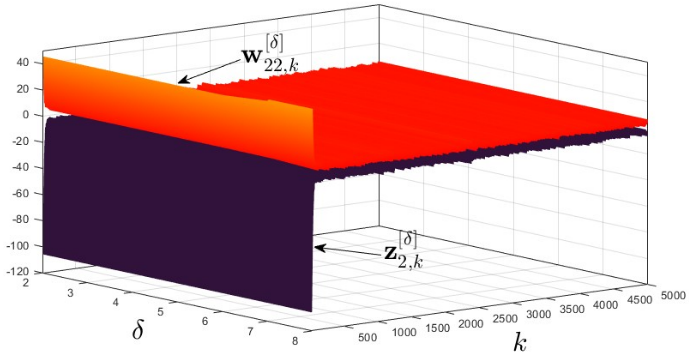

Figure 10. 3D asymptotical anti-synchronization of states $\mathbf{w}_{22}$ and $\mathbf{z}_2$.

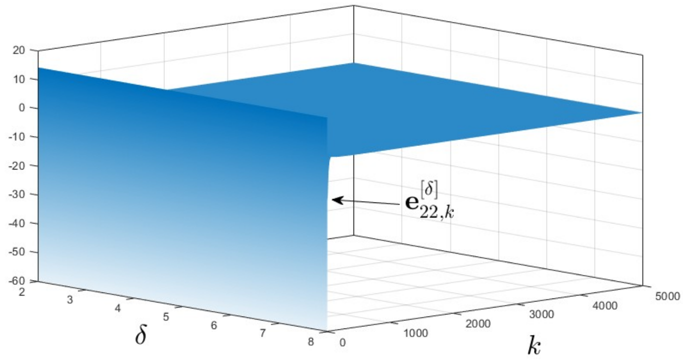

Figure 11. The error of states $\mathbf{w}_{22}$ and $\mathbf{z}_2$ in 3D space.

In Figures 4–11, we picture the 3-dimensional asymptotical anti-synchronization of the drive-response CDNs (4.1) and (4.2). Specifically, by taking the space variable $\delta = 3$, we also show the trajectories of the errors between the drive-response CDNs (4.1) and (4.2) in the 2-dimensional space; see Figures 12–19. Finally, based upon the update law (4.3) for adaptive parameter $\tilde{\gamma}_k$, we give a variational curve in Figure 20, which tends to $\gamma = 0.1$ over time.

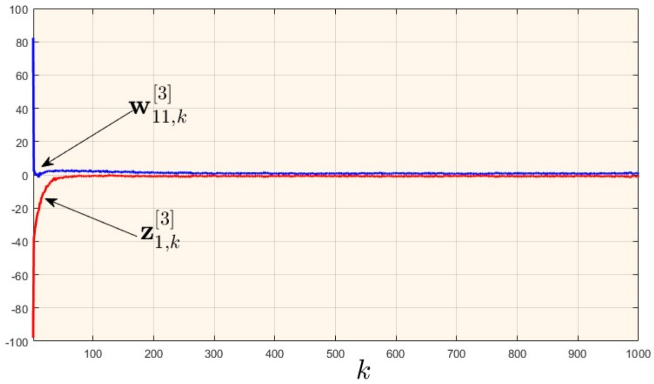

Figure 12. 2D anti-synchronization of states $\mathbf{w}_{11}$ and $\mathbf{z}_1$.

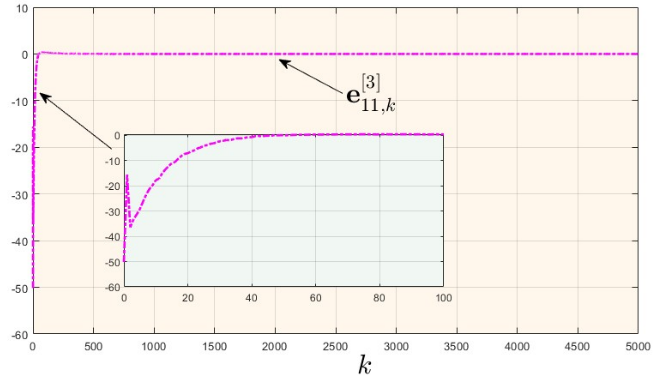

Figure 13. The error of states $\mathbf{w}_{11}$ and $\mathbf{z}_1$ in 2D space.

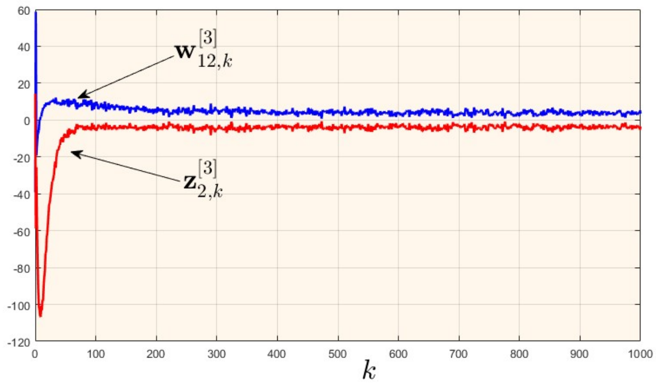

Figure 14. 2D anti-synchronization of states $\mathbf{w}_{12}$ and $\mathbf{z}_2$.

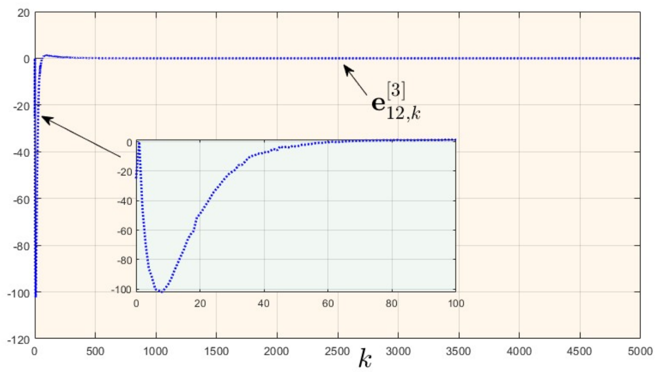

Figure 15. The error of states $\mathbf{w}_{12}$ and $\mathbf{z}_2$ in 2D space.

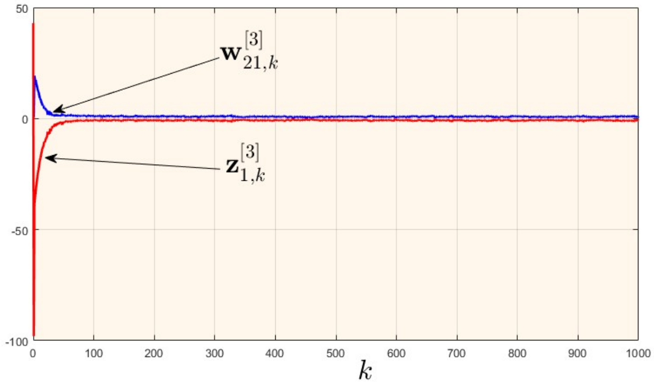

Figure 16. 2D anti-synchronization of states $\mathbf{w}_{21}$ and $\mathbf{z}_1$.

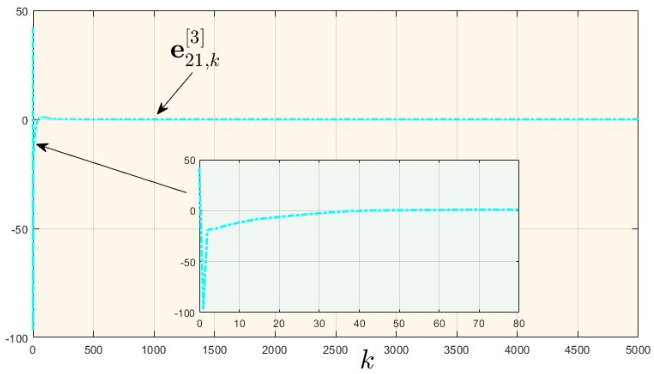

Figure 17. The error of states $\mathbf{w}_{21}$ and $\mathbf{z}_1$ in 2D space.

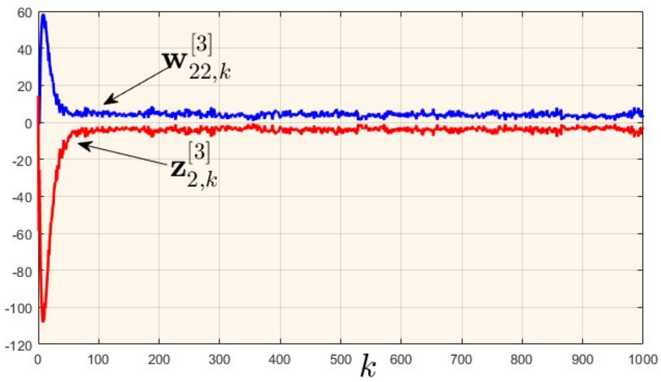

Figure 18. 2D anti-synchronization of states $\mathbf{w}_{22}$ and $\mathbf{z}_2$.

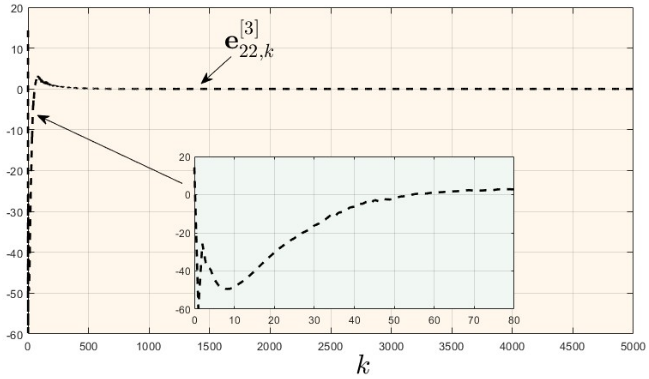

Figure 19. The error of states $\mathbf{w}_{22}$ and $\mathbf{z}_2$ in 2D space.

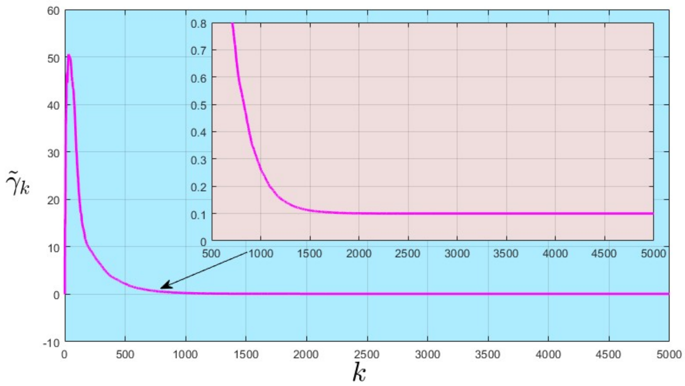

Figure 20. The variation of adaptive parameter $\tilde{\gamma}$.

5. Conclusions

In the first place, by using mixed approaches of Euler time-difference and finite central space-difference, we consider a delayed multivariable discrete stochastic transportation networks with unknown parameter. In the second place, based on the theory of matrix, the compact drive-response nonidentical stochastic transportation networks with the adaptive parameter are built. Via the method of the Lyapunov-Krasovskii functional and the designation of the update law for the unknown parameter in the response networks, the article performs the mean squared asymptotic anti-synchronization for the drive-response nonidentical stochastic transportation networks. Besides, by designing an adaptive parameter updated rule, the adaptive parameter in the response networks can achieve the accuracy identification to the unknown information in the drive networks. The current work establishes a theory of the problem of asymptotic anti-synchronization for the drive-response nonidentical stochastic transportation networks under the framework of space-time discrete schemes.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the Scientific Research Funds of Educational Department of Yunnan Province under Grant No. 2024J1169.

Conflict of interest

The authors declare there is no conflicts of interest.

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