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Research article

Normalized solution for a kind of coupled Kirchhoff systems

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Abstract: In this paper, we investigate the existence of a normalized solution for the following Kirchhoff system in the entire space $\mathbb{R}^N$ ($N \geq 3$):

$$\begin{cases} -\left(1 + \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u = \lambda_1 u + \mu_1 |u|^{p-2} u + \beta r_1 |u|^{r_1 - 2} u |v|^{r_2}, \\ -\left(1 + \int_{\mathbb{R}^N} |\nabla v|^2 dx\right) \Delta v = \lambda_2 v + \mu_2 |v|^{q-2} v + \beta r_2 |u|^{r_1} |v|^{r_2 - 2} v, \end{cases} \tag{P}$$

under the constraints $\int_{\mathbb{R}^N} |u|^2 dx = m_1$ and $\int_{\mathbb{R}^N} |v|^2 dx = m_2$, where $m_1, m_2 > 0$ are prescribed. The parameters $\mu_1, \mu_2, \beta > 0$, $2 \leq p, q < 2 + \frac{8}{N}$, $r_1, r_2 > 1$, and satisfy $r_1 + r_2 = 2^* = \frac{2N}{N-2}$. The frequencies λ_1, λ_2 appear as Lagrange multipliers. With the help of the Pohožaev manifold and the minimization of the energy functional over a combination of the mass constraints and the closed balls, we obtain a positive ground state solution to (P). We mainly extend the results of Yang (Normalized ground state solutions for Kirchhoff-type systems) concerning the above problem from a single critical to a coupled critical nonlinearity.

Keywords: normalized solutions; nonlinearity; coupled Kirchhoff equation; Pohožaev manifold

1. Introduction

In the present paper, we study the following Kirchhoff system with a coupled critical nonlinearity

$$\begin{cases} -\left(1 + \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u = \lambda_1 u + \mu_1 |u|^{p-2} u + \beta r_1 |u|^{r_1 - 2} u |v|^{r_2}, \\ -\left(1 + \int_{\mathbb{R}^N} |\nabla v|^2 dx\right) \Delta v = \lambda_2 v + \mu_2 |v|^{q-2} v + \beta r_2 |u|^{r_1} |v|^{r_2 - 2} v, \end{cases} \tag{1.1}$$

having prescribed mass

$$\int_{\mathbb{R}^N} |u|^2 dx = m_1 \quad \text{and} \quad \int_{\mathbb{R}^N} |v|^2 dx = m_2, \tag{1.2}$$

where $m_1, m_2 > 0$, $\lambda_1, \lambda_2, \beta > 0$ and $N \geq 3$, λ_1, λ_2 are unknown parameters that will appear as Lagrange multipliers.

Problem (1.1) originates from the steady-state analogy of the equation:

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.3}$$

which was proposed by Kirchhoff in 1883 in [1] as the existence of the classical D'Alembert wave equation for the free vibration of elastic strings. The Kirchhoff model takes into consideration the changes in the length of the string that are caused by transverse vibrations.

In recent years, lots of interesting results on the normalized solutions for the Kirchhoff type problem that has been obtained. From a physical perspective, the mass $\int_{\mathbb{R}^N} |u|^2 dx = m$ may represent the number of the power supply in the framework of nonlinear optics or Bose-Einstein condensates. Alternatively, finding normalized solutions seems to be particularly meaningful because the L^2-norm of such solutions is a preserved quantity of the evolution, and their variational characterization can help to analyze the orbital stability or instability, e.g., see [2–4]. In Bose-Einstein condensates, the parameters μ_i and β both describe the interactions between particles. When $\beta > 0$, the two components attract each other, while $\beta < 0$, the two components repel each other.

Based on the above important background, the problem like (1.1) has been studied in numerous papers. For example, Yang [5] has obtained a couple of positive solutions to the following equation:

$$\begin{cases} -\left(a_1 + b_1 \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u = \lambda_1 u + \mu_1 |u|^{p-2} u + \beta r_1 |u|^{r_1 - 2} u |v|^{r_2}, \\ -\left(a_2 + b_2 \int_{\mathbb{R}^N} |\nabla v|^2 dx\right) \Delta v = \lambda_2 v + \mu_2 |v|^{q-2} v + \beta r_2 |u|^{r_1} |v|^{r_2 - 2} v, \end{cases} \tag{1.4}$$

where $a_i, b_i > 0 (i = 1, 2)$ and $2 \leq N \leq 4$. By proving that (1.4) satisfies the mountain pass structure, they obtained a couple of positive solutions. In particular, as $\beta > 0$, Cao et al. [6] considered the L^2-subcritical case and L^2-critical case of the problem by the bifurcation method and showed the existence of normalized solutions when $N \leq 3$. Eq (1.1) can also be formally transformed into the following fractional Kirchhoff equation

$$\begin{cases} \left(a_1 + b_1 \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u|^2 \mathrm{d}x\right) (-\triangle)^s u + \lambda u = f(u) + \gamma v, & \text{in } \mathbb{R}^3, \\ \left(a_2 + b_2 \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} v|^2 \mathrm{d}x\right) (-\triangle)^s v + \mu v = g(v) + \gamma u, & \text{in } \mathbb{R}^3, \\ u, v \in H^s\left(\mathbb{R}^3\right), \end{cases} \tag{1.5}$$

where $a_i, b_i (i = 1, 2), \lambda, \mu > 0$. When $s \in [\frac{3}{4}, 1)$ and $\gamma > 0$, by assuming that the nonlinear terms f and g satisfy Berestycki-Lions conditions, and combining with Pohožaev identity, Che and Chen in [7] proved problem (1.5) has positive ground state solutions, and the asymptotic behavior of the solution was also studied when $\gamma \to 0^+$. When $s = 1$, Lü and Peng [8] proved that (1.5) has vector solutions. We refer readers to [9, 10] for multiplicity solutions. However, to our knowledge, there are few articles discussing the results regarding $N \geq 5$ for the Kirchhoff-type system . This motivates us to consider the solution of the Kirchhoff system (1.1) for $N \geq 3$ and with a coupled critical nonlinearity, where $2 \leq p, q < 2 + \frac{8}{N}$ and $r_1 + r_2 = 2^* = \frac{2N}{N-2}$.

Other forms of (1.1), such as the Schrödinger equation, have also been extensively studied. For example, Li and Zou [11] considered the case with $2 < p, r_1 + r_2 < 2^*, q \leq 2^*$ of the following equation:

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 |u|^{p-2}u + \beta r_1 |u|^{r_1-2}|v|^{r_2}u & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2 |v|^{q-2}v + \beta r_2 |u|^{r_1}|v|^{r_2-2}v & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = a_1^2 \quad \text{and} \quad \int_{\mathbb{R}^N} v^2 dx = a_2^2. \end{cases} \tag{1.6}$$

When $2 < r_1 + r_2 < 2^* = p = q$, Bartsch et al. in [12] have proved (1.6) has a normalized ground state solution and have also investigated the asymptotic behavior by the symmetric decreasing rearrangement and the Ekeland variational principle. When $2 + \frac{4}{N} < p, q < r_1 + r_2 < 2^*$ and $N \geq 3$, Liu and Fang [13] obtained the existence of positive normalized solutions of (1.6) by revealing the basic behavior of mountain-pass energy. Compared with Schrödinger equations, it is more challenging and interesting to study problem (1.1) due to the nonlocal term $\int_{\mathbb{R}^N} |\nabla u|^2 dx \Delta u$ and $\int_{\mathbb{R}^N} |\nabla v|^2 dx \Delta v$.

In order to study the solution of Eq (1.1) satisfying the normalized condition (1.2), it suffices to consider the critical points of the functional

$$\begin{aligned} \mathcal{I}(u,v) = {} & \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx + \frac{1}{4}\left[\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx\right]^2 - \frac{\mu_1}{p}\int_{\mathbb{R}^N} |u|^p dx \\ & - \frac{\mu_2}{q}\int_{\mathbb{R}^N} |v|^q dx - \beta \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} dx, \end{aligned} \tag{1.7}$$

on the constraint $S(m_1, m_2) = S(m_1) \times S(m_2)$, where $S(m) = \{u \in H^1(\mathbb{R}^N) : \|u\|_2^2 = m\}$ for $m > 0$. In this paper, we employ the Pohožaev manifold, which is defined by (1.8) and plays a crucial role, encompassing all solutions that satisfy the condition $(u, v) \in S(m_1, m_2)$

$$P(m_1, m_2) = \{(u, v) \in S(m_1, m_2) : \vartheta(u, v) = 0\}, \tag{1.8}$$

where

$$\begin{aligned} \vartheta(u,v) = {} & \int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right) dx + \left[\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right) dx\right]^2 - \mu_1 \delta_p \int_{\mathbb{R}^N} |u|^p dx \\ & - \mu_2 \delta_q \int_{\mathbb{R}^N} |v|^q dx - \beta 2^* \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} dx, \end{aligned}$$

where $\delta_t = \frac{N(t-2)}{2t}$. To accommodate the constraint $S(m)$, it becomes crucial to define dilation

$$(t * u)(x) = e^{\frac{Nt}{2}} u(e^t x), \quad \text{for} \quad \text{a.e. } x \in \mathbb{R}^N.$$

Consider the following functionals $\mathcal{I}(u, v)$ and $\mathcal{L}_{u,v}(t)$

$$\begin{aligned} \mathcal{L}_{u,v}(t) = \mathcal{I}(t * u, t * v) = {} & \frac{1}{2}e^{2t}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right) dx + \frac{1}{4}e^{4t}\left[\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right) dx\right]^2 \\ & - \frac{\mu_1}{p}e^{p\delta_p t}\int_{\mathbb{R}^N} |u|^p dx - \frac{\mu_2}{q}e^{q\delta_q t}\int_{\mathbb{R}^N} |v|^q dx - \beta e^{2^* t}\int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} dx, \end{aligned}$$

for any $(u, v) \in S(m_1, m_2)$.

Remark 1.1. *As in [5], if (u,v) is a solution of (1.1), then* $(u, v) \in P(m_1, m_2)$*. We can also see that if*

$(u,v) \in S(m_1,m_2)$, then $(e^{\frac{Nt}{2}}u(e^t x), e^{\frac{Nt}{2}}v(e^t x)) \in S(m_1,m_2)$. Furthermore, for fixed $(u,v) \in S(m_1,m_2)$, by performing a simple calculation, we can obtain $(\mathcal{L}_{u,v})'(0) = \vartheta(u,v)$. Then we have that $(t * u, t * v) \in P(m_1,m_2)$ if and only if t is a critical point of $\mathcal{L}_{u,v}(t)$. In addition, $(u,v) \in P(m_1,m_2)$ if $t = 0$ is a critical point of $\mathcal{L}_{u,v}(t)$.

To prove the existence of a normalized solution to (1.1), we use the following assumptions:
(H_1) $N \in \{3,4\}$, $2 < p,q < 2 + \frac{8}{N}$, $r_1 + r_2 = 2^*$.
(H_2) $N \geq 5$, $2 < p,q < 2 + \frac{2}{N-2}$, $r_1 + r_2 = 2^*$.
Here comes our main result:

Theorem 1.2. *Assume that (H_1) or (H_2) is established. Then, there exist $\beta_\tau = \beta_\tau(m_1,m_2) > 0$ and $\rho_\tau = \rho_\tau(m_1,m_2) > 0$ such that for arbitrary $0 < \beta < \beta_\tau$, (1.1) has a positive ground state solution (u,v) for $\lambda_1, \lambda_2 < 0$, which satisfies*

$$\mathcal{I}(u,v) = \inf_{(u,v)\in P(m_1,m_2)} \mathcal{I}(u,v) = \inf_{(u,v)\in S(m_1,m_2)\cap V(\rho_\tau)} \mathcal{I}(u,v) < 0,$$

where

$$V(r) = \{(u,v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) : \|\nabla u\|_2^2 + \|\nabla v\|_2^2 < r^2\}.$$

Remark 1.3. *Due to the additional difficulties caused by the combined effect of the nonlocal term $\int_{\mathbb{R}^N} |\nabla u|^2 dx \Delta u$, $\int_{\mathbb{R}^N} |\nabla v|^2 dx \Delta v$ and multiple powers, the study is much more challenging; for example, the functional $\mathcal{I}(u,v)$ is composed of several distinct terms that exhibit varying scaling behavior with respect to the dilation $e^{\frac{Nt}{2}}u(e^t x)$. The intricate interplay among these terms makes it more difficult to ascertain the types of critical points for $\mathcal{I}(u,v)$ on $S(m_1,m_2)$. Furthermore, when proving $(\tilde{u}_n, \tilde{v}_n) \to (u,v)$ in $D^{1,2}(\mathbb{R}^N;\mathbb{R}^2)$, the inequalities that need to be estimated will also be more difficult.*

Remark 1.4. *From a variational point of view, besides the Sobolev critical exponent $2^* := \frac{2N}{N-2}$ for $N \geq 3$ and $2^* = \infty$ for $N = 1,2$, a new L^2-critical exponent $P_N := 2 + \frac{8}{N}$ arises that plays a pivotal role in the study of normalized solutions to (1.1). This threshold determines whether the constrained functional $\mathcal{I}(u,v)$ remains bounded from below on $S(m_1,m_2)$.*

Definition 1.5. *We say that $(\tilde{u},\tilde{v})$ is a couple of ground state solutions to (1.1) on $S(m_1,m_2)$ if it is a couple of solutions to (1.1) having minimal energy among all the solutions, i.e., $d\mathcal{I}|_{S(m_1,m_2)}(\tilde{u},\tilde{v}) = 0$ and*

$$\mathcal{I}(\tilde{u},\tilde{v}) = \inf\{\mathcal{I}(u,v) : d\mathcal{I}|_{S(m_1,m_2)}(u,v) = 0 \text{ and } (u,v) \in S(m_1,m_2)\}.$$

2. Preliminary results

In this section, we recall some preliminary results that will be used later. Throughout this paper, we represent the norms on $L^t(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$ with $\|\cdot\|_t$ and $\|\cdot\|$, respectively. Denote $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ by $\mathcal{V}$ with the norm

$$\|(u,v)\|_{\mathcal{V}}^2 = \|u\|^2 + \|v\|^2.$$

Let $L^t(\mathbb{R}^N;\mathbb{R}^2)$ be the space $L^t(\mathbb{R}^N \times \mathbb{R}^N)$ with the norm

$$\|(u,v)\|_{L^t}^t = \|u\|_t^t + \|v\|_t^t.$$

$D^{1,2}(\mathbb{R}^N)$ represents the closure of the $C_c^\infty(\mathbb{R}^N)$ with norm

$$\|u\|_{D^{1,2}} = \|\nabla u\|_2.$$

For $N \geq 3$, the best Sobolev constant is given by

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}. \tag{2.1}$$

For all $u \in H^1(\mathbb{R}^N)$, we consider the Gagliardo-Nirenberg-Sobolev inequality:

$$\|u\|_p^p \leq C_p^p \|u\|_2^{p(1-\delta_p)} \|\nabla u\|_2^{p\delta_p}, \text{ where } \delta_p = \frac{N(p-2)}{2p}. \tag{2.2}$$

For any $u, v \in H^1(\mathbb{R}^N)$, by the Young's inequality, we can prove:

$$\begin{aligned} \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} dx &\leq \int_{\mathbb{R}^N} \frac{r_1}{2^*}|u|^{2^*} dx + \int_{\mathbb{R}^N} \frac{r_2}{2^*}|v|^{2^*} dx \\ &\leq S^{-\frac{2^*}{2}} \left(\frac{r_1}{2^*}\|\nabla u\|_2^{2^*} + \frac{r_2}{2^*}\|\nabla v\|_2^{2^*} \right) \\ &\leq S^{-\frac{2^*}{2}} \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right)^{\frac{2^*}{2}}. \end{aligned} \tag{2.3}$$

Furthermore, taking into consideration the existing results of the Kirchhoff equation as follows:

$$\begin{cases} -(1 + \int_{\mathbb{R}^N} |\nabla u|^2)\triangle u = \lambda u + \mu |u|^{p-2}u, \text{ in } \mathbb{R}^N; \\ \int_{\mathbb{R}^N} |u|^2 = m > 0. \end{cases} \tag{P_m}$$

Solution u of (P_m) can be found as critical points of the functional $\mathcal{I}_\mu(u)$ defined by

$$\mathcal{I}_\mu(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 - \frac{\mu}{p}\int_{\mathbb{R}^N} |u|^p dx$$

constrained to the L^2-sphere $S(m)$.
Similar to [14] and [6], we can get the following lemma.
Lemma 2.1 *([6]). Assume that* $p \in (2, 2 + \frac{8}{N})$, $m > 0$, *and* $\mu > 0$. *Set*

$$\zeta_p^\mu(m) := \inf_{u \in S(m)} \mathcal{I}_\mu(u).$$

Then,
(i) *there exists a unique couple* $(u_{m,\mu}, \lambda_m) \in \mathbb{R}^+ \times H^1(\mathbb{R}^N)$ *satisfying* (P_m)*;*
(ii) $\mathcal{I}_\mu(u_{m,\mu}) = \zeta_p^\mu(m) < 0$*;*
(iii) *the map* $m \mapsto \zeta_p^\mu(m)$ *is strictly decreasing with respect to m, and* $\zeta_p^\mu(m) \to -\infty$ *as* $m \to +\infty$.

3. Proof of Theorem 1.2

To begin with, we set

$$\gamma_1 = u_{m_1,\mu_1}, \quad \gamma_2 = u_{m_2,\mu_2}$$

and

$$\zeta_1 = \mathcal{I}_\mu(\gamma_1), \ \zeta_2 = \mathcal{I}_\mu(\gamma_2).$$

Lemma 3.1. *Let* $m_1, m_2, \mu_1, \mu_2 > 0$ *be given and assume* (H_1) or (H_2) *holds. Then there exists* $\beta_\tau = \beta_\tau(m_1, m_2) > 0$ *and* $\rho_\tau = \rho_\tau(m_1, m_2) > (\|\nabla\gamma_1\|_2^2 + \|\nabla\gamma_2\|_2^2)^{\frac{1}{2}}$ *such that*

$$\mathcal{I}(u, v) > 0 \quad \text{on} \quad S(m_1, m_2) \cap V(2\rho_\tau)\setminus V(\rho_\tau) \quad \text{for any } 0 < \beta < \beta_\tau.$$

Proof. For $(u,v)\in\mathcal{V}$, let $\rho=(\|\nabla u\|_2^2+\|\nabla v\|_2^2)^{\frac{1}{2}}$. From (2.2) and (2.3), we derive that

$$\begin{aligned}\mathcal{I}(u,v)&\geq\frac{1}{2}(\|\nabla u\|_2^2+\|\nabla v\|_2^2)+\frac{1}{4}(\|\nabla u\|_2^2+\|\nabla v\|_2^2)^2-\frac{\mu_1}{p}C_p^p\|u\|_2^{p(1-\delta_p)}\|\nabla u\|_2^{p\delta_p}\\&\quad-\frac{\mu_2}{q}C_q^q\|v\|_2^{q(1-\delta_q)}\|\nabla v\|_2^{q\delta_q}-\beta S^{-\frac{2^*}{2}}\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)^{\frac{2^*}{2}}\\&\geq\frac{1}{2}\rho^2+\frac{1}{4}\rho^4-\frac{\mu_1}{p}C_p^pm_1^{\frac{p(1-\delta_p)}{2}}\rho^{p\delta_p}-\frac{\mu_2}{q}C_q^qm_2^{\frac{q(1-\delta_q)}{2}}\rho^{q\delta_q}-\beta S^{-\frac{2^*}{2}}\rho^{2^*}\\&=\rho^2[\frac{1}{2}+\frac{1}{4}\rho^2-\frac{\mu_1}{p}C_p^pm_1^{\frac{p(1-\delta_p)}{2}}\rho^{p\delta_p-2}-\frac{\mu_2}{q}C_q^qm_2^{\frac{q(1-\delta_q)}{2}}\rho^{q\delta_q-2}-\beta S^{-\frac{2^*}{2}}\rho^{2^*-2}].\end{aligned}\tag{3.1}$$

Recalling that $p\delta_q<2$ and $q\delta_q<2$, we can take a large enough

$$\rho_\tau>\max\{\|\nabla\gamma_1\|_2,\|\nabla\gamma_2\|_2\},$$

such that

$$\frac{\mu_1}{p}C_p^pm_1^{\frac{p(1-\delta_p)}{2}}\rho_\tau^{p\delta_p-2}+\frac{\mu_2}{q}C_q^qm_2^{\frac{q(1-\delta_q)}{2}}\rho_\tau^{q\delta_q-2}\leq\frac{1}{4}.\tag{3.2}$$

Due to the fact that $2^*-2>0$, there exists a $\beta_\tau>0$ such that

$$\beta_\tau S^{-\frac{2^*}{2}}(2\rho_\tau)^{2^*-2}\leq\frac{1}{8}.\tag{3.3}$$

We conclude that $\mathcal{I}(u,v)>0$ follows from (3.1)–(3.3). □

Define

$$\mathcal{M}(m_1,m_2):=\inf_{(u,v)\in S(m_1,m_2)\cap V(2\rho_\tau)}\mathcal{I}(u,v),$$

where ρ_τ is defined in Lemma 3.1.

Lemma 3.2. *Let $m_1,m_2,\mu_1,\mu_2>0$ be given, and (H_1) or (H_2) is true. Then for arbitrary $0<\beta<\beta_\tau$, the following statements are true:*
(i) $\mathcal{M}(m_1,m_2)<\zeta_1+\zeta_2<0$;
(ii) $\mathcal{M}(m_1,m_2)\leq\mathcal{M}(m_{\alpha1},m_{\alpha2})$, for any $0<m_{\alpha1}<m_1$ and $0<m_{\alpha2}<m_2$.

Proof. (*i*) From Lemma 3.1, we know that $(\gamma_1,\gamma_2)\in V(\rho_\tau)$. Moreover, we deduce that

$$\begin{aligned}\mathcal{M}(m_1,m_2)&\leq\mathcal{I}(\gamma_1,\gamma_2)=\mathcal{I}_{\mu_1}(\gamma_1)+\mathcal{I}_{\mu_2}(\gamma_2)-\beta\int_{\mathbb{R}^N}|\gamma_1|^{r_1}|\gamma_2|^{r_2}dx\\&<\zeta_1+\zeta_2<0.\end{aligned}$$

(*ii*) The proof is similar to that of [15]. We just need to prove that for arbitrary $\epsilon>0$,

$$\mathcal{M}(m_1,m_2)\leq\mathcal{M}(m_{\alpha1},m_{\alpha2})+\epsilon$$

for any $0<m_{\alpha1}<m_1$ and $0<m_{\alpha2}<m_2$.
By Lemma 3.1 and the definition of $\mathcal{M}(m_{\alpha1},m_{\alpha2})$, there exist $u,v\in S(m_{\alpha1},m_{\alpha2})\cap V(\rho_\tau)$ such that

$$\mathcal{I}(u,v)\leq\mathcal{M}(m_{\alpha1},m_{\alpha2})+\frac{\epsilon}{2}.$$

Define a cut-off function: $\omega \in C_m^\infty(\mathbb{R}^N)$ such that

$$0 \le \omega(t) \le 1 \ \text{ and } \omega(t) = \begin{cases} 1, & |t| \le 1; \\ 0, & |t| \ge 2. \end{cases} \tag{3.4}$$

For any $\iota > 0$, we define $(u_\iota(t), v_\iota(t))$=$(u\omega(\iota t), v\omega(\iota t))$. Clearly, $(u_\iota, v_\iota) \to (u, v)$ in $\mathcal{V}$ as $\iota \to 0^+$. As a consequence, for $\eta > 0$ small enough, there exists a sufficiently small ι such that

$$\mathcal{I}(u_\iota, v_\iota) \le \mathcal{I}(u, v) + \frac{\varepsilon}{4} \ \text{ and } \ (\|\nabla u_\iota\|_2^2 + \|\nabla v_\iota\|_2^2)^{\frac{1}{2}} < \rho_\tau - \eta. \tag{3.5}$$

Let $\chi(t) \in C_m^\infty(\mathbb{R}^N)$ such that $\text{supp}(\chi) \subset \{t \in \mathbb{R}^N : \frac{4}{\iota} \le |t| \le 1 + \frac{4}{\iota}\}$ and set

$$(u_{m_1}, v_{m_2}) = \left(\frac{\sqrt{m_1 - \|u_\iota\|_2}}{\|\chi\|_2}\chi, \frac{\sqrt{m_2 - \|v_\iota\|_2}}{\|\chi\|_2}\chi \right).$$

And observe that

$$\text{supp}(u_\iota) \cap \text{supp}(t * u_{m_1}) = \emptyset \ \text{ and } \ \text{supp}(v_\iota) \cap \text{supp}(t * v_{m_2}) = \emptyset$$

for any $t \le 0$, hence,

$$(u_\iota + t * u_{m_1}, v_\iota + t * v_{m_2}) \in S_m.$$

Next, since

$$\mathcal{I}(t * u_{m_1}, t * v_{m_2}) \to 0 \quad \text{and} \quad (\|\nabla t * u_{m_1}\|_2^2 + \|\nabla t * v_{m_2}\|_2^2)^{\frac{1}{2}} \to 0,$$

as $t \to -\infty$, we can obtain

$$\mathcal{I}(t * u_{m_1}, t * v_{m_2}) \le \frac{\varepsilon}{4} \text{ and } \left(\|\nabla t * u_{m_1}\|_2^2 + \|\nabla t * v_{m_2}\|_2^2\right)^{1/2} \le \frac{\eta}{2}, \ \text{ for } t \ll 0. \tag{3.6}$$

It follows that

$$\left(\nabla\|(u_\iota + t * u_{m_1})\|_2^2 + \nabla\|(v_\iota + t * v_{m_2})\|_2^2\right)^{1/2} < \rho_\tau.$$

Using (3.5) and (3.6), we conclude

$$\begin{aligned} \mathcal{M}(m_1, m_2) &\le \mathcal{I}(u_\iota + t * u_{m_1}, v_\iota + t * v_{m_2}) = \mathcal{I}(u_\iota, v_\iota) + \mathcal{I}(t * u_{m_1}, t * v_{m_2}) \\ &\le \mathcal{I}(u, v) + \frac{\varepsilon}{2} \\ &\le \mathcal{M}(m_{\alpha 1}, m_{\alpha 2}) + \varepsilon \end{aligned}$$

for $t \ll 0$. □

Lemma 3.3. *Let $m_1, m_2, \mu_1, \mu_2 > 0$, and assume that either (H_1) is true or (H_2) is true. Then, for arbitrary $0 < \beta < \beta_\tau$ and $(u, v) \in S(m_1, m_2)$, $\mathcal{L}_{u,v}(t)$ has two critical points $\tau_{u_1 v_1} < \tau_{u_2 v_2} \in \mathbb{R}$ and two zero points $\varphi_1 < \varphi_2$ with $\tau_{u_1 v_1} < \varphi_1 < \tau_{u_2 v_2} < \varphi_2$. Moreover,*
(i) if $(t * u, t * v) \in P(m_1, m_2)$, then $t = \tau_{u_1 v_1}$ or $t = \tau_{u_2 v_2}$;
(ii) $(\|\nabla t * u\|_2^2 + \|\nabla t * v\|_2^2)^{\frac{1}{2}} \le \rho_\tau$ for all $t \le \varphi_1$ and

$$\mathcal{I}(\tau_{u_1 v_1} * u, \tau_{u_1 v_1} * v) = \min\left\{\mathcal{I}(t * u, t * v) : t \in \mathbb{R} \text{ and } (\|\nabla t * u\|_2^2 + \|\nabla t * v\|_2^2)^{\frac{1}{2}} \le \rho_\tau\right\} < 0,$$

where ρ_τ is given in Lemma 3.1;
(iii) $\mathcal{I}(\tau_{u_2 v_2} * u, \tau_{u_2 v_2} * v) = \max\{\mathcal{I}(t * u, t * v) : t \in \mathbb{R}\}$.

Proof. (*i*) Since $q\delta_q, p\delta_p < 2 < 2^*$, it can be seen that $\mathcal{L}_{u,v}(-\infty) = 0^-$ and $\mathcal{L}_{u,v}(+\infty) = -\infty$. According to Lemma 3.1, we obtain that $\mathcal{L}_{u,v}(t)$ has at least two critical points $\tau_{u_1v_1} < \tau_{u_2v_2}$, with $\tau_{u_1v_1}$ local minimum point of $\mathcal{L}_{u,v}(t)$ at a negative level and $\tau_{u_2v_2}$ global maximum point at a positive level. Secondly, similar to [5], it is not difficult to check that there are no other critical points. On the other hand,

$$\mathcal{L}'_{uv}(t) = e^{2t}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + e^{4t}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^2 - e^{p\delta_{p}t}\mu_1\delta_p\|u\|_p^p$$
$$- e^{q\delta_{q}t}\mu_2\delta_q\|v\|_q^q - e^{2^*t}2^*\beta\||u|^{r_1}|v|^{r_2}\|_1.$$

Putting together all the considerations mentioned above, we conclude that $\mathcal{L}_{u,v}$ has exactly two critical points. By monotonicity and recalling the behavior at infinity, $\mathcal{L}_{u,v}$ has moreover exactly two zeros points $\varphi_1 < \varphi_2$ with $\tau_{u_1v_1} < \varphi_1 < \tau_{u_2v_2} < \varphi_2$. From Lemma 3.1 and (*i*), we can deduce the (*ii*) and (*iii*). □

Corollary 3.4. *Let* $m_1, m_2, \mu_1, \mu_2 > 0$, *and assume that either* (H_1) *is true or* (H_2) *is true. Then, for arbitrary* $0 < \beta < \beta_\tau$, *the following inequality holds:*

$$-\infty < \mathcal{M}(m_1, m_2) = \inf_{P(m_1,m_2)} \mathcal{I}(u, v) < 0.$$

Next, we establish a necessary condition for the existence of a non-negative solution to (1.1). This Liouville-type result will be used to prove the existence of a positive solution.

Lemma 3.5.([16]) Suppose $0 < p \leq \frac{N}{N-2}$ when $N \geq 3$ and $0 < p < \infty$ when $N = 1, 2$. Let $u \in L^p(\mathbb{R}^N)$ be a smooth, nonnegative function and satisfy $-\Delta u \geq 0$ in $\mathbb{R}^N$. Then $u \equiv 0$ holds.

Lemma 3.6. *Let* $(u, v) \in S(m_1, m_2)$, $u, v \geq 0$, *and* $u, v \not\equiv 0$, *if (u,v) satisfies*

$$\begin{cases} -(1 + \int_{\mathbb{R}^N} |\nabla u|^2 dx)\Delta u = \lambda_1 u + \mu_1 |u|^{p-2}u + \beta r_1 |v|^{r_2}|u|^{r_1-2}u, \\ -(1 + \int_{\mathbb{R}^N} |\nabla v|^2 dx)\Delta v = \lambda_2 v + \mu_2 |v|^{q-2}v + \beta r_2 |u|^{r_1}|v|^{r_2-2}v, \end{cases} \tag{3.7}$$

then $\lambda_1, \lambda_2 < 0$.

Proof. Arguing by contradiction, we assume that $\lambda_1 \geq 0$. Since $u \geq 0$, we have that all components on the right-hand side of

$$-(1 + \int_{\mathbb{R}^N} |\nabla u|^2 dx)\Delta u = \lambda_1 u + \mu_1 |u|^{p-2}u + \beta r_1 |v|^{r_2}|u|^{r_1-2}u$$

are nonnegative. Hence,

$$-(1 + \int_{\mathbb{R}^N} |\nabla u|^2 dx)\Delta u \geq 0,$$

it is easy to see that

$$-\Delta u \geq 0.$$

Moreover, modifying the standard elliptic regularity theorems, we can ensure that the smoothness of (u, v) is up to C^2. Hence, it follows from Lemma 3.5 that $u = 0$. This contradicts with $u \not\equiv 0$; thus, $\lambda_1 < 0$. The proof of $\lambda_2 < 0$ is the same as that of $\lambda_1 < 0$. □

Lemma 3.7.([17]) *Let* $(u_n)_{n\geq 0} \subset H^1(\mathbb{R}^N)$ *be a bounded sequence of spherically symmetric functions. If* $N \geq 2$ *or if* $u_n(x)$ *is a nonincreasing function of* $|x|$ *for every* $n \geq 0$*, then there exist a subsequence* $(u_{n_k})_{k\geq 0}$ *and* $u \in H^1(\mathbb{R}^N)$ *such that* $u_{n_k} \to u$ *as* $k \to \infty$ *in* $L^p(\mathbb{R}^N)$ *for every* $2 < p < \frac{2N}{N-2}$*.*

Proof of Theorem 1.2. Let us consider a minimizing sequence $\{(u_n, v_n)\}$ for $\mathcal{I}|_{S(m_1,m_2)\cap V(2\rho_\tau)}$ and $\{(u_n, v_n)\} \subset \mathcal{V} \cap S(m_1, m_2)$. Without loss of generality, we can assume that $(u_n, v_n) \subset \mathcal{V}$ are nonnegative and radially decreasing for every n[Otherwise, we replace (u_n, v_n) with $(|u_n|^*, |v_n|^*)$, which is the Schwarz rearrangement of $(|u_n|, |v_n|)$]. Furthermore, by Lemma 3.3 (ii), $(\|\nabla s * u\|_2^2 + \|\nabla s * v\|_2^2)^{\frac{1}{2}} \leq \rho_\tau$, and $\{\tau_{u_n v_n} * u, \tau_{u_n v_n} * v\}$ is still a minimizing sequence for $\mathcal{I}|_{S(m_1,m_2)\cap V(2\rho_\tau)}$. And hence, by the Ekeland variational principle [18], it yields that there exists a new minimizing sequence $\{(\tilde{u}_n, \tilde{v}_n)\}$ satisfying

$$\begin{cases} \|\tilde{u}_n - \tau_{u_n v_n} * \tilde{u}_n\| + \|\tilde{v}_n - \tau_{u_n v_n} * \tilde{u}_n\| \to 0, & \text{as } n \to \infty, \\ \mathcal{I}(\tilde{u}_n, \tilde{v}_n) \to \mathcal{M}(m_1, m_2), & \text{as } n \to \infty, \\ \vartheta(\tilde{u}_n, \tilde{v}_n) \to 0, & \text{as } n \to \infty, \\ \mathcal{I}'|_{S(m_1,m_2)}(\tilde{u}_n, \tilde{v}_n) \to 0, & \text{as } n \to \infty. \end{cases} \tag{3.8}$$

In the sequel, we divide the proof into three steps.
Step 1: $(\tilde{u}_n, \tilde{v}_n) \to (u, v)$ in $L^t(\mathbb{R}^N; \mathbb{R}^2)$ for arbitrarily $t \in (2, 2^*)$.
In fact, from (3.8), we can know that $\mathcal{I}'|_{S(m_1,m_2)}(\tilde{u}_n, \tilde{v}_n) \to 0$. By the Lagrange multipliers theorem, there exist two sequences $\{\lambda_{1,n}\} \subset \mathbb{R}$ and $\{\lambda_{2,n}\} \subset \mathbb{R}$ satisfying the following equation

$$\begin{aligned} &\int_{\mathbb{R}^N} (\nabla\tilde{u}_n\nabla\phi + \nabla\tilde{v}_n\nabla\psi)\,dx + \left(\int_{\mathbb{R}^N} (\nabla\tilde{u}_n\nabla\phi + \nabla\tilde{v}_n\nabla\psi)\,dx\right)^2 \\ &- \int_{\mathbb{R}^N} \left(\mu_1|\tilde{u}_n|^{p-2}\tilde{u}_n\phi + \mu_2|\tilde{v}_n|^{p-2}\tilde{u}_n\psi\right)dx - \beta r_1 \int_{\mathbb{R}^N} |\tilde{u}_n|^{r_1-2}|\tilde{v}_n|^{r_2}\tilde{u}_n\phi dx \\ &- \beta r_2 \int_{\mathbb{R}^N} |\tilde{u}_n|^{r_1}|\tilde{v}_n|^{r_2-2}\tilde{v}_n\psi dx \\ = &\int_{\mathbb{R}^N} (\lambda_{1,n}\tilde{u}_n\phi + \lambda_{2,n}\tilde{v}_n\psi)\,dx + o_n(1)(\|\phi\| + \|\psi\|), \end{aligned} \tag{3.9}$$

for arbitrarily $(\phi, \psi) \in \mathcal{V}$. By substituting $(\tilde{u}_n, 0)$ and $(0, \tilde{v}_n)$ into (3.9), we can derive

$$\lambda_{1,n}m_1 = \|\nabla\tilde{u}_n\|_2^2 + \|\nabla\tilde{u}_n\|_2^4 - \mu_1\|\tilde{u}_n\|_p^p$$

and

$$\lambda_{2,n}m_2 = \|\nabla\tilde{v}_n\|_2^2 + \|\nabla\tilde{v}_n\|_2^4 - \mu_2\|\tilde{v}_n\|_q^q.$$

Since $\{\tilde{u}_n, \tilde{v}_n\} \subset S(m_1, m_2) \cap V(2\rho_\tau)$, up to a subsequence, $(\lambda_{1,n}, \lambda_{2,n}) \to (\lambda_1, \lambda_2) \in \mathbb{R}^2$ and $(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (u, v) \in \mathcal{V}$, where both u and v are non-negative. Combined with that, $\vartheta(u, v) = 0$, then (u, v) is a weak solution of (1.1). By Lemma 3.7, we obtain that $(\tilde{u}_n, \tilde{v}_n) \to (u, v)$ in $L^t(\mathbb{R}^N, \mathbb{R}^2)$ for any $t \in (2, 2^*)$.
Step 2: $(\tilde{u}_n, \tilde{v}_n) \to (u, v)$ in $D^{1,2}(\mathbb{R}^N; \mathbb{R}^2)$.
Let $(u_n, v_n) = (\tilde{u}_n - u, \tilde{v}_n - v)$. Then $u_n \to 0$ in $L^p(\mathbb{R}^N)$ and $v_n \to 0$ in $L^q(\mathbb{R}^N)$. Moreover, from the Brézis-Lieb Lemma, we have

$$\int_{\mathbb{R}^N} [|\tilde{u}_n|^{r_1}|\tilde{v}_n|^{r_2} - |u|^{r_1}|v|^{r_2}]dx = \int_{\mathbb{R}^N} |u_n|^{r_1}|v_n|^{r_2}dx + o_n(1). \tag{3.10}$$

Since $\vartheta(\tilde{u}_n, \tilde{v}_n) - \vartheta(u, v) \to 0$, we can infer from (2.3) and (3.10) that

$$\begin{aligned}
&\|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2 + \left(\|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2\right)^2 \\
=&\beta 2^* \int_{\mathbb{R}^N} |u_n|^{r_1}|v_n|^{r_2}dx + o_n(1) \\
\leq&\beta 2^* S^{-\frac{2^*}{2}} \left(\|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2\right)^{\frac{2^*}{2}} + o_n(1).
\end{aligned} \tag{3.11}$$

Up to a subsequence, we assume that $\|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2 \to R \geq 0$. Then $R = 0$ or $R \geq \left(\frac{1}{\beta 2^*}\right)^{\frac{N-2}{2}} S^{\frac{N}{2}}$. If $R \geq \left(\frac{1}{\beta 2^*}\right)^{\frac{N-2}{2}} S^{\frac{N}{2}}$, from (3.8), (3.10), and (3.11), we have

$$\begin{aligned}
\mathcal{M}(m_1, m_2) = \lim_{n\to\infty} \mathcal{I}(\tilde{u}_n, \tilde{v}_n) &= \mathcal{I}(u, v) + \lim_{n\to\infty} \mathcal{I}(u_n, v_n) \\
&\geq \mathcal{M}(\|u\|_2^2, \|v\|_2^2) + \lim_{n\to\infty} \Big[\frac{1}{2}\left(\|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2\right) \\
&\quad + \frac{1}{4}\left(\|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2\right)^2 - \beta \int_{\mathbb{R}^N} |u_n|^{r_1}|v_n|^{r_2}\Big] \\
&\geq m(\|u\|_2^2, \|v\|_2^2) + \frac{1}{N}\lim_{n\to\infty}\left(\|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2\right) \\
&= m(\|u\|_2^2, \|v\|_2^2) + \frac{1}{N}\left(\frac{1}{\beta 2^*}\right)^{\frac{N-2}{2}} S^{\frac{N}{2}}.
\end{aligned}$$

This contradicts with Lemma 3.2 (*ii*). Then $\|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2 \to 0$. Thus, we conclude $(\tilde{u}_n, \tilde{v}_n) \to (u, v)$ in $D^{1,2}(\mathbb{R}^N; \mathbb{R}^2)$.

Step 3: $(\tilde{u}_n, \tilde{v}_n) \to (u, v)$ in $\mathcal{V}$.

From Step 1, then, as in [19], we know that there exists $(u, v) \in \mathcal{V}$ that is a weak solution of

$$\begin{cases}
-(1 + \int_{\mathbb{R}^N} |\nabla u|^2 dx)\Delta u = \lambda_1 u + \mu_1 |u|^{p-2}u + \beta r_1 |v|^{r_2}|u|^{r_1-2}u, \\
-(1 + \int_{\mathbb{R}^N} |\nabla v|^2 dx)\Delta v = \lambda_2 v + \mu_2 |v|^{q-2}v + \beta r_2 |u|^{r_1}|v|^{r_2-2}v,
\end{cases} \tag{3.12}$$

with

$$\|u\|_2^2 \leq \liminf \|\tilde{u}_n\|_2^2 = m_1 \quad \text{and} \quad \|v\|_2^2 \leq \liminf \|\tilde{v}_n\|_2^2 = m_2.$$

We claim that $u \neq 0$ and $v \neq 0$. Indeed, if $v = 0$, then u satisfies

$$\begin{cases}
-(1 + \int_{\mathbb{R}^N} |\nabla u|^2 dx)\Delta u = \lambda u + \mu |u|^{p-2}u, \text{ in } \mathbb{R}^N, \\
\|u\|_2^2 \leq m_1.
\end{cases}$$

By applying Lemma 2.1, we know that $\zeta_p^{\mu}(m)$ is strictly decreasing with respect to m. So

$$\zeta_p^{\mu_1}(m_1) \leq \zeta_p^{\mu_1}(\|u\|_2^2) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\|\nabla u\|_2^4 - \frac{\mu_1}{p}\|u\|_p^p.$$

However,

$$\mathcal{M}(m_1, m_2) = \lim_{n\to\infty} \mathcal{I}(\tilde{u}_n, \tilde{v}_n)$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{1}{2}(\|\nabla \tilde{u}_n\|_2^2 + \|\nabla \tilde{v}_n\|_2^2) + \frac{1}{4}(\|\nabla \tilde{u}_n\|_2^2 + \|\nabla \tilde{v}_n\|_2^2)^2 \\
&\quad - \frac{\mu_1}{p}\|\tilde{u}_n\|_p^p - \frac{\mu_2}{q}\|\tilde{v}_n\|_q^q - \beta \int_{\mathbb{R}^N} |\tilde{u}_n|^{r_1}|\tilde{v}_n|^{r_2} \\
&\geq \frac{1}{2}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) + \frac{1}{4}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)^2 \\
&\quad - \frac{\mu_1}{p}\|u\|_p^p - \frac{\mu_2}{q}\|v\|_q^q \\
&\geq \zeta_p^{\mu_1}(m_1) + \zeta_p^{\mu_2}(m_2),
\end{aligned}
$$

which contradicts to Lemma 3.2 (i). Hence, $v \neq 0$. Similarly, we have $u \neq 0$. Thus, from Lemma 3.6, we know $\lambda_1, \lambda_2 < 0$. Then, by substituting $(\tilde{u}_n, 0)$ and $(u, 0)$ into (3.9), we can derive

$$
\|\nabla \tilde{u}_n\|_2^2 + \|\nabla \tilde{u}_n\|_2^4 + \mu_1\|\tilde{u}_n\|_p^p = \lambda_1\|\tilde{u}_n\|_2^2 + o_n(1)
$$

and

$$
\|\nabla u\|_2^2 + \|\nabla u\|_2^4 + \mu_1\|u\|_p^p = \lambda_1\|u\|_2^2,
$$

which implies that $\tilde{u}_n \to u$ in $H^1(\mathbb{R}^N)$ as $\lambda_1 < 0$. Similarly, we obtain $\tilde{v}_n \to v$ in $H^1(\mathbb{R}^N)$. Therefore, we have $(\tilde{u}_n, \tilde{v}_n) \to (u, v)$ in $\mathcal{V}$ and by Corollary 3.4, we have

$$
\mathcal{I}(u, v) = \inf_{(u,v)\in P(a,b)} \mathcal{I}(u, v) = \inf_{(u,v)\in S(m_1,m_2)\cap V(\rho_\tau)} \mathcal{I}(u, v) < 0.
$$

Therefore, we deduce that (u, v) is a normalized solution. By the maximum principle, we conclude that (u, v) is a positive solution.

4. Conclusions

In this paper, we establish the existence of a ground state solution for a nonlinear Kirchhoff-type system using the minimization of the energy functional over a combination of the mass-constrained and the closed balls. To the best of our knowledge, there are few articles that deal with a coupled critical nonlinearity of the Kirchhoff system. Especially, our assumptions on the parameters are different from the previous related works. Therefore, we need to use some new analytical tricks to estimate the critical value. Our results in this article improve and generalize the related ones in the literature. In addition, condition $2 \leq p, q < 2 + \frac{8}{N}$ means that our results are established in a critical setting. Therefore, a new research direction closely related to problem (1.1) is to replace $2 \leq p, q < 2 + \frac{8}{N}$ with the following L^2-supercritical condition: $2 + \frac{8}{N} \leq p, q < 2^*$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. G. Kirchhoff, *Mechanik*, Teubner, Leipzigl, 1883.
2. W. Bao, Y. Cai, Mathematical theory and numerical methods for Bose-Einstein condensation, *Kinet. Relat. Models*, **6** (2013), 1–135. https://doi.org/10.3934/krm.2013.6.1
3. N. Soave, Normalized ground states for the NLS equation with combined nonlinearities, *J. Differ. Equations*, **269** (2020), 6941–6987. https://doi.org/10.1016/j.jde.2020.05.016
4. B. D. Esry, C. H. Greene, J. P. Burke Jr, J. L. John, Hartree-fock theory for double condensates, *Phys. Rev. Lett.*, **78** (1997), 3594–3597. https://doi.org/10.1103/PhysRevLett.78.3594
5. Z. Yang, Normalized ground state solutions for Kirchhoff type systems, *J. Math. Phys.*, **62** (2021), 031504. https://doi.org/10.1063/5.0028551
6. X. Cao, J. Xu, J. Wang, The existence of solutions with prescribed L^2-norm for Kirchhoff type system, *J. Math. Phys.*, **58** (2017), 041502. https://doi.org/10.1063/1.4982037
7. G. Che, H. Chen, Existence and asymptotic behavior of positive ground state solutions for coupled nonlinear fractional Kirchhoff-type systems, *Comput. Math. Appl.*, **77** (2019), 173–188. https://doi.org/10.1016/j.camwa.2018.09.020
8. D. Lü, S. Peng, Existence and asymptotic behavior of vector solutions for coupled nonlinear Kirchhoff-type systems, *J. Differ. Equations*, **263** (2017), 8947–8978. https://doi.org/10.1016/j.jde.2017.08.062
9. Y. Jalilian, Existence and multiplicity of solutions for a coupled system of Kirchhoff type equations, *Acta Math. Sci.*, **40** (2020), 1831–1848. https://doi.org/10.1007/s10473-020-0614-7
10. G. Che, H. Chen, Existence and multiplicity of systems of Kirchhoff-type equations with general potentials, *Math. Methods Appl. Sci.*, **40** (2017), 775–785. https://doi.org/10.1002/mma.4007
11. H. Li, W. Zou, Normalized ground states for semilinear elliptic systems with critical and subcritical nonlinearities, *J. Fixed Point Theory Appl.*, **23** (2021), 43. https://doi.org/10.1007/s11784-021-00878-w
12. T. Bartsch, H. Li, W. Zou, Existence and asymptotic behavior of normalized ground states for Sobolev critical Schrödinger systems, *Calculus Var. Partial Differ. Equations*, **62** (2023), 9. https://doi.org/10.1007/s00526-022-02355-9
13. M. Liu, X. Fang, Normalized solutions for the Schrödinger systems with mass supercritical and double Sobolev critical growth, *Z. Angew. Math. Phys.*, **73** (2022), 108. https://doi.org/10.1007/s00033-022-01757-1
14. T. Bartsch, L. Jeanjean, Normalized solutions for nonlinear Schrödinger systems, *Proc. R. Soc. Edinburgh Sect. A: Math.*, **148** (2018), 225–242. https://doi.org/10.1017/s0308210517000087
15. L. Jeanjean, S. Lu, A mass supercritical problem revisited, *Calculus Var. Partial Differ. Equations*, **59** (2020), 174. https://doi.org/10.1007/s00526-020-01828-z

16. N. Ikoma, Compactness of minimizing sequences in nonlinear Schrödinger systems under multiconstraint conditions, *Adv. Nonlinear Stud.*, **14** (2014), 115–136. https://doi.org/10.1515/ans-2014-0104

17. T. Cazenave, *Semilinear Schrödinger Equations*, American Mathematical Society, Rhode Island, 2003. https://doi.org/10.1090/cln/010

18. N. Ghoussoub, *Duality and Perturbation Methods in Critical Point Theory*, Cambridge University Press, Cambridge, 1993. https://doi.org/10.1017/CBO9780511551703

19. X. Luo, Q. Wang, Existence and asymptotic behavior of high energy normalized solutions for the Kirchhoff type equations in $\mathbb{R}^3$, *Nonlinear Anal. Real World Appl.*, **33** (2017), 19–32. https://doi.org/10.1016/j.nonrwa.2016.06.001