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Research article

Second gradient thermoelasticity with microtemperatures

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Abstract: This research was concerned with a linear theory of thermoelasticity with microtemperatures where the second thermal displacement gradient and the second gradient of microtemperatures are included in the classical set of independent constitutive variables. The master balance laws of micromorphic continua, the theory of the strain gradient of elasticity, and Green-Naghdi thermomechanics were used to derive a second gradient theory. The semigroup theory of linear operators allowed us to prove that the problem of the second gradient thermoelasticity with microtemperatures is well-posed. For the equations of isotropic rigids, we presented a natural extension of the Cauchy-Kovalevski-Somigliana solution of isothermal theory. In the case of stationary vibrations, the fundamental solutions of the basic equations were obtained. Uniqueness and instability of the solutions were obtained in the case of antiplane shear deformations.

Keywords: elastic solids with microtemperatures; solids with microstructure; second gradient theory; constitutive equations; well-posed problems

1. Introduction

In recent years, Green-Naghdi thermodynamics [1–3] has been used to establish some theories of thermoelasticity that take into account the second-order temperature gradient [4–6]. On the other hand, the balance laws of the continua with microstructure [7–10] led to a theory of thermodynamics of elastic materials where the inner structure has microelements with microtemperatures. At the begin of this paper, we use the theory of non-simple elastic solids [11–13] and results from thermodynamics of multipolar continua [14] to obtain a second gradient theory of thermoelasticity with microtemperatures. An introduction of the concepts of thermal displacement and thermal microdisplacements as well as the theory of multipolar continua allows us to derive the local form of energy balance and constitutive equations. In [3], the authors established a theory of thermoelasticity characterized by constitutive equations that depend on the first gradient of the displacement vector, on temperature, and on the first gradient of thermal displacement. In the present work, we have consider the following new independent constitutive variables: the second gradient of thermal displacement, the second gradient of thermal microdisplacements, as well as microtemperatures. To simplify the writing, we limit our attention only to the introduction of the second-order spatial derivatives of the thermal variables.

We express the field equations of the linear case in terms of components of the displacement vector, thermal displacement, and thermal microdisplacements, and obtain a fourth-order system of equations. The boundary-initial-value problems are also formulated. The semigroup theory of linear operators allows us to prove that the problem of the second gradient thermoelasticity with microtemperatures is well-posed. For the equations of isotropic rigids, we present a natural extension of the Cauchy-Kovalevski-Somigliana solution of the isothermal theory. In the case of stationary vibrations, we establish the fundamental solutions of the basic equations. Anti-plane shear deformations are also considered and uniqueness and instability results are obtained.

The relevance in introducing temperature gradient effects in thermomechanics can be recalled from [15].

2. Balance equations

In this section, we propose a second gradient theory of solids with microtemperatures by using the basic laws of mechanics of materials with microstructure and Green-Naghdi thermomechanics. Throughout this paper, the motion of the body is referred to the reference configuration B, occupied by the body at time t_0, and to a fixed system of rectangular Cartesian coordinates Ox_j, (j = 1, .., 3). Latin subscripts range over the integers (1, 2, 3), and Greek subscripts range over the integers (1, 2). Cartesian tensor notation is used throughout. In what follows, x_j are reference coordinates, y_j are spatial coordinates, a superposed dot denotes material time differentiation, and $f_{,j}$ denotes partial differentiation of f with respect to x_j . We denote by ∂B the boundary of B. Following [1,2], we get the following local balance of entropy:

$$\rho\dot{\eta} = S_{i,i} + \rho(s + \xi). \tag{2.1}$$

By using the theory of continua with microstructure [9], we can obtain the balance of the first moment of entropy in the form:

$$\rho\dot{\eta}_j = \Lambda_{ki,k} + S_i - H_i + \rho(Q_i + \xi_i), \tag{2.2}$$

In the relations (2.1) and (2.2), we have used the following notations: ρ is the reference mass density; η is the entropy per unit mass of the body; S_i is the entropy flux vector; s is the external rate of supply of entropy per unit mass; ξ is the internal rate of production of entropy per unit mass; η_j is the first entropy moment vector; Λ_{ij} is the first entropy flux moment tensor; H_i is the mean entropy flux vector; Q_i is the first moment of the external rate of supply of entropy, and ξ_j is the first moment of the internal rate of production of entropy. The entropy flux Σ and the first entropy moment flux vector σ_j at regular points of ∂B are given by [1,2,8],

$$\Sigma = S_j n_j, \quad \sigma_k = \Lambda_{jk} n_j, \tag{2.3}$$

where n_i is the outward unit normal of ∂B. Let θ be the absolute temperature. We denote by $\mathbf{x}$ the center of mass of a generic microelement V. We assume that for $\mathbf{x}' \in V$, we have

$$\theta(\mathbf{x}', t) = \theta(\mathbf{x}, t) + T_i(\mathbf{x}, t)(x_i' - x_i). \tag{2.4}$$

We call the functions T_i microtemperatures. As in [1], we consider the thermal displacement α and the thermal microdisplacements β_j by

$$\dot{\alpha} = \theta, \ \ \dot{\beta}_j = T_j. \tag{2.5}$$

We now consider a domain $\mathcal{P}$ at time t, bounded by a surface $\partial\mathcal{P}$, and let P be the respective domain in reference configuration, with the boundary ∂P. In view of [1, 9, 11, 14], we propose an energy balance in the form:

$$\int_P \rho(\ddot{u}_i\dot{u}_i + \dot{e})dv = \int_P \rho(f_i\dot{u}_i + s\theta + Q_iT_i)dv \tag{2.6}$$

$$+ \int_{\partial P}(t_i\dot{u}_i + \Sigma\theta + \sigma_jT_j + G_j\theta_{,j} + \Pi_{ji}T_{i,j})da$$

for every region P of B and every time. Here u_i is the displacement vector, e is the internal energy per unit mass f_i is the body force per unit mass, t_i the stress vector associated with the surface $\partial\mathcal{P}$ but measured per unit area of ∂P, and G_i and Π_{ij} are the monopolar and dipolar entropy flux per unit area, respectively. We impose that the dipolar body force and the spin inertia per unit mass are not present (see [14]). From (2.6), we can derive the balance of linear momentum so that, by the well-known method, we obtain

$$t_j = t_{ij}n_i \tag{2.7}$$

and

$$t_{ji,j} + \rho f_i = \rho\ddot{u}_i \tag{2.8}$$

where t_{ij} is the stress tensor. After the use of the divergence theorem and the equalities (2.1)–(2.3), (2.7), and (2.8), the relation (2.6) can be written in the form:

$$\int_P \rho\dot{e}dv = \int_P [t_{ji}\dot{u}_{i,j} + \rho\dot{\eta}\theta + \rho\dot{\eta}_jT_j + S_j\theta_{,j} + \Lambda_{kj}T_{j,k} - (S_i - H_i)T_i - \rho\xi\theta - \rho\xi_jT_j]dv \tag{2.9}$$

$$+ \int_{\partial P}(G_j\theta_{,j} + \Pi_{ji}T_{i,j})da.$$

With an argument similar to that used to derive the relation (2.7), from (2.9), we obtain

$$(G_j - G_{kj}n_k)\theta_{,j} + (\Pi_{ji} - \Pi_{kji}n_k)T_{i,j} = 0, \tag{2.10}$$

where G_{kj} and Π_{kji} are tensors associated to the surface loads G_i and Π_{ji}, respectively. With the help of (2.10), we obtain the local expression form of the energy balance

$$\rho\dot{e} = t_{ji}\dot{u}_{i,j} + \rho\dot{\eta}\theta + \rho\dot{\eta}_jT_j + F_j\theta_{,j} + \Gamma_{kj}T_{j,k} + (H_i - S_i)T_i \tag{2.11}$$

$$+ G_{kj}\theta_{,jk} + \Pi_{kji}T_{,ijk} - \rho\xi\theta - \rho\xi_jT_j,$$

where the following notation

$$F_j = S_j + G_{kj,k}, \quad \Gamma_{kj} = \Lambda_{kj} + \Pi_{mkj,m}, \tag{2.12}$$

is used.

Following [14], we assume a motion of the body that is different from the given motion by a superposed uniform rigid body angular velocity, and that $\rho, e, t_{ij}, \eta, \theta, \eta_j, T_j, S_j, \Lambda_{kj}, H_j, F_j, \Gamma_{kj}, \xi$, and ξ_j do not change by such motion. The equality (2.11) implies that

$$t_{ji} = t_{ij}. \tag{2.13}$$

If we consider the Helmholtz free energy ψ by

$$\psi = e - \theta\eta - T_j\eta_j, \tag{2.14}$$

and we see that the energy balance may be written in the form:

$$\rho(\dot{\psi} + \theta\dot{\eta} + T_j\dot{\eta}_j) = t_{ij}\dot{e}_{ij} + F_j\theta_{,j} \tag{2.15}$$

$$+ \Gamma_{kj}T_{j,k} + (H_i - S_i)T_i + G_{kj}\theta_{,jk} + \Pi_{kij}T_{i,jk} - \rho\xi\theta - \rho\xi_jT_j,$$

where we have introduced the strain tensor

$$2e_{ij} = u_{i,j} + u_{j,i}. \tag{2.16}$$

3. Constitutive equations

From now on, we define the constitutive equations for $\psi, t_{ij}, \eta, \eta_j, S_j, H_j, G_{kj}, F_j, \Gamma_{kj}, \Pi_{kj}, \xi$, and ξ_j, and we suppose that these are functions of the set $V = (e_{ij}, \theta, T_j, \alpha_{,j}, \beta_{k,j}, \alpha_{,ij}, \beta_{k,ij})$. To simplify the writting, we omit the explicit dependence of x_k and then the material should be homogeneous and assume that there is no kinematical constraint. In the theory established in [3], the constitutive variables are e_{ij}, θ, and $\alpha_{,j}$. If we introduce the notation $A = \rho\psi$, then Eq (2.15) becomes

$$(\frac{\partial A}{\partial e_{ij}} - t_{ij})\dot{e}_{ij} + (\frac{\partial A}{\partial \theta} + \rho\eta)\dot{\theta} + (\frac{\partial A}{\partial T_i} + \rho\eta_i)\dot{T}_i \tag{3.1}$$

$$+ (\frac{\partial A}{\partial \alpha_{,i}} - F_i)\theta_{,i} + (\frac{\partial A}{\partial \beta_{j,i}} - \Gamma_{ij})T_{j,i} + (\frac{\partial A}{\partial \alpha_{,ij}} - G_{ji})\theta_{,ij} + (\frac{\partial A}{\partial \beta_{i,jk}} - \Pi_{kji})T_{i,jk}$$

$$+ \rho\theta\xi + (\rho\xi_i + S_i - H_i)T_i = 0.$$

From (3.1), we find that [3]

$$t_{ij} = \frac{\partial A}{\partial e_{ij}}, \ \ \rho\eta = -\frac{\partial A}{\partial \theta}, \ \ \rho\eta_j = -\frac{\partial A}{\partial T_j}, \tag{3.2}$$

$$F_i = \frac{\partial A}{\partial \alpha_{,i}}, \ \ \Gamma_{ij} = \frac{\partial A}{\partial \beta_{j,i}}, \ \ G_{ji} = \frac{\partial A}{\partial \alpha_{,ij}}, \ \ \Pi_{kji} = \frac{\partial A}{\partial \beta_{i,jk}},$$

and

$$\rho\xi\theta + (\rho\xi_i + S_i - H_i)T_i = 0. \tag{3.3}$$

We introduce the notations

$$\theta(\mathbf{x}, t_0) = T_0, \ \ T_j(\mathbf{x}, t_0) = T_j^0, \ \ \alpha(\mathbf{x}, t_0) = \alpha_0, \ \ \beta_j(\mathbf{x}, t_0) = \beta_j^0, \tag{3.4}$$

where t_0 is a reference time and T_0, T_j^0, α_0, and β_j^0 are given constants. As in [3], we consider new thermal variables

$$T = \theta - T_0,\ \theta_i = T_i - T_i^0,\ \ \chi = \int_0^t T ds,\ \ \varphi_i = \int_0^t \theta_i ds. \tag{3.5}$$

From (3.4) and (3.5), we get

$$\alpha = \chi + T_0(t - t_0) + \alpha_0, \beta_j = \varphi_j + T_j^0(t - t_0) + \beta_j^0, \alpha_{,i} = \chi_{,i}, \beta_{i,j} = \varphi_{i,j}, \tag{3.6}$$

$$\dot{\chi} = T,\ \dot{\varphi}_i = \theta_i.$$

From now on, we restrict our attention to the linear theory where the functions u_i, T, and θ_j can be written as

$$u_i = \epsilon u_i',\ \ T = \epsilon T',\ \ \theta_j = \epsilon\theta_j'$$

where ϵ is a parameter small enough for squares and higher powers to be neglected, and u_i', T', and θ_j' are independent of ϵ. As usual, we assume that A is a quadratic form of the variables $e_{ij}, T, \theta_j, \alpha_{,j}, \beta_{k,j}, \alpha_{,ij}$, and $\beta_{k,ij}$ and that H_i, ξ, and ξ_j are linear functions of the same variables. We consider the case of a material with a center of symmetry. Thus, we have

$$2A = A_{ijrs}e_{ij}e_{rs} - 2b_{ij}e_{ij}T + 2C_{ijrs}e_{ij}\varphi_{r,s} + 2D_{ijrs}e_{ij}\chi_{,rs} - aT^2 - 2L_{ij}T\varphi_{i,j} \tag{3.7}$$

$$- 2N_{ij}T\chi_{,ij} - B_{ij}\theta_i\theta_j - 2C_{ij}\theta_i\chi_{,j} - 2d_{ipqr}\varphi_{p,qr}\theta_i + K_{ij}\chi_{,i}\chi_{,j} + 2M_{ipqr}\chi_{,i}\varphi_{p,qr}$$

$$+ E_{ijrs}\varphi_{i,j}\varphi_{r,s} + 2H_{ijrs}\varphi_{i,j}\chi_{,rs} + U_{ijkpqr}\varphi_{i,jk}\varphi_{p,qr} + Q_{ijrs}\chi_{,ij}\chi_{,rs}.$$

The following symmetries are satisfied:

$$A_{ijrs} = A_{jirs} = A_{rsij},\ \ b_{ij} = b_{ji},\ \ C_{ijrs} = C_{jirs}, D_{ijrs} = D_{jirs} = D_{jisr}, \tag{3.8}$$

$$B_{ij} = B_{ji},\ N_{ij} = N_{ji},\ d_{ipqr} = d_{iprq},\ K_{ij} = K_{ji},\ \ E_{ijrs} = E_{rsij},\ M_{ipqr} = M_{iprq},$$

$$H_{ijrs} = H_{ijsr},\ \ U_{ijkpqr} = U_{pqrijk} = U_{ikjpqr},\ \ Q_{ijrs} = Q_{rsij} = Q_{jirs}.$$

It follows from (3.2), (3.7), and (3.8) that

$$\begin{aligned}
t_{ij} &= A_{ijrs}e_{rs} - b_{ij}T + C_{ijrs}\varphi_{r,s} + D_{ijrs}\chi_{,rs},\\
\rho\eta &= b_{ij}e_{ij} + aT + L_{ij}\varphi_{i,j} + N_{ij}\chi_{,ij},\\
\rho\eta_i &= B_{ij}\theta_j + C_{ij}\chi_{,j} + d_{ipqr}\varphi_{p,qr},\\
F_j &= -C_{ij}\theta_i + K_{ij}\chi_{,i} + M_{jpqr}\varphi_{p,qr},\\
\Gamma_{ij} &= C_{rsji}e_{rs} - L_{ji}T + E_{jirs}\varphi_{r,s} + H_{jirs}\chi_{,rs},\\
G_{ji} &= D_{rsij}e_{rs} - N_{ij}T + Q_{ijrs}\chi_{,rs} + H_{rsij}\varphi_{r,s},\\
\Pi_{kji} &= -d_{rijk}\theta_r + M_{sijk}\chi_{,s} + U_{ijkpqr}\varphi_{p,qr}.
\end{aligned} \tag{3.9}$$

For isotropic materials, the number of constitutive coefficients is drastically reduced (see [12]). From (3.3), we see that the response function ξ vanishes when the microtemperatures T_j vanish. In the linear case, the function ξ that satisfies this requirement must be $\xi = c_j T_j$, where c_j are constants. Since the body has a center of symmetry, we get $\xi = 0$. Thus, from (2.12), (3.3), and (3.9), we find that

$$\rho\xi_i = H_i - S_i. \tag{3.10}$$

If we use these results, then Eqs (2.1) and (2.2) take the form

$$\rho\eta = S_{k,k} + \rho s, \quad \rho\eta_i = \Lambda_{ki,k} + \rho Q_i. \tag{3.11}$$

The equations of the linear theory consist of the equations of motion (2.8), the energy equations (3.11), the constitutive equations (3.9), and the geometrical equations (2.16). In view of (2.12), we can write the Eqs (3.11) in the form

$$F_{k,k} - G_{kj,kj} - \rho\dot{\eta} = -\rho s, \; \Gamma_{ij,i} - \Pi_{rkj,rk} - \rho\dot{\eta}_j = -\rho Q_j. \tag{3.12}$$

Equations (2.8) and (3.11) can be expressed in terms of the unknowns u_j, χ, and φ_i. Thus, we obtain the equations

$$\begin{gathered} A_{ijrs}u_{r,sj} - b_{ij}\dot{\chi}_{,j} + D_{jirk}\chi_{,rkj} + C_{ijrs}\varphi_{r,sj} + \rho f_i = \rho\ddot{u}_i, \\ -D_{rpqk}u_{r,pqk} - b_{ij}\dot{u}_{i,j} + K_{ij}\chi_{,ij} - Q_{ijrs}\chi_{,ijrs} - a\ddot{\chi} - R_{pqjr}\varphi_{p,qrj} \\ -p_{ij}\dot{\varphi}_{i,j} = -\rho s, \\ C_{rsjk}u_{r,sk} - p_{jk}\dot{\chi}_{,k} + \zeta_{pjqr}\dot{\varphi}_{p,qr} + R_{jkrs}\chi_{,rsk} \\ +E_{jkrs}\varphi_{r,sk} - U_{jrspqm}\varphi_{p,qmsr} - B_{jk}\ddot{\varphi}_k = -\rho Q_j, \end{gathered} \tag{3.13}$$

where

$$R_{jkrs} = H_{jkrs} - M_{rjks}, \quad \zeta_{pjqr} = d_{pjqr} - d_{jpqr}, \quad p_{jk} = L_{jk} + C_{jk}. \tag{3.14}$$

To the system (3.13), we have to adjoin boundary and initial conditions.

4. Boundary-initial-value problems

Now, we study the boundary conditions and formulate the basic boundary-initial-value problems. We assume that the boundary of B consists of the union of a finite number of smooth surfaces, smooth curves (edges), and points (corners). Let C be the union of the edges. As in [11, 12], to obtain the form of the boundary conditions, we must study the surface integral in (2.5). By using (2.3), (2.7), and (2.10), we find that

$$\begin{aligned} &\int_{\partial P}(t_i\dot{u}_i + \Sigma\theta + \sigma_j T_j + G_j\theta_{,j} + \Pi_{ji}T_{i,j})da \\ &= \int_{\partial P}[t_{ki}\dot{u}_i + (F_k - G_{rk,r})\theta + (\Gamma_{kj} - \Pi_{mkj,m})T_j + G_{kj}\theta_{,j} + \Pi_{kji}T_{i,j}]n_k da. \end{aligned} \tag{4.1}$$

We will use the notations

$$Df = f_{,j}n_j, \quad D_i = (\delta_{ij} - n_i n_j)\frac{\partial}{\partial x_j}, \tag{4.2}$$

where δ_{ij} is the Kronecker delta. Then we obtain

$$G_{kj}\theta_{,j}n_k = G_{kj}\, n_k n_l D\theta - \theta(G_{kj}n_k) + D_j(G_{kj}n_k\theta), \tag{4.3}$$

$$\Pi_{kji}T_{i,j}n_k = \Pi_{kji}n_kn_jDT_i - T_iD_j(\Pi_{kji}n_k) + D_j(\Pi_{kji}n_kT_i).$$

As in [11, 12], from (4.1) and (4.3), we get

$$\int_{\partial P}(t_i\dot{u}_i + \Sigma\theta + \sigma_jT_j + G_j\theta_{,j} + \Pi_{ji}T_{i,j})da \tag{4.4}$$

$$= \int_{\partial P}(t_i\dot{u}_i + \Phi_1\theta + \Phi_2D\theta + \Psi_jT_j + W_iDT_i)da$$

$$+ \int_C(Y\theta + \Omega_iT_i)ds$$

where

$$\Phi_1 = (F_k - G_{rk,r})n_k - D_j(n_sG_{sj}) + (D_jn_j)n_sn_pG_{sp}, \quad \Phi_2 = G_{rs}n_rn_s, \tag{4.5}$$

$$\Psi_i = (\Gamma_{ki} - \Pi_{rki,r})n_k - D_j(n_s\Pi_{sji}) + (D_jn_j)n_sn_p\Pi_{spi}, \quad W_i = \Pi_{rsi}n_rn_s,$$

$$Y =< G_{rs}n_ry_s >, \quad \Omega_i =< \Pi_{rsi}n_ry_s >, \quad y_i = \epsilon_{irk}s_rn_k.$$

Here, s_k are the components of the unit vector tangent to C, $< f >$ denotes the difference of the limits of f from both sides of C, and ϵ_{jrk} is the alternating symbol. The first boundary-initial-value problem is characterized by the boundary conditions

$$u_i = u_i^*, \ \chi = \chi^*, \ \varphi_i = \varphi_i^*, \ D\chi = \zeta^*, \ D\varphi_i = \gamma_i^* \text{ on } \partial B \times I, \tag{4.6}$$

where $u_i^*, \chi^*, \varphi_i^*, \zeta^*$, and γ_i^* are prescribed functions.

For the second boundary-initial-value problem, the boundary conditions are [11]

$$t_i = t_i^*, \Phi_1 = \Phi_1^*, \Phi_2 = \Phi_2^*, \Psi_i = \Psi_i^*, W_i = W_i^* \text{ on } \partial B \times I, \tag{4.7}$$

$$Y = Y^*, \quad \Omega_i = \Omega_i^* \text{ on } C \times I,$$

where $t_i^*, \Phi_1^*, \Phi_2^*, \Psi_i^*, W_i^*, Y^*$, and Ω_i^* are given. The initial conditions are

$$u_i(\mathbf{x}, 0) = u_i^0(\mathbf{x}), \ \dot{u}_i(\mathbf{x}, 0) = v_i^0(\mathbf{x}), \ \chi(\mathbf{x}, 0) = \chi^0(\mathbf{x}), \ \dot{\chi}(\mathbf{x}, 0) = \chi^1(\mathbf{x}), \tag{4.8}$$

$$\varphi(\mathbf{x}, 0) = \varphi_i^0(\mathbf{x}), \quad \dot{\varphi}_i(\mathbf{x}, 0) = \nu_i^0(\mathbf{x}), \quad \mathbf{x} \in B,$$

where $u_i^0, v_i^0, \chi^0, \chi^1, \varphi_i^0$, and ν_i^0 are given.

5. An existence result

Now, we provide an existence and uniqueness result for the problem determined by the system of Eqs (3.13), with the initial condition (4.8) and the homogeneous version of the boundary conditions (4.6). We will use the theory of contractive linear semigroups [16].

We assume once and for all that:

(i) The mass density ρ and the thermal capacity a are strictly positive.

(ii) The matrix B_{ij} is positive definite.

(iii) The quadratic form

$$W(e_{ij}, \kappa_{ijk}, \chi_{,i}, \chi_{,ji}, \varphi_{,i}, \varphi_{,ij}) =$$

$$A_{ijrs}e_{ij}e_{rs} + 2C_{ijrs}e_{ij}\varphi_{r,s} + 2D_{ijrs}e_{ij}\chi_{,rs} + K_{ij}\chi_{,i}\chi_{,j} + 2M_{ipqr}\chi_{,i}\varphi_{p,qr}$$
$$+ E_{ijrs}\varphi_{i,j}\varphi_{r,s} + 2H_{ijrs}\varphi_{i,j}\chi_{,rs} + U_{ijkpqr}\varphi_{i,jk}\varphi_{p,qr} + Q_{ijrs}\chi_{,ij}\chi_{,rs},$$

is positive definite, i.e., there exists a positive constant C such that:

$$W \geq C(e_{ij}e_{ij} + \chi_{,r}\chi_{,r} + \chi_{,rs}\chi_{,rs} + \varphi_{r,s}\varphi_{r,s} + \varphi_{p,qr}\varphi_{p,qr}).$$

Let us to propose the problem as an abstract problem in a suitable Hilbert space. We will work on the space

$$\mathcal{H} = \mathbf{W}_0^{1,2}(B) \times \mathbf{L}^2(B) \times W_0^{2,2}(B) \times L^2(B) \times \mathbf{W}_0^{2,2}(B) \times \mathbf{L}^2(B),$$

where $W_0^{1,2}$, $W_0^{2,2}$, and L^2 are the usual Sobolev spaces, $\mathbf{W}_0^{2,2} = [W_0^{2,2}]^3$ and $\mathbf{L}^2 = [L^2]^3$. The elements in this space can be denoted by $\mathcal{U} = (\mathbf{u}, \mathbf{v}, \chi, \theta, \boldsymbol{\varphi}, \boldsymbol{\phi})$.

We consider the scalar product associated to the norm

$$\|(\mathbf{u}, \mathbf{v}, \chi, \theta, \boldsymbol{\varphi}, \boldsymbol{\phi})\|^2 \tag{5.1}$$
$$= \int_B [\rho v_i v_i + a\theta^2 + B_{ij}\phi_i\phi_j + W(e_{ij}, \chi_{,i}, \chi_{,ji}, \varphi_{,i}, \varphi_{,ij})]dv.$$

Now, we want to see our problem as a Cauchy problem in $\mathcal{H}$. We define the operator

$$\mathcal{A}\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \chi \\ \theta \\ \boldsymbol{\varphi} \\ \boldsymbol{\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{v} \\ \mathbf{M} \\ \theta \\ \upsilon \\ \boldsymbol{\phi} \\ \boldsymbol{\Omega} \end{pmatrix} \tag{5.2}$$

where

$$\mathbf{M} = (M_i), \quad \boldsymbol{\Omega} = (\omega_i),$$

and

$$M_i = \rho^{-1}(A_{ijrs}u_{r,sj} - b_{ij}\theta_{,j} + C_{ijrs}\varphi_{r,sj} + D_{jirk}\chi_{,rkj}),$$
$$\omega_i = F_{ij}(C_{rsjk}u_{r,sk} - p_{jk}\theta_{,k} + \zeta_{pjqr}\phi_{p,qr} + R_{jkrs}\chi_{,rsk}$$
$$+ E_{jkrs}\varphi_{r,sk} - U_{jrspqm}\varphi_{p,qmsr}),$$
$$\upsilon = a^{-1}(-D_{rpqk}u_{r,pqk} - b_{ij}v_{i,j} + K_{ij}\chi_{,ij} - Q_{ijrs}\chi_{,ijrs} - R_{pqjr}\varphi_{p,qrj}$$
$$-p_{ij}\phi_{i,j}),$$

where $F_{ij}B_{jk} = \delta_{ik}$.

We note that our problem can be written as

$$\frac{d\mathcal{U}}{dt} = \mathcal{A}\mathcal{U} + \mathcal{F}(t), \quad \mathcal{U}(0) = (\mathbf{u}_0, \mathbf{v}_0, \chi^0, \chi^1, \boldsymbol{\varphi}^0, \boldsymbol{\nu}^0), \tag{5.3}$$

where

$$\mathcal{F}(t) = (0, \mathbf{f}(t), 0, \rho a^{-1}s, 0, \rho F_{ij}Q_j).$$

The domain of the operator $\mathcal{A}$ is the subspace of elements of our Hilbert space such that

$$\mathbf{v} \in \mathbf{W}_0^{1,2}, \boldsymbol{\phi} \in \mathbf{W}_0^{2,2}, \theta \in W^{2,2},$$

$$A_{ijrs}u_{r,sj} + D_{jirk}\chi_{,rkj} - b_{ij}\theta_{,j} \in L^2,$$

$$-D_{rpqk}u_{r,pqk} - Q_{ijrs}\chi_{,ijrs} - R_{pqjr}\varphi_{p,qrj} \in L^2,$$

and

$$C_{rsjk}u_{r,sk} + R_{jkrs}\chi_{,rsk} - U_{jrspqm}\varphi_{p,qmsr} \in L^2.$$

This domain is a dense subset of our space.

After an easy but laborious calculation, we can see that

$$< \mathcal{A}\mathcal{U}, \mathcal{U} > = 0,$$

for every element $\mathcal{U}$ at the domain of the operator.

The next step in our approach is to show that zero belongs to the resolvent of the operator. Let $\mathcal{U}^*$ be in $\mathcal{H}$. We must prove that the equation

$$\mathcal{A}\mathcal{U} = \mathcal{U}^*$$

admits a solution. That is

$$\mathbf{v} = \mathbf{u}^*, \quad \theta = \chi^*, \quad \boldsymbol{\phi} = \boldsymbol{\varphi}^*,$$

$$\mathbf{M} = \mathbf{v}^*, \quad \upsilon = \theta^*, \quad \boldsymbol{\Omega} = \boldsymbol{\phi}^*.$$

We substitute the first three equations into the others to find that

$$A_{ijrs}u_{r,sj} + C_{ijrs}\varphi_{r,sj} + D_{jirk}\chi_{,rkj} = \rho v_i^* + b_{ij}\chi^*_{,j},$$

$$-D_{rpqk}u_{r,pqk} + K_{ij}\chi_{,ij} - Q_{ijrs}\chi_{,ijrs} - R_{pqjr}\varphi_{p,qrj}$$
$$= a\theta^* + b_{ij}u^*_{i,j} + p_{ij}\varphi^*_{i,j},$$

$$C_{rsjk}u_{r,sk} + R_{jkrs}\chi_{,rsk} + E_{jkrs}\varphi_{r,sk}$$
$$-U_{jrspqm}\varphi_{p,qmsr} = B_{jk}\phi_k^* + p_{jk}\chi^*_{,k} - \zeta_{pjqr}\varphi^*_{p,qr}.$$

If we denote

$$\alpha_{i1} = \rho v_i^* + b_{ij}\chi^*_{,j}, \quad \alpha_2 = a\theta^* + b_{ij}u^*_{i,j} + p_{ij}\varphi^*_{i,j},$$

$$\alpha_{3j} = B_{jk}\phi_k^* + p_{jk}\chi^*_{,k} - \zeta_{pjqr}\varphi^*_{p,qr}, \quad A_{1i} = A_{ijrs}u_{r,sj} + C_{ijrs}\varphi_{r,sj} + D_{jirk}\chi_{,rkj},$$

$$A_2 = -D_{rpqk}u_{r,pqk} + K_{ij}\chi_{,ij} - Q_{ijrs}\chi_{,ijrs} - R_{pqjr}\varphi_{p,qrj},$$

$$A_{3j} = C_{rsjk}u_{r,sk} + R_{jkrs}\chi_{,rsk} + E_{jkrs}\varphi_{r,sk} - U_{jrspqm}\varphi_{p,qmsr},$$

then our system can be written as

$$A_{i1} = \alpha_{i1}, \; A_2 = \alpha_2, \; A_{i3} = \alpha_{i3}.$$

To solve this last system, we note that $(\alpha_{i1}, \alpha_2, \alpha_{i3}) \in [W^{-1,2}]^3 \times [W^{-2,2}]^4$ and that the form

$$\mathcal{B}(\mathbf{u}, \chi, \boldsymbol{\varphi}), (\mathbf{u}^*, \chi^*, \boldsymbol{\varphi}^*)] = \int_B (A_{i1} u_i^* + A_2 \chi^* + A_{i3} \varphi_i) dv$$

defines a form in $[W_0^{1,2}]^3 \times [W_0^{2,2}]^4$ which is bounded and coercive. In view of the Lax-Migram lemma [17], we can guarantee the existence of solutions to our system as well as the existence of a positive constant K (independent of the point) such that

$$\|\mathcal{U}\| \leq K \|\mathcal{U}^*\|.$$

In view of the previous arguments, we can conclude that $\mathcal{A}$ generates a contractive semigroup. Therefore, we obtained the following result:

Theorem 1. Let us assume that f_i, s, and Q_i are functions continuous at the domain of the operator $\mathcal{A}$ and of class C^1 in L^2. Then, there exists a unique solution to the problem (5.3) that is continuous in the domain of the operator and is of class C^1 in the Hilbert space $\mathcal{H}$.

It is well-known that we also obtain

$$\|\mathcal{U}(t)\| \leq \|\mathcal{U}(0)\| + \int_0^t \|\mathcal{F}(s)\| ds,$$

which establishes the continuous dependence of solutions on initial and supply terms. Thus, we can say that under the conditions (i), (ii), and (iii), the problem of the second gradient thermoelasticity with microtemperatures is well-posed.

6. Basic equations for isotropic solids

Now, we state the constitutive equations for isotropic materials and express the basic equations in terms of the unknown functions u_i, χ, and φ_i. For centrosymmetric and isotropic materials, the number of independent parameters is greatly reduced. Using the general forms of isotropic tensors [12] and taking into account the symmetries (3.8), we obtain

$$\begin{aligned} A = {} & \frac{\lambda}{2} e_{ii} e_{jj} + \mu e_{ij} e_{ij} + C_1 e_{ii} \varphi_{r,r} + 2C_2 e_{ij} \varphi_{i,j} + d_1 e_{rr} \chi_{,jj} + 2d_2 e_{ij} \chi_{,ij} - \frac{a}{2} T^2 \\ & - \beta e_{ii} T - \gamma T \varphi_{r,r} - \nu T \chi_{,rr} - \frac{b}{2} \theta_j \theta_j - \zeta \theta_j \chi_{,j} + \frac{k}{2} \chi_{,j} \chi_{,j} \\ & + \mu_1 \chi_{,j} \varphi_{j,ii} + 2\mu_2 \chi_{,j} \varphi_{i,ji} + \frac{1}{2} (E_1 \varphi_{r,r} \varphi_{i,i} + E_2 \varphi_{i,j} \varphi_{i,j} + E_3 \varphi_{i,j} \varphi_{j,i}) \\ & + \gamma_1 \varphi_{i,ij} \varphi_{j,kk} + \gamma_2 \varphi_{i,ij} \varphi_{r,rj} + \gamma_3 \varphi_{j,ii} \varphi_{j,kk} + \gamma_4 \varphi_{i,jk} \varphi_{i,jk} + \gamma_5 \varphi_{i,jk} \varphi_{k,ij} \\ & + \frac{\kappa_1}{2} \chi_{,ii} \chi_{,jj} + \kappa_2 \chi_{,ij} \chi_{,ij}, \end{aligned} \tag{6.1}$$

where $\lambda, \mu, \beta, C_\alpha, d_\alpha, a, \gamma, \nu, b, \zeta, k, \mu_\alpha, E_j, \gamma_r$, and $\kappa_\alpha, (\alpha = 1, 2; r = 1, 2, .., 5)$, are material coefficients. By using (3.6) and the relations

$$t_{ij} = \partial A / \partial e_{ij}, \quad \rho \eta = -\partial A / \partial T, \quad \rho \eta_j = -\partial A / \partial \theta_j,$$

$$F_j = \partial A/\partial \chi_{,j}, \;\; \Gamma_{ij} = \partial A/\partial \varphi_{j,i}, \;\; \Pi_{kji} = \partial A/\partial \varphi_{i,jk},$$

we obtain the following constitutive equations

$$\begin{aligned}
t_{ij} &= \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} - \beta T\delta_{ij} + C_1\varphi_{r,r}\delta_{ij} + C_2(\varphi_{i,j} + \varphi_{j,i}) + d_1\chi_{,rr}\delta_{ij} + 2d_2\chi_{,ij},\\
\rho\eta &= \beta e_{rr} + aT + \gamma\varphi_{j,j} + \nu\chi_{,ii}, \;\; \rho\eta_j = b\theta_j + \zeta\chi_{,j},\\
F_j &= k\chi_{,j} - \zeta\theta_j + \mu_1\varphi_{j,ii} + 2\mu_2\varphi_{i,ij},\\
\Gamma_{ij} &= C_1 e_{rr}\delta_{ij} + 2C_2 e_{ij} - \gamma T\delta_{ij} + E_1\varphi_{r,r}\delta_{ij} + E_2\varphi_{j,i} + E_3\varphi_{i,j},\\
G_{ji} &= d_1 e_{rr}\delta_{ij} + 2d_2 e_{ij} - \nu T\delta_{ij} + \kappa_1\chi_{,kk}\delta_{ij} + 2\kappa_2\chi_{,ij},\\
\Pi_{ijk} &= \frac{\gamma_1}{2}(\varphi_{i,rr}\delta_{jk} + 2\varphi_{r,rk}\delta_{ij} + \varphi_{j,rr}\delta_{ik}) + \gamma_2(\varphi_{r,ri}\delta_{jk} + \varphi_{r,rj}\delta_{ik}) + 2\gamma_3\varphi_{k,rr}\delta_{ij}\\
&\quad + 2\gamma_4\varphi_{k,ij} + \gamma_5(\varphi_{i,jk} + \varphi_{j,ki}) + \mu_1\chi_{,k}\delta_{ij} + \mu_2(\chi_{,j}\delta_{ik} + \chi_{,i}\delta_{jk}).
\end{aligned} \tag{6.2}$$

For isotropic solids, the equations for the unknown functions u_j, χ, and φ_i are established by substituting the functions in (6.2) into Eqs (2.8) and (3.12). Thus we obtain the equations:

$$\begin{aligned}
&\mu\Delta u_i + (\lambda + \mu)u_{r,ri} - \beta\dot{\chi}_{,i} + C_2\Delta\varphi_i + C_3\varphi_{r,ri} + d_3\Delta\chi_{,i} + \rho f_i = \rho\ddot{u}_i,\\
&k\Delta\chi + \mu_3\Delta\varphi_{r,r} - (\gamma + \zeta)\dot{\varphi}_{r,r} - d_3\Delta u_{r,r} - \kappa_3\Delta^2\chi - \partial/\partial t(\beta u_{r,r} + a\dot{\chi}) = -\rho s,\\
&C_2\Delta u_j + C_3 u_{r,rj} - (\gamma + \zeta)\dot{\chi}_{,j} + E_2\Delta\varphi_j + E_4\varphi_{r,rj} - b_1\Delta^2\varphi_j - b_2\Delta\varphi_{r,rj}\\
&\quad - \mu_3\Delta\chi_{,j} - b\ddot{\varphi}_j = -\rho Q_j,
\end{aligned} \tag{6.3}$$

where

$$C_3 = C_1 + C_2, \;\; d_3 = d_1 + 2d_2, \;\; \mu_3 = \mu_1 + 2\mu_2, \tag{6.4}$$

$$\kappa_3 = \kappa_1 + 2\kappa_2, \;\; E_4 = E_1 + E_3, \;\; b_1 = 2(\gamma_3 + \gamma_4), \;\; b_2 = 2(\gamma_1 + \gamma_2 + \gamma_5).$$

The problem of heat flow in rigid materials with microtemperatures is characterized by the following equations:

$$\begin{aligned}
&k\Delta\chi + \mu_3\Delta\varphi_{r,r} - (\gamma + \zeta)\dot{\varphi}_{r,r} - \kappa_3\Delta^2\chi - a\ddot{\chi} = -\rho s,\\
&-(\gamma + \zeta)\dot{\chi}_{,j} + E_2\Delta\varphi_j + E_4\varphi_{r,rj} - b_1\Delta^2\varphi_j - b_2\Delta\varphi_{r,rj}\\
&\quad - \mu_3\Delta\chi_{,j} - b\ddot{\varphi}_j = -\rho Q_j.
\end{aligned} \tag{6.5}$$

7. A representation theorem

In this section, we establish a representation theorem for solutions of the system (6.5). We will use the notations

$$\begin{aligned}
&A_1 = k\Delta - \kappa_3\Delta^2 - a\partial^2/\partial t^2, \;\; A_2 = \mu_3\Delta - (\gamma + \zeta)\partial/\partial t, \;\; A_3 = -\mu_3\Delta - (\gamma + \zeta)\partial/\partial t,\\
&A_4 = E_2\Delta - b_1\Delta^2 - b\partial^2/\partial t^2, \;\; A_5 = E_4 - b_2\Delta,\\
&P_1 = A_1 A_4 + \Delta P_2, \;\; P_2 = A_1 A_5 - A_2 A_3.
\end{aligned} \tag{7.1}$$

Equations (6.5) can be written in the form

$$A_1\chi + A_2\varphi_{r,r} = -\rho s,$$

$$A_3\chi_{,j} + A_4\varphi_j + A_5\varphi_{r,rj} = -\rho Q_j. \tag{7.2}$$

A counterpart of the Cauchy-Kovalevski-Somigliana solution of the classical elastodynamics is given by the following theorem.

Theorem 2. Let

$$\chi = -A_4(P_1\Phi + A_2A_3\Delta\Phi) + A_1A_2A_4V_{j,j}, \tag{7.3}$$

$$\varphi_j = A_1A_3A_4\Phi_{,j} + A_1(P_2V_{r,rj} - P_1V_j),$$

where the functions Φ and V_j satisfy

$$A_1A_4P_1\Phi = \rho s, \quad A_1A_4P_1V_j = \rho Q_j. \tag{7.4}$$

Then χ and φ_j satisfy the Eqs (7.2).

Proof. We note that

$$A_1A_4 + P_2\Delta - P_1 = 0,$$

$$A_1A_3A_4 + A_1A_3A_5\Delta - A_3^2A_2\Delta - A_3P_1 = 0, \tag{7.5}$$

$$A_1^2A_2A_4 + A_1A_2\Delta P_2 - A_1A_2P_1 = 0,$$

$$(A_4 + A_5\Delta)P_2 + A_2A_3A_4 - A_5P_1 = 0,$$

$$A_1A_3A_4A_5\Delta + A_1A_3A_4^2 - A_3A_4P_1 - A_2A_3^2A_4\Delta = 0.$$

After substitution of χ and φ_j given by (7.3) in (7.2), then in view of (7.5) and (7.4), we see

$$A_1\chi + A_2\varphi_{r,r} = -A_1A_4P_1\Phi + A_1A_2(A_1A_4 + P_2\Delta - P_1)V_{j,j} = -A_1A_4P_1\Phi,$$

$$A_3\chi_{,j} + A_4\varphi_j + A_5\varphi_{r,rj} = -A_1A_4P_1V_j + (A_1A_3A_4^2 + A_1A_5A_3A_4\Delta$$

$$-A_2A_3^2A_4\Delta - A_3A_4P_1)\Phi_{,j} + A_1(A_4P_2 + A_2A_3A_4 - A_5P_1)V_{r,rj}$$

$$= -A_1A_4P_1V_j.$$

With the help of (7.4), we obtain the desired result.

In continuum mechanics, such solution representations have been used to establish the fundamental solutions of the field equations. In classical thermoelasticity, these solutions led to the introduction of the single layer potential and the double layer potential. The method of potentials has been used to reduce boundary value problems of steady vibration theory to singular integral equations and to prove existence theorems (see, e.g., [18]).

8. Fundamental solutions

In this section, we use the representation (7.1) to establish the fundamental solutions in the case of steady vibrations. We assume that

$$s = \mathrm{Re}[s'(\mathbf{x})\exp(-i\omega t)], \quad Q_j = \mathrm{Re}[Q'_j(\mathbf{x})\exp(-i\omega t)], \tag{8.1}$$
$$\chi = \mathrm{Re}[\chi'(\mathbf{x})\exp(-i\omega t)], \quad \varphi_j = \mathrm{Re}[\varphi'_j(\mathbf{x})\exp(-i\omega t)],$$

where ω is a given frequency, $\mathbf{x} = (x_1, x_2, x_3), i = (-1)^{1/2}$, and $\mathrm{Re}[f]$ is the real part of the function f. Let us introduce the notations

$$A_1^* = k\Delta - \kappa_3\Delta^2 + a\omega^2, \;\; A_2^* = \mu_3\Delta + i\omega(\gamma+\zeta), \;\; A_3^* = -\mu_3\Delta + i\omega(\gamma+\zeta),$$
$$A_4^* = E_2\Delta - b_1\Delta^2 + b\omega^2, \;\; A_5^* = E_4 - b_2\Delta, \tag{8.2}$$
$$P_1^* = A_1^*A_4^* + \Delta P_2^*, \;\; P_2^* = A_1^*A_5^* - A_2^*A_3^*.$$

We obtain a differential system for the amplitudes χ' and φ'_j. To simplify the notation, we omit the primes so that these equations can be written as

$$A_1^*\chi + A_2^*\varphi_{r,r} = -\rho s, \quad A_3^*\chi_{,j} + A_4^*\varphi_j + A_5^*\varphi_{r,rj} = -\rho Q_j. \tag{8.3}$$

The next theorem is a consequence of previous results.

Theorem 3. Let

$$\chi = -A_4^*(P_1^*F + A_2^*A_3^*\Delta F) + A_1^*A_2^*A_4^*U_{j,j}, \tag{8.4}$$
$$\varphi_j = A_1^*A_3^*A_4^*F_j + A_1^*(P_2^*U_{r,rj} - P_1^*U_j),$$

where the functions F and U_j satisfy

$$A_1^*A_4^*P_1^*F = \rho s, \quad A_1^*A_4^*P_1^*U_j = \rho Q_j. \tag{8.5}$$

Then χ and φ_j satisfy the Eqs (8.3).

If we introduce the notations

$$b_3 = b_1 + b_2, \;\; \alpha_1^* = -\mu_3^2 + kb_3 + \kappa_3(E_2 + E_4),$$
$$\alpha_2^* = k(E_2+E_4) - \omega^2(\kappa_3 b + ab_3), \quad \alpha_3^* = \omega^2(a(E_2+E_4) + kb + (\gamma+\zeta)^2), \tag{8.6}$$

then from (8.2), we get

$$P_1^* = \kappa_3 b_3\Delta^4 - \alpha_1^*\Delta^3 + \alpha_2^*\Delta^2 + \alpha_3^*\Delta + ab\omega^4. \tag{8.7}$$

It is easy to see that if $k_j^2 (j = 1, ..., 4)$ are the roots of the equation

$$\kappa_3 b_3 x^4 + \alpha_1^* x^3 + \alpha_2^* x^2 - \alpha_3^* x + ab\omega^4 = 0, \tag{8.8}$$

then we have

$$P_1^* = \kappa_3 b_3(\Delta + k_1^2)(\Delta + k_2^2)(\Delta + k_3^2)(\Delta + k_4^2). \tag{8.9}$$

We denote by k_5^2 and k_6^2 the roots of the following equation:

$$\kappa_3 x^2 + kx - a\omega^2 = 0. \tag{8.10}$$

From (8.6) and (8.10), we obtain

$$A_1^* = -\kappa_3(\Delta + k_5^2)(\Delta + k_6^2). \tag{8.11}$$

Similarly we find

$$A_3^* = -b_1(\Delta + k_7^2)(\Delta + k_8^2), \tag{8.12}$$

where k_7^2 and k_8^2 are the roots of the equation

$$b_1 x^2 + E_2 x - b\omega^2 = 0. \tag{8.13}$$

Let us assume that

$$s = f, \quad Q_j = 0, \tag{8.14}$$

where f is a given function. From (8.5), we see that in this case we can take $F = e$ and $U_j = 0$, where e satisfies the equation

$$P_1^* A_1^* A_4^* e = f. \tag{8.15}$$

We denote by k_j $(j = 1, 2, .., 8)$ the roots with positive real parts and assume that they are different. If the functions g_j satisfy the equations:

$$\kappa_3^2 b_1 b_3(\Delta + k_j^2) g_j = f \text{ (no sum; } j = 1, 2, ..., 8), \tag{8.16}$$

then the function e can expressed as

$$e = \sum_{j=1}^{8} m_j g_j, \tag{8.17}$$

where

$$m_r^{-1} = \prod_{j=1(j\neq r)}^{8} (k_r^2 - k_j^2). \tag{8.18}$$

We now assume that $s = \delta(\mathbf{x} - \mathbf{y})$, where $\delta(.)$ is the Dirac delta and $\mathbf{y}$ is a fixed point. In this case, the solution of the Eq (8.15) is given by

$$e_0 = (4\pi\kappa_3^2 b_1 b_3)^{-1} \sum_{j=1}^{8} m_j \exp(ik_j r), \tag{8.19}$$

where $r = |\mathbf{x} - \mathbf{y}|$. If we take in (8.4), $U_j = 0$, and $F = e_0$, then we obtain the solution

$$\chi^{(1)} = -A_4^*\{P_1^* + A_2^* A_3^* \Delta\} e_0, \quad \varphi_j^{(1)} = A_1^* A_2^* A_4^* e_{0,j}. \tag{8.20}$$

Let us assume that

$$s = 0, \;\; Q_i = \delta_{ij}\delta(\mathbf{x} - \mathbf{y}), \tag{8.21}$$

where j is fixed. We see that in this case, Eqs (8.5) are satisfied if we take $F = 0$ and $U_i = e_0\delta_{ij}$. From (8.4), we find the following solution:

$$\chi^{(1+j)} = A_1^* A_2^* A_4^* e_{0,j}, \quad \varphi_i^{(1+j)} = A_1^*(P_2^* e_{0,ij} - \delta_{ij} P_1^* e_0). \tag{8.22}$$

The functions $\chi^{(k)}$ and $\varphi_i^{(k)}$ $(k = 1, ..., 4)$, given by (8.20) and (8.22), represent the fundamental solutions of the system of equations describing steady vibrations.

9. Anti-plane shear deformations

In this last section, we use the previous results to study the problem of anti-plane shear deformations. We assume that the domain B from here on refers to a right cylinder of length h with the cross-section $\mathcal{D}$. We select the coordinate frame in such a way that the x_3-axis is parallel to the generator of the cylinder. Body loads are assumed to be of the form

$$f_\alpha = 0,\ \ Q_\alpha = 0,\ \ f_3 = f(x_1, x_2, t),\ \ Q_3 = Q(x_1, x_2, t),\ \ s = s(x_1, x_2, t), \alpha = 1, 2, t \in I,$$

where $I = (0, T^*)$. We look for a solution in the form

$$u_\alpha = 0, \varphi_\alpha = 0, u_3 = u(x_1, x_2, t), \varphi_3 = \varphi(x_1, x_2, t), \chi = \chi(x_1, x_2, t), \alpha = 1, 2.$$

We can observe that these functions satisfy our system in the case that the following system

$$\mu\Delta u + C_2\Delta\varphi + \rho f = \rho\ddot{u},$$

$$C_2\Delta u + E_2\Delta\varphi - b_1\Delta^2\varphi + \rho Q = b\ddot{\varphi},$$
$$k\Delta\chi - \kappa_3\Delta^2\chi + \rho s = a\ddot{\chi},$$

holds. Throughout this section, the Laplacian operator is considered in the two-dimensional case. We can see that the first two equations are coupled to each other, but both are decoupled from the temperature equation. First of all, we will concentrate our attention on the first two equations. We impose the boundary conditions

$$u(x_1, x_2, t) = \varphi(x_1, x_2, t) = \Delta\varphi(x_1, x_2, t) = 0,\ \ (x_1, x_2) \in \partial\mathcal{D},\ \ t > 0$$

as well as the initial conditions

$$u(x_1, x_2, 0) = u^0(x_1, x_2),\ \ \dot{u}(x_1, x_2, 0) = v^0(x_1, x_2),$$

$$\varphi(x_1, x_2, 0) = \varphi^0(x_1, x_2),\ \ \dot{\varphi}(x_1, x_2, 0) = \nu^0(x_1, x_2).$$

For this problem, the energy satisfies:

$$E(t) = \frac{1}{2}\int_{\mathcal{D}} (\rho|\dot{u}|^2 + b|\dot{\varphi}|^2 + \mu|\nabla u|^2 + 2C_2\nabla u\nabla\varphi + E_2|\nabla\varphi|^2 + b_1|\Delta\varphi|^2) d\mathbf{x} = E(0),$$

whenever we assume $f = Q = 0$, because

$$\dot{E}(t) = \int_{\mathcal{D}} (\rho\dot{u}\ddot{u} + b\dot{\varphi}\ddot{\varphi} + \mu\nabla u\nabla\dot{u} + C_2(\nabla\dot{u}\nabla\varphi + \nabla u\nabla\dot{\varphi}) + E_2\nabla\varphi\nabla\dot{\varphi} + b_1\Delta\varphi\Delta\dot{\varphi}) d\mathbf{x}$$

$$= \int_{\mathcal{D}} (\dot{u}(\mu\Delta u + C_2\Delta\varphi) + \dot{\varphi}(C_2\Delta u + E_2\Delta\varphi - b_1\Delta^2\varphi) d\mathbf{x} \tag{9.1}$$

$$+ \int_{\mathcal{D}} (\mu\nabla u\nabla\dot{u} + C_2(\nabla\dot{u}\nabla\varphi + \nabla u\nabla\dot{\varphi}) + E_2\nabla\varphi\nabla\dot{\varphi} + b_1\Delta\varphi\Delta\dot{\varphi}) d\mathbf{x} = 0,$$

where the last equality follows after the use of the divergence theorem and boundary conditions.

We do not assume any condition on the coefficients μ, C_2, E_2, and b_1, but we are going to obtain a couple of qualitative results for our system. Nevertheless we will need to suppose that ρ and b are two positive real numbers. To do this, it will be convenient to work with a function that will allow us to conclude the desired results. We define this function in the form:

$$\mathcal{H}(t) = \int_{\mathcal{D}} (\rho u^2 + b\varphi^2) d\mathbf{x} + \omega^*(t + t_0)^2, \tag{9.2}$$

where ω^* and t_0 are two non-negative real numbers to be selected. We have

$$\dot{\mathcal{H}}(t) = 2\int_{\mathcal{D}} (\rho u\dot{u} + b\varphi\dot{\varphi}) d\mathbf{x} + 2\omega^*(t + t_0),$$

and

$$\ddot{\mathcal{H}}(t) = 2\int_{\mathcal{D}} (\rho u\ddot{u} + b\varphi\ddot{\varphi}) d\mathbf{x} + 2\int_{D} (\rho|\dot{u}|^2 + b|\dot{\varphi}|^2) d\mathbf{x} + 2\omega^*.$$

We can notice that

$$\int_{\mathcal{D}} (\rho u\ddot{u} + b\varphi\ddot{\varphi}) d\mathbf{x} = -\int_{\mathcal{D}} (\mu|\nabla u|^2 + 2C_2\nabla\varphi\nabla\varphi + E_2|\nabla\varphi|^2 + b_1|\Delta\varphi|^2) d\mathbf{x}$$

$$\int_{\mathcal{D}} (\rho|\dot{u}|^2 + b|\dot{\varphi}|^2) d\mathbf{x} - 2E(0).$$

Therefore

$$\ddot{\mathcal{H}}(t) = 4\int_{\mathcal{D}} (\rho|\dot{u}|^2 + b|\dot{\varphi}|^2) d\mathbf{x} + 2(\omega^* - 2E(0)).$$

A simple use of Holder's inequality allows us to conclude

$$\mathcal{H}(t)\ddot{\mathcal{H}}(t) - (\dot{\mathcal{H}}(t))^2 \geq -2(\omega^* + 2E(0))\mathcal{H}(t).$$

If we put homogeneous initial conditions, we have that $E(0) = 0$ and if we take $\omega^* = 0$, we can conclude that

$$\mathcal{H}(t)\ddot{\mathcal{H}}(t) - (\dot{\mathcal{H}})^2 \geq 0. \tag{9.3}$$

This inequality allows us to establish (see [19], p. 19) that

$$\mathcal{H}(t) \leq \mathcal{H}(0)^{1-t/T^*}\mathcal{H}(T^*)^{t/T^*}$$

for all t between 0 and T^*. Thus, in the case where we impose null initial data, we obtain that $\mathcal{H}(t) = 0$ for all t in the interval and ,consequently we obtain the null solution. This allows us to conclude the uniqueness of the solutions.

If we now go back to the general case and suppose that the initial energy is negative, we can take $\omega^* = -2E(0)$ and again conclude the previous inequality. We can also get (see [19], p. 20)

$$\mathcal{H}(t) \geq \mathcal{H}(0)\exp\Big(\frac{t\dot{\mathcal{H}}(0)}{\mathcal{H}(0)}\Big). \tag{9.4}$$

We note that we can always select t_0 large enough to guarantee that $\dot{\mathcal{H}}(0) > 0$. When $E(0) = 0$ and $\dot{\mathcal{H}}(0) > 0$, we also conclude the growth estimator.

Theorem 4. Let us to suppose that ρ and b are positive. Then:

(i) The initial-boundary-value problem for the anti-plane shear deformations has a unique solution.

(ii) When $E(0) < 0$ or $(E(0) = 0,\ \dot{\mathcal{H}}(0) > 0)$, then the solution is exponentially unstable.

A similar argument could prove the uniqueness and instability of the solutions for the temperature equation in the case that we only assume that a is strictly positive.

10. Conclusions

The results obtained in this paper can be summarized as follows:

(a) We present a linear theory of thermoelasticity with microtemperatures where the second thermal displacement gradient and the second gradient of microtemperatures are included in the classical set of independent constitutive variables.

(b) We express the field equations of the linear theory in terms of components of the displacement vector, thermal displacement, and thermal microdisplacement,s and obtain a fourth-order system of equations. The boundary-initial-value problems are also formulated.

(c) The semigroup theory of linear operators is used to prove that the problem of the second gradient thermoelasticity with microtemperatures is well-posed.

(d) We establish a counterpart of the Cauchy-Kovalevski-Somigliana solution of the isothermal theory.

(e) In the case of stationary vibrations, we establish the fundamental solutions of the field equations.

(f) Uniqueness and instability of the solutions are obtained in the case of anti-plane shear deformations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Ramón Quintanilla is an editorial board member for Electronic Research Archive and was not involved in the editorial review or the decision to publish this article. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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