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Research article

On exponential decay properties of solutions of the (3 + 1)-dimensional modified Zakharov-Kuznetsov equation

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Abstract: We study here the special decay properties of real solutions to the initial value problem associated with the (3 + 1)-dimensional modified Zakharov-Kuznetsov equation. More precisely, we prove the properties of exponential decay of order 3/2 above the plane $x + y + z = 0$ as time evolves. This property is related with the persistence properties of the solution flow in weighted Sobolev spaces and sharp unique continuation properties of solutions to this problem.

Keywords: (3 + 1)-dimensional modified Zakharov-Kuznetsov equation; exponential decay; energy estimate

1. Introduction

In this paper, we would like to investigate the initial value problem (IVP) associated with the (3+1)-dimensional modified Zakharov-Kuznetsov (mZK) equation

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x \partial_y^2 u + \partial_x \partial_z^2 u + \gamma u^2 \partial_x u = 0, & (x, y, z) \in \mathbb{R}^3 \quad t \geq 0, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $u = u(x, y, z, t)$ is a real-valued function, $u_0 = u_0(x, y, z)$, and γ is a nonzero constant. Furthermore, it is proved that its solutions $u(x, y, z, t)$ have the properties of exponential decay above the plane $x + y + z = 0$.

Equation (1.1) was proposed by Zakharov and Kuznetsov [1] as a three-dimensional generalization of the Korteweg-de Vries (KdV) equation, which was derived from the Euler-Poisson system with magnetic field by Lannes et al. in [2]. This equation describes the unidirectional propagation of ionic-acoustic waves in magnetized plasma.

It is easy to see that the ZK equation can be regarded as a multidimensional generalization of the one-dimensional KdV equation

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0, & x \in \mathbb{R}, \quad t \geq 0, \\ u(0) = u_0. \end{cases}$$

It is worth mentioning that two dimensional versions of the KdV equation and modified Korteweg-de Vries (mKdV) equation are the ZK equation and the mZK equation, respectively. Up to now, to the best of our knowledge, for the two-dimensional ZK equation, the well-posedness and uniqueness results have been studied extensively. For some related works, refer to [3–8] and references therein. At the same time, the Cauchy problem for the 2D mZK equation has also been discussed. The local well-posedness in $H^1(\mathbb{R}^2)$ was obtained by Biagioni and Linares in [9]. The local result was generalized to the data in $H^s(\mathbb{R}^2)$, $s > 3/4$, by Linares and Pastor in [10]. In terms of a smallness assumption on the L^2-norm of the data [11], they proved the global well-posedness in $H^s(\mathbb{R}^2)$, $s > 53/63$. In [12], Ribaud and Vento investigated the local well-posedness in $H^s(\mathbb{R}^2)$, $s > 1/4$.

On the other hand, for the three-dimensional ZK equation, many interesting results have been obtained. For initial data in $H^s(\mathbb{R}^3)$ with $s > 9/8$, Linares and Saut [13] showed the local well-posedness of this initial problem. The local well-posedness theory of the Benjamin-Ono equation was established in [14] by utilizing similar techniques in [13]. For more discussions of the 3D ZK equation, see [15–17] and reference therein. Moreover, for the 3D mZK equation, in [18], Grünrock proved the local well-posedness of the Cauchy problem (1.1) for initial data in $H^s(\mathbb{R}^3)$ with $s > 1/2$. Kinoshita [19] established the well-posedness in the critical space $H^{1/2}(\mathbb{R}^3)$ for the Cauchy problem of the mZK equation. Ali et al. [20] developed the propagation of dispersive wave solutions for (3+1)-dimensional nonlinear mZK equation in plasma physics. Further analysis results can be also found in [21] and references therein.

The properties of decay preservation are of great interest. In an innovative paper, Isaza and León [22] studied the optimal exponential decay properties of solutions to the KdV equation. Larkin and Tronco [23] derived the decay properties of small solutions for the ZK equation posed on a half-strip. In [24], Larkin further established the exponential decay of the H^1-norm for the 2D ZK equation. Recently, the decay properties for solutions of the ZK equation were also obtained in [25].

It is obvious that the decay properties are closely related to the aspect of unique continuation. It is noted that Bustamante et al. [15] derived the unique continuation property of the solutions of the 3D Zakharov-Kuznetsov equation. In recent years, the unique continuation principles of several models arising in nonlinear dispersive equations were investigated, see references [26–30] for example.

The well-posedness for the two dimensional generalized ZK equation in anisotropic weighted Sobolev spaces was discussed in [31]. In [32], Bustamante et al. established the well-posedness of the IVP for the 2D ZK equation in weighted Sobolev spaces $H^s(\mathbb{R}^2) \cap L^2((1 + x^2 + y^2)^r dxdy)$ for $s, r \in \mathbb{R}$. Furthermore, they also showed in [15] that, for some small $\varepsilon > 0$,

$$u_1, u_2 \in C([0, 1]; H^4(\mathbb{R}^3) \cap L^2((1 + x^2 + y^2 + z^2)^{\frac{8}{5}+\varepsilon} dxdydz)) \cap C^1([0, 1]; L^2(\mathbb{R}^3)),$$

are solutions of the IVP for the three-dimensional ZK equation. Then, there exists a constant $a_0 > 0$ such that if for some $a > a_0$,

$$u_1(0) - u_2(0), u_1(1) - u_2(1) \in L^2(e^{a(x^2+y^2+z^2)^{3/4}} dxdydz),$$

then $u_1 \equiv u_2$.

The main goal of the present paper is to formally derive the decay properties of exponential type solutions $u(x, y, z, t)$ to the IVP (1.1). In order to achieve this goal, we shall utilize Kato's approach to prove Kato's estimation in three-dimensional form.

Now, we are in the position to state our main results.

Theorem 1.1. *Let a_0 be a positive constant. For any given data*

$$u_0 \in H^2(\mathbb{R}^3) \cap L^2(e^{a_0(x+y+z)_+^{3/2}} dxdydz), \tag{1.2}$$

the unique solution $u(\cdot, \cdot, \cdot)$ of the IVP (1.1) *provided in [18]*

$$u \in C([0, T]; H^2(\mathbb{R}^3))$$

satisfies

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3} e^{a(t)(x+y+z)_+^{3/2}} |u(x, y, z, t)|^2 dxdydz \leq c^*, \tag{1.3}$$

where

$$c^* = c^* \left(a_0; \|u_0\|_{H^1(\mathbb{R}^3)}; \|e^{a_0(x+y+z)_+^{3/2}} u_0\|_{L^2(\mathbb{R}^3)}; T \right),$$

with

$$a(t) = \frac{a_0}{\sqrt{1 + \frac{81}{4} a_0^2 t}}.$$

Let us consider weighted spaces with symmetric weight, which take the form

$$L^2(\langle x + y + z\rangle^b dxdydz) = L^2((1 + (x + y + z)^2)^{\frac{b}{2}} dxdydz).$$

Regardless of whether the time direction is forward $t > 0$ or backward $t < 0$, its persistent properties should hold.

Theorem 1.2. *Let a_0 be a positive constant. Let u_1, u_2 be solutions of the IVP* (1.1) *such that*

$$\begin{aligned} &u_1 \in C([0, T]; H^3(\mathbb{R}^3)) \cap L^2(\langle x + y + z\rangle^2 dxdydz), \\ &u_2 \in C([0, T]; H^3(\mathbb{R}^3)). \end{aligned}$$

If

$$\Lambda = \int_{\mathbb{R}^3} e^{a_0(x+y+z)_+^{3/2}} |u_{0,1}(x, y, z) - u_{0,2}(x, y, z)|^2 dxdydz < \infty,$$

then

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3} e^{a(t)(x+y+z)_+^{3/2}} |u_1(x, y, z, t) - u_2(x, y, z, t)|^2 dxdydz \leq c^{**}, \tag{1.4}$$

where $c^{**} = c^{**} \left(a_0; \|u_{0,1}\|_{H^4(\mathbb{R}^3)}; \|u_{0,2}\|_{H^4(\mathbb{R}^3)}; \|x^2 u_{0,1}\|_{L^2(\mathbb{R}^3)}; \|x u_{0,2}\|_{L^2(\mathbb{R}^3)}; \Lambda; T \right)$ and

$$a(t) = \frac{a_0}{\sqrt{1 + \frac{81}{4} a_0^2 t}}.$$

Let us denote the norm of the functional space X^s by

$$\|f\|_{X^s} = \|J_x^s f\|_{L^2(\mathbb{R}^3)} + \|f\|_{H^1_{yz}L^2_x},$$

for $s > 1/2$, where $\widehat{J_x^s}(\xi, \eta, \gamma) = (1 + \xi^2)^{s/2}\widehat{f}(\xi, \eta, \gamma)$.

In terms of the context, the previous result shows that it is necessary to have a similar property in a suitable Sobolev space $H^s(\mathbb{R}^3)$ for a solution of the IVP (1.1) to satisfy the persistent property in $L^2(\langle x + y + z\rangle^b dxdydz)$.

Theorem 1.3. *Let $u_0 \in X^s$, $s > 9/8$. There exists $T = T(\|u_0\|_{X^s})$ and a unique solution of the IVP* (1.1) *such that $u \in C([0, T]; X^s)$ provided by Lemma 2.1 below. If there exist $\alpha > 0$ and two different instants of time $t_0, t_1 \in [0, T]$ such that*

$$\langle x + y + z\rangle^\alpha u(x, y, z, t_0), \langle x + y + z\rangle^\alpha u(x, y, z, t_1) \in L^2(\mathbb{R}^3),$$

then for any $t \in [0, T]$,

$$u(t) \in L^2(\langle x + y + z\rangle^\alpha dxdydz),$$

$$\left(\partial_x u(t) + \partial_y u(t)\right), (\partial_x u(t) + \partial_z u(t)) \in L^2(\langle x + y + z\rangle^{\alpha-1/2} dxdydz).$$

The rest of this paper is organized as follows. In Section 2, some details on known results of the three-dimensional mZK equation will be introduced. In Section 3, the weights will be constructed to put forward the theory. Section 4 is devoted to proving Theorems 1.1 and 1.2. Finally, in Section 5, we demonstrate Theorem 1.3.

2. Preliminaries

Attention in this section is now turned to prove some preliminary estimates which we often use in our analysis. We first give the following result.

Lemma 2.1. *Given $u_0 \in X^s$, $s > 9/8$, there exists $T = T(\|u_0\|_{X^s})$ and a unique solution of the IVP* (1.1) *such that $u \in C([0, T]; X^s)$, $u, \partial_x u \in L^1_T L^\infty_{xyz}$. Moreover, the map $u_0 \mapsto u$ is continuous from a neighborhood of $u_0 \in X^s$ into $C([0, T]; X^s)$.*

The proof is similar to Theorem 3.9 in [13]. Meanwhile, using the assumptions of Theorem 1.3, we deduce that

$$\int_0^T \|u\|_{L^\infty_{xyz}} dt + \int_0^T \|\partial_x u\|_{L^\infty_{xyz}} dt \le c_T, \tag{2.1}$$

where c_T is a constant.

Lemma 2.2. *Let $u \in C([0, T]; H^2(\mathbb{R}^3))$ be a solution of the IVP* (1.1)*, corresponding to data $u_0 \in H^2(\mathbb{R}^3) \cap L^2(e^{\beta(x+y+z)} dxdydz)$, $\beta > 0$. Then,*

$$e^{\beta(x+y+z)} u \in C([0, T]; L^2(\mathbb{R}^3))$$

and

$$\|e^{\beta(x+y+z)} u(t)\|_{L^2(\mathbb{R}^3)} \le c\|e^{\beta(x+y+z)} u_0\|_{L^2(\mathbb{R}^3)}, \qquad t \in [0, T].$$

Proof. Applying Kato's approach in [33], let us now prove this lemma. First, we consider the equation

$$\partial_t u + \partial_x^3 u + \partial_x\partial_y^2 u + \partial_x\partial_z^2 u + \gamma u^2\partial_x u = 0, \qquad (x, y, z) \in \mathbb{R}^3, \quad t \geq 0. \tag{2.2}$$

Next, multiplying by $u\varphi_\delta$ on both sides of Eq (2.2) and integrating by parts, a direct computation gives rise to

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} u^2\varphi_\delta dxdydz + \int_{\mathbb{R}^3}(\partial_x u + \partial_y u)^2\partial_y\varphi_\delta dxdydz + \int_{\mathbb{R}^3}(\partial_x u + \partial_z u)^2\partial_z\varphi_\delta dxdydz \\
&\quad + \int_{\mathbb{R}^3}(\partial_x u)^2(3\partial_x\varphi_\delta - \partial_y\varphi_\delta - \partial_z\varphi_\delta)dxdydz + \int_{\mathbb{R}^3}(\partial_y u)^2(\partial_x\varphi_\delta - \partial_y\varphi_\delta)dxdydz \\
&\quad + \int_{\mathbb{R}^3}(\partial_z u)^2(\partial_x\varphi_\delta - \partial_z\varphi_\delta)dxdydz \\
&= \int_{\mathbb{R}^3} u^2\left(\partial_x^3\varphi_\delta + \partial_{xyy}\varphi_\delta + \partial_{xzz}\varphi_\delta\right)dxdydz + \frac{\gamma}{2}\int_{\mathbb{R}^3} u^4\partial_x\varphi_\delta dxdydz.
\end{aligned}$$

For $\beta > 0$, we define

$$\varphi_\delta(x, y, z) = \frac{e^{\beta(x+y+z)}}{1 + \delta e^{\beta(x+y+z)}} \qquad \text{for} \quad \delta \in (0, 1), \quad \delta \ll 1.$$

Thus, we see that

$$\varphi_\delta \in L^\infty(\mathbb{R}^3) \qquad \text{and} \qquad \|\varphi_\delta\|_{L^\infty(\mathbb{R}^3)} = \frac{1}{\delta}. \tag{2.3}$$

$$0 \leq \partial_x\varphi_\delta(x, y, z) = \partial_y\varphi_\delta(x, y, z) = \partial_z\varphi_\delta(x, y, z) = \frac{\beta e^{\beta(x+y+z)}}{(1 + \delta e^{\beta(x+y+z)})^2} \leq \beta\varphi_\delta(x, y, z),$$

$$\partial_x^2\varphi_\delta(x, y, z) = \partial_{xy}\varphi_\delta(x, y, z) = \partial_{xz}\varphi_\delta(x, y, z) = \frac{\beta^2 e^{\beta(x+y+z)}(1 - \delta e^{\beta(x+y+z)})}{(1 + \delta e^{\beta(x+y+z)})^3},$$

and then

$$|\partial_x^2\varphi_\delta(x, y, z)| = |\partial_{xy}\varphi_\delta(x, y, z)| = |\partial_{xz}\varphi_\delta(x, y, z)| \leq \beta^2\frac{e^{\beta(x+y+z)}}{(1 + \delta e^{\beta(x+y+z)})^2}.$$

$$\begin{aligned}
\partial_x^3\varphi_\delta(x, y, z) &= \partial_{xyy}\varphi_\delta(x, y, z) = \partial_{xzz}\varphi_\delta(x, y, z) \\
&= \frac{\beta^3 e^{\beta(x+y+z)}(1 - 4\delta e^{\beta(x+y+z)} + \delta^2 e^{2\beta(x+y+z)})}{(1 + \delta e^{\beta(x+y+z)})^4},
\end{aligned}$$

hence

$$|\partial_x^3\varphi_\delta(x, y, z)| = |\partial_{xyy}\varphi_\delta(x, y, z)| = |\partial_{xzz}\varphi_\delta(x, y, z)| \leq 2\beta^3\frac{e^{\beta(x+y+z)}}{(1 + \delta e^{\beta(x+y+z)})^2}.$$

Therefore,

$$\partial_x^3\varphi_\delta(x,y,z) + \partial_{xyy}\varphi_\delta(x,y,z) + \partial_{xzz}\varphi_\delta(x,y,z) \le c_0\beta^3\varphi_\delta(x,y,z). \tag{2.4}$$

Moreover,

$$\varphi_\delta(x,y,z) \le \varphi_{\delta'}(x,y,z), \qquad (x,y,z)\in\mathbb{R}^3 \quad \text{if} \quad 0<\delta'<\delta,$$

and

$$\lim_{\delta\downarrow 0}\varphi_\delta(x,y,z) = e^{\beta(x+y+z)}.$$

We apply properties (2.3) and (2.4) to obtain the estimate

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3} u^2\varphi_\delta(x,y,z)dxdydz\\ &\le c_0\beta^3\int_{\mathbb{R}^3} u^2\varphi_\delta(x,y,z)dxdydz + \frac{\gamma}{2}\int_{\mathbb{R}^3} u^4\partial_x\varphi_\delta(x,y,z)dxdydz.\end{aligned} \tag{2.5}$$

In the case of $u\in C([0,T];H^2(\mathbb{R}^3))$, there exists a positive constant c such that

$$\|u\|_{L^\infty(\mathbb{R}^3)} \le c.$$

Next, we consider the last term of (2.5). We write

$$\begin{aligned}\int_{\mathbb{R}^3} u^4\partial_x\varphi_\delta(x,y,z)dxdydz &\le \beta\|u\|^2_{L^\infty(\mathbb{R}^3)}\int_{\mathbb{R}^3} u^2\varphi_\delta(x,y,z)dxdydz\\ &\le c\int_{\mathbb{R}^3} u^2\varphi_\delta(x,y,z)dxdydz.\end{aligned}$$

Inserting this estimate into (2.5), one has

$$\frac{d}{dt}\int_{\mathbb{R}^3} u^2\varphi_\delta(x,y,z)dxdydz \le c\int_{\mathbb{R}^3} u^2\varphi_\delta(x,y,z)dxdydz. \tag{2.6}$$

Using Gronwall's lemma and integrating (2.6) in $t\in[0,T]$, we deduce that

$$\begin{aligned}&\sup_{t\in[0,T]}\int_{\mathbb{R}^3} u^2(x,y,z,t)\varphi_\delta(x,y,z)dxdydz\\ &\le c\int_{\mathbb{R}^3} u_0^2(x,y,z)\varphi_\delta(x,y,z)dxdydz\\ &\le c\int_{\mathbb{R}^3} u_0^2(x,y,z)\varphi_0(x,y,z)dxdydz,\end{aligned}$$

where c is a constant.

Letting $\delta\downarrow 0$, this completes the proof of Lemma 2.2.

□

3. Construction of weights

We multiply $u\phi_N$ on both sides of (2.2). Then, for a fixed $t \in [0, T]$, integrating over $\mathbb{R}^3$ with x, y, and z, and making use of integration by parts yields

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} u^2\phi_N dxdydz + \int_{\mathbb{R}^3}(\partial_x u + \partial_y u)^2\partial_y\phi_N dxdydz + \int_{\mathbb{R}^3}(\partial_x u + \partial_z u)^2\partial_z\phi_N dxdydz \\
&\quad + \int_{\mathbb{R}^3}(\partial_x u)^2(3\partial_x\phi_N - \partial_y\phi_N - \partial_z\phi_N)dxdydz + \int_{\mathbb{R}^3}(\partial_y u)^2(\partial_x\phi_N - \partial_y\phi_N)dxdydz \\
&\quad + \int_{\mathbb{R}^3}(\partial_z u)^2(\partial_x\phi_N - \partial_z\phi_N)dxdydz \qquad (3.1)\\
&= \int_{\mathbb{R}^3} u^2\left(\partial_x^3\phi_N + \partial_{xyy}\phi_N + \partial_{xzz}\phi_N + \partial_t\phi_N\right)dxdydz + \frac{\gamma}{2}\int_{\mathbb{R}^3} u^4\partial_x\phi_N dxdydz.
\end{aligned}$$

A sequence of the weights $\{\phi_N\}_{N=1}^{\infty}$ will be constructed, which plays an important role in the proof of our main theorems.

Theorem 3.1. *Given $a_0 > 0$, there exists a sequence $\{\phi_N\}_{N=1}^{\infty}$ of functions with*

$$\phi_N : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}$$

satisfying for any $N \in \mathbb{Z}^+$:
(i) $\phi_N \in C^2(\mathbb{R}^3 \times [0, \infty))$ with $\partial_x^3\phi_N(\cdot,\cdot,\cdot,t)$, $\partial_{xyy}\phi_N(\cdot,\cdot,\cdot,t)$, $\partial_{xzz}\phi_N(\cdot,\cdot,\cdot,t)$ having a jump discontinuity at $x + y + z = N$.
(ii) $\phi_N(x, y, z, t) > 0$ for all $(x, y, z, t) \in \mathbb{R}^3 \times [0, \infty)$.
(iii) $\partial_x\phi_N(x, y, z, t) = \partial_y\phi_N(x, y, z, t) = \partial_z\phi_N(x, y, z, t) > 0$ for all $(x, y, z, t) \in \mathbb{R}^3 \times [0, \infty)$.
(iv) There exist constants $c_N = c(N) > 0$ and $c_0 = c_0(a_0) > 0$ such that

$$\phi_N(x, y, z, t) \le c_N c_0 \langle (x + y + z)_+ \rangle^2,$$

with

$$(x + y + z)_+ = \max\{0; x + y + z\}, \qquad \langle x + y + z\rangle = (1 + (x + y + z)^2)^{1/2}.$$

(v) For $T > 0$, there is $N_0 \in \mathbb{Z}^+$ such that

$$\phi_N(x, y, z, 0) \le e^{a_0(x+y+z)_+^{3/2}} \qquad if \quad N > N_0.$$

Also,

$$\lim_{N\uparrow\infty}\phi_N(x, y, z, t) = e^{a(t)(x+y+z)_+^{3/2}},$$

for any $t > 0$ and $x + y + z \in (-\infty, 0) \cap (1, \infty)$, where

$$a(t) = \frac{a_0}{\sqrt{1 + \frac{81}{4}a_0^2 t}}.$$

(vi) There exists a constant $c_0 = c_0(a_0) > 0$ such that

$$\partial_t\phi_N + \partial_x^3\phi_N + \partial_{xyy}\phi_N + \partial_{xzz}\phi_N \le c_0\phi_N, \tag{3.2}$$

for any $(x,y,z,t) \in \mathbb{R}^3 \times [0,\infty)$.
(vii) There exists a constant $c_1 = c_1(a_0) > 0$ such that

$$|\partial_x\phi_N(x,y,z,t)| \le c_1\langle x+y+z\rangle^{1/2}\phi_N(x,y,z,t), \tag{3.3}$$

for any $(x,y,z,t) \in \mathbb{R}^3 \times [0,\infty)$.

Proof. For $N \in \mathbb{Z}^+$, given $a_0 > 0$, let us first define

$$\phi_N(x,y,z,t) = \begin{cases} e^{a(t)\varphi(x,y,z)}, & -\infty < x+y+z \le 1, \\ e^{a(t)(x+y+z)^{3/2}}, & 1 \le x+y+z \le N, \\ P_N(x,y,z,t), & x+y+z \ge N, \end{cases}$$

where

$$a(t) = \frac{a_0}{\sqrt{1+\frac{81}{4}a_0^2 t}} \in (0,a_0], \qquad t \ge 0, \tag{3.4}$$

a_0 being the initial parameter.

$$\varphi(x,y,z) = (1-\eta(x+y+z))(x+y+z)_+^3 + \eta(x+y+z)(x+y+z)^{3/2},$$
$$(x+y+z)_+ = \max\{0; x+y+z\},$$

for $x+y+z \in (-\infty,1]$ where $\eta \in C^\infty(\mathbb{R}^3)$, $\eta_x = \eta_y = \eta_z \ge 0$, and

$$\eta(x+y+z) = \begin{cases} 0, & x+y+z \le 1/2, \\ 1, & x+y+z \ge 3/4, \end{cases} \tag{3.5}$$

i.e., for each $x+y+z \in [0,1]$, $\varphi(x,y,z)$ is a convex combination of $(x+y+z)^3$ and $(x+y+z)^{3/2}$. $P_N(x,y,z,t)$ is a polynomial of order 2 in $(x+y+z)$, which matches the value of $e^{a(t)(x+y+z)^{3/2}}$ and its partial derivatives up to order 2 at $x+y+z = N$:

$$P_N(x,y,z,t) = \left[1 + \frac{3}{2}aN^{1/2}(x+y+z-N) + \left(\frac{9}{4}a^2N + \frac{3}{4}aN^{-1/2}\right)\frac{(x+y+z-N)^2}{2}\right]e^{aN^{3/2}},$$

with $a = a(t)$ as in (3.4).

Thus, to prove Theorem 3.1, let us consider the regions $x+y+z \in (-\infty,0]$, $[0,1]$, $[1,N]$, and $[N,\infty)$, respectively.

In the first region $x+y+z \le 0$, we get

$$\phi_N(x,y,z,t) = e^{a(t)\cdot 0} = 1,$$

which clearly satisfies Theorem 3.1.

In the region $x+y+z \in [0,1]$, we deduce that

$$\phi_N(x,y,z,t) = e^{a(t)\varphi(x,y,z)},$$

with

$$\varphi(x,y,z) = (1-\eta(x+y+z))(x+y+z)^3 + \eta(x+y+z)(x+y+z)^{3/2} \geq 0, \qquad x+y+z \in [0,1],$$

with η as in (3.5). Since in this region $(x+y+z)^{3/2} \geq (x+y+z)^3$, it follows that

$$\begin{aligned}\partial_x\varphi(x,y,z) &= (1-\eta(x+y+z))3(x+y+z)^2 + \eta(x+y+z)\frac{3}{2}(x+y+z)^{1/2} \\ &\quad + \partial_x\eta(x+y+z)((x+y+z)^{3/2} - (x+y+z)^3) \\ &\geq (1-\eta(x+y+z))3(x+y+z)^2 + \eta(x+y+z)\frac{3}{2}(x+y+z)^{1/2} \geq 0,\end{aligned}$$

likewise

$$\partial_y\varphi(x,y,z) \geq 0, \qquad \partial_z\varphi(x,y,z) \geq 0,$$

with

$$\partial_x\varphi(x,y,z) = \partial_y\varphi(x,y,z) = \partial_z\varphi(x,y,z), \tag{3.6}$$

and there exists $c > 0$ such that

$$\begin{aligned}&\partial_x\varphi(x,y,z), \quad \partial_{xx}\varphi(x,y,z), \quad \partial_{xy}\varphi(x,y,z), \quad \partial_{xz}\varphi(x,y,z) \leq c, \\ &\partial_{xxx}\varphi(x,y,z), \quad \partial_{xyy}\varphi(x,y,z), \quad \partial_{xzz}\varphi(x,y,z) \leq c, \quad x+y+z \in [0,1].\end{aligned}$$

Note that for $a'(t) \leq 0$, it is found that

$$a(t) \leq a_0 \quad \text{for} \quad t \geq 0, \tag{3.7}$$

and it is deduced that

$$\partial_t\phi_N(x,y,z,t) = a'(t)\varphi(x,y,z)\phi_N(x,y,z,t) \leq 0.$$

Next, let us prove that there exists $c_0 = c_0(a_0) > 0$ in this region such that

$$\partial_x^3\phi_N + \partial_{xyy}\phi_N + \partial_{xzz}\phi_N \leq c_0\phi_N.$$

Since

$$\begin{aligned}&\partial_x\phi_N = a\varphi_x\phi_N, \quad \partial_y\phi_N = a\varphi_y\phi_N, \quad \partial_z\phi_N = a\varphi_z\phi_N, \\ &\partial_x^2\phi_N = \left(a\varphi_{xx} + (a\varphi_x)^2\right)\phi_N, \quad \partial_{xy}\phi_N = \left(a\varphi_{xy} + a^2\varphi_x\varphi_y\right)\phi_N, \\ &\partial_{xz}\phi_N = \left(a\varphi_{xz} + a^2\varphi_x\varphi_z\right)\phi_N, \quad \partial_x^3\phi_N = \left(a\varphi_{xxx} + 3a^2\varphi_{xx}\varphi_x + (a\varphi_x)^3\right)\phi_N,\end{aligned}$$

$$\partial_{xyy}\phi_N = \left(a\varphi_{xyy} + 2a^2\varphi_{xy}\varphi_y + a^2\varphi_x\varphi_{yy} + a^3\varphi_x\varphi_y^2\right)\phi_N,$$
$$\partial_{xzz}\phi_N = \left(a\varphi_{xzz} + 2a^2\varphi_{xz}\varphi_z + a^2\varphi_x\varphi_{zz} + a^3\varphi_x\varphi_z^2\right)\phi_N.$$

From (3.6), we derive that

$$\partial_x\phi_N = \partial_y\phi_N = \partial_z\phi_N,$$

thus, for $x + y + z \sim 0$ ($x + y + z \geq 0$),

$$\partial_x\phi_N = \partial_y\phi_N = \partial_z\phi_N \sim 3a(x + y + z)^2\phi_N,$$
$$\partial_x^2\phi_N = \partial_{xy}\phi_N = \partial_{xz}\phi_N \sim \left(6a(x + y + z) + 9a^2(x + y + z)^4\right)\phi_N,$$
$$\partial_x^3\phi_N = \partial_{xyy}\phi_N = \partial_{xzz}\phi_N \sim \left(6a + 54a^2(x + y + z)^3 + 27a^3(x + y + z)^6\right)\phi_N.$$

Hence, for $x + y + z \sim 0$ ($0 \leq x + y + z \leq 1$),

$$\partial_x^3\phi_N(x, y, z, t) \leq c(a + a^3)\phi_N, \quad \partial_{xyy}\phi_N(x, y, z, t) \leq c(a + a^3)\phi_N,$$
$$\partial_{xzz}\phi_N(x, y, z, t) \leq c(a + a^3)\phi_N.$$

Using (3.7) (i.e., $a(t) \leq a_0$ for $t \geq 0$), it is easy to see that there exist $\delta > 0$ and a universal constant $c > 0$ such that

$$\partial_x^3\phi_N + \partial_{xyy}\phi_N + \partial_{xzz}\phi_N \leq c(a_0 + a_0^3)\phi_N, \quad \text{for} \quad x \in [0, \delta], \quad t \geq 0. \tag{3.8}$$

In the region $x + y + z \in [1, \delta]$, we conclude that (3.8) still holds (with a possible large $c > 0$). Applying the above estimates, it then follows that Theorem 3.1 holds in this region.

In the domain $x + y + z \in [1, N]$, we observe that

$$\phi_N(x, y, z, t) = e^{a(t)(x+y+z)^{3/2}}, \qquad x + y + z \in [1, N], \quad t \geq 0.$$

Then, a direct computation gives rise to

$$\begin{aligned}
\partial_x\phi_N &= \frac{3}{2}a(x + y + z)^{1/2}\phi_N > 0,\\
\partial_x^2\phi_N &= \left[\frac{9}{4}a^2(x + y + z) + \frac{3}{4}a(x + y + z)^{-1/2}\right]\phi_N,\\
\partial_{xy}\phi_N &= \left[\frac{9}{4}a^2(x + y + z) + \frac{3}{4}a(x + y + z)^{-1/2}\right]\phi_N,\\
\partial_{xz}\phi_N &= \left[\frac{9}{4}a^2(x + y + z) + \frac{3}{4}a(x + y + z)^{-1/2}\right]\phi_N,\\
\partial_x^3\phi_N &= \left[\frac{27}{8}a^3(x + y + z)^{3/2} + \frac{27}{8}a^2 - \frac{3}{8}a(x + y + z)^{-3/2}\right]\phi_N,\\
\partial_{xyy}\phi_N &= \left[\frac{27}{8}a^3(x + y + z)^{3/2} + \frac{27}{8}a^2 - \frac{3}{8}a(x + y + z)^{-3/2}\right]\phi_N,\\
\partial_{xzz}\phi_N &= \left[\frac{27}{8}a^3(x + y + z)^{3/2} + \frac{27}{8}a^2 - \frac{3}{8}a(x + y + z)^{-3/2}\right]\phi_N.
\end{aligned} \tag{3.9}$$

Hence, $\phi_N > 0$ and

$$\begin{aligned}&\partial_t\phi_N + \partial_x^3\phi_N + \partial_{xyy}\phi_N + \partial_{xzz}\phi_N \\ &= \left[a'(x+y+z)^{\frac{3}{2}} + \frac{81}{8}a^3(x+y+z)^{\frac{3}{2}} + \frac{81}{8}a^2 - \frac{9}{8}a(x+y+z)^{-\frac{3}{2}}\right]\phi_N.\end{aligned} \tag{3.10}$$

Taking advantage of

$$a'(t) + \frac{81}{8}a^3(t) = 0,$$

we eliminate the terms with power $3/2$ on the right-hand side of (3.10). Therefore,

$$a(t) = \frac{a_0}{\sqrt{1+\frac{81}{4}a_0^2 t}}. \tag{3.11}$$

We show that

$$\partial_t\phi_N + \partial_x^3\phi_N + \partial_{xyy}\phi_N + \partial_{xzz}\phi_N \le c_0\phi_N,$$

with $c_0 = c_0(a_0) > 0$, and it is easy to find that for $1 \le x+y+z \le N$,

$$\frac{81}{8}a^2 - \frac{9}{8}a(x+y+z)^{-\frac{3}{2}} \le c_0,$$

since $a(t) = a \le a_0$, $-\frac{9}{8}a(x+y+z)^{-\frac{3}{2}} \le 0$.

Next, it follows from (3.9) that

$$\begin{aligned}&\partial_x\phi_N = \partial_y\phi_N = \partial_z\phi_N = \frac{3}{2}a(x+y+z)^{1/2}\phi_N \le ca_0\langle x+y+z\rangle^{1/2}\phi_N, \\ &\partial_x^2\phi_N = \partial_{xy}\phi_N = \partial_{xz}\phi_N \le c(a_0^2 + a_0)\langle x+y+z\rangle\phi_N, \\ &\partial_x^3\phi_N = \partial_{xyy}\phi_N = \partial_{xzz}\phi_N \le c(a_0^3 + a_0)\langle x+y+z\rangle\phi_N.\end{aligned}$$

Lastly, we remark that

$$\phi_N(x,y,z,t) = e^{a(t)(x+y+z)^{3/2}} \le e^{a_0 N^{3/2}} \qquad \text{for} \quad t \ge 0, \quad x+y+z \in [1,N],$$

which completes the proof of Theorem 3.1 in this region.

Finally, let us consider the last region $x+y+z \in [N,\infty]$. In this domain,

$$\begin{aligned}&\phi_N(x,y,z,t) = P_N(x,y,z,t) \\ &= \left[1 + \frac{3}{2}aN^{1/2}(x+y+z-N) + \left(\frac{9}{4}a^2N + \frac{3}{4}aN^{-1/2}\right)\frac{(x+y+z-N)^2}{2}\right]e^{aN^{3/2}},\end{aligned} \tag{3.12}$$

with $a = a(t)$ as in (3.11). Hence, we obtain that there exists $c > 0$ such that for $x+y+z \ge N$,

$$P_N(x,y,z,t)$$

$$\geq c\left[1 + aN^{1/2}(x+y+z-N) + (a^2N + aN^{-1/2})\frac{(x+y+z-N)^2}{2}\right]e^{aN^{3/2}}$$
$$\geq ce^{aN^{3/2}} > 0, \tag{3.13}$$

which proves (ii) in Theorem 3.1 in this region. Furthermore, one gets

$$\partial_x P_N(x,y,z,t) = \partial_y P_N(x,y,z,t) = \partial_z P_N(x,y,z,t)$$
$$\geq \left[\frac{3}{2}aN^{1/2} + \frac{9}{4}a^2N(x+y+z-N)\right]e^{aN^{3/2}} \geq 0,$$

which proves (iii) in Theorem 3.1 in this domain, and

$$\partial_t P_N(x,y,z,t) = a'(t)S_N(x,y,z,t)e^{aN^{3/2}} + a'(t)N^{3/2}P_N(x,y,z,t),$$

where

$$S_N(x,y,z,t) = \frac{3}{2}N^{1/2}(x+y+z-N) + \left(\frac{9}{4}aN + \frac{3}{8}N^{-1/2}\right)(x+y+z-N)^2 \geq 0.$$

Next, we shall prove that if $x+y+z \geq N$,

$$\partial_t\phi_N + \partial_x^3\phi_N + \partial_{xyy}\phi_N + \partial_{xzz}\phi_N \leq c_0\phi_N. \tag{3.14}$$

Note that

$$\partial_x^3\phi_N = \partial_{xyy}\phi_N = \partial_{xzz}\phi_N \equiv 0, \quad \partial_t\phi_N(x,y,z,t) = \partial_t P_N(x,y,z,t) < 0.$$

Combining the above estimates completes the proof of (3.14). Then, (3.13) yields (3.3) in this region $x+y+z \geq N$.

In order to complete the proof, it is necessary to prove (v) in the region $x+y+z \in [N,\infty)$. Taking advantage of (3.9) with $t=0$, we need only prove that for $x+y+z \geq N$,

$$\phi_N(x,y,z,0) = P_N(x,y,z,0) \leq e^{a_0(x+y+z)_+^{3/2}}. \tag{3.15}$$

Let $x+y+z = \omega$ to prove (3.15). We need to prove that

$$\phi_N(\omega,0) = P_N(\omega,0) \leq e^{a_0\omega_+^{3/2}}. \tag{3.16}$$

Considering $P_N(\omega,0)$ and $e^{a_0\omega_+^{3/2}}$, and their derivatives up to the second order, which coincide at $\omega = N$, to prove (3.16), it is sufficient to prove that

$$\partial_\omega^2 P_N(\omega,0) \leq \frac{d^2}{d\omega^2}e^{a_0\omega^{3/2}}, \quad \text{for} \quad \omega \geq N. \tag{3.17}$$

For this purpose, we deduce from (3.12) that the constant value of $\partial_\omega^2 P_N(\omega,0)$ is given by

$$\partial_\omega^2 P_N(\omega,0) = \left(\frac{3}{4}a_0N^{-1/2} + \frac{9}{4}a_0^2N\right)e^{a_0N^{3/2}}$$

and coincides at $\omega = N$ with

$$\frac{d^2}{d\omega^2}e^{a_0\omega^{3/2}} = \left(\frac{3}{4}a_0\omega^{-1/2} + \frac{9}{4}a_0^2\omega\right)e^{a_0\omega^{3/2}}.$$

Let us observe that

$$\begin{aligned}\frac{d^3}{d\omega^3}e^{a_0\omega^{3/2}} &= \left(-\frac{3}{8}a_0\omega^{-3/2} + \frac{9}{8}a_0^2 + \frac{9}{4}a_0^2 + \frac{27}{8}a_0^3\omega^{3/2}\right)e^{a_0\omega^{3/2}}\\ &= \frac{27}{8}a_0^3\omega^{3/2}\left(-\frac{1}{9a_0^2\omega^3} + \frac{1}{a_0\omega^{3/2}} + 1\right)e^{a_0\omega^{3/2}} > 0,\end{aligned}$$

if $\omega > N$ and $N > (9a_0^2)^{-1/3} \equiv N_0(a_0)$. Therefore, if $N \geq N_0$, $\frac{d^2}{d\omega^2}e^{a_0\omega^{3/2}}$ is an increasing function with regard to the variable ω for $\omega \geq N$. According to this fact, we obtain (3.17). Hence, (3.16) and (3.15) hold, which gives the proof of (v) in this region.

Thus, the proof of Theorem 3.1 has been completed. □

4. Decay of solutions

4.1. Proof of Theorem 1.1

By virtue of Lemma 2.2, we deduce that the solution u of IVP (1.1) satisfies

$$u \in C([0,T]; H^2(\mathbb{R}^3) \cap L^2(e^{\beta(x+y+z)}dxdydz)), \qquad \text{for} \quad \text{any} \quad \beta > 0. \tag{4.1}$$

In general, if for some $\beta > 0$, $e^{\beta(x+y+z)}f, \partial_x^2 f \in L^2(\mathbb{R}^3)$, then $e^{\beta(x+y+z)/2}\partial_x f \in L^2(\mathbb{R}^3)$ since

$$\int_{\mathbb{R}^3} e^{\beta(x+y+z)}(\partial_x f)^2 dxdydz \leq \beta^2 \int_{\mathbb{R}^3} e^{\beta(x+y+z)} f^2 dxdydz + \left|\int_{\mathbb{R}^3} e^{\beta(x+y+z)} f\partial_x^2 f dxdydz\right|. \tag{4.2}$$

To prove (4.2), one initially assumes that $f \in H^2(\mathbb{R}^3)$ with compact support to obtain (4.2) by integration by parts, and then the density of this class is employed to achieve the desired result.

Therefore, applying the last argument and (4.1), it follows that

$$\partial_x^j u, \partial_y^j u, \partial_z^j u \in C([0,T]; H^{2-j}(\mathbb{R}^3) \cap L^2(e^{\beta(x+y+z)}dxdydz)), \qquad j = 0, 1, 2. \tag{4.3}$$

In particular, for any k, $u \in C([0,T]; L^2(\langle x+y+z\rangle^k dxdydz))$, we assume that u is sufficiently regular, that is, $u \in C([0,T]; H^3(\mathbb{R}^3))$. Then, we derive energy estimates on u applying the weights $\{\phi_N\}$ (since $\phi_N \leq c\langle x+y+z\rangle^2$). Thus, multiplying by $u\phi_N$ on both sides of (2.2), and integrating the result in $\mathbb{R}^3$ with x, y, z, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3} \partial_t uu\phi_N dxdydz + \int_{\mathbb{R}^3} \partial_x^3 uu\phi_N dxdydz + \int_{\mathbb{R}^3} \partial_{xyy} uu\phi_N dxdydz\\ &\quad + \int_{\mathbb{R}^3} \partial_{xzz} uu\phi_N dxdydz + \gamma\int_{\mathbb{R}^3} u^2\partial_x uu\phi_N dxdydz = 0.\end{aligned}$$

Applying (3.1) and property (iii) in Theorem 3.1, it is inferred that

$$\frac{d}{dt}\int_{\mathbb{R}^3} u^2\phi_N dxdydz \le \int_{\mathbb{R}^3} u^2(\partial_t\phi_N + \partial_x^3\phi_N + \partial_{xyy}\phi_N + \partial_{xzz}\phi_N)dxdydz \\ + \frac{\gamma}{2}\int_{\mathbb{R}^3} u^4\partial_x\phi_N dxdydz.$$

From (3.2), it follows that

$$\frac{d}{dt}\int_{\mathbb{R}^3} u^2\phi_N dxdydz \le c_0\int_{\mathbb{R}^3} u^2\phi_N dxdydz + \frac{\gamma}{2}\int_{\mathbb{R}^3} u^4\partial_x\phi_N dxdydz. \tag{4.4}$$

with $c_0 = c_0(a_0)$.

Let us estimate the second term on the right-hand side of (4.4). To bound the contribution of the second term using (3.3), we write

$$0 \le \partial_x\phi_N(x,y,z,t) \le c_1\langle x+y+z\rangle^{1/2}\phi_N(x,y,z,t) \le c(1+e^{x+y+z})\phi_N(x,y,z,t),$$

hence

$$\int_{\mathbb{R}^3} u^4\partial_x\phi_N dxdydz \le c\left(\|e^{\frac{1}{2}(x+y+z)}u\|^2_{L^\infty(\mathbb{R}^3)} + \|u\|^2_{L^\infty(\mathbb{R}^3)}\right)\int_{\mathbb{R}^3} u^2\phi_N dxdydz.$$

Combining the Sobolev embedding theorem and (4.3), one has

$$\int_0^T \|e^{\frac{1}{2}(x+y+z)}u\|^2_{L^\infty(\mathbb{R}^3)}(t)dt < \infty.$$

Inserting the above estimates into (4.4), for any $N \in \mathbb{Z}^+$, we deduce that

$$\frac{d}{dt}\int_{\mathbb{R}^3} u^2(x,y,z,t)\phi_N(x,y,z,t)dxdydz \le L(t)\int_{\mathbb{R}^3} u^2(x,y,z,t)\phi_N(x,y,z,t)dxdydz,$$

with $L(t) \in L^\infty([0,T])$, where $L(\cdot) = L(a_0; \|e^{\frac{1}{2}(x+y+z)}u_0\|_{L^2(\mathbb{R}^3)}; \|u_0\|_{H^1(\mathbb{R}^3)})$. In view of property ($v$) in Theorem 3.1, and using Gronwall's lemma, it follows that for $t \in [0,T]$,

$$\begin{aligned} &\int_{\mathbb{R}^3} u^2(x,y,z,t)\phi_N(x,y,z,t)dxdydz \\ &\le c\left(\int_{\mathbb{R}^3} u_0^2(x,y,z)\phi_N(x,y,z,0)dxdydz\right)e^{\int_0^T L(t')dt'} \\ &\le c\left(a_0; \|e^{\frac{1}{4}a_0(x+y+z)_+^{3/2}}u_0\|_{L^2(\mathbb{R}^3)}; \|u_0\|_{H^1(\mathbb{R}^3)}; T\right)\int_{\mathbb{R}^3} u_0^2(x,y,z)e^{a_0(x+y+z)_+^{3/2}}dxdydz. \end{aligned} \tag{4.5}$$

We are now in a position to establish (4.5) for our less regular solution $u \in C([0,T]; H^2(\mathbb{R}^3))$. For this purpose, let us consider IVP (1.1) with regularized initial data $u_{0,\delta} := \rho_\delta * u(\cdot+\delta, \cdot+\delta, \cdot+\delta, 0)$, where $\delta > 0$, $\rho_\delta = \frac{1}{\delta^3}\rho(\frac{\cdot}{\delta}, \frac{\cdot}{\delta}, \frac{\cdot}{\delta})$, $\rho \in C^\infty(\mathbb{R}^3)$ is supported in $(-1,1)\times(-1,1)\times(-1,1)$, and $\int_{\mathbb{R}^3}\rho d\xi d\eta d\zeta = 1$. Since

$$u_{0,\delta} \to u_0 \quad \text{in} \quad H^2(\mathbb{R}^3) \quad \text{as} \quad \delta \to 0, \tag{4.6}$$

for IVP (1.1) in $H^2(\mathbb{R}^3)$, according to the well-posedness result in [18], the corresponding solutions u_δ satisfy $u_\delta(t) \to u(t)$ in $H^2(\mathbb{R}^3)$ uniformly for $t \in [0, T]$ as $\delta \to 0$. Furthermore, in terms of the Sobolev embedding theorem, for fixed t,

$$u_\delta(x, y, z, t) \to u(x, y, z, t) \qquad \text{for} \quad \text{all} \quad (x, y, z) \in \mathbb{R}^3 \quad \text{as} \quad \delta \to 0.$$

Meanwhile, by applying Minkowski's integral inequality, we prove that

$$\|e^{\frac{1}{2}a_0(x+y+z)_+^{3/2}} u_{0,\delta}\|_{L^2(\mathbb{R}^3)} \le \|e^{\frac{1}{2}a_0(x+y+z)_+^{3/2}} u_0\|_{L^2(\mathbb{R}^3)}. \tag{4.7}$$

Note that u_δ is sufficiently regular, and we obtain (4.5) with u_δ and $u_{0,\delta}$ instead of u and u_0. For fixed t, taking account of (4.6), (4.7) and using Fatou's lemma, one can deduce that

$$\begin{aligned} &\int_{\mathbb{R}^3} u^2(x, y, z, t)\phi_N(x, y, z, t)dxdydz \\ &\le c\left(a_0; \|e^{\frac{1}{4}a_0(x+y+z)_+^{3/2}} u_0\|_{L^2(\mathbb{R}^3)}; \|u_0\|_{H^1(\mathbb{R}^3)}; T\right) \int_{\mathbb{R}^3} u_0^2(x, y, z)e^{a_0(x+y+z)_+^{3/2}} dxdydz. \end{aligned}$$

Taking $N \uparrow \infty$, and making use of Fatou's lemma and the property (v) in Theorem 3.1, it follows that

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3} e^{a(t)(x+y+z)_+^{3/2}} |u(x, y, z, t)|^2 dxdydz \le c^*,$$

which completes the proof of Theorem 1.1.

4.2. Proof of Theorem 1.2

Let us consider the difference of the two solutions to the equation

$$w(x, y, z, t) = (u_1 - u_2)(x, y, z, t),$$

that is,

$$\partial_t w + \partial_x^3 w + \partial_{xyy} w + \partial_{xzz} w + \gamma(u_1^2 \partial_x w + (u_1 + u_2)w\partial_x u_2) = 0. \tag{4.8}$$

Taking advantage of the argument developed in Theorem 1.1, we multiply (4.8) by $w\phi_N$, integrate the result in $\mathbb{R}^3$ with x, y, z, and formally use integration by parts to deduce that

$$\int_{\mathbb{R}^3} u_1^2 \partial_x w w\phi_N dxdydz = -\int_{\mathbb{R}^3} u_1\partial_x u_1 w^2 \phi_N dxdydz - \frac{1}{2}\int_{\mathbb{R}^3} u_1^2 \partial_x \phi_N w^2 dxdydz,$$

where

$$\left|\int_{\mathbb{R}^3} u_1\partial_x u_1 w^2 \phi_N dxdydz\right| \le \|u_1\partial_x u_1\|_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} w^2 \phi_N dxdydz.$$

Using (3.3), it follows that

$$\left|\int_{\mathbb{R}^3} u_1^2 \partial_x \phi_N w^2 dxdydz\right| \le c\int_{\mathbb{R}^3} u_1^2 \langle x + y + z\rangle^{\frac{1}{2}} w^2 \phi_N dxdydz$$

$$\leq c\|u_1\langle x+y+z\rangle^{\frac{1}{4}}\|^2_{L^\infty(\mathbb{R}^3)}\int_{\mathbb{R}^3} w^2\phi_N dxdydz.$$

Altogether,

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}\partial_x u_2(u_1+u_2)w^2\phi_N dxdydz\right|\\ &\leq \|u_1\partial_x u_2\|_{L^\infty(\mathbb{R}^3)}\int_{\mathbb{R}^3} w^2\phi_N dxdydz+\|u_2\partial_x u_2\|_{L^\infty(\mathbb{R}^3)}\int_{\mathbb{R}^3} w^2\phi_N dxdydz.\end{aligned}$$

Thus, combining the equality

$$\begin{aligned}&\int_{\mathbb{R}^3}\partial_t ww\phi_N dxdydz+\int_{\mathbb{R}^3}\partial_x^3 ww\phi_N dxdydz+\int_{\mathbb{R}^3}\partial_{xyy} ww\phi_N dxdydz\\ &\quad+\int_{\mathbb{R}^3}\partial_{xzz} ww\phi_N dxdydz+\gamma\int_{\mathbb{R}^3}(u_1^2\partial_x w+(u_1+u_2)w\partial_x u_2)w\phi_N dxdydz=0,\end{aligned}$$

and the above estimates, one has

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3} w^2(x,y,z,t)\phi_N(x,y,z,t)dxdydz\\ &\leq c_0\int_{\mathbb{R}^3} w^2(x,y,z,t)\phi_N(x,y,z,t)dxdydz+\gamma\|u_1\partial_x u_1\|_{L^\infty(\mathbb{R}^3)}\int_{\mathbb{R}^3} w^2\phi_N dxdydz\\ &\quad+c\gamma\|u_1\langle x+y+z\rangle^{\frac{1}{4}}\|^2_{L^\infty(\mathbb{R}^3)}\int_{\mathbb{R}^3} w^2\phi_N dxdydz+\gamma\|u_1\partial_x u_2\|_{L^\infty(\mathbb{R}^3)}\int_{\mathbb{R}^3} w^2\phi_N dxdydz\\ &\quad+\gamma\|u_2\partial_x u_2\|_{L^\infty(\mathbb{R}^3)}\int_{\mathbb{R}^3} w^2\phi_N dxdydz,\end{aligned}$$

i.e.,

$$\frac{d}{dt}\int_{\mathbb{R}^3} w^2(x,y,z,t)\phi_N(x,y,z,t)dxdydz\leq G(t)\int_{\mathbb{R}^3} w^2(x,y,z,t)\phi_N(x,y,z,t)dxdydz,$$

where

$$G(t)=c(\|u_1\partial_x u_1\|_{L^\infty(\mathbb{R}^3)}+\|u_1\langle x+y+z\rangle^{\frac{1}{4}}\|^2_{L^\infty(\mathbb{R}^3)}+\|u_1\partial_x u_2\|_{L^\infty(\mathbb{R}^3)}+\|u_2\partial_x u_2\|_{L^\infty(\mathbb{R}^3)}),$$

with $G(t)\in L^\infty([0,T])$. Therefore,

$$\begin{aligned}&\sup_{t\in[0,T]}\int_{\mathbb{R}^3} w^2(x,y,z,t)\phi_N(x,y,z,t)dxdydz\\ &\qquad\leq c\left(\int_{\mathbb{R}^3} w^2(x,y,z,0)\phi_N(x,y,z,0)dxdydz\right)e^{\int_0^T G(t)dt},\end{aligned}$$

which completes the proof of Theorem 1.2.

5. Proof of Theorem 1.3

Proof. We suppose $t_0 = 0$ and $0 < t_1 < T$. First, let us consider $\alpha \in (0, 1/2]$. For $x + y + z \geq 0, N \in \mathbb{Z}^+$, and $\alpha > 0$, we are in a position to define

$$\varphi_{N,\alpha}(x, y, z) = \begin{cases} [1 + (x + y + z)^4]^{\frac{\alpha}{2}} - 1, & x + y + z \in [0, N], \\ (2N)^{2\alpha}, & x + y + z \geq 10N, \end{cases} \tag{5.1}$$

with $\varphi_{N,\alpha}(x, y, z) \in C^3(x + y + z \geq 0)$, $\varphi_{N,\alpha}(x, y, z) \geq 0$, and $\partial_x\varphi_{N,\alpha} = \partial_y\varphi_{N,\alpha} = \partial_z\varphi_{N,\alpha} \geq 0$, and for $\alpha \in (0, 1/2]$,

$$|\partial_x\varphi_{N,\alpha}(x, y, z)| = |\partial_y\varphi_{N,\alpha}(x, y, z)| = |\partial_z\varphi_{N,\alpha}(x, y, z)| \leq C,$$
$$|\partial_x^3\varphi_{N,\alpha}(x, y, z)| = |\partial_{xyy}\varphi_{N,\alpha}(x, y, z)| = |\partial_{xzz}\varphi_{N,\alpha}(x, y, z)| \leq C,$$

where C is independent of N.

Let $\theta_{N,\alpha}$ be defined as the following:

$$\theta_{N,\alpha}(x, y, z) = \theta_N(x, y, z) = \begin{cases} \varphi_{N,\alpha}(x, y, z), & x + y + z \geq 0, \\ -\varphi_{N,\alpha}(-x, -y, -z), & x + y + z \leq 0. \end{cases} \tag{5.2}$$

Note that

$$\partial_x\theta_N(x, y, z) = \partial_y\theta_N(x, y, z) = \partial_z\theta_N(x, y, z) \geq 0, \quad \forall (x, y, z) \in \mathbb{R}^3,$$
$$\theta_N \in C^3(\mathbb{R}^3), \quad \|\theta_N\|_{L^\infty(\mathbb{R}^3)} = (2N)^{2\alpha}.$$

Then, let $(u_{0,m})_{m\in\mathbb{Z}^+}$ be a sequence in $C_0^\infty(\mathbb{R}^3)$ such that

$$u_{0,m} \to u_0 \quad \text{in} \quad H^2(\mathbb{R}^3) \quad \text{as} \quad m \uparrow \infty, \tag{5.3}$$

and let $u_m \in C([0, T]; H^\infty(\mathbb{R}^3))$ be the solution of Eq (1.1) corresponding to the initial data $u_{0,m}$. We have

$$u_m \to u \quad \text{in} \quad C([0, T]; H^2(\mathbb{R}^3)). \tag{5.4}$$

From the continuous dependence of the solution upon the data (see Lemma 2.1), (5.3), and (5.4), there exists $T > 0$ such that

$$\begin{aligned} &(a) \quad \sup_{t\in[0,T]} \|u(t) - u_m(t)\|_{L^2(\mathbb{R}^3)} \to 0 \quad \text{as} \quad m \uparrow \infty, \\ &(b) \quad \int_0^T \|u(t) - u_m(t)\|_{L^\infty(\mathbb{R}^3)} dt \to 0 \quad \text{as} \quad m \uparrow \infty. \end{aligned} \tag{5.5}$$

Owing to $u_m \in C([0, T], H^\infty(\mathbb{R}^3))$ satisfying Eq (1.1), we multiply it by $u_m\theta_N$. Then integrating the result and formally using integration by parts (justified since θ_N is bounded), we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^3} u_m^2\theta_N dxdydz + \int_{\mathbb{R}^3} (\partial_x u_m + \partial_y u_m)^2\partial_y\theta_N dxdydz + \int_{\mathbb{R}^3} (\partial_x u_m + \partial_z u_m)^2\partial_z\theta_N dxdydz$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^3}(\partial_x u_m)^2(3\partial_x\theta_N-\partial_y\theta_N-\partial_z\theta_N)dxdydz+\int_{\mathbb{R}^3}(\partial_y u_m)^2(\partial_x\theta_N-\partial_y\theta_N)dxdydz\\
&+\int_{\mathbb{R}^3}(\partial_z u_m)^2(\partial_x\theta_N-\partial_z\theta_N)dxdydz \qquad (5.6)\\
=&\int_{\mathbb{R}^3}u_m^2\partial_x^3\theta_N dxdydz+\int_{\mathbb{R}^3}u_m^2\partial_{xyy}\theta_N dxdydz+\int_{\mathbb{R}^3}u_m^2\partial_{xzz}\theta_N dxdydz\\
&+\frac{\gamma}{2}\int_{\mathbb{R}^3}u_m^4\partial_x\theta_N dxdydz.
\end{aligned}
$$

Notice that $\partial_x\theta_N=\partial_y\theta_N=\partial_z\theta_N$, $3\partial_x\theta_N-\partial_y\theta_N-\partial_z\theta_N>0$, $\partial_x\theta_N-\partial_y\theta_N=0$, and $\partial_x\theta_N-\partial_z\theta_N=0$. We rewrite (5.6) as follows:

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}u_m^2\theta_N dxdydz+\int_{\mathbb{R}^3}(\partial_x u_m+\partial_y u_m)^2\partial_y\theta_N dxdydz+\int_{\mathbb{R}^3}(\partial_x u_m+\partial_z u_m)^2\partial_z\theta_N dxdydz\\
\le&\int_{\mathbb{R}^3}u_m^2\partial_x^3\theta_N dxdydz+\int_{\mathbb{R}^3}u_m^2\partial_{xyy}\theta_N dxdydz+\int_{\mathbb{R}^3}u_m^2\partial_{xzz}\theta_N dxdydz \qquad (5.7)\\
&+\frac{\gamma}{2}\int_{\mathbb{R}^3}u_m^4\partial_x\theta_N dxdydz.
\end{aligned}
$$

For m large enough, thanks to the boundedness of the partial derivatives of θ_N, the L^2-norm conservation law, and the convergence of the sequence $\{u_{0,m}\}$, we deduce that

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3}(u_m)^2\partial_x^3\theta_N dxdydz\right|\le C\|u_{0,m}\|^2_{L^2(\mathbb{R}^3)}\le 2C\|u_0\|^2_{L^2(\mathbb{R}^3)},\\
&\left|\int_{\mathbb{R}^3}(u_m)^2\partial_{xyy}\theta_N dxdydz\right|\le C\|u_{0,m}\|^2_{L^2(\mathbb{R}^3)}\le 2C\|u_0\|^2_{L^2(\mathbb{R}^3)},\\
&\left|\int_{\mathbb{R}^3}(u_m)^2\partial_{xzz}\theta_N dxdydz\right|\le C\|u_{0,m}\|^2_{L^2(\mathbb{R}^3)}\le 2C\|u_0\|^2_{L^2(\mathbb{R}^3)}, \qquad (5.8)\\
&\left|\int_{\mathbb{R}^3}(u_m)^4\partial_x\theta_N dxdydz\right|\le C\|u_m(t)\|^2_{L^\infty(\mathbb{R}^3)}\|u_{0,m}\|^2_{L^2(\mathbb{R}^3)}\\
&\qquad\le 2C\|u_m(t)\|^2_{L^\infty(\mathbb{R}^3)}\|u_0\|^2_{L^2(\mathbb{R}^3)}.
\end{aligned}
$$

Integrating (5.7) with regard to t in $[0,t_1]$ and using (5.8), it follows that

$$
\begin{aligned}
&\int_0^{t_1}\int_{\mathbb{R}^3}(\partial_x u_m+\partial_y u_m)^2(x,y,z,t)\partial_y\theta_N(x,y,z)dxdydzdt\\
&\quad+\int_0^{t_1}\int_{\mathbb{R}^3}(\partial_x u_m+\partial_z u_m)^2(x,y,z,t)\partial_z\theta_N(x,y,z)dxdydzdt\\
\le&\|u_m^2(t_1)\theta_N\|_{L^1(\mathbb{R}^3)}+\|u_{0,m}^2\theta_N\|_{L^1(\mathbb{R}^3)}+Ct_1\|u_0\|^2_{L^2(\mathbb{R}^3)}\\
&\quad+C\|u_0\|^2_{L^2(\mathbb{R}^3)}\int_0^{t_1}\|u_m(t)\|^2_{L^\infty(\mathbb{R}^3)}dt,
\end{aligned}
$$

where C represents a constant, and its value may change from line to line. Meanwhile, it does not depend on the initial parameters of the problem. Setting $m\uparrow\infty$ and making use of (5.5), Lemma 2.1, (2.1), and the assumptions of Theorem 1.3, we deduce that

$$
\overline{\lim_{m\uparrow\infty}}\int_0^{t_1}\int_{\mathbb{R}^3}(\partial_x u_m+\partial_y u_m)^2(x,y,z,t)\partial_y\theta_N(x,y,z)dxdydzdt
$$

$$
\begin{aligned}
&+\overline{\lim_{m\uparrow\infty}}\int_0^{t_1}\int_{\mathbb{R}^3}(\partial_x u_m+\partial_z u_m)^2(x,y,z,t)\partial_z\theta_N(x,y,z)dxdydzdt\\
\leq\ &\|u^2(t_1)\theta_N\|_{L^1(\mathbb{R}^3)}+\|u_0^2\theta_N\|_{L^1(\mathbb{R}^3)}+Ct_1\|u_0\|^2_{L^2(\mathbb{R}^3)}\\
&+C\|u_0\|^2_{L^2(\mathbb{R}^3)}\int_0^{t_1}\|u(t)\|^2_{L^\infty(\mathbb{R}^3)}dt\leq M,
\end{aligned}\tag{5.9}
$$

with $M=M\left(\|\langle x+y+z\rangle^\alpha u_0\|_{L^2(\mathbb{R}^3)},\|\langle x+y+z\rangle^\alpha u(t_1)\|_{L^2(\mathbb{R}^3)}\right)$. Next, we use (5.4) and (5.5) to conclude that for any fixed $\bar{N}\in\mathbb{Z}^+$ and $\bar{N}>10N$,

$$
\partial_x u_m\to\partial_x u,\quad \partial_y u_m\to\partial_y u,\quad \partial_z u_m\to\partial_z u,\tag{5.10}
$$

in $L^2\left([0,t_1]\times\{x+y+z\in[-\overline{N},\overline{N}]\}\right)$ as $m\uparrow\infty$.

Thanks to $\partial_y\theta_N$ and $\partial_z\theta_N$ having compact support, one has

$$
\begin{aligned}
&\int_0^{t_1}\int_{\mathbb{R}^3}(\partial_x u+\partial_y u)^2(x,y,z,t)\partial_y\theta_N(x,y,z)dxdydzdt\leq M,\\
&\int_0^{t_1}\int_{\mathbb{R}^3}(\partial_x u+\partial_z u)^2(x,y,z,t)\partial_z\theta_N(x,y,z)dxdydzdt\leq M.
\end{aligned}\tag{5.11}
$$

At last, notice that $\partial_y\theta_N=\partial_z\theta_N\geq 0$, and for $x+y+z>1$,

$$
\partial_y\theta_N=\partial_z\theta_N\to\frac{2\alpha(x+y+z)^3}{[1+(x+y+z)^4]^{1-\frac{\alpha}{2}}}\sim\langle x+y+z\rangle^{2\alpha-1}.
$$

Using Fatou's lemma in (5.11), we obtain

$$
\begin{aligned}
&\int_0^{t_1}\int_{|x+y+z|>1}(\partial_x u+\partial_y u)^2(x,y,z,t)\langle x+y+z\rangle^{2\alpha-1}dxdydzdt\leq M,\\
&\int_0^{t_1}\int_{|x+y+z|>1}(\partial_x u+\partial_z u)^2(x,y,z,t)\langle x+y+z\rangle^{2\alpha-1}dxdydzdt\leq M.
\end{aligned}\tag{5.12}
$$

From Lemma 2.1 and (2.1), it follows that

$$
\begin{aligned}
&\int_0^{t_1}\int_{|x+y+z|\leq 1}(\partial_x u+\partial_y u)^2(x,y,z,t)dxdydzdt\leq M,\\
&\int_0^{t_1}\int_{|x+y+z|\leq 1}(\partial_x u+\partial_z u)^2(x,y,z,t)dxdydzdt\leq M.
\end{aligned}
$$

We derive that

$$
\begin{aligned}
&\int_0^{t_1}\int_{\mathbb{R}^3}(\partial_x u+\partial_y u)^2(x,y,z,t)\langle x+y+z\rangle^{2\alpha-1}dxdydzdt\leq M,\\
&\int_0^{t_1}\int_{\mathbb{R}^3}(\partial_x u+\partial_z u)^2(x,y,z,t)\langle x+y+z\rangle^{2\alpha-1}dxdydzdt\leq M.
\end{aligned}\tag{5.13}
$$

With (5.13), we reapply the above argument with $\psi_{N,\alpha}(x,y,z) = \psi_N(x,y,z)$:

$$\psi_{N,\alpha}(x,y,z) = \psi_N(x,y,z) = \begin{cases} \varphi_{N,\alpha}(x,y,z), & x+y+z \geq 0, \\ \varphi_{N,\alpha}(-x,-y,-z), & x+y+z \leq 0. \end{cases} \tag{5.14}$$

Note that

$$|\partial_x \psi_N(x,y,z)| = \left|\partial_y \psi_N(x,y,z)\right| = |\partial_z \psi_N(x,y,z)| \leq C\langle x+y+z\rangle^{2\alpha-1}.$$

In Eq (5.6) with $\psi_N(x,y,z)$ instead of $\theta_N(x,y,z)$, similar computations to that in (5.8) lead us to

$$\begin{aligned}
&\overline{\lim_{m\uparrow\infty}} \int_0^{t_1} \int_{\mathbb{R}^3} (\partial_x u_m + \partial_y u_m)^2(x,y,z,t)\partial_y \psi_N(x,y,z)dxdydzdt \\
&= \int_0^{t_1} \int_{\mathbb{R}^3} (\partial_x u + \partial_y u)^2(x,y,z,t)\partial_y \psi_N(x,y,z)dxdydzdt, \\
&\overline{\lim_{m\uparrow\infty}} \int_0^{t_1} \int_{\mathbb{R}^3} (\partial_x u_m + \partial_z u_m)^2(x,y,z,t)\partial_z \psi_N(x,y,z)dxdydzdt \\
&= \int_0^{t_1} \int_{\mathbb{R}^3} (\partial_x u + \partial_z u)^2(x,y,z,t)\partial_z \psi_N(x,y,z)dxdydzdt,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_0^{t_1} \int_{\mathbb{R}^3} (\partial_x u + \partial_y u)^2(x,y,z,t)\partial_y \psi_N(x,y,z)dxdydzdt\right| \\
&\leq \int_0^{t_1} \int_{\mathbb{R}^3} (\partial_x u + \partial_y u)^2(x,y,z,t)\langle x+y+z\rangle^{2\alpha-1}dxdydzdt \leq M, \\
&\left|\int_0^{t_1} \int_{\mathbb{R}^3} (\partial_x u + \partial_z u)^2(x,y,z,t)\partial_z \psi_N(x,y,z)dxdydzdt\right| \\
&\leq \int_0^{t_1} \int_{\mathbb{R}^3} (\partial_x u + \partial_z u)^2(x,y,z,t)\langle x+y+z\rangle^{2\alpha-1}dxdydzdt \leq M.
\end{aligned}$$

Collecting all the above estimates and integrating in $[0,t] \subset [0,t_1]$ yields

$$\langle x+y+z\rangle^{\alpha} u(t) \in L^2(\mathbb{R}^3) \qquad t \in [0,t_1]. \tag{5.15}$$

Taking advantage of (5.13), for $t \in [0,t_1]$, it is inferred that

$$\begin{aligned}
&(\partial_x u(t) + \partial_y u(t))\langle x+y+z\rangle^{\alpha-1/2} \in L^2(\mathbb{R}^3), \\
&(\partial_x u(t) + \partial_z u(t))\langle x+y+z\rangle^{\alpha-1/2} \in L^2(\mathbb{R}^3).
\end{aligned}$$

Hence, the desired result holds.

Next, let us consider the case $\alpha \in (1/2, 1]$. A direct computation gives rise to

$$\begin{aligned}
&\partial_x \theta_{N,\alpha}(x,y,z) + \partial_y \theta_{N,\alpha}(x,y,z) + \partial_z \theta_{N,\alpha}(x,y,z) + |\partial_x \psi_{N,\alpha}(x,y,z)| \\
&\quad + |\partial_y \psi_{N,\alpha}(x,y,z)| + |\partial_z \psi_{N,\alpha}(x,y,z)| + |\theta_{N,\alpha-\frac{1}{2}}(x,y,z)| \leq C\langle x+y+z\rangle,
\end{aligned}$$

$$|\partial_x^3\theta_{N,\alpha}(x,y,z)| = |\partial_{xyy}\theta_{N,\alpha}(x,y,z)| = |\partial_{xzz}\theta_{N,\alpha}(x,y,z)| \leq C.$$

As before, we deduce that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} u_m^2\theta_{N,\alpha}dxdydz + \int_{\mathbb{R}^3}(\partial_x u_m + \partial_y u_m)^2\partial_y\theta_{N,\alpha}dxdydz \\
&\quad + \int_{\mathbb{R}^3}(\partial_x u_m + \partial_z u_m)^2\partial_z\theta_{N,\alpha}dxdydz \\
&\leq \int_{\mathbb{R}^3} u_m^2\partial_x^3\theta_{N,\alpha}dxdydz + \int_{\mathbb{R}^3} u_m^2\partial_{xyy}\theta_{N,\alpha}dxdydz + \int_{\mathbb{R}^3} u_m^2\partial_{xzz}\theta_{N,\alpha}dxdydz \\
&\quad + \frac{\gamma}{2}\int_{\mathbb{R}^3} u_m^4\partial_x\theta_{N,\alpha}dxdydz,
\end{aligned} \tag{5.16}$$

with $u_m \in C([0,T];H^\infty(\mathbb{R}^3))$.

In (5.16), we first utilize that

$$\begin{aligned}
\left|\int_{\mathbb{R}^3}(u_m)^2\partial_x^3\theta_{N,\alpha}dxdydz\right| &\leq C\|u_{0,m}\|^2_{L^2(\mathbb{R}^3)} \leq 2C\|u_0\|^2_{L^2(\mathbb{R}^3)}, \\
\left|\int_{\mathbb{R}^3}(u_m)^2\partial_{xyy}\theta_{N,\alpha}dxdydz\right| &\leq C\|u_{0,m}\|^2_{L^2(\mathbb{R}^3)} \leq 2C\|u_0\|^2_{L^2(\mathbb{R}^3)}, \\
\left|\int_{\mathbb{R}^3}(u_m)^2\partial_{xzz}\theta_{N,\alpha}dxdydz\right| &\leq C\|u_{0,m}\|^2_{L^2(\mathbb{R}^3)} \leq 2C\|u_0\|^2_{L^2(\mathbb{R}^3)}.
\end{aligned} \tag{5.17}$$

Next, for the last term on the right-hand side of (5.16), it follows that

$$\left|\partial_x\theta_{N,\alpha}(x,y,z)\right| = \left|\partial_y\theta_{N,\alpha}(x,y,z)\right| = \left|\partial_z\theta_{N,\alpha}(x,y,z)\right| \leq C|\theta_{N,\alpha-\frac{1}{2}}(x,y,z)|^2,$$

with C independent of N. Accordingly,

$$\left|\int_{\mathbb{R}^3} u_m^4\partial_x\theta_{N,\alpha}dxdydz\right| \leq C\|u_m(t)\|^2_{L^\infty(\mathbb{R}^3)}\|u_m(t)\theta_{N,\alpha-\frac{1}{2}}\|^2_{L^2(\mathbb{R}^3)}. \tag{5.18}$$

For each fixed N, the $\theta_{N,\alpha}$'s are bounded, and

$$\sup_{t\in[0,t_1]}\|(u-u_m)(t)\|_{L^2(\mathbb{R}^3)} \longrightarrow 0 \qquad \text{as} \quad m\uparrow\infty,$$

we conclude that

$$\sup_{t\in[0,t_1]}\|(u-u_m)(t)\theta_{N,\alpha-\frac{1}{2}}\|_{L^2(\mathbb{R}^3)} \longrightarrow 0 \qquad \text{as} \quad m\uparrow\infty.$$

Hence,

$$\begin{aligned}
\sup_{t\in[0,t_1]}\|u_m(t)\theta_{N,\alpha-\frac{1}{2}}\|_{L^2(\mathbb{R}^3)} &\leq 2\sup_{t\in[0,t_1]}\|u(t)\theta_{N,\alpha-\frac{1}{2}}\|_{L^2(\mathbb{R}^3)} \\
&\leq 2\sup_{t\in[0,t_1]}\|\langle x+y+z\rangle^{\frac{1}{2}}u(t)\|_{L^2(\mathbb{R}^3)} \leq M,
\end{aligned} \tag{5.19}$$

with $M = M\left(\|\langle x+y+z\rangle^{1/2}u_0\|_{L^2(\mathbb{R}^3)}, \|\langle x+y+z\rangle^{1/2}u(t_1)\|_{L^2(\mathbb{R}^3)}\right)$ for $m \gg 1$.

Plugging the above estimates into (5.16) and applying the same argument in the previous case $\alpha \in (0, 1/2]$, we obtain

$$\begin{aligned}
&\int_0^{t_1}\int_{\mathbb{R}^3}(\partial_x u_m + \partial_y u_m)^2(x,y,z,t)\partial_y\theta_N(x,y,z)dxdydzdt \le (1+t_1)M,\\
&\int_0^{t_1}\int_{\mathbb{R}^3}(\partial_x u_m + \partial_z u_m)^2(x,y,z,t)\partial_z\theta_N(x,y,z)dxdydzdt \le (1+t_1)M,
\end{aligned} \tag{5.20}$$

for $m \gg 1$, and

$$\begin{aligned}
&\int_0^{t_1}\int_{\mathbb{R}^3}(\partial_x u + \partial_y u)^2(x,y,z,t)\langle x+y+z\rangle^{2\alpha-1}dxdydzdt \le (1+t_1)M,\\
&\int_0^{t_1}\int_{\mathbb{R}^3}(\partial_x u + \partial_z u)^2(x,y,z,t)\langle x+y+z\rangle^{2\alpha-1}dxdydzdt \le (1+t_1)M,
\end{aligned} \tag{5.21}$$

where $M = M\left(\|\langle x+y+z\rangle^{1/2}u_0\|_{L^2(\mathbb{R}^3)}, \|\langle x+y+z\rangle^{1/2}u(t_1)\|_{L^2(\mathbb{R}^3)}\right)$.

In the following, we use $\psi_{N,\alpha}$ instead of $\theta_{N,\alpha}$. Then we obtain the similar formulation to (5.16). From (5.20) and (5.21), the desired result is valid.

For the case $\alpha \in (1, 3/2]$ and higher α, we may apply a similar bootstrap technique to get the desired result. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest that may influence the publication of this paper.

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