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Research article

Event-triggered synchronization control for neural networks against DoS attacks

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Abstract: This paper studied the event-triggered synchronization problem for time-delay neural networks under DoS attacks. A novel event-triggered scheme based on switching between periodic sampling and a continuous event-triggered scheme was proposed, which not only cuts down the number of data transmissions but also offsets cyberattacks. By choosing a suitable piecewise Lyapunov-Krasovskii functional and using several free-weighting matrices, sufficient conditions were established to ensure the exponential stability of the synchronization error system in the occurrence of DoS attacks. Furthermore, a co-design method was provided to acquire the desired non-fragile output-feedback control gain and event-triggering parameter. Finally, a numerical example was given to illustrate the usefulness of the proposed approach.

Keywords: event-triggered scheme; synchronization control; DoS attacks; neural networks

1. Introduction

Neural networks (NNs) have provoked ever-increasing attention because of their ability to self-learn and self-adapt. To date, a variety of NN architectures and NN learning methods have been utilized for secure communication, trajectory prediction, system identification and control, and practical applications in many areas to solve pattern recognition, classification, regression, and optimization problems [1–3]. In these practical applications, time delays are often inevitable due to inherently limited delivery time between neurons, which leads to NNs generating more complex dynamic behavior or even chaos [4]. Therefore, the synchronization issue of time-delay NNs (TDNNs) has always been an active subject of study. The prime objective of synchronization control is to force a slave system to follow the master system by a suitable control method, which means that the slave system is affected by the behavior of the master system, while the master system is independent of the slave one. For this purpose, many effective means of synchronization control have been proposed, such as event-triggered control [5, 6], impulsive control [7, 8], and adaptive control [9]. Along with the evolvement of the research, a large number of interesting and remarkable results have been reported on the synchronization of TDNNs, please refer to the literature [10, 11].

With the advancement of information technology, networked systems have been utilized widely owing to timeliness, facility, and flexibility in practical applications. Therefore, the signal transfer under the networked communication scheme is achieved via a shared communication network channel. However, due to the openness and sharing of communication networks, networked systems are vulnerable to malicious cyberattacks. Cyberattacks that can be commonly divided into denial-of-service (DoS) attacks, replay attacks, and deception attacks aim to operate the data sent by communication channels to interfere with or damage a system [12]. Compared with deception attacks and replay attacks [13,14], DoS attacks are the most likely type as they release massive requests to jam the communication network channel such that the filter/controller is inaccessible to the required data [15]. Thus, many actions have been taken to handle networked system security under DoS attacks [16–19]. Periodic DoS attacks were presented based on a state-feedback control scheme in [16], which characterized the effect of jamming attacks on systems by using a switched system model. Given that the system state cannot be fully measured, the result in [16] was extended to the observer-based, event-triggered controller to settle the resilient control issue under periodic DoS attacks in [17]. When the period of the jammer signal is unknown, the periodic DoS attacks are developed into nonperiodic DoS jammer signals, which are more practical than periodic ones. In [20–23], nonperiodic DoS jamming attacks were considered in the filtering and control issues for the switched system, respectively. While, for the synchronization of TDNNs, the aperiodic DoS attacks have rarely been discussed, which is one of our motivations.

Besides the issue of network security, another significant issue of networked systems that needs to be considered is the limitation of network bandwidth. In order to solve the problem, two different transmission strategies have been widely used. The first one is periodic sampling, where the signals are usually sampled at a fixed period [24]. In this circumstance, signals are transmitted even when the output fluctuation is small, which may result in network burden and waste network resources. To overcome the drawbacks of periodic sampling, the event-triggered scheme (ETS) was developed [25–27]. In this way, the sampling signals are released if and only if a pre-defined condition is violated, rather than at a fixed time. Therefore, it is desirable to use the ETS to decide whether the sampling signal should be transmitted out or not. A crucial difficulty in the ETS design is finding a suitable condition so that the signal release rate can be validly reduced while the system performance can be maintained at a certain level. For instance, in [28], a continuous control strategy implicitly defined by the continuous ETS was proposed; nevertheless, the Zeno phenomenon occurred. To avoid the Zeno behavior, the periodic ETS, which checks periodically the event-triggering condition in discrete-time instants, and the switching ETS, which combines the ideas of both periodic sampling and the continuous ETS were proposed [29, 30]. It is noteworthy that most of the existing results mainly focus on the combination of DoS attacks with the periodic ETS [20–22, 31]. However, to the best of our knowledge, there are no relevant reports on the design of the switching ETS under DoS attacks for synchronization of TDNNs, which is the other main motivation.

Impelled by the above discussions, we endeavor to cope with the master-slave synchronization problem for TDNNs in the cases of limited network resources and DoS attacks, where the controller is permitted to be limited by norm-bounded perturbations. The main contributions include the following:

1) Different from the existing results based on the periodic ETS under DoS attacks [16, 20–22, 31], the proposed ETS is described as switching between periodic sampling and the continuous ETS

counteracting the effect of DoS attacks.

2) In the presence of aperiodic DoS attacks, the proposed control scheme is based on the output-feedback control with the fluctuation in control gain rather than state-feedback control in [16, 22], which is more realistic.

3) Sufficient conditions including the restriction of jamming attacks are proposed guaranteeing exponential stability of the resulting error system by utilizing a piecewise Lyapunov-Krasovskii functional (PLKF) and adopting several free-weighting matrices.

Notation: In this article, $\mathbb{R}^{n_i}$ denotes the n_i-dimensional Euclidean space; $\mathbb{R}^{n_i \times n_p}$ stands for the set of real $n_i \times n_p$ matrices; $\mathbb{R}_+^{n_i}$ is the set of real symmetric positive-definite $n_i \times n_i$ matrices; $\mathbb{N}$ means the set of non-negative integers; $S > 0$ (respectively, $S \geq 0$) represents that S is symmetric positive definite (respectively, semi-definite); $\lambda_{max}(S)$ (respectively, $\lambda_{min}(S)$) is the largest (respectively, smallest) eigenvalue of matrix S; I (respectively, 0) means the identity (respectively, zero) matrix with compatible dimension; S^T is the transpose of matrix S; $He(S)$ denotes $S + S^T$; $diag\{\cdots\}$ is a diagonal matrix; $col\{\cdots\}$ represents a column vector; and $*$ is a symmetric block.

2. Preliminaries

Consider the following master-slave NN model:

$$M : \begin{cases} \dot{x}(t) = -\mathcal{A}x(t) + \mathcal{W}_0\phi(x(t)) + \mathcal{W}_1\phi(x(t-\theta)) + \mathcal{J}(t), \\ \check{y}(t) = Cx(t), \end{cases} \tag{2.1}$$

and

$$S : \begin{cases} \dot{\hat{x}}(t) = -\mathcal{A}\hat{x}(t) + \mathcal{W}_0\phi(\hat{x}(t)) + \mathcal{J}(t) + \mathcal{W}_1\phi(\hat{x}(t-\theta)) + \mathcal{B}u(t), \\ \hat{y}(t) = C\hat{x}(t), \end{cases} \tag{2.2}$$

where $\mathcal{M}$ stands for the master NN; $\mathcal{S}$ is the slave NN; $x(t) \in \mathbb{R}^{n_i}$ is the state of $\mathcal{M}$; $\hat{x}(t) \in \mathbb{R}^{n_i}$ is the state of $\mathcal{S}$; $\check{y}(t) \in \mathbb{R}^{n_q}$ and $\hat{y}(t) \in \mathbb{R}^{n_q}$ are the output vectors of $\mathcal{M}$ and $\mathcal{S}$; $\mathcal{J}(t) \in \mathbb{R}^{n_i}$ represents the external input; $u(t) \in \mathbb{R}^{n_p}$ means the control input of $\mathcal{S}$; $\mathcal{A} = diag\{a_1, \ldots, a_{n_i}\} \in \mathbb{R}^{n_i \times n_i}$ with $a_k > 0, k = 1, \ldots, n_i$; $\mathcal{W}_0 \in \mathbb{R}^{n_i \times n_i}$ and $\mathcal{W}_1 \in \mathbb{R}^{n_i \times n_i}$ are the connection weight matrices of NN; $\mathcal{B} \in \mathbb{R}^{n_i \times n_p}$ and $C \in \mathbb{R}^{n_q \times n_i}$ are known constant matrices; θ signifies the time delay; $\phi(\cdot) = col\{\phi_1(\cdot), \phi_2(\cdot), \ldots, \phi_{n_i}(\cdot)\}$ represents the neuron activation function satisfying:

$$\Phi_l^- \leq \frac{\phi_j(\kappa_1) - \phi_j(\kappa_2)}{\kappa_1 - \kappa_2} \leq \Phi_l^+, l = 1, 2, \ldots, n_i, \tag{2.3}$$

where $\forall \kappa_1, \kappa_2 \in \mathbb{R}$, $\kappa_1 \neq \kappa_2$; Φ_l^- and Φ_l^+ are known constants.

Figure 1 illustrates the synchronization control strategy. Accordingly, we define the synchronization error $\eta(t) = \hat{x}(t) - x(t)$ and $y(t) = \hat{y}(t) - \check{y}(t)$. Then the synchronization error system can be derived:

$$\begin{cases} \dot{\eta}(t) = -\mathcal{A}\eta(t) + \mathcal{W}_0\phi(\eta(t)) + \mathcal{W}_1\phi(\eta(t-\theta)) + \mathcal{B}u(t), \\ y(t) = C\eta(t), \end{cases} \tag{2.4}$$

where $\phi(r(t) = \phi(\hat{x}(t)) - \phi(x(t))$.

When DoS attacks are absent, set t_ν ($\nu = 0, 1, \ldots, t_0 = 0$) as the triggering times. Then the time sequence $\{t_\nu\}_{\nu \in \mathbb{N}}$ can be acquired by the event trigger. In addition, considering the effect of gain perturbations, the output feedback $u(t)$ is expressed as

$$u(t) = -(K + \Delta K)y(t_\nu), \forall t \in [t_\nu, t_{\nu+1}), \nu \in \mathbb{N}, \tag{2.5}$$

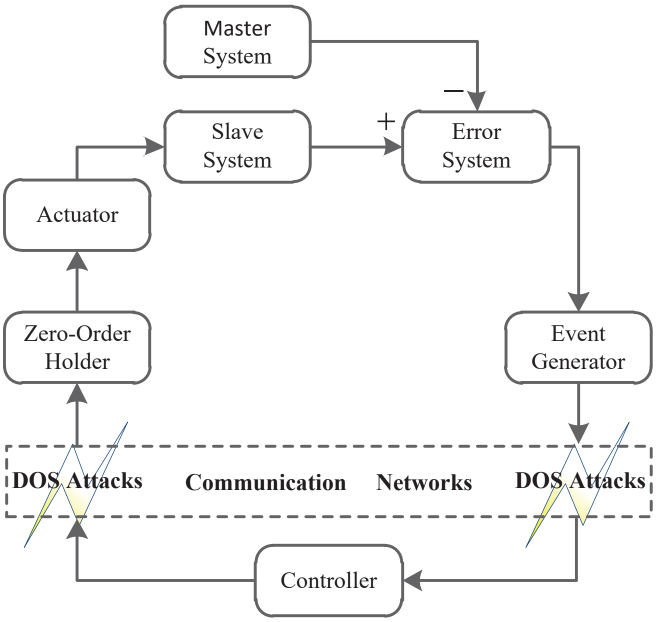

Figure 1. A block diagram of synchronization control under DoS attacks.

where K denotes the controller gain matrix to be designed and ΔK is an unknown matrix representing the gain perturbation. It is assumed that ΔK has the following norm-bounded structure:

$$\Delta K = M\mathfrak{F}N, \tag{2.6}$$

where M and N are known constant matrices, and $\mathfrak{F}$ stands for an uncertain matrix satisfying $\mathfrak{F}^T\mathfrak{F} \leq I$.

As a follow-up, to offset the effect of DoS attacks on the system quantitatively, following [20], a type of power-constraint aperiodic DoS jamming attack is introduced for $i \in \mathbb{N}$:

$$\mathfrak{D}_{DoS}(t) = \begin{cases} 1, & t \in \mathcal{G}^i_{off}, \\ 0, & t \in \mathcal{G}^i_{on}, \end{cases} \tag{2.7}$$

where $\mathcal{G}^i_{off} = [t_{0,i}, t_{\sigma_i})$ represents the ith DoS sleeping time interval, in which $t_{0,i}$ is the time instant at which the ith DoS attack is off and the signal can be sent. $\mathcal{G}^i_{on} = [t_{\sigma_i}, t_{0,i+1})$ means the ith DoS time interval, where t_{σ_i} is the time instant at which the ith DoS attack is active and the signal transmission is interrupted.

When there are no DoS attacks, the event-triggering condition is defined as [30]:

$$t_{v+1} = \min\{t \geq t_v + h \left| y_v^T(t)\mathcal{F}y_v(t) > \mu y^T(t)\mathcal{F}y(t)\right\}, \tag{2.8}$$

where $h > 0$ is a given integer; $y_v(t) = y(t) - y(t_v)$ with $y(t)$ being the current sampling measurement and $y(t_v)$ being the latest one; $\mu \in (0, 1)$ is a threshold; and $\mathcal{F} \in \mathbb{R}^{n_i}_+$ is a weighting matrix.

On another note, under DoS attacks (2.7), the communication channel is stopped over DoS time intervals $\cup_{i\in\mathbb{N}}\mathcal{G}^i_{on}$. Therefore, the above condition in (2.8) is no longer satisfied. Therefore, we will modify the above switching ETS.

Definition 1. *The event-triggering instant subject to DoS attacks is as follows:*

$$t_{\nu+1,i} = \min\{t \in [t_{\nu,i} + h, t_{\sigma_i}) \left| \bar{y}_{\nu,i}(t) > \mu y^T(t)\mathcal{F}y(t)\right.\} \tag{2.9}$$

where $i \in \mathbb{N}$, $\bar{y}_{\nu,i}(t) = (y(t) - y(t_{\nu,i}))^T \mathcal{F}(y(t) - y(t_{\nu,i}))$, *and* $\nu \in \mathbb{N}$ *is the number of events occurring in the ith attack period.*

Remark 1. *Unlike an adaptive ETS with the relative threshold strategy [32, 33], the proposed ETS in (2.9) can be regarded as switching between the periodic sampling method and the continuous ETS, that is, a signal needs to wait for at least h to be sent. Then the sensor starts to continuously check the event-triggering condition and sends signals when it is disobeyed. Moreover, ETS (2.9) can not only cut down the auxiliary consumption of computational resources and save communication resources, but also adapt to the DoS attacks.*

Under DoS attack (2.7) and ETS (2.9), the controller $u(t)$ in (2.5) can be represented as

$$u(t) = \begin{cases} -(K + \Delta K)y(t_{\nu,i}), \ t \in [t_{\nu,i}, t_{\nu+1,i}) \cap \mathcal{G}^i_{off}, \\ 0, \ t \in \mathcal{G}^i_{on}, \end{cases} \tag{2.10}$$

where $i \in \mathbb{N}$, $\nu \in \{0, 1, ..., \nu(i)\}$, $\nu(i) = \sup\{\nu \in \mathbb{N} \left| t_{\sigma_i} \geq t_{\nu,i}\right.\}$, and $\{t_{\nu,i}\}$ represents the sequence of successful control update instants.

Remark 2. *Considering the effect of DoS attacks in (2.7), we give an output-feedback controller in (2.10) to offset DoS attacks. The data is updated at every trigger instant during the dormant period and is blocked when DoS attacks are active. Specifically, when there are DoS attacks, data is inaccessible to target users. In this case, controller (2.10) becomes invalid, i.e.,* $u(t) = 0$. *While DoS attacks are absent, the controller can successfully update, i.e.,* $u(t) = -(K + \Delta K)y(t_{\nu,i})$. *Moreover, controller (2.10) has non-fragility, which can be limited by norm-bounded perturbations.*

In what follows, utilizing the time delay and switched system approaches, we divide the trigger interval $[t_{\nu,i}, t_{\nu+1,i})$ into two sub-intervals, i.e.,

$$[t_{\nu,i}, t_{\nu,i} + h) \cup [t_{\nu,i} + h, t_{\nu+1,i}).$$

Set

$$\begin{cases} \rho(t) = t - t_{\nu,i}, t \in [t_{\nu,i}, t_{\nu,i} + h) \cap \mathcal{G}^i_{off}, \\ z(t) = y(t_{\nu,i}) - y(t), t \in [t_{\nu,i} + h, t_{\nu+1,i}) \cap \mathcal{G}^i_{off}. \end{cases} \tag{2.11}$$

Then by (2.10) and (2.11), the synchronization error system in (2.4) can be further cast into the following switched error system:

$$\begin{cases} \dot{\eta}(t) = \begin{cases} -\bar{\mathcal{A}}\eta(t) + \mathcal{W}_0\phi(\eta(t)) + \mathcal{W}_1\phi(\eta(t-\theta)) + \bar{\mathcal{B}}\int_{t-\rho(t)}^{t} \dot{\eta}(s)ds, \ t \in [t_{\nu,i}, t_{\nu,i} + h) \cap \mathcal{G}^i_{off}, \\ -\bar{\mathcal{A}}\eta(t) + \mathcal{W}_0\phi(\eta(t)) + \mathcal{W}_1\phi(\eta(t-\theta)) - \bar{\mathcal{B}}z(t), \ t \in [t_{\nu,i} + h, t_{\nu+1,i}) \cap \mathcal{G}^i_{off}, \\ -\mathcal{A}\eta(t) + \mathcal{W}_0\phi(\eta(t)) + \mathcal{W}_1\phi(\eta(t-\theta)), \ t \in \mathcal{G}^i_{on}, \end{cases} \\ y(t) = C\eta(t), \\ \eta(t) = \varpi(t), t \in [-\max\{\theta, h\}, 0], \end{cases} \tag{2.12}$$

where $\bar{\mathcal{A}} = \mathcal{A} + \mathcal{B}(K + \Delta K)C$, $\bar{\mathcal{B}} = \mathcal{B}(K + \Delta K)C$, and $z(t)$ satisfies

$$z^T(t)\mathcal{F}z(t) \leq \mu y^T(t)\mathcal{F}y(t). \tag{2.13}$$

In virtue of the above analysis, we construct the following PLKF for switched error system (2.12):

$$V(t) = \begin{cases} V_0(t) = V_{P_1}(t) + V_{H_1}(t), \ t \in [t_{\nu,i}, t_{\nu,i} + h) \cap \mathcal{G}^i_{off}, \\ V_1(t) = V_{P_1}(t), \ t \in [t_{\nu,i} + h, t_{\nu+1,i}) \cap \mathcal{G}^i_{off}, \\ V_2(t) = V_{P_2}(t) + V_{H_2}(t), \ t \in \mathcal{G}^i_{on}, \end{cases} \tag{2.14}$$

where

$$\begin{aligned} V_{P_j}(t) &= \eta^T(t)P_j\eta(t) + \int_{t-\theta}^{t} e^{2(-1)^j\varepsilon_j(t-s)}\eta^T(s)Q_j\eta(s)ds, \\ V_{H_1}(t) &= (h-\rho(t))\int_{t-\rho(t)}^{t} e^{-2\varepsilon_1(t-s)}\dot{\eta}^T(s)H_{11}\dot{\eta}(s)ds \\ &\quad + (h-\rho(t))\int_{t-\rho(t)}^{t}\int_{\pi}^{t} e^{-2\varepsilon_1(t-s)}\dot{\eta}^T(s)H_{12}\dot{\eta}(s)dsd\pi, \\ V_{H_2}(t) &= (t_{0,i+1}-t)\int_{t_{\sigma_i}}^{t} e^{2\varepsilon_2(t-s)}\dot{\eta}^T(s)H_{21}\dot{\eta}(s)ds \\ &\quad + (t_{0,i+1}-t)\int_{t_{\sigma_i}}^{t}\int_{\pi}^{t} e^{2\varepsilon_2(t-s)}\dot{\eta}^T(s)H_{22}\dot{\eta}(s)dsd\pi, \end{aligned}$$

and $P_j \in \mathbb{R}^{n_i}_+$, $j = \{1,2\}$, $Q_j \in \mathbb{R}^{n_i}_+$, $H_{jl} \in \mathbb{R}^{n_i}_+$, $l = \{1,2\}$, and $\varepsilon_j > 0$.

Now, in this paper, the event-triggered, non-fragile control issue subject to DoS attacks to be addressed is formulated: Given DoS parameters $T_{on} = \max\{|\mathcal{G}^i_{on}|\}$ and $T_{off} = \min\{|\mathcal{G}^i_{off}|\}$ for $i \in \mathbb{N}$, co-design an event-triggering parameter $\mathcal{F}$ in (2.9) and a non-fragile, output-feedback controller in (2.10) to ensure the exponential synchronization of $\mathcal{M}$ and $\mathcal{S}$ under DoS attacks, that is, there exist two constants $b > 0$ and $a > 0$ such that error system (2.12) meets

$$\|\eta(t)\| \le be^{-at} \sup_{-\bar{\tau}\le\tau\le 0} \|\eta(\tau), \dot{\eta}(\tau)\|, \tag{2.15}$$

where $\bar{\tau} = \max\{\theta, h\}$, ϵ is the decay coefficient, and a is the decay rate.

3. Main results

In this section, based on the switching ETS, we will provide a criterion to achieve the synchronization of the NN with time delay under DoS attacks, and then, an output-feedback control method is given by considering the gain perturbation. Before starting our main results, we introduce the following useful lemmas, which will be used in our later derivation.

Lemma 1. *[34] Given $E = E^T$, matrices $\mathfrak{E}$ and $\mathfrak{L}$ having suitable dimensions, and then*

$$E + \mathfrak{E}\Lambda\mathfrak{L} + \mathfrak{L}^T\Lambda^T\mathfrak{E}^T \le 0$$

with Λ meeting $\Lambda^T\Lambda \le I$, if there exists $\upsilon > 0$ such that

$$\begin{bmatrix} E & \mathfrak{E} & \upsilon\mathfrak{L}^T \\ * & -\upsilon I & 0 \\ * & * & -\upsilon I \end{bmatrix} \le 0.$$

Lemma 2. *[35] For vector function* $\varsigma : [a,b) \rightarrow \mathbb{R}^{n_i}$, *matrices* $H \in \mathbb{R}_+^{n_i}, L_1, L_3 \in \mathbb{R}_+^{3n_i}, L_2 \in \mathbb{R}^{3n_i \times 3n_i}$, *and* $S_1, S_3 \in \mathbb{R}^{3n_i \times n_i}$ *satisfy*

$$\begin{bmatrix} L_1 & L_2 & S_1 \\ * & L_3 & S_2 \\ * & * & H \end{bmatrix} \geq 0,$$

and one has

$$-\int_a^b \dot{\varsigma}^T(s) H \dot{\varsigma}(s) ds \leq \psi^T(t) \Phi \psi(t),$$

where

$$\begin{aligned} \psi(t) &= col\Big\{\varsigma(b), \varsigma(a), \frac{1}{b-a}\int_a^b \varsigma(s)ds\Big\}, \\ \Phi &= (b-a)(L_1 + \frac{1}{3}L_3) + He(S_1\Pi_1 + S_2\Pi_2), \\ \Pi_1 &= \bar{d}_1 - \bar{d}_2, \Pi_2 = 2\bar{d}_3 - \bar{d}_1 - \bar{d}_2, \\ \bar{d}_1 &= [I, 0, 0], \bar{d}_2 = [0, I, 0], \bar{d}_3 = [0, 0, I]. \end{aligned}$$

Lemma 3. *[36,37] Given a real scalar* $b > 0$, *if there exist real matrices* Z, Λ_a, Γ_a, *and* F_a $(a = 1, \dots n_i)$ *such that*

$$\begin{bmatrix} Z & \Lambda_1 + b\Gamma_1 + \cdots + \Lambda_{n_i} + b\Gamma_{n_i} \\ * & diag\{-bF_1 - bF_1^T - \cdots - bF_{n_i} - bF_{n_i}^T\} \end{bmatrix} < 0$$

holds, then one has

$$Z + \sum_{a=1}^{n_i} He(\Lambda_a F_a^{-1} \Gamma_a^T) < 0.$$

Lemma 4. *For known parameters* T_{on} *and* T_{off}, *the output-feedback control gain matrix* K, *and some scalars* $0 < \mu < 1$, $0 < h < T_{off}$, *and* $\varepsilon_j > 0, j = \{1,2\}$, *if we have* $P_j \in \mathbb{R}_+^{n_i}, Q_j \in \mathbb{R}_+^{n_i}, R_1, R_3 \in \mathbb{R}_+^{3n_i}, H_{jl} \in \mathbb{R}_+^{n_i}, l = \{1,2\}, \mathcal{F} \in \mathbb{R}_+^{n_i}$, *diagonal matrices* $\Upsilon_{j\omega} > 0, \omega = \{0,1,2\}$, *and matrices* $R_2 \in \mathbb{R}^{3n_i \times 3n_i}, Z_j \in \mathbb{R}^{3n_i \times n_i}, \Gamma_{1\omega}$ *and* $\Gamma_{2\omega}$ *such that the following inequalities hold*

$$\Theta_0 + h\Sigma_s < 0, s = 1, 2, \tag{3.1}$$

$$\Theta_1 < 0, \tag{3.2}$$

$$\Theta_2 + T_{on}\Sigma_s < 0, s = 3, 4, < 0, \tag{3.3}$$

$$\begin{bmatrix} R_1 & R_2 & Z_1 \\ * & R_3 & Z_2 \\ * & * & H_{jl} \end{bmatrix} \geq 0, \tag{3.4}$$

where

$$\begin{aligned} d_m &= [0_{n_i \times (m-1)n_i}, I, 0_{n_i \times (7-m)n_i}], m = 1, \cdots, 7, \\ \Theta_0 &= d_1^T \Theta_{01} d_1 + d_5^T \Theta_{02} d_5 + He(\Theta_{03}) - 2e^{-2\varepsilon_1 h} U_4^T H_{12} U_4 \\ &\quad - d_6^T \Upsilon_{10} d_6 - d_7^T \Upsilon_{20} d_7 + d_2^T \frac{h^2}{4} H_{12} d_2 + \bar{\Gamma}_0, \end{aligned}$$

$$
\begin{aligned}
\Theta_{01} &= 2\varepsilon_1 P_1 + Q_1 - \Phi_1\Upsilon_{10},\ \Theta_{02} = -e^{-2\varepsilon_1\theta}Q_1 - \Phi_1\Upsilon_{20},\\
\Theta_{03} &= d_1^T P_1 d_2 + d_1^T\Phi_2\Upsilon_{10}d_6 + d_5^T\Phi_2\Upsilon_{20}d_7 + e^{-2\varepsilon_1 h}(U_1^T Z_1 U_2 + U_1^T Z_2 U_3),\\
\bar{\Gamma}_0 &= He\Big([d_1^T\Gamma_{10}^T + d_2^T\Gamma_{20}^T][-d_2 - \mathcal{A}d_1 + \mathcal{W}_0 d_6 + \mathcal{W}_1 d_7 - \bar{\mathcal{B}}d_3]\Big),\\
\Sigma_1 &= d_2^T H_{11} d_2,\ \Sigma_2 = e^{-2\varepsilon_1 h}U_1^T(R_1 + \frac{1}{3}R_3)U_1,\\
U_1 &= col\{d_1, d_3, d_4\}, U_2 = d_1 - d_3,\\
U_3 &= 2d_4 - d_1 - d_3, U_4 = d_1 - d_4,\\
\Theta_1 &= d_1^T\Theta_{11}d_1 + d_4^T\Theta_{12}d_4 + He(\Theta_{13}) - d_3^T\mathcal{F}d_3 - d_5^T\Upsilon_{11}d_5 - d_6^T\Upsilon_{21}d_6 + \bar{\Gamma}_1,\\
\Theta_{11} &= 2\varepsilon_1 P_1 + Q_1 + \mu C^T\mathcal{F}C - \Phi_1\Upsilon_{11},\\
\Theta_{12} &= -e^{-2\varepsilon_1\theta}Q_1 - \Phi_1\Upsilon_{21},\\
\Theta_{13} &= d_1^T P_1 d_2 + d_1^T\Phi_2\Upsilon_{11}d_5 + d_4^T\Phi_2\Upsilon_{21}d_6,\\
\bar{\Gamma}_1 &= He\Big([d_1^T\Gamma_{11}^T + d_2^T\Gamma_{21}^T][-d_2 - \bar{\mathcal{A}}d_1 + \mathcal{W}_0 d_5 + \mathcal{W}_1 d_6 - \bar{\mathcal{B}}d_3]\Big),\\
\Theta_2 &= d_1^T\Theta_{21}d_1 + d_5^T\Theta_{22}d_5 + He(\Theta_{23}) - 2U_4^T H_{22}U_4\\
&\quad -d_6^T\Upsilon_{12}d_6 - d_7^T\Upsilon_{22}d_7 + \bar{\Gamma}_2,\\
\Theta_{21} &= -2\varepsilon_2 P_2 + Q_2 - \Phi_1\Upsilon_{12},\ \Theta_{22} = -e^{2\varepsilon_1\theta}Q_2 - \Phi_1\Upsilon_{22},\\
\Theta_{23} &= d_1^T P_2 d_2 + d_1^T\Phi_2\Upsilon_{12}d_6 + d_5^T\Phi_2\Upsilon_{22}d_7 + U_1^T Z_1 U_2 + U_1^T Z_2 U_3,\\
\bar{\Gamma}_2 &= He\Big([d_1^T\Gamma_{12}^T + d_2^T\Gamma_{22}^T][-d_2 - \mathcal{A}d_1 + \mathcal{W}_0 d_6 + \mathcal{W}_1 d_7]\Big),\\
\Sigma_3 &= d_2^T H_{21} d_2,\ \Sigma_4 = U_1^T(R_1 + \frac{1}{3}R_3)U_1,\\
\Phi_1 &= diag\{\Phi_1^-\Phi_1^+, \Phi_2^-\Phi_2^+, \ldots, \Phi_{n_i}^-\Phi_{n_i}^+\},\\
\Phi_2 &= diag\left\{\frac{\Phi_1^- + \Phi_1^+}{2}, \frac{\Phi_2^- + \Phi_2^+}{2}, \ldots, \frac{\Phi_{n_i}^- + \Phi_{n_i}^+}{2}\right\},
\end{aligned}
$$

then for $t \in \mathcal{G}_{off}^i \cup \mathcal{G}_{on}^i$, the functionals given in (2.14) meet

$$
\begin{cases}
V_0(t) \le e^{-2\varepsilon_1(t-t_{v,i})}V_0(t_{v,i}),\ t \in [t_{v,i}, t_{v,i}+h) \cap \mathcal{G}_{off}^i,\\
V_1(t) \le e^{-2\varepsilon_1(t-t_{v,i}-h)}V_1(t_{v,i}+h),\ t \in [t_{v,i}+h, t_{v+1,i}) \cap \mathcal{G}_{off}^i,\\
V_2(t) \le e^{2\varepsilon_2(t-t_{\sigma_i})}V_2(t_{\sigma_i}),\ t \in \mathcal{G}_{on}^i.
\end{cases} \tag{3.5}
$$

Proof. By (2.3), for positive diagonal matrices $\Upsilon_{1\omega}$ and $\Upsilon_{2\omega}$ with $\omega = 0, 1, 2$, it can be obtained that

$$\xi^T(t)\bar{\Upsilon}_\omega\xi(t) \ge 0, \tag{3.6}$$

where

$$
\begin{aligned}
\xi(t) &= col\{\eta(t), \eta(t-\theta), \phi(\eta(t)), \phi(\eta(t-\theta))\}\\
\bar{\Upsilon}_\omega &= \begin{bmatrix} -\Phi_1\Upsilon_{1\omega} & 0 & \Phi_2\Upsilon_{1\omega} & 0\\ * & -\Phi_1\Upsilon_{2\omega} & 0 & \Phi_2\Upsilon_{2\omega}\\ * & * & -\Upsilon_{1\omega} & 0\\ * & * & * & -\Upsilon_{2\omega} \end{bmatrix}.
\end{aligned}
$$

DoS-free case: For $t \in [t_{v,i}, t_{v,i} + h) \cap \mathcal{G}^i_{off}$ $(i, v \in \mathbb{N})$ along the trajectories of error system (2.12) with $j = 1$, we have

$$\dot{V}_0(t) = \dot{V}_{P_1}(t) + \dot{V}_{H_1}(t),$$

where

$$\begin{aligned} \dot{V}_{P_1}(t) &= -2\varepsilon_1 V_{P_1}(t) + 2\varepsilon_1 \eta^T(t) P_1 \eta(t) + 2\eta^T(t) P_1 \dot{\eta}(t) + \eta^T(t) Q_1 \eta(t) \\ &\quad -\eta^T(t+\theta) e^{-2\varepsilon_1 \theta} Q_1 \eta(t+\theta), \end{aligned} \tag{3.7}$$

by $\dot{\eta}(t-\rho(t)) = (1-\dot{\rho}(t))\dot{\eta}(t-\rho(t)) = 0, 0$, which yields

$$\begin{aligned} \dot{V}_{H_1}(t) &= -2\varepsilon_1 V_{H_1}(t) + (h-\rho(t))\dot{\eta}^T(t) H_{11}\dot{\eta}(t) + (h-\rho(t))\rho(t)\dot{\eta}^T(t) H_{12}\dot{\eta}(t) \\ &\quad - \int_{t-\rho(t)}^{t} e^{-2\varepsilon_1(t-s)}\dot{\eta}^T(s) H_{11}\dot{\eta}(s)ds - \int_{t-\rho(t)}^{t}\int_{\pi}^{t} e^{-2\varepsilon_1(t-s)}\dot{\eta}^T(s) H_{12}\dot{\eta}(s)dsd\pi. \end{aligned} \tag{3.8}$$

By Lemma 2, the fourth term in (3.8) can be reformed as

$$\begin{aligned} & - \int_{t-\rho(t)}^{t} e^{-2\varepsilon_1(t-s)}\dot{\eta}^T(s) H_{11}\dot{\eta}(s)ds \\ \leq\ & -e^{-2\varepsilon_1 h}\zeta_0^T(t)\Big[\rho(t)U_1^T(R_1 + \frac{1}{3}R_3)U_1 + He(U_1^T Z_1 U_2 + U_1^T Z_2 U_3)\Big]\zeta_0(t), \end{aligned} \tag{3.9}$$

where

$$\zeta_0(t) = col\Big\{\eta(t), \dot{\eta}(t), \eta(t-\rho(t)), \frac{1}{\rho(t)}\int_{t-\rho(t)}^{t}\dot{\eta}(s)ds, \eta(t-\theta), \phi(\eta(t)), \phi(\eta(t-\theta))\Big\}.$$

Furthermore, based on Lemma 1 in [38], we get

$$\int_{t-\rho(t)}^{t}\int_{\pi}^{t} e^{-2\varepsilon_1(t-s)}\dot{\eta}^T(s) H_{12}\dot{\eta}(s)dsd\pi \leq -2e^{-2\varepsilon_1 h}\zeta_0^T(t)U_4^T H_{12} U_4 \zeta_0(t). \tag{3.10}$$

For free-weighting matrices Γ_{10} and Γ_{20} with appropriate dimensions, the following expression holds:

$$\begin{aligned} 0 &= 2[\eta^T(t)\Gamma_{10}^T + \dot{\eta}^T(t)\Gamma_{20}^T][-\dot{\eta}(t) - \bar{\mathcal{A}}\eta(t) + \mathcal{W}_0\phi(\eta(t)) \\ &\quad + \mathcal{W}_1\phi(\eta(t-\theta)) + \bar{\mathcal{B}}\int_{t-\rho(t)}^{t}\dot{\eta}(s)ds]. \end{aligned} \tag{3.11}$$

By $(h-\rho(t))\rho(t) \leq \frac{[(h-\rho(t))+\rho(t)]^2}{4} = \frac{h^2}{4}$ and $\omega = 0$, we have from (3.6)–(3.10) that

$$\dot{V}_0(t) \leq -2\varepsilon_1 V_0(t) + \zeta_0^T(t)\Big(\Theta_0 + (h-\rho(t))\Sigma_1 + \rho(t)\Sigma_2\Big)\zeta_0(t). \tag{3.12}$$

$\Theta_0 + (h-\rho(t))\Sigma_1 + \rho(t)\Sigma_2 < 0$ in inequality (3.12) is equivalent to $\Theta_0 + h\Sigma_1 < 0$ and $\Theta_0 + h\Sigma_2 < 0$, which can be ensured by (3.1). This means that

$$\dot{V}_0(t) \leq -2\varepsilon_1 V_0(t).$$

For $t \in [t_{\nu,i} + h, t_{\nu+1,i}) \cap \mathcal{G}^i_{off}$ $(i, \nu \in \mathbb{N})$, (2.13) implies that

$$0 \leq \mu\eta^T(t)C^T\mathcal{F}C\eta(t) - z^T(t)\mathcal{F}z(t). \tag{3.13}$$

Following the similar lines of the previous $\dot{V}_0(t)$, we have

$$0 = 2[\eta^T(t)\Gamma_{11}^T + \dot{\eta}^T(t)\Gamma_{21}^T][-\dot{\eta}(t) - \bar{\mathcal{A}}\eta(t) + \mathcal{W}_0\phi(\eta(t)) + \mathcal{W}_1\phi(\eta(t-\theta)) - \bar{\mathcal{B}}z(t)], \tag{3.14}$$

and for matrices $\Upsilon_{1\omega}$ and $\Upsilon_{2\omega}$ with $\omega = 1$,

$$\xi^T(t)\bar{\Upsilon}_1\xi(t) \geq 0. \tag{3.15}$$

By combining (3.13)–(3.15) with (3.2), it can be acquired that

$$\dot{V}_1(t) + 2\varepsilon_1 V_1(t) \leq \zeta_1^T(t)\Theta_1\zeta_1(t) < 0,$$

where $\zeta_1(t) = col\{\eta(t), \dot{\eta}(t), z(t), \eta(t-\theta), \phi(\eta(t)), \phi(\eta(t-\theta))\}$.

DoS case: Same as the proof of $\dot{V}_0(t)$, for any matrices Γ_{12} and Γ_{22} with appropriate dimensions, we have

$$0 = 2[\eta^T(t)\Gamma_{12}^T + \dot{\eta}^T(t)\Gamma_{22}^T][-\dot{\eta}(t) - \mathcal{A}\eta(t) + \mathcal{W}_0\phi(\eta(t)) + \mathcal{W}_1\phi(\eta(t-\theta))], \tag{3.16}$$

and for matrices $\Upsilon_{1\omega}$ and $\Upsilon_{2\omega}$ with $\omega = 2$,

$$\xi^T(t)\bar{\Upsilon}_2\xi(t) \geq 0. \tag{3.17}$$

Along the trajectories of error system (2.12) with $j = 2$ and by integrating (3.16) and (3.17), we have

$$\dot{V}_2(t) \leq 2\varepsilon_2 V_2(t) + \zeta_2^T(t)\Big(\Theta_2 + (t_{0,i+1} - t)\Sigma_3 + (t - t_{\sigma_i})\Sigma_4 + (t_{0,i+1} - t)(t - t_{\sigma_i})H_{22}\Big)\zeta_2(t), \tag{3.18}$$

where $\zeta_2(t) = col\Big\{\eta(t), \dot{\eta}(t), \eta(t_{\sigma_i}), \frac{1}{t-t_{\sigma_i}}\int_{t_{\sigma_i}}^t \eta(s)ds, \eta(t-\theta), \phi(\eta(t)), \phi(\eta(t-\theta))\Big\}$. Further, $\Theta_2 + (t_{0,i+1} - t)\Sigma_3 + (t - t_{\sigma_i})\Sigma_4 + (t_{0,i+1} - t)(t - t_{\sigma_i})H_{22} < 0$ is equivalent to $\Theta_2 + T_{on}\Sigma_3 < 0$ and $\Theta_2 + T_{on}\Sigma_4 < 0$, which is guaranteed by (3.3). It can be concluded that

$$\dot{V}_2(t) \leq 2\varepsilon_2 V_2(t).$$

Thereby, based on the above discussions, (3.1)–(3.4) can guarantee that (3.5) holds. This completes the proof. □

Remark 3. *Owing to the introduction of mechanism (2.9), a time-dependent PLKF (i.e., $V(t)$) is designed in this study to match the switched system in (2.12). More specifically, the time-dependent function $V_0(t)$ is constructed for the sub-interval $[t_{\nu,i}, t_{\nu,i} + h) \cap \mathcal{G}^i_{off}$, the time-independent function $V_1(t)$ is employed for the sub-interval $[t_{\nu,i} + h, t_{\nu+1,i}) \cap \mathcal{G}^i_{off}$, and the time-dependent function $V_2(t)$ is devised for the active-attack interval $\mathcal{G}^i_{on}$. In the light of Lemma 2, the free-weighting matrices method, and Schur's complement, the exponential decay estimate of $V(t)$ along the trajectories of the synchronization error system in (2.12) is given in Lemma 4.*

3.1. Exponential synchronization analysis

Provided that feedback gain K is known, we provide the following theorem for the exponential stability of the error system in (2.12) with jamming attacks (2.7). Further, the following result is obtained.

Theorem 1. *For known parameters T_{on} and T_{off}, the output-feedback gain matrix K, and some scalars $0 < \mu < 1$, $0 < h < T_{off}$, $\alpha_j > 1$, and $\varepsilon_j > 0$, $j = \{1, 2\}$, the event-based switched synchronization error system in (2.12) is exponentially stable under DoS attacks, if there exist matrices $P_j \in \mathbb{R}_+^{n_i}$, $Q_j \in \mathbb{R}_+^{n_i}$, $R_1, R_3 \in \mathbb{R}_+^{3n_i}$, $H_{jl} \in \mathbb{R}_+^{n_i}$, $l = \{1, 2\}$, $\mathcal{F} \in \mathbb{R}_+^{n_i}$, diagonal matrices $\Upsilon_{j\omega} > 0, \omega = \{0, 1, 2\}$, and matrices $R_2 \in \mathbb{R}^{3n_i \times 3n_i}$, $Z_j \in \mathbb{R}^{3n_i \times n_i}$, $\Gamma_{1\omega}$ and $\Gamma_{2\omega}$ such that (3.1)–(3.4) and the following inequalities hold:*

$$\begin{cases} P_1 \leq \alpha_2 P_2, \\ P_2 \leq \alpha_1 P_1, \end{cases} \tag{3.19}$$

$$0 < \delta = 2\varepsilon_1 T_{off} - 2\varepsilon_2 T_{on} - \ln(\alpha_1 \alpha_2), \tag{3.20}$$

and the decay rate $\varphi = \frac{\delta}{2(T_{off}+T_{on})}$.

Proof. In the light of (3.5) in Lemma 4, it is obtained that for $\forall t \geq 0$,

$$V(t) \leq \begin{cases} e^{-2\varepsilon_1(t-t_{\nu,i})} V_0(t_{\nu,i}), \ t \in [t_{\nu,i}, t_{\nu,i} + h) \cap \mathcal{G}_{off}^i, \\ e^{-2\varepsilon_1(t-t_{\nu,i}-h)} V_1(t_{\nu,i} + h), \ t \in [t_{\nu,i} + h, t_{\nu+1,i}) \cap \mathcal{G}_{off}^i, \\ e^{2\varepsilon_2(t-t_{\sigma_i})} V_2(t_{\sigma_i}), \ t \in \mathcal{G}_{on}^i. \end{cases} \tag{3.21}$$

It is observed from (3.21) that the values of $V_0(t)$ and $V_1(t)$ coincide at the switching instants $t_{\nu,i}$ and $t_{\nu,i} + h$, which implies that $V(t)$ is continuous over the whole T_{off} time interval.

When $t_{\sigma_i} - t_{\nu,i} \geq h$, $V(t)$ will switch from $V_1(t)$ to $V_2(t)$. By (3.19), we have

$$\begin{cases} V_0(t_{0,i}) = V_1(t_{0,i}) \leq \alpha_2 V_2(t_{0,i}^-), \\ V_2(t_{\sigma_i}) \leq \alpha_1 V_1(t_{\sigma_i}^-). \end{cases} \tag{3.22}$$

When $t_{\sigma_i} - t_{\nu,i} < h$, $V(t)$ will switch from $V_0(t)$ to $V_2(t)$. We obtain from (3.19) that

$$\begin{cases} V_0(t_{0,i}) = V_1(t_{0,i}) \leq \alpha_2 V_2(t_{0,i}^-), \\ V_2(t_{\sigma_i}) \leq \alpha_1 V_1(t_{\sigma_i}^-) \leq \alpha_1 V_0(t_{\sigma_i}^-), \end{cases} \tag{3.23}$$

where $V(t_{0,i}^-) = \lim_{t \to t_{0,i}^-} V(t)$ and $V(t_{\sigma_i}^-) = \lim_{t \to t_{\sigma_i}^-} V(t)$.

We can find an $i \in \mathbb{N}$ for $\forall t \geq 0$ to guarantee $t \in [t_{0,i}, t_{\sigma_i})$ or $t \in [t_{\sigma_i}, t_{0,i+1})$. Therefore, we consider the following two cases:

DoS-free case: For $t \in [t_{0,i}, t_{\sigma_i})$, we get from (3.21)–(3.23) that

$$\begin{aligned} V(t) &\leq e^{-2\varepsilon_1(t-t_{0,i})} V_0(t_{0,i}) \\ &\leq \alpha_2 e^{-2\varepsilon_1(t-t_{0,i})} V_2(t_{0,i}^-) \\ &\leq \alpha_2 e^{-2\varepsilon_1(t-t_{0,i})} e^{2\varepsilon_2(t_{0,i}-t_{\sigma_{i-1}})} V_2(t_{\sigma_{i-1}}) \\ &\leq \alpha_1 \alpha_2 e^{-2\varepsilon_1(t-t_{0,i})} e^{2\varepsilon_2(t_{0,i}-t_{\sigma_{i-1}})} V_1(t_{\sigma_{i-1}}^-) \\ &\leq \alpha_1 \alpha_2 e^{-2\varepsilon_1(t-t_{0,i}+t_{\sigma_{i-1}}-t_{0,i-1})} \times e^{2\varepsilon_2(t_{0,i}-t_{\sigma_{i-1}})} V_0(t_{0,i-1}) \end{aligned}$$

$$\cdots$$
$$\leq \ e^{-\delta i}V_0(0).$$

Notice that $t < t_{\sigma_i} = i(T_{off} + T_{on}) + T_{off}$, i.e., $i > \frac{t-T_{off}}{T_{off}+T_{on}}$, and it follows that

$$V(t) \leq V_0(0)e^{\frac{\delta T_{off}}{T_{off}+T_{on}}}e^{\frac{-\delta t}{T_{off}+T_{on}}}. \tag{3.24}$$

DoS case: For $t \in [t_{\sigma_i}, t_{0,i+1})$, following the similar lines of the previous **DoS-free case,** it is easy to get

$$V(t) \leq \frac{V_0(0)}{\alpha_2}e^{\frac{-\delta t}{T_{off}+T_{on}}}. \tag{3.25}$$

Set $\lambda_0 = \max\{\frac{\delta T_{off}}{T_{off}+T_{on}}, \frac{1}{\alpha_2}\}$, $\lambda_1 = \min\{\lambda_{\min}(P_j)\}$, $\lambda_2 = \max\{\lambda_{\max}(P_j)\}$, and $\lambda_3 = \lambda_2 + \theta\lambda_{\max}(Q_1)$. Then we can deduce that

$$V(t) \leq \lambda_0 e^{\frac{-\delta t}{T_{off}+T_{on}}}V_0(0). \tag{3.26}$$

According to the definition of $V(t)$ in (2.14), we have for $\forall t \geq 0$,

$$V(t) \geq \lambda_1 \|r(t)\|^2, V_0(0) \leq \lambda_3 \|\varpi(0)\|_\tau^2. \tag{3.27}$$

Combining (3.26) and (3.27), we have

$$\|r(t)\| \leq \sqrt{\frac{\lambda_0\lambda_3}{\lambda_1}}e^{-\varphi t}\|\varpi(0)\|_\tau, \forall t \geq 0. \tag{3.28}$$

From (2.15), (3.20) and (3.28), it can be concluded that synchronization error system (2.12) is exponentially stable with the decay rate φ. This completes the proof. □

3.2. Exponential synchronization synthesis

This section concentrates on solving the non-fragile exponential synchronization problem. A co-design method for the desired non-fragile controller K and event-triggering parameter $\mathcal{F}$ is presented.

Theorem 2. *For known parameters T_{on} and T_{off}, matrices J_j, M and N, some scalars $0 < \mu < 1$, $0 < h < T_{off}$, $\gamma > 0$, and $\varepsilon_j > 0$, $j = \{1, 2\}$, the event-based switched synchronization error system in (2.12) is exponentially stable under DoS attacks, if there exist matrices $P_j \in \mathbb{R}_+^{n_i}$, $Q_j \in \mathbb{R}_+^{n_i}$, $R_1, R_3 \in \mathbb{R}_+^{3n_i}$, $H_{jl} \in \mathbb{R}_+^{n_i}$, $l = \{1, 2\}$, $\mathcal{F} \in \mathbb{R}_+^{n_i}$, diagonal matrices $\Upsilon_{j\omega} > 0$, $\omega = \{0, 1, 2\}$, and matrices $R_2 \in \mathbb{R}^{3n_i\times 3n_i}$, $Z_j \in \mathbb{R}^{3n_i\times n_i}$, $\Gamma_{1\omega}$, $\Gamma_{2\omega}$, X, Y, and $\upsilon_m > 0$, $m = \{0, 1\}$, such that (3.3), (3.4), (3.19), (3.20), and the following inequalities hold:*

$$\Omega_m = \begin{bmatrix} \Omega_{11}^m & \Omega_{12}^m \\ * & -\gamma X - \gamma X^T \end{bmatrix} < 0, \tag{3.29}$$

where

$$\begin{aligned}
\Omega_{11}^m &= \bar{\Theta}_m + He(\Gamma_{J_m}Y\bar{\Gamma}_{C_m}),\ \Omega_{12}^m = \bar{\Gamma}_{B_m} - \Gamma_{J_m}X + \gamma\bar{\Gamma}_{C_m}^T Y^T, \\
\bar{\Theta}_m &= \begin{bmatrix} \hat{\Theta}_m & \Gamma_{B_\Delta}^m & \upsilon_m\Gamma_{C_\Delta}^{mT} \\ * & -\upsilon_m I & 0 \\ * & * & -\upsilon_m I \end{bmatrix},
\end{aligned}$$

$$
\begin{aligned}
\hat{\Theta}_0 &= d_1^T\Theta_{01}d_1 + d_5^T\Theta_{02}d_5 + He(\Theta_{03}) - 2e^{-2\varepsilon_1 h}U_4^T H_{12}U_4 \\
&\quad -d_6^T\Upsilon_{10}d_6 - d_7^T\Upsilon_{20}d_7 + d_2^T\frac{h^2}{4}H_{12}d_2 + \hat{\Gamma}_0 + h\Sigma_s,\ s=\{1,2\}, \\
\hat{\Gamma}_0 &= He\Big([d_1^T\Gamma_{10}^T + d_2^T\Gamma_{20}^T][-d_2 - \mathcal{A}d_1 + \mathcal{W}_0 d_6 + \mathcal{W}_1 d_7 - \mathcal{B}d_3]\Big), \\
\hat{\Theta}_1 &= d_1^T\Theta_{11}d_1 + d_4^T\Theta_{12}d_4 + He(\Theta_{13}) - d_3^T\mathcal{F}d_3 - d_5^T\Upsilon_{11}d_5 - d_6^T\Upsilon_{21}d_6 + \hat{\Gamma}_1, \\
\hat{\Gamma}_1 &= He\Big([d_1^T\Gamma_{11}^T + d_2^T\Gamma_{21}^T][-d_2 - \mathcal{A}d_1 + \mathcal{W}_0 d_5 + \mathcal{W}_1 d_6 - \mathcal{B}d_3]\Big)
\end{aligned}
$$

with

$$
\begin{aligned}
\Gamma_{J_0} &= col\{J_1^T, J_2^T, 0,0,0,0,0,0,0\},\ \bar{\Gamma}_{C_0} = [0,0,-C,0,0,0,0,0,0], \\
\bar{\Gamma}_{B_0} &= col\{\Gamma_{10}^T\mathcal{B}, \Gamma_{20}^T\mathcal{B}, 0,0,0,0,0,0,0\},\ \Gamma_{B_\Delta}^0 = col\{\Gamma_{10}^T\mathcal{B}M, \Gamma_{20}^T\mathcal{B}M, 0,0,0,0,0\}, \\
\Gamma_{C_\Delta}^0 &= [0,0,-NC,0,0,0,0],\ \Gamma_{J_1} = col\{J_1^T, J_2^T, 0,0,0,0,0,0\}, \\
\bar{\Gamma}_{C_1} &= [-C,0,-C,0,0,0,0,0],\ \bar{\Gamma}_{B_1} = col\{\Gamma_{11}^T\mathcal{B}, \Gamma_{21}^T\mathcal{B}, 0,0,0,0,0,0\}, \\
\Gamma_{B_\Delta}^1 &= col\{\Gamma_{11}^T\mathcal{B}M, \Gamma_{21}^T\mathcal{B}M, 0,0,0,0\},\ \Gamma_{C_\Delta}^1 = [-NC,0,-NC,0,0,0],
\end{aligned}
$$

and the other symbols are the same as those in Lemma 4. On such ground, the output-feedback controller in (2.5) is given by $K = X^{-1}Y$.

Proof. For $t \in [t_{v,i}, t_{v,i}+h) \cap \mathcal{G}_{off}^i$, using Shur's complement, (3.1) can be rewritten as

$$
\check{\Theta}_0^s + He(\Gamma_{B_0}\Delta K\Gamma_{C_0}) < 0,
$$

where

$$
\begin{aligned}
\check{\Theta}_0^s &= d_1^T\Theta_{01}d_1 + d_5^T\Theta_{02}d_5 + He(\Theta_{03}) - 2e^{-2\varepsilon_1 h}U_4^T H_{12}U_4 - d_6^T\Upsilon_{10}d_6 \\
&\quad -d_7^T\Upsilon_{20}d_7 + d_2^T\frac{h^2}{4}H_{12}d_2 + \check{\Gamma}_0 + h\Sigma_s,\ s=1,2, \\
\check{\Gamma}_0 &= He\Big([d_1^T\Gamma_{10}^T + d_2^T\Gamma_{20}^T][-d_2 - \mathcal{A}d_1 + \mathcal{W}_0 d_6 + \mathcal{W}_1 d_7 - \mathcal{B}KCd_3]\Big), \\
\Gamma_{B_0} &= col\{\Gamma_{10}^T\mathcal{B}, \Gamma_{20}^T\mathcal{B}, 0,0,0,0,0\},\ \Gamma_{C_0} = [0,0,-C,0,0,0,0].
\end{aligned}
$$

By applying Lemma 1 and (2.6), there exists a scalar υ_0 such that

$$
\begin{bmatrix} \check{\Theta}_0^s & \Gamma_{B_\Delta}^0 & \upsilon_0\Gamma_{C_\Delta}^{0T} \\ * & -\upsilon_0 I & 0 \\ * & * & -\upsilon_0 I \end{bmatrix} < 0. \tag{3.30}
$$

Then (3.30) can be reorganized as

$$
\bar{\Theta}_0 + He(\bar{\Gamma}_{B_0}K\bar{\Gamma}_{C_0}) < 0. \tag{3.31}
$$

By $K = X^{-1}Y$, (3.31) can be rearranged as

$$
\bar{\Theta}_0 + He(\Gamma_{J_0}Y\bar{\Gamma}_{C_0} + (\bar{\Gamma}_{B_0} - \Gamma_{J_0}X)X^{-1}Y\bar{\Gamma}_{C_0}) < 0. \tag{3.32}
$$

Similar, for $t \in [t_{v,i}+h, t_{v+1,i}) \cap \mathcal{G}_{off}^i$, $\Theta_1 < 0$ in (3.2) can be reorganized as

$$
\bar{\Theta}_1 + He(\Gamma_{J_1}Y\bar{\Gamma}_{C_1} + (\bar{\Gamma}_{B_1} - \Gamma_{J_1}X)X^{-1}Y\bar{\Gamma}_{C_1}) < 0. \tag{3.33}
$$

By Lemma 3, (3.32) and (3.33) can be ensured by (3.29). This completes the proof. □

Remark 4. *By using Lemmas 1 and 3, a novel method is proposed to co-design the non-fragile output-feedback controller and event-triggering parameter for guaranteeing synchronization error system (2.12) subject to DoS attacks to be exponentially stable in Theorem 2 is presented. Such as in Algorithm 1, if conditions (3.3), (3.4), (3.19), (3.20), and (3.29) are feasible, then the desired control gain and event-triggering parameter can be acquired.*

Algorithm 1 Solution Algorithm for Theorem 2

Step 1. Given parameters $T_{on}, T_{off}, \gamma > 0$, matrices J_j, $j = \{1, 2\}$, M, N, and choosing small parameters $0 < \mu < 1, 0 < h < T_{off}$, $\alpha_j > 1$, and $\varepsilon_j > 0$ to satisfy (3.20), if there is a solution for (3.3), (3.4), (3.19), (3.20), and (3.29), the algorithm ends. If there is no solution, we turn to Step 2.
Step 2. Adjust the values of $T_{on}, T_{off}, \gamma, \mu, h, \alpha_j, \varepsilon_j > 0$, $j = \{1, 2\}$, to satisfy (3.20). If the solution is found, the algorithm ends. If there is no solution, turn to Step 3.
Step 3. Reselect a larger and returning Step 2 until the solution is found, i.e., obtain the controller gain $K = X^{-1}Y$ and event-triggering parameter $\mathcal{F}$.

Remark 5. *In Theorems 1 and 2, the event-triggered synchronization control problem of NNs under DoS attacks is studied, in which the proposed control scheme can not only reduce the burden of the network but also offset network attacks. Therefore, the theoretical results obtained in this paper make the synchronization of NNs more reliable in many applications, such as secure communication, system identification and control, image encryption, and others.*

4. Numerical example

Consider $\mathcal{M}$ in (2.1) with the following parameters [39]:

$$\begin{aligned}
\mathcal{A} &= diag\{1, 1\},\ \theta = 1,\ \mathcal{J}(t) = 0, \\
\mathcal{W}_0 &= \begin{bmatrix} 1.8 & -0.1 \\ -2.0 & 0.4 \end{bmatrix},\ \mathcal{W}_1 = \begin{bmatrix} -1.7 & -0.6 \\ 0.5 & -2.5 \end{bmatrix}, \\
\phi(x(t)) &= \begin{bmatrix} \tanh(x_1(t)) \\ \tanh(x_2(t)) \end{bmatrix},\ \phi(\hat{x}(t)) = \begin{bmatrix} \tanh(\hat{x}_1(t)) \\ \tanh(\hat{x}_2(t)) \end{bmatrix}.
\end{aligned}$$

Set the initial values $x(s) = \begin{bmatrix} 0 & 0.3 \end{bmatrix}^T$ and $\hat{x}(s) = \begin{bmatrix} -0.1 & -0.3 \end{bmatrix}^T$ ($s \in [-\theta, 0]$). When there is no controller, the state responses of $\mathcal{M}$ and $\mathcal{S}$ are shown in Figure 2, in which we can observe that the state $x(t)$ of $\mathcal{M}$ in (2.1) and the state $\hat{x}(t)$ of $\mathcal{S}$ in (2.2) cannot be synchronized. This means the system is unstable.

Table 1. Comparison of the amount of sent measurements (SMs) by different ETS methods.

h	SMs of the ETS (2.9)	SMs of the ETS in [40]
0.06	84	
0.09	58	119
0.12	50	
0.15	47	

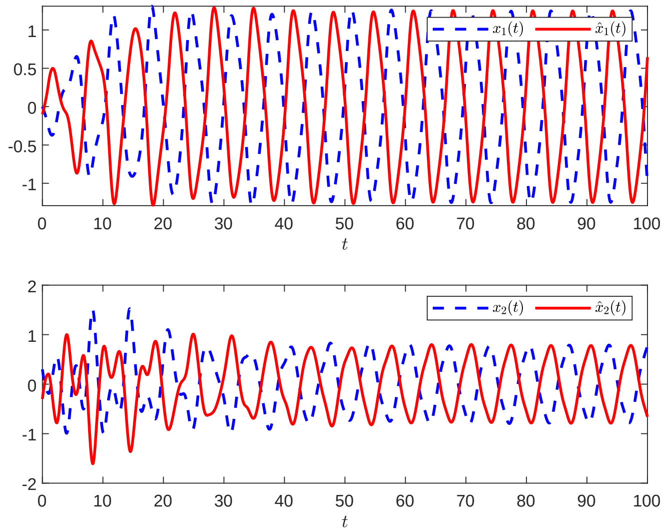

Figure 2. State responses of $x(t)$ and $\hat{x}(t)$ without control.

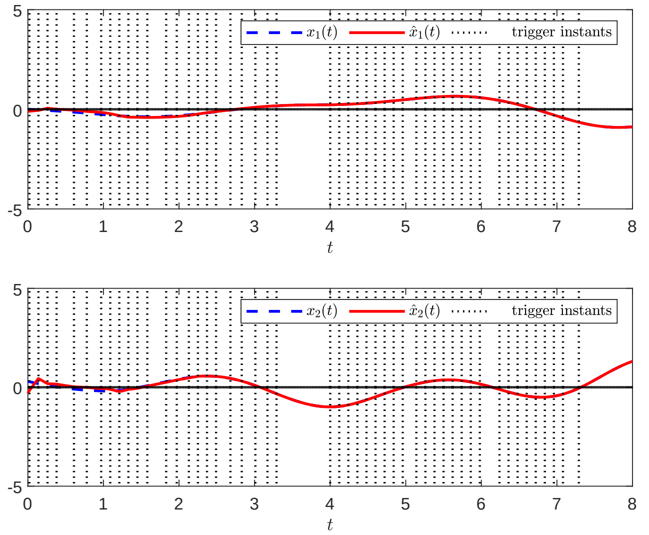

Figure 3. State responses of $x(t)$ and $\hat{x}(t)$ under the DoS attacks.

In what follows, we display the validity of the proposed method. Choose

$$\begin{aligned}
\mathcal{B} &= \begin{bmatrix} 0.7 & -0.2 \\ 0.1 & 0.5 \end{bmatrix}, \ C = diag\{1, 1\}, \\
M &= diag\{0.5, 0.5\}, \ N = diag\{0.2, 0.2\}, \\
\Phi_1 &= 0, \ \Phi_2 = diag\{0.5, 0.5\}, \ J_1 = J_2 = \mathcal{B}, \\
h &= 0.12, \ \varepsilon_1 = 0.10, \ \varepsilon_2 = 0.50, \ \alpha_1 = \alpha_2 = 1.01, \\
\mu &= 0.05, \ \gamma = 0.01, \ T_{on} = 0.65, \ T_{off} = 3.35.
\end{aligned}$$

Then, by solving the LMIs (3.3), (3.4), (3.19), (3.20), and (3.29), the corresponding controller gain and event-triggering parameter can be obtained:

$$K = \begin{bmatrix} 10.0921 & 3.3191 \\ -5.3805 & 19.1021 \end{bmatrix},$$
$$\mathcal{F} = \begin{bmatrix} 8.9110 & -1.4354 \\ -1.4354 & 6.9984 \end{bmatrix}.$$

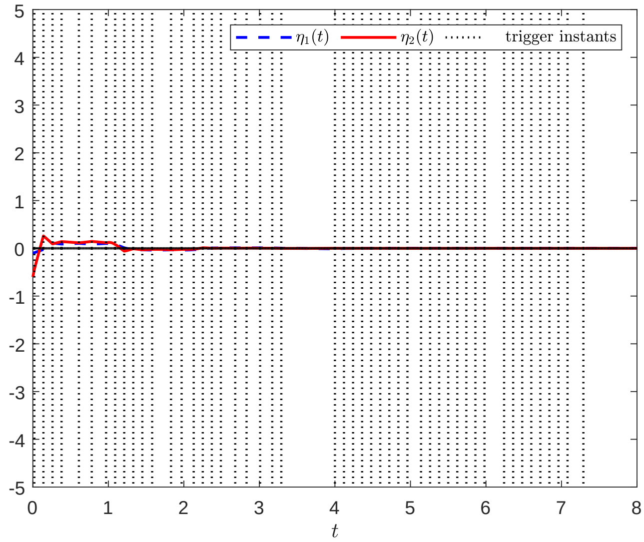

Figure 4. State response of $\beta(t)$ under DoS attacks.

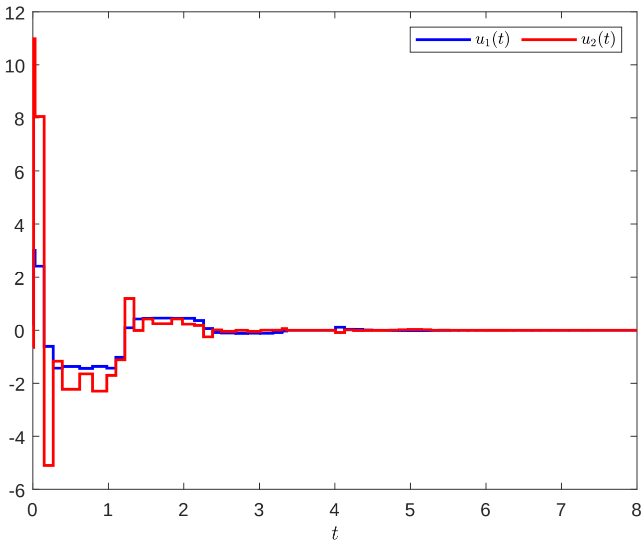

Figure 5. Control input $u(t)$.

When the system has controller (2.10), the state responses of $\mathcal{M}$ and $\mathcal{S}$ are shown in Figure 3, from which we can see that the state $x(t)$ of $\mathcal{M}$ in (2.1) and the state $\hat{x}(t)$ of $\mathcal{S}$ in (2.2) can achieve synchronization as time increases under DoS attacks. Moreover, the state response of error system (2.12) and control input $u(t)$ (2.10) are depicted in Figures 4 and 5. Therefore, we conclude that the designed non-fragile, output-feedback controller and the ETS are valid. In addition, we give Table 1 to compare the amount of sent measurements (SMs) by different ETS methods. From Table 1, we see that as h increases, the amount of sent measurements (SMs) decreases. Moreover, the proposed ETS in (2.9) can reduce the number of false triggers compared with the continuous ETS in [40].

5. Conclusions

The synchronization problem for time-delay NNs under the coaction of non-fragility, ETS, and DoS attacks has been addressed in this paper. A novel ETS based on switching between periodic sampling and the continuous ETS has been proposed, which not only can cut down the number of data transmissions but also offset cyberattacks. By choosing a suitable PLKF and using some free-weighting matrices, sufficient conditions have been established to ensure the synchronization error system to be exponentially stable. Finally, a numerical example has been given to illustrate the usefulness of the proposed approach. In the near future, we will discuss the finite-time synchronization control of neural networks with an adaptive event-triggered scheme under other attacks, such as deception attacks or hybrid attacks [41–43].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this paper.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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