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http://www.aimspress.com/journal/era

ERA, 33(1): 87–101.
DOI: 10.3934/era.2025005
Received: 01 November 2024
Revised: 02 January 2025
Accepted: 07 January 2025
Published: 14 January 2025

Research article

The properties on F-manifold color algebras and pre-F-manifold color algebras

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Abstract: The concept of F-manifold algebras is an algebraic expression of F-manifolds. In this paper, we provide the definition of an F-manifold color algebra, which can be viewed as a natural generalization of an F-manifold algebra. We develop the representation theory of an F-manifold color algebra and show that F-manifold color algebras admitting non-degenerate symmetric bilinear forms are coherence F-manifold color algebras. The concept of pre-F-manifold color algebras is also presented, and using this definition one can construct F-manifold color algebras. These results extend some properties of F-manifold algebras.

Keywords: Lie color algebra; F-manifold algebra; F-manifold color algebra; representation theory; pre-F-manifold color algebra

1. Introduction

Dubrovin [1] invented the notion of Frobenius manifolds in order to give geometrical expressions associated with WDVV equations. In 1999, Hertling and Manin [2] introduced the concept of F-manifolds as a relaxation of the conditions of Frobenius manifolds. Inspired by the investigation of describing F-manifolds algebraically, Dotsenko [3] defined F-manifold algebras in 2019 to relate operad F-manifold algebras to operad pre-Lie algebras. By definition, an F-manifold algebra is a triple $(F, \cdot, [\,,\,])$ satisfying the following Hertling–Manin relation:

$$H_{x_1 \cdot x_2}(x_3, x_4) = x_1 \cdot H_{x_2}(x_3, x_4) + x_2 \cdot H_{x_1}(x_3, x_4), \quad \forall x_1, x_2, x_3, x_4 \in F,$$

where $(F, \cdot)$ is a commutative associative algebra, $(F, [,])$ is a Lie algebra, and $H_{x_1}(x_2, x_3) = [x_1, x_2 \cdot x_3] - [x_1, x_2] \cdot x_3 - x_2 \cdot [x_1, x_3]$.

A vector space F admitting a linear map $\cdot$ is called a pre-Lie algebra if the following holds:

$$(x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3) = (x_2 \cdot x_1) \cdot x_3 - x_2 \cdot (x_1 \cdot x_3), \quad \forall x_1, x_2, x_3 \in F.$$

In recent years, pre-Lie algebras have attracted a great deal of attention in many areas of mathematics and physics (see [4–7] and so on).

Liu et al. [8] introduced the concept of pre-F-manifold algebras. Note that these algebras allow us to construct F-manifold algebras. They also studied representations of F-manifold algebras and constructed many other examples of these algebras. The definition of super F-manifold algebras and related categories was stated by Cruz Morales et al. [9]. Chen et al. [10] discussed the classification of three-dimensional F-manifold algebras over the complex field $\mathbb{C}$, which was based on the results of the classifications of low-dimensional commutative associative algebras and low-dimensional Lie algebras. Recently, the concept of Hom-F-manifold algebras and their properties have been given by Ben Hassine et al. [11].

In this paper, we provide the concepts of an F-manifold color algebra and a pre-F-manifold color algebra, respectively. We extend some properties of F-manifold algebras that were obtained in [8] to the color case. In Section 2, we summarize some concepts of Lie color algebras, pre-Lie color algebras, and representations of χ-commutative associative algebras and Lie color algebras, respectively. In Section 3, we provide the concept of an F-manifold color algebra and then study its representation. The concept of a coherence F-manifold color algebra is also introduced, and it follows that an F-manifold color algebra admitting a non-degenerate symmetric bilinear form is a coherence F-manifold color algebra. The concept of pre-F-manifold color algebras is defined in Section 4, and using these algebras, one can construct F-manifold color algebras.

Throughout this paper, we assume that k is a field with char $k = 0$ and all vector spaces are finite dimensional over k.

A preprint of this paper was posted on arXiv [12].

2. Lie color algebras and relative algebraic structures

The concept of a Lie color algebra was introduced in [13] and systematically studied in [14]. Since then, Lie color algebras have been studied from different aspects: Lie color ideals [15], generalized derivations [16], representations [17, 18], T^*-extensions of Lie color algebras [19, 20] and hom-Lie color algebras [21], cohomology groups [22] and the color left-symmetric structures on Lie color algebras [23]. In this section, we collect some basic definitions that will be needed in the remainder of the paper. In the following, we assume that G is an abelian group and denote $k\backslash\{0\}$ by k^*.

Definition 2.1. A skew-symmetric bicharacter is a map $\chi : G \times G \to k^*$ satisfying

(i) $\chi(g_1, g_2) = \chi(g_2, g_1)^{-1}$,

(ii) $\chi(g_1, g_2)\chi(g_1, g_3) = \chi(g_1, g_2 + g_3)$,

(iii) $\chi(g_1, g_3)\chi(g_2, g_3) = \chi(g_1 + g_2, g_3)$,

for all $g_1, g_2, g_3 \in G$.

By the definition, it is obvious that for any $a \in G$, we have $\chi(a, 0) = \chi(0, a) = 1$ and $\chi(a, a) = \pm 1$.

Definition 2.2. A pre-Lie color algebra is the G-graded vector space

$$F = \bigoplus_{g \in G} F_g$$

with a bilinear multiplication operation $\cdot$ satisfying

1) $F_{g_1} \cdot F_{g_2} \subseteq F_{g_1+g_2}$,

2) $(x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3) = \chi(g_1, g_2)((x_2 \cdot x_1) \cdot x_3 - x_2 \cdot (x_1 \cdot x_3))$,

for all $x_1 \in F_{g_1}, x_2 \in F_{g_2}, x_3 \in F_{g_3}$, and $g_1, g_2, g_3 \in G$.

Definition 2.3. A Lie color algebra is the G-graded vector space

$$F = \bigoplus_{g \in G} F_g$$

with a bilinear multiplication [,] satisfying

(i) $[F_{g_1}, F_{g_2}] \subseteq F_{g_1+g_2}$,

(ii) $[x_1, x_2] = -\chi(g_1, g_2)[x_2, x_1]$,

(iii) $\chi(g_3, g_1)[x_1, [x_2, x_3]] + \chi(g_1, g_2)[x_2, [x_3, x_1]] + \chi(g_2, g_3)[x_3, [x_1, x_2]] = 0$,

for all $x_1 \in F_{g_1}, x_2 \in F_{g_2}, x_3 \in F_{g_3}$, and $g_1, g_2, g_3 \in G$.

Remark Given a pre-Lie algebra $(F, \cdot)$, if we define the bracket $[x_1, x_2] = x_1 \cdot x_2 - x_2 \cdot x_1$, then $(F, [,])$ becomes a Lie algebra. Similarly, one has a pre-Lie color algebra's version, that is to say, a pre-Lie color algebra $(A, \cdot, \chi)$ with the bracket $[x_1, x_2] = x_1 \cdot x_2 - \chi(x_1, x_2)x_2 \cdot x_1$ becomes a Lie color algebra.

Let the vector space F be G-graded. An element $x \in F$ is called homogeneous with degree $g \in G$ if $x \in F_g$. In the rest of this paper, for any $x_1 \in F_{g_1}, x_2 \in F_{g_2}, x_3 \in F_{g_3}$, we will write $\chi(x_1, x_2)$ instead of $\chi(g_1, g_2)$, $\chi(x_1 + x_2, x_3)$ instead of $\chi(g_1 + g_2, g_3)$, and so on. Furthermore, when we write the skew-symmetric bicharacter $\chi(x_1, x_2)$, it is always assumed that the elements x_1 and x_2 are both homogeneous.

For a χ-commutative associative algebra $(F, \cdot, \chi)$, we mean that $(F, \cdot)$ is a G-graded associative algebra with the following χ-commutativity:

$$x_1 \cdot x_2 = \chi(x_1, x_2)x_2 \cdot x_1$$

for all $x_1 \in F_{g_1}$ and $x_2 \in F_{g_2}$.

Now, we assume that the vector space V is G-graded. A representation (V, μ) of the algebra $(F, \cdot, \chi)$ is a linear map $\mu : F \longrightarrow \mathrm{End}_k(V)_G := \bigoplus_{g \in G} \mathrm{End}_k(V)_g$ satisfying

$$\mu(x_2)v \in V_{g_1+g_2}, \quad \mu(x_2 \cdot x_3) = \mu(x_2) \circ \mu(x_3)$$

for all $v \in V_{g_1}, x_2 \in F_{g_2}, x_3 \in F_{g_3}$, where $\mathrm{End}_k(V)_g := \{f \in \mathrm{End}_k(V) | f(V_h) \subseteq V_{h+g}\}$. Given a Lie color algebra $(F, [,], \chi)$, its representation (V, ρ) is a linear map $\rho : F \longrightarrow \mathrm{End}_k(V)_G$ satisfying

$$\rho(x_2)v \in V_{g_1+g_2}, \quad \rho([x_2, x_3]) = \rho(x_2) \circ \rho(x_3) - \chi(x_2, x_3)\rho(x_3) \circ \rho(x_2)$$

for all $v \in V_{g_1}, x_2 \in F_{g_2}, x_3 \in F_{g_3}$.

The dual space $V^* = \bigoplus_{g\in G} V^*_g$ is also G-graded, where

$$V^*_{g_1} = \{\xi \in V^* | \xi(x) = 0, g_2 \neq -g_1, \forall x \in V_{g_2}, g_2 \in G\}.$$

Define a linear map $\mu^* : F \longrightarrow \mathrm{End}_k(V^*)_G$ satisfying

$$\mu^*(x_1)\xi \in V^*_{g_1+g_3}, \quad \langle \mu^*(x_1)\xi, v\rangle = -\chi(x_1, \xi)\langle \xi, \mu(x_1)v\rangle$$

for all $x_1 \in F_{g_1}, v \in V_{g_2}, \xi \in V^*_{g_3}$.

It is easy to see that

1) If (V, μ) is one representation of the algebra $(F, \cdot, \chi)$, then $(V^*, -\mu^*)$ is also its representation;

2) If (V, μ) is one representation of the algebra $(F, [,], \chi)$, then (V^*, μ^*) is also its representation.

3. F-manifold color algebras and representations

The concept of F-manifold color algebras is presented, and some results in [8] to the color case are established.

Definition 3.1. Let $(F, [,], \chi)$ be a Lie color algebra and $(F, \cdot, \chi)$ be a χ-commutative associative algebra. A quadruple $(F, \cdot, [,], \chi)$ is called an F-manifold color algebra if the following holds for any homogeneous element $x_1, x_2, x_3, x_4 \in F$,

$$H_{x_1\cdot x_2}(x_3, x_4) = x_1 \cdot H_{x_2}(x_3, x_4) + \chi(x_1, x_2)x_2 \cdot H_{x_1}(x_3, x_4), \tag{3.1}$$

where $H_{x_1}(x_2, x_3)$ is the color Leibnizator given by

$$H_{x_1}(x_2, x_3) = [x_1, x_2 \cdot x_3] - [x_1, x_2] \cdot x_3 - \chi(x_1, x_2)x_2 \cdot [x_1, x_3]. \tag{3.2}$$

Remark It is noticed that if we set $G = \{0\}$ and $\chi(0, 0) = 1$, then $(F, \cdot, [,], \chi)$ is exactly an F-manifold algebra.

Definition 3.2. Let $(F, \cdot, [,], \chi)$ be an F-manifold color algebra, (V, μ) be a representation of the algebra $(F, \cdot, \chi)$, and (V, ρ) be a representation of the algebra $(F, [,], \chi)$. A representation of $(F, \cdot, [,], \chi)$ is a triple (V, ρ, μ) if the following holds for any homogeneous element $x_1, x_2, x_3 \in F$,

$$M_1(x_1 \cdot x_2, x_3) = \mu(x_1)M_1(x_2, x_3) + \chi(x_1, x_2)\mu(x_2)M_1(x_1, x_3),$$
$$\mu(H_{x_1}(x_2, x_3)) = \chi(x_1, x_2 + x_3)M_2(x_2, x_3)\mu(x_1) - \mu(x_1)M_2(x_2, x_3),$$

where the linear maps M_1 and M_2 from $F \otimes F$ to $\mathrm{End}_k(V)_G$ are given by

$$M_1(x_1, x_2) = \rho(x_1)\mu(x_2) - \chi(x_1, x_2)\mu(x_2)\rho(x_1) - \mu([x_1, x_2]), \tag{3.3}$$
$$M_2(x_1, x_2) = \mu(x_1)\rho(x_2) + \chi(x_1, x_2)\mu(x_2)\rho(x_1) - \rho(x_1 \cdot x_2). \tag{3.4}$$

Example 3.1. *Let $(F, \cdot, [,], \chi)$ be an F-manifold color algebra. We have that $(F, ad, \mathcal{L})$ is a representation of $(F, \cdot, [,], \chi)$, where $ad : F \longrightarrow End_k(F)_G$ is given by*

$$ad_{x_1}x_2 = [x_1, x_2]$$

and the left multiplication operator $\mathcal{L} : F \longrightarrow End_k(F)_G$ *is given by*

$$\mathcal{L}_{x_1} x_2 = x_1 \cdot x_2$$

for any homogeneous element $x_1, x_2 \in F$.

Proof. Note that $(F, \mathcal{L})$ is a representation of the algebra $(F, \cdot, \chi)$ and (F, ad) is a representation of the algebra $(F, [,], \chi)$.

Now, for any homogeneous element $x_1, x_2, x_3, x_4 \in F$, we obtain

$$\begin{aligned} M_1(x_1, x_2)x_3 &= (ad_{x_1}\mathcal{L}_{x_2} - \chi(x_1, x_2)\mathcal{L}_{x_2} ad_{x_1} - \mathcal{L}_{[x_1,x_2]})x_3 \\ &= [x_1, x_2 \cdot x_3] - \chi(x_1, x_2)x_2 \cdot [x_1, x_3] - [x_1, x_2] \cdot x_3 \\ &= H_{x_1}(x_2, x_3). \end{aligned}$$

Thus

$$H_{x_1 \cdot x_2}(x_3, x_4) = x_1 \cdot H_{x_2}(x_3, x_4) + \chi(x_1, x_2)x_2 \cdot H_{x_1}(x_3, x_4)$$

implies the equation

$$M_1(x_1 \cdot x_2, x_3)x_4 = \mathcal{L}_{x_1} M_1(x_2, x_3)x_4 + \chi(x_1, x_2)\mathcal{L}_{x_2} M_1(x_1, x_3)x_4.$$

On the other hand, we obtain

$$\begin{aligned} & M_2(x_2, x_3)x_4 = (\mathcal{L}_{x_2} ad_{x_3} + \chi(x_2, x_3)\mathcal{L}_{x_3} ad_{x_2} - ad_{x_2 \cdot x_3})x_4 \\ = \ & x_2 \cdot [x_3, x_4] + \chi(x_2, x_3)x_3 \cdot [x_2, x_4] - [x_2 \cdot x_3, x_4] \\ = \ & -\chi(x_3, x_4)x_2 \cdot [x_4, x_3] - \chi(x_2, x_4)\chi(x_3, x_4)[x_4, x_2] \cdot x_3 + \chi(x_2 + x_3, x_4)[x_4, x_2 \cdot x_3] \\ = \ & \chi(x_2 + x_3, x_4)([x_4, x_2 \cdot x_3] - [x_4, x_2] \cdot x_3 - \chi(x_4, x_2)x_2 \cdot [x_4, x_3]) \\ = \ & \chi(x_2 + x_3, x_4)H_{x_4}(x_2, x_3). \end{aligned}$$

Thus

$$\begin{aligned} & \chi(x_1, x_2 + x_3)M_2(x_2, x_3)\mathcal{L}_{x_1} x_4 - \mathcal{L}_{x_1} M_2(x_2, x_3)x_4 \\ = \ & \chi(x_1, x_2 + x_3)M_2(x_2, x_3)(x_1 \cdot x_4) - x \cdot M_2(x_2, x_3)x_4 \\ = \ & \chi(x_1, x_2 + x_3)\chi(x_2 + x_3, x_1 + x_4)H_{x_1 \cdot x_4}(x_2, x_3) - \chi(x_2 + x_3, x_4)x_1 \cdot H_{x_4}(x_2, x_3) \\ = \ & \chi(x_2 + x_3, x_4)\{H_{x_1 \cdot x_4}(x_2, x_3) - x \cdot H_{x_4}(x_2, x_3)\} \\ = \ & \chi(x_2 + x_3, x_4)\chi(x_1, x_4)x_4 \cdot H_{x_1}(x_2, x_3) \\ = \ & \chi(x_1 + x_2 + x_3, x_4)x_4 \cdot H_{x_1}(x_2, x_3) \\ = \ & H_{x_1}(x_2, x_3) \cdot x_4. \end{aligned}$$

Hence, the proof is completed. □

Let (V, ρ, μ) be a representation of the F-manifold color algebra $(F, \cdot, [,], \chi)$. Note that $F \oplus V$ is a G-graded vector space. In the following, if we write $x + v \in F \oplus V$ as a homogeneous element for

$x \in F, v \in V$, it means that x and v are of the same degree as $x+v$. Now assume that x_1+v_1 and x_2+v_2 are both homogeneous elements in $F \oplus V$. Define

$$[x_1+v_1, x_2+v_2]_\rho = [x_1, x_2] + \rho(x_1)v_2 - \chi(x_1, x_2)\rho(x_2)v_1.$$

Then we obtain that $(F \oplus V, [,]_\rho, \chi)$ is a Lie color algebra. Moreover, define

$$(x_1+v_1)\cdot_\mu (x_2+v_2) = x_1 \cdot x_2 + \mu(x_1)v_2 + \chi(x_1, x_2)\mu(x_2)v_1.$$

It is easy to see that $(F \oplus V, \cdot_\mu, \chi)$ is a χ-commutative associative algebra. In fact, we have

Proposition 3.2. *With the above notations, $(F \oplus V, \cdot_\mu, [,]_\rho, \chi)$ is an F-manifold color algebra.*

Proof. It is sufficient to check that the relation in Definition 3.1 holds.

For any homogeneous element $x_1+v_1, x_2+v_2, x_3+v_3 \in F \oplus V$, we have

$$\begin{aligned}
& H_{x_1+v_1}(x_2+v_2, x_3+v_3) \\
= \; & [x_1+v_1, (x_2+v_2)\cdot_\mu (x_3+v_3)]_\rho - [x_1+v_1, x_2+v_2]_\rho \cdot_\mu (x_3+v_3) \\
& -\chi(x_1, x_2)(x_2+v_2)\cdot_\mu [x_1+v_1, x_3+v_3]_\rho \\
= \; & [x_1, x_2\cdot x_3] + \rho(x_1)\{\mu(x_2)v_3 + \chi(x_2, x_3)\mu(x_3)v_2\} - \chi(x_1, x_2+x_3)\rho(x_2\cdot x_3)v_1 - I - II.
\end{aligned}$$

where

$$\begin{aligned}
I = \; & \{[x_1, x_2] + \rho(x_1)v_3 - \chi(x_1, x_2)\rho(x_2)v_1\}\cdot_\mu (x_3+v_3) \\
= \; & [x_1, x_2]\cdot x_3 + \mu([x_1, x_2])v_3 + \chi(x_1+x_2, x_3)\mu(x_3)\{\rho(x_1)v_2 - \chi(x_1, x_2)\rho(x_2)v_1\},
\end{aligned}$$

and

$$\begin{aligned}
II = \; & \chi(x_1, x_2)(x_2+v_2)\cdot_\mu \{[x_1, x_3] + \rho(x_1)v_3 - \chi(x_1, x_3)\rho(x_3)v_1\} \\
= \; & \chi(x_1, x_2)\{x_2\cdot [x_1, x_3] + \mu(x_2)(\rho(x_1)v_3 - \chi(x_1, x_3)\rho(x_3)v_1) \\
& +\chi(x_2, x_1+x_3)\mu([x_1, x_3])v_2\}.
\end{aligned}$$

Thus

$$\begin{aligned}
& H_{x_1+v_1}(x_2+v_2, x_3+v_3) \\
= \; & H_{x_1}(x_2, x_3) + \{\rho(x_1)\mu(x_2) - \mu([x_1, x_2]) - \chi(x_1, x_2)\mu(x_2)\rho(x_1)\}v_3 \\
& +\{\chi(x_2, x_3)\rho(x_1)\mu(x_3) - \chi(x_1+x_2, x_3)\mu(x_3)\rho(x_1) \\
& -\chi(x_1, x_2)\chi(x_2, x_1+x_3)\mu([x_1, x_3])\}v_2 + \{-\chi(x_1, x_2+x_3)\rho(x_2\cdot x_3) \\
& +\chi(x_1+x_2, x_3)\chi(x_1, x_2)\mu(x_3)\rho(x_2) + \chi(x_1, x_2)\chi(x_1, x_3)\mu(x_2)\rho(x_3)\}v_1 \\
= \; & H_{x_1}(x_2, x_3) + M_1(x_1, x_2)v_3 + \chi(x_2, x_3)M_1(x_1, x_3)v_2 + \chi(x_1, x_2+x_3)M_2(x_2, x_3)v_1.
\end{aligned}$$

Hence, for any homogeneous element $x_4+v_4 \in F \oplus V$, we have

$$H_{(x_1+v_1)\cdot_\mu (x_2+v_2)}(x_3+v_3, x_4+v_4)$$

$$
\begin{aligned}
&= H_{x_1\cdot x_2+\mu(x_1)v_2+\chi(x_1,x_2)\mu(x_2)v_1}(x_3+v_3,x_4+v_4)\\
&= H_{x_1\cdot x_2}(x_3,x_4)+M_1(x_1\cdot x_2,x_3)v_4+\chi(x_3,x_4)M_1(x_1\cdot x_2,x_4)v_3\\
&\quad +\chi(x_1+x_2,x_3+x_4)M_2(x_3,x_4)(\mu(x_1)v_2+\chi(x_1,x_2)\mu(x_2)v_1).
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&(x_1+v_1)\cdot_\mu H_{x_2+v_2}(x_3+v_3,x_4+v_4)\\
= \ &(x_1+v_1)\cdot_\mu \{H_{x_2}(x_3,x_4)+M_1(x_2,x_3)v_4+\chi(x_3,x_4)M_1(x_2,x_4)v_3+\chi(x_2,x_3+x_4)M_2(x_3,x_4)v_2\}\\
= \ &x_1\cdot H_{x_2}(x_3,x_4)+\mu(x_1)\{M_1(x_2,x_3)v_4+\chi(x_3,x_4)M_1(x_2,x_4)v_3+\chi(x_2,x_3+x_4)M_2(x_3,x_4)v_2\}\\
&+\chi(x_1,x_2+x_3+x_4)\mu(H_{x_2}(x_3,x_4))v_1,
\end{aligned}
$$

and

$$
\begin{aligned}
&\chi(x_1,x_2)(x_2+v_2)\cdot_\mu H_{x_1+v_1}(x_3+v_3,x_4+v_4)\\
= \ &\chi(x_1,x_2)\{x_2\cdot H_{x_1}(x_3,x_4)+\mu(x_2)\{M_1(x_1,x_3)v_4+\chi(x_3,x_4)M_1(x_1,x_4)v_3\\
&+\chi(x_1,x_3+x_4)M_2(x_3,x_4)v_1\}+\chi(x_2,x_1+x_3+x_4)\mu(H_{x_1}(x_3,x_4))v_2\}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&(x_1+v_1)\cdot_\mu H_{x_2+v_2}(x_3+v_3,x_4+v_4)+\chi(x_1,x_2)(x_2+v_2)\cdot_\mu H_{x_1+v_1}(x_3+v_3,x_4+v_4)\\
= \ &x_1\cdot H_{x_2}(x_3,x_4)+\chi(x_1,x_2)x_2\cdot H_{x_1}(x_3,x_4)\\
&+\{\mu(x_1)M_1(x_2,x_3)+\chi(x_1,x_2)\mu(x_2)(M_1(x_1,x_3))\}v_4\\
&+\{\chi(x_3,x_4)\mu(x_1)M_1(x_2,x_4)+\chi(x_1,x_2)\chi(x_3,x_4)\mu(x_2)M_1(x_1,x_4)\}v_3\\
&+\{\chi(x_2,x_3+x_4)\mu(x_1)M_2(x_3,x_4)+\chi(x_1,x_2)\chi(x_2,x_1+x_3+x_4)\mu(H_{x_1}(x_3,x_4))\}v_2\\
&+\chi(x_1,x_2+x_3+x_4)\{\mu(x_2)M_2(x_3,x_4)+\mu(H_{x_2}(x_3,x_4))\}v_1\\
= \ &H_{(x_1+v_1)\cdot_\mu(x_2+v_2)}(x_3+v_3,x_4+v_4),
\end{aligned}
$$

which satisfies the relation in Definition 3.1. Hence, the conclusion follows immediately. □

It is noticed that, given a representation (V,ρ,μ) of an F-manifold algebra, Liu, Sheng, and Bai [8] asserted that $(V^*,\rho^*,-\mu^*)$ may not be its representation. Now, assume that $(F,\cdot,[\,,\,],\chi)$ is an F-manifold color algebra, together with a representation (V,μ) of the algebra $(F,\cdot,\chi)$ and a representation (V,ρ) of the algebra $(F,[\,,\,],\chi)$. In order to prove the following proposition associated with an F-manifold color algebra, we need to define the linear map M_3 from $F\otimes F$ to $\mathrm{End}_k(V)_G$ by

$$M_3(x_1,x_2)=-\chi(x_1,x_2)\rho(x_2)\mu(x_1)-\rho(x_1)\mu(x_2)+\rho(x_1\cdot x_2),$$

and the linear maps M_1^*, M_2^* from $F\otimes F$ to $\mathrm{End}_k(V^*)_G$ by

$$
\begin{aligned}
M_1^*(x_1,x_2)&=-\rho^*(x_1)\mu^*(x_2)+\chi(x_1,x_2)\mu^*(x_2)\rho^*(x_1)+\mu^*([x_1,x_2]),\\
M_2^*(x_1,x_2)&=-\mu^*(x_1)\rho^*(x_2)-\chi(x_1,x_2)\mu^*(x_2)\rho^*(x_1)-\rho^*(x_1\cdot x_2)
\end{aligned}
$$

for any homogeneous element $x_1,x_2\in F$.

Proposition 3.3. *With the above notations, assume that for any homogeneous element $x_1, x_2, x_3 \in F$, the following holds:*

$$M_1(x_1 \cdot x_2, x_3) = \chi(x_1, x_2 + x_3)M_1(x_2, x_3)\mu(x_1) + \chi(x_2, x_3)M_1(x_1, x_3)\mu(x_2),$$
$$\mu(H_{x_1}(x_2, x_3)) = -\chi(x_1, x_2 + x_3)M_3(x_2, x_3)\mu(x_1) + \mu(x_1)M_3(x_2, x_3).$$

Then $(V^*, \rho^*, -\mu^*)$ is a representation of $(F, \cdot, [,], \chi)$.

Proof. Suppose that $x_1, x_2, x_3 \in F, v \in V, \xi \in V^*$ are all homogeneous elements. First, we claim the following two identities:

$$\begin{aligned} \langle M_1^*(x_1, x_2)(\xi), v\rangle &= \langle \xi, \chi(x_1 + x_2, \xi)M_1(x_1, x_2)v\rangle; \\ \langle M_2^*(x_1, x_2)(\xi), v\rangle &= \langle \xi, \chi(x_1 + x_2, \xi)M_3(x_1, x_2)v\rangle. \end{aligned}$$

The claims follow from some direct calculations, respectively:

$$\begin{aligned} & \langle M_1^*(x_1, x_2)(\xi), v\rangle \\ = & \langle(-\rho^*(x_1)\mu^*(x_2) + \chi(x_1, x_2)\mu^*(x_2)\rho^*(x_1) + \mu^*([x_1, x_2]))\xi, v\rangle \\ = & \chi(x_1, x_2 + \xi)\langle \mu^*(x_2)\xi, \rho(x_1)v\rangle - \chi(x_1, x_2)\chi(x_2, x_1 + \xi)\langle(\rho^*(x_1)\xi, \mu(x_2)v\rangle \\ & -\chi(x_1 + x_2, \xi)\langle \xi, \mu([x_1, x_2])v\rangle \\ = & -\chi(x_1, x_2)\chi(x_1 + x_2, \xi)\langle \xi, \mu(x_2)\rho(x_1)v\rangle + \chi(x_2, \xi)\chi(x_1, \xi)\langle \xi, \rho(x_1)\mu(x_2)v\rangle \\ & -\chi(x_1 + x_2, \xi)\langle \xi, \mu([x_1, x_2])v\rangle \\ = & \langle \xi, \chi(x_1 + x_2, \xi)\{-\chi(x_1, x_2)\mu(x_2)\rho(x_1) + \rho(x_1)\mu(x_2) - \mu([x_1, x_2])\}v\rangle \\ = & \langle \xi, \chi(x_1 + x_2, \xi)M_1(x_1, x_2)v\rangle, \end{aligned}$$

and

$$\begin{aligned} & \langle M_2^*(x_1, x_2)(\xi), v\rangle \\ = & \langle\{-\mu^*(x_1)\rho^*(x_2) - \chi(x_1, x_2)\mu^*(x_2)\rho^*(x_1) - \rho^*(x_1 \cdot x_2)\}\xi, v\rangle \\ = & -\chi(x_1, x_2 + \xi)\chi(x_2, \xi)\langle \xi, \rho(x_2)\mu(x_1)v\rangle - \chi(x_2, \xi)\chi(x_1, \xi)\langle \xi, \rho(x_1)\mu(x_2)v\rangle \\ & +\chi(x_1 + x_2, \xi)\langle \xi, \rho(x_1 \cdot x_2)v\rangle \\ = & \langle \xi, \chi(x_1 + x_2, \xi)\{-\chi(x_1, x_2)\rho(x_2)\mu(x_1) - \rho(x_1)\mu(x_2) + \rho(x_1 \cdot x_2)\}v\rangle \\ = & \langle \xi, \chi(x_1 + x_2, \xi)M_3(x_1, x_2)v\rangle. \end{aligned}$$

With the above identities, we have

$$\begin{aligned} & \langle\{M_1^*(x_1 \cdot x_2, x_3) + \mu^*(x_1)M_1^*(x_2, x_3) + \chi(x_1, x_2)\mu^*(x_2)M_1^*(x_1, x_3)\}\xi, v\rangle \\ = & \langle \xi, \chi(x_1 + x_2 + x_3, \xi)M_1(x_1 \cdot x_2, x_3)v\rangle - \chi(x_1, x_2 + x_3 + \xi)\chi(x_2 + x_3, \xi)\langle \xi, M_1(x_2, x_3)\mu(x_1)v\rangle \\ & -\chi(x_1 + x_3, \xi)\chi(x_2, x_3 + \xi)\langle \xi, M_1(x_1, x_3)\mu(x_2)v\rangle \\ = & \chi(x_1 + x_2 + x_3, \xi)\langle \xi, \{M_1(x_1 \cdot x_2, x_3) - \chi(x_1, x_2 + x_3)M_1(x_2, x_3)\mu(x_1) - \chi(x_2, x_3)M_1(x_1, x_3)\mu(x_2)\}v\rangle \\ = & 0, \end{aligned}$$

and

$$\langle\{-\mu^*(H_{x_1}(x_2, x_3)) + \chi(x_1, x_2 + x_3)M_2^*(x_2, x_3)\mu^*(x_1) - \mu^*(x_1)M_2^*(x_2, x_3)\}\xi, v\rangle$$

$$
\begin{aligned}
&= \chi(x_1+x_2+x_3,\xi)\langle\xi,\mu(H_{x_1}(x_2,x_3))v\rangle+\chi(x_1,x_2+z)\chi(x_2+x_3,x_1+\xi)\langle\mu^*(x_1)\xi,M_3(x_2,x_3)v\rangle\\
&\quad+\chi(x_1,x_2+x_3+\xi)\langle M_2^*(x_2,x_3)\xi,\mu(x_1)v\rangle\\
&= \chi(x_1+x_2+x_3,\xi)\langle\xi,\mu(H_{x_1}(x_2,x_3))v\rangle-\chi(x_2+x_3,\xi)\chi(x,\xi)\langle\xi,\mu(x_1)M_3(x_2,x_3)v\rangle\\
&\quad+\chi(x,x_2+x_3+\xi)\chi(x_2+x_3,\xi)\langle\xi,M_3(x_2,x_3)\mu(x_1)v\rangle\\
&= \chi(x_1+x_2+x_3,\xi)\langle\xi,\{\mu(H_{x_1}(x_2,x_3))-\mu(x_1)M_3(x_2,x_3)+\chi(x_1,x_2+x_3)M_3(x_2,x_3)\mu(x_1)\}v\rangle\\
&= 0.
\end{aligned}
$$

Therefore, the conclusion follows immediately from the hypothesis and Definition 3.2. □

Given an F-manifold color algebra $(F,\cdot,[,],\chi)$, we define the linear map T from $F\otimes F$ to $\mathrm{End}_k(F)_G$ by

$$T(x_1,x_2)(x_3)=-\chi(x_1,x_2)[x_2,x_1\cdot x_3]-[x_1,x_2\cdot x_3]+[x_1\cdot x_2,x_3]$$

for any homogeneous elements $x_1,x_2,x_3\in F$.

Definition 3.3. An F-manifold color algebra $(F,\cdot,[,],\chi)$ is called a coherence one if for any homogeneous elements $x_1,x_2,x_3,x_4\in F$, the following hold:

$$
\begin{aligned}
H_{x_1\cdot x_2}(x_3,x_4) &= \chi(x_1,x_2+x_3)H_{x_2}(x_3,x_1\cdot x_4)+\chi(x_2,x_3)H_{x_1}(x_3,x_2\cdot x_4),\\
H_{x_1}(x_2,x_3)x_4 &= -\chi(x_1,x_2+x_3)T(x_2,x_3)(x_1\cdot x_4)+x_1T(x_2,x_3)(x_4).
\end{aligned}
$$

Proposition 3.4. *Assume that* $(,)$ *is a non-degenerate symmetric bilinear form on the F-manifold color algebra* $(F,\cdot,[,],\chi)$ *satisfying*

$$(x_1\cdot x_2,x_3)=(x_1,x_2\cdot x_3)\ \ \textit{and}\ \ ([x_1,x_2],x_3)=(x_1,[x_2,x_3])$$

for any homogeneous elements $x_1,x_2,x_3\in F$. *Then* $(F,\cdot,[,],\chi)$ *is a coherence F-manifold color algebra.*

Proof. First, we prove that

$$(H_{x_1}(x_2,x_3),x_4)=\chi(x_1+x_2,x_3)(x_3,H_{x_1}(x_2,x_4))$$

for any homogeneous elements $x_1,x_2,x_3,x_4\in F$.

In fact, we obtain

$$
\begin{aligned}
&(H_{x_1}(x_2,x_3),x_4)\\
&= ([x_1,x_2\cdot x_3]-[x_1,x_2]\cdot x_3-\chi(x_1,x_2)x_2\cdot[x_1,x_3],x_4)\\
&= -\chi(x_1,x_2+x_3)([x_2\cdot x_3,x_1],x_4)-\chi(x_1+x_2,x_3)(x_3,[x_1,x_2]\cdot x_4)\\
&\quad-\chi(x_1,x_2)\chi(x_2,x_1+x_3)([x_1,x_3],x_2\cdot x_4)\\
&= -\chi(x_1,x_2+x_3)(x_2\cdot x_3,[x_1,x_4])-\chi(x_1+x_2,x_3)(x_3,[x_1,x_2]\cdot x_4)+\chi(x_2,x_3)\chi(x_1,x_3)(x_3,[x_1,x_2\cdot x_4])\\
&= -\chi(x_1,x_2+x_3)\chi(x_2,x_3)(x_3,x_2\cdot[x_1,x_4])-\chi(x_1+x_2,x_3)(x_3,[x_1,x_2]\cdot x_4)\\
&\quad+\chi(x_1+x_2,x_3)(x_3,[x_1,x_2\cdot x_4])\\
&= \chi(x_1+x_2,x_3)(x_3,-\chi(x_1,x_2)x_2\cdot[x_1,x_4]-[x_1,x_2]\cdot x_4+[x_1,x_2\cdot x_4])\\
&= \chi(x_1+x_2,x_3)(x_3,H_{x_1}(x_2,x_4)).
\end{aligned}
$$

By the above relation, for every homogeneous element $x_1, x_2, x_3, w_1, w_2 \in F$, we have

$$\begin{aligned}
& (H_{x_1\cdot x_2}(x_3, w_1) - \chi(x_1, x_2 + x_3)H_{x_2}(x_3, x_1 \cdot w_1) - \chi(x_2, x_3)H_{x_1}(x_3, x_2 \cdot w_1), w_2) \\
= \; & \chi(x_1 + x_2 + x_3, w_1)(w_1, H_{x_1\cdot x_2}(x_3, w_2)) - \chi(x_1, x_2 + x_3)\chi(x_2 + x_3, x_1 + w_1)(x_1 \cdot w_1, H_{x_2}(x_3, w_2)) \\
& -\chi(x_2, x_3)\chi(x_1 + x_3, x_2 + w_1)(x_2 \cdot w_1, H_{x_1}(x_3, w_2)) \\
= \; & \chi(x_1 + x_2 + x_3, w_1)(w_1, H_{x_1\cdot x_2}(x_3, w_2)) - \chi(x_1, x_2 + x_3)\chi(x_2 + x_3, x_1 + w_1)\chi(x_1, w_1) \\
& (w_1, x_1 \cdot H_{x_2}(x_3, w_2)) - \chi(x_2, x_3)\chi(x_1 + x_3, x_2 + w_1)\chi(x_2, w_1)(w_1, x_2 \cdot H_{x_1}(x_3, w_2)) \\
= \; & \chi(x_1 + x_2 + x_3, w_1)(w_1, H_{x_1\cdot x_2}(x_3, w_2)) - \chi(x_1 + x_2 + x_3, w_1)(w_1, x_1 \cdot H_{x_2}(x_3, w_2)) \\
& -\chi(x_2, x_3)\chi(x_1 + x_3, x_2)\chi(x_1 + x_2 + x_3, w_1)(w_1, x_2 \cdot H_{x_1}(x_3, w_2)) \\
= \; & \chi(x_1 + x_2 + x_3, w_1)(w_1, H_{x_1\cdot x_2}(x_3, w_2)) - \chi(x_1 + x_2 + x_3, w_1)(w_1, x_1 \cdot H_{x_2}(x_3, w_2)) \\
& -\chi(x_1, x_2)\chi(x_1 + x_2 + x_3, w_1)(w_1, x_2 \cdot H_{x_1}(x_3, w_2)) \\
= \; & \chi(x_1 + x_2 + x_3, w_1)(w_1, H_{x_1\cdot x_2}(x_3, w_2) - x_1 \cdot H_{x_2}(x_3, w_2) - \chi(x_1, x_2)x_2 \cdot H_{x_1}(x_3, w_2)) \\
= \; & 0.
\end{aligned}$$

We claim the following identity:

$$(T(x_2, x_3)(w_1), w_2) = \chi(x_2 + x_3, w_1 + w_2)(w_1, H_{w_2}(x_2, x_3)).$$

In fact, we have

$$\begin{aligned}
& (T(x_2, x_3)(w_1), w_2) \\
= \; & (-\chi(x_2, x_3)[x_3, x_2 \cdot w_1] - [x_2, x_3 \cdot w_1] + [x_2 \cdot x_3, w_1], w_2) \\
= \; & \chi(x_2, x_3)\chi(x_3, x_2 + w_1)(x_2 \cdot w_1, [x_3, w_2]) + \chi(x_2, x_3 + w_1)(x_3 \cdot w_1, [x_2, w_2]) \\
& -\chi(x_2 + x_3, w_1)(w_1, [x_2 \cdot x_3, w_2]) \\
= \; & \chi(x_3, w_1)\chi(x_2, w_1)(w_1, x_2 \cdot [x_3, w_2]) + \chi(x_2, x_3 + w_1)\chi(x_3, w_1)(w_1, x_3 \cdot [x_2, w_2]) \\
& -\chi(x_2 + x_3, w_1)(w_1, [x_2 \cdot x_3, w_2]) \\
= \; & \chi(x_2 + x_3, w_1)(w_1, x_2 \cdot [x_3, w_2]) + \chi(x_2 + x_3, w_1)\chi(x_2, x_3)(w_1, x_3 \cdot [x_2, w_2]) \\
& -\chi(x_2 + x_3, w_1)(w_1, [x_2 \cdot x_3, w_2]) \\
= \; & \chi(x_2 + x_3, w_1)(w_1, x_2 \cdot [x_3, w_2] + \chi(x_2, x_3)x_3 \cdot [x_2, w_2] - [x_2 \cdot x_3, w_2]) \\
= \; & \chi(x_2 + x_3, w_1)(w_1, \chi(x_2 + x_3, w_2)H_{w_2}(x_2, x_3)) \\
= \; & \chi(x_2 + x_3, w_1 + w_2)(w_1, H_{w_2}(x_2, x_3)).
\end{aligned}$$

With the above identity, we have

$$\begin{aligned}
& (H_{x_1}(x_2, x_3) \cdot w_1 + \chi(x_1, x_2 + x_3)T(x_2, x_3)(x_1 \cdot w_1) - x_1 \cdot T(x_2, x_3)(w_1), w_2) \\
= \; & \chi(x_1 + x_2 + x_3, w_1)(w_1, H_{x_1}(x_2, x_3)w_2) + \chi(x_1, x_2 + x_3)\chi(x_2 + x_3, x_1 + w_1 + w_2) \\
& (x_1 \cdot w_1, H_{w_2}(x_2, x_3)) - \chi(x_1, x_2 + x_3 + w_1)(T(x_2, x_3)w_1, x_1 \cdot w_2) \\
= \; & \chi(x_1 + x_2 + x_3, w_1)(w_1, H_{x_1}(x_2, x_3)w_2) + \chi(x_1, w_1)\chi(x_2 + x_3, w_1 + w_2)(w_1, x_1 \cdot H_{w_2}(x_2, x_3)) \\
& -\chi(x_1, x_2 + x_3 + w_1)\chi(x_2 + x_3, x + w_1 + w_2)(w_1, H_{x_1\cdot w_2}(x_2, x_3))
\end{aligned}$$

$$
\begin{aligned}
&= \chi(x_1+x_2+x_3,w_1)(w_1,H_{x_1}(x_2,x_3)w_2)+\chi(x_1,w_1)\chi(x_2+x_3,w_1+w_2)(w_1,x_1\cdot H_{w_2}(x_2,x_3))\\
&\quad -\chi(x_1,w_1)\chi(x_2+x_3,w_1+w_2)(w_1,H_{x_1\cdot w_2}(x_2,x_3))\\
&= \chi(x_1+x_2+x_3,w_1)(w_1,H_{x_1}(x_2,x_3)w_2+\chi(x_2+x_3,w_2)x_1\cdot H_{w_2}(x_2,x_3)\\
&\quad -\chi(x_2+x_3,w_2)H_{x_1\cdot w_2}(x_2,x_3))\\
&= \chi(x_1+x_2+x_3,w_1)(w_1,H_{x_1}(x_2,x_3)w_2+\chi(x_2+x_3,w_2)x_1\cdot H_{w_2}(x_2,x_3)\\
&\quad -(H_{x_1}(x_2,x_3)w_2+\chi(x_2+x_3,w_2)x_1\cdot H_{w_2}(x_2,x_3)))\\
&= 0.
\end{aligned}
$$

Then, according to the assumption that the symmetric bilinear form $(,)$ is non-degenerate, the conclusion is obtained. □

4. Pre-F-manifold color algebras

The concept of pre-F-manifold color algebras is presented in this section, and using these algebras we construct F-manifold color algebras.

Definition 4.1. Let the vector space F be G-graded and $\bullet$ be a bilinear multiplication operator on F. A triple $(F,\bullet,\chi)$ is called a Zinbiel color algebra if the following hold:

(i) $F_{g_1}\bullet F_{g_2}\subseteq F_{g_1+g_2}$,

(ii) $x_1\bullet(x_2\bullet x_3)=(x_1\bullet x_2)\bullet x_3+\chi(x_1,x_2)(x_2\bullet x_1)\bullet x_3$,

for any homogeneous elements $x_1\in F_{g_1}, x_2\in F_{g_2}, x_3\in F_{g_3}$, and $g_1,g_2,g_3\in G$.

Given a Zinbiel color algebra $(F,\bullet,\chi)$, define

$$x_1\cdot x_2 = x_1\bullet x_2+\chi(x_1,x_2)x_2\bullet x_1, \tag{4.1}$$

for any homogeneous elements $x_1,x_2\in F$. Then it is not difficult to see that the algebra $(F,\cdot,\chi)$ is both χ-commutative and associative.

Define a linear map $\mathfrak{L}: F\longrightarrow \mathrm{End}_k(F)_G$ by

$$\mathfrak{L}_{x_1}x_2 = x_1\bullet x_2, \tag{4.2}$$

for any homogeneous elements $x_1,x_2\in F$. Then one has the following result.

Lemma 4.1. *With the above notations, $(F,\mathfrak{L})$ is a representation of $(F,\cdot,\chi)$.*

Proof. According to the definition of $\mathfrak{L}$, we get

$$\mathfrak{L}_{x_1\cdot x_2}x_3=(x_1\cdot x_2)\bullet x_3=(x_1\bullet x_2+\chi(x_1,x_2)(x_2\bullet x_1))\bullet x_3=x_1\bullet(x_2\bullet x_3)=\mathfrak{L}_{x_1}\mathfrak{L}_{x_2}x_3.$$

Thus, the proof follows. □

Let $(F,\bullet,\chi)$ be a Zinbiel color algebra and $(F,*,\chi)$ be a pre-Lie color algebra. For any homogeneous elements $x_1,x_2,x_3\in F$, define two linear maps $Q_1,Q_2: F\otimes F\otimes F\longrightarrow F$ by

$$Q_1(x_1,x_2,x_3) = x_1*(x_2\bullet x_3)-\chi(x_1,x_2)x_2\bullet(x_1*x_3)-[x_1,x_2]\bullet x_3,$$

$$Q_2(x_1,x_2,x_3) \;=\; x_1\bullet(x_2*x_3)+\chi(x_1,x_2)x_2\bullet(x_1*x_3)-(x_1\cdot x_2)*x_3,$$

where the operation $\cdot$ is given by (4.1) and the bracket $[,]$ is given by

$$[x_1,x_2]=x_1*x_2-\chi(x_1,x_2)x_2*x_1. \tag{4.3}$$

Definition 4.2. With the above notations, $(F,\bullet,*,\chi)$ is called a pre-F-manifold color algebra if the following hold

$$\begin{aligned}&(Q_1(x_1,x_2,x_3)+\chi(x_2,x_3)Q_1(x_1,x_3,x_2)+\chi(x_1,x_2+x_3)Q_2(x_2,x_3,x_1))\bullet x_4\\ =\;&\chi(x_1,x_2+x_3)Q_2(x_2,x_3,x_1\bullet x_4)-x_1\bullet Q_2(x_2,x_3,x_4),\end{aligned}$$

$$Q_1(x_1\cdot x_2,x_3,x_4)=x_1\bullet Q_1(x_2,x_3,x_4)+\chi(x_1,x_2)x_2\bullet Q_1(x_1,x_3,x_4)$$

for any homogeneous elements $x_1,x_2,x_3,x_4\in F$.

Since $(F,[,],\chi)$ is a Lie color algebra, it is known that (F,L) is a representation of $(F,[,],\chi)$ if one defines the linear map $L:F\longrightarrow \mathrm{End}_k(F)_G$ by

$$L_{x_1}x_2=x_1*x_2, \tag{4.4}$$

for any homogeneous elements $x_1,x_2\in F$.

Theorem 4.2. *Suppose that $(F,\bullet,*,\chi)$ is a pre-F-manifold color algebra; then*

(1) $(F,\cdot,[,],\chi)$ *is an F-manifold color algebra, where the operation $\cdot$ is given by (4.1) and the bracket $[,]$ is given by (4.3);*
(2) $(F;L,\mathfrak{L})$ *is a representation of $(F,\cdot,[,],\chi)$, where the map L is given by (4.4) and the map $\mathfrak{L}$ is given by (4.2).*

Proof. (1) It is known that $(F,[,],\chi)$ is a Lie color algebra and $(F,\cdot,\chi)$ is a χ-commutative associative algebra. Thus, we only need to prove that the relation in Definition 3.1 is satisfied.

Assume that $x_1,x_2,x_3,x_4\in F$ are all homogeneous elements. We claim the following identity:

$$H_{x_1}(x_2,x_3)=Q_1(x_1,x_2,x_3)+\chi(x_2,x_3)Q_1(x_1,x_3,x_2)+\chi(x_1,x_2+x_3)Q_2(x_2,x_3,x_1). \tag{4.5}$$

In fact, we have

$$\begin{aligned}&H_{x_1}(x_2,x_3)=[x_1,x_2\cdot x_3]-[x_1,x_2]\cdot x_3-\chi(x_1,x_2)x_2\cdot[x_1,x_3]\\ =\;&x_1*(x_2\cdot x_3)-\chi(x_1,x_2+x_3)(x_2\cdot x_3)*x_1-[x_1,x_2]\bullet x_3-\chi(x+x_2,x_3)x_3\bullet[x_1,x_2]\\ &-\chi(x_1,x_2)\{x_2\bullet[x_1,x_3]+\chi(x_2,x_1+x_3)[x_1,x_3]\bullet x_2\}\\ =\;&x_1*(x_2\bullet x_3)-\chi(x_1,x_2)x_2\bullet(x_1*x_3)-[x_1,x_2]\bullet x_3\\ &+\chi(x_2,x_3)\{x_1*(x_3\bullet x_2)-\chi(x_1,x_3)x_3\bullet(x_1*x_2)-[x_1,x_3]\bullet x_2\}\\ &+\chi(x_1,x_2+x_3)\{x_2\bullet(x_3*x_1)+\chi(x_2,x_3)x_3\bullet(x_2*x_1)-(x_2\cdot x_3)*x_1\}\\ =\;&Q_1(x_1,x_2,x_3)+\chi(x_2,x_3)Q_1(x_1,x_3,x_2)+\chi(x_1,x_2+x_3)Q_2(x_2,x_3,x_1).\end{aligned}$$

With the above identity, we obtain

$$H_{x_1\cdot x_2}(x_3,x_4)-x_1\cdot H_{x_2}(x_3,x_4)-\chi(x_1,x_2)x_2\cdot H_{x_1}(x_3,x_4)$$

$$
\begin{aligned}
&= Q_1(x_1\cdot x_2,x_3,x_4)+\chi(x_3,x_4)Q_1(x_1\cdot x_2,x_4,x_3)+\chi(x_1+x_2,x_3+x_4)Q_2(x_3,x_4,x_1\cdot x_2)\\
&\quad -x_1\cdot\{Q_1(x_2,x_3,x_4)+\chi(x_3,x_4)Q_1(x_2,x_4,x_3)+\chi(x_2,x_3+x_4)Q_2(x_3,x_4,x_2)\}\\
&\quad -\chi(x_1,x_2)x_2\cdot\{Q_1(x_1,x_3,x_4)+\chi(x_3,x_4)Q_1(x_1,x_4,x_3)+\chi(x_1,x_3+x_4)Q_2(x_3,x_4,x_1)\}\\
&= \{Q_1(x_1\cdot x_2,x_3,x_4)-x_1\bullet Q_1(x_2,x_3,x_4)-\chi(x_1,x_2)x_2\bullet Q_1(x_1,x_3,x_4)\}\\
&\quad +\{\chi(x_3,x_4)Q_1(x_1\cdot x_2,x_4,x_3)-\chi(x_3,x_4)x_1\bullet Q_1(x_2,x_4,x_3)-\chi(x_1,x_2)\chi(x_3,x_4)x_2\bullet Q_1(x_1,x_4,x_3)\}\\
&\quad +\{\chi(x_1+x_2,x_3+x_4)Q_2(x_3,x_4,x_1\bullet x_2)-\chi(x_1,x_2)\chi(x_2,x_1+x_3+x_4)Q_1(x_1,x_3,x_4)\bullet x_2\\
&\quad -\chi(x_1,x_2)\chi(x_3,x_4)\chi(x_2,x_1+x_3+x_4)Q_1(x_1,x_4,x_3)\bullet x_2\\
&\quad -\chi(x_1,x_2)\chi(x_1,x_3+x_4)\chi(x_2,x_1+x_3+x_4)Q_2(x_3,x_4,x_1)\bullet x_2\\
&\quad -\chi(x_2,x_3+x_4)x_1\bullet Q_2(x_3,x_4,x_2)\}+\{\chi(x_1+x_2,x_3+x_4)\chi(x_1,x_2)Q_2(x_3,x_4,x_2\bullet x_1)\\
&\quad -\chi(x_1,x_2+x_3+x_4)Q_1(x_2,x_3,x_4)\bullet x_1-\chi(x_3,x_4)\chi(x_1,x_2+x_3+x_4)Q_1(x_2,x_4,x_3)\bullet x_1\\
&\quad -\chi(x_2,x_3+x_4)\chi(x_1,x_3+x_4+x_2)Q_2(x_3,x_4,x_2)\bullet x_1-\chi(x_1,x_2)\chi(x_1,x_3+x_4)x_2\bullet Q_2(x_3,x_4,x_1)\}\\
&= \chi(x_1+x_2,x_3+x_4)\{Q_2(x_3,x_4,x_1\bullet x_2)-\chi(x_3+x_4,x_1)Q_1(x_1,x_3,x_4)\bullet x_2\\
&\quad -\chi(x_3,x_4)\chi(x_3+x_4,x_1)Q_1(x_1,x_4,x_3)\bullet x_2-Q_2(x_3,x_4,x_1)\bullet x_2-\chi(x_3+x_4,x_1)x_1\bullet Q_2(x_3,x_4,x_2)\}\\
&\quad +\chi(x_1,x_2+x_3+x_4)\{\chi(x_2,x_3+x_4)Q_2(x_3,x_4,x_2\bullet x_1)-Q_1(x_2,x_3,x_4)\bullet x_1\\
&\quad -\chi(x_3,x_4)Q_1(x_2,x_4,x_3)\bullet x_1-\chi(x_2,x_3+x_4)Q_2(x_3,x_4,x_2)\bullet x_1-x_2\bullet Q_2(x_3,x_4,x_1)\}\\
&= \chi(x_2,x_3+x_4)\{\chi(x_1,x_3+x_4)Q_2(x_3,x_4,x_1\bullet x_2)-Q_1(x_1,x_3,x_4)\bullet x_2\\
&\quad -\chi(x_3,x_4)Q_1(x_1,x_4,x_3)\bullet x_2-\chi(x_1,x_3+x_4)Q_2(x_3,x_4,x_1)\bullet x_2-x_1\bullet Q_2(x_3,x_4,x_2)\}\\
&= 0.
\end{aligned}
$$

Hence, $(F,\cdot,[,],\chi)$ is an F-manifold color algebra.

(2) It is known that (F,L) is a representation of the Lie color algebra $(F,[,],\chi)$. According to Lemma 4.1, $(F,\mathfrak{L})$ is a representation of the χ-commutative associative algebra $(F,\cdot,\chi)$. Define the linear map M_4 from $F\otimes F$ to $\mathrm{End}_k(F)_G$ by

$$M_4(x_1,x_2) = L_{x_1}\mathfrak{L}_{x_2}-\chi(x_1,x_2)\mathfrak{L}_{x_2}L_{x_1}-\mathfrak{L}_{[x_1,x_2]}.$$

Thus $Q_1(x_1,x_2,x_3)=M_4(x_1,x_2)(x_3)$, and the equation

$$Q_1(x_1\cdot x_2,x_3,x_4)=x_1\bullet Q_1(x_2,x_3,x_4)+\chi(x_1,x_2)x_2\bullet Q_1(x_1,x_3,x_4)$$

implies

$$M_4(x_1\cdot x_2,x_3)=\mathfrak{L}_{x_1}M_4(x_2,x_3)+\chi(x_1,x_2)\mathfrak{L}_{x_2}M_4(x_1,x_3).$$

On the other hand, define the linear map M_5 from $F\otimes F$ to $\mathrm{End}_k(F)_G$ by

$$M_5(x_1,x_2) = \mathfrak{L}_{x_1}L_{x_2}+\chi(x_1,x_2)\mathfrak{L}_{x_2}L_{x_1}-L_{x_1\cdot x_2}.$$

Thus $Q_2(x_1,x_2,x_3)=M_5(x_1,x_2)(x_3)$. Combining (4.5), the equation

$$
\begin{aligned}
&(Q_1(x_1,x_2,x_3)+\chi(x_2,x_3)Q_1(x_1,x_3,x_2)+\chi(x_1,x_2+x_3)Q_2(x_2,x_3,x_1))\bullet x_4\\
=\ &\chi(x_1,x_2+x_3)Q_2(x_2,x_3,x_1\bullet x_4)-x_1\bullet Q_2(x_2,x_3,x_4)
\end{aligned}
$$

implies

$$\mathfrak{L}_{H_{x_1}(x_2,x_3)}=\chi(x_1,x_2+x_3)M_5(x_2,x_3)\mathfrak{L}_{x_1}-\mathfrak{L}_{x_1}M_5(x_2,x_3).$$

Hence, the proof is completed. □

5. Conclusions

An F-manifold is "locally" an F-manifold algebra. We generalize the definition of an F-manifold algebra by introducing an F-manifold color algebra and study its representation theory. Then we provide the concept of a coherence F-manifold color algebra and obtain that an F-manifold color algebra admitting a non-degenerate symmetric bilinear form is a coherence F-manifold color algebra. The concept of a pre-F-manifold color algebra is also defined, and with the help of these algebras, one can construct F-manifold color algebras.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the Guangdong Basic and Applied Basic Research Foundation (2023A1515011739) and the Basic Research Joint Funding Project of University and Guangzhou City under Grant 202201020103.

Conflict of interest

The authors declare there are no conflicts of interest.

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