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Research article

L^p-theory for the $\partial\bar{\partial}$-equation and isomorphisms results

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Abstract: We establish an L^p_{loc}-existence theorem for the $\partial\bar{\partial}$-equation on a half-space of $\mathbb{C}^n$. The result is achieved for forms of class L^p_{loc} as well as for those forms in the scale of $W^{1,p}_{\text{loc}}$-Sobolev spaces and admitting distributional boundary values. Some isomorphisms and regularity results in relation to de Rham, Bott–Chern, and Aeppli cohomology groups are moreover obtained.

Keywords: $\partial\bar{\partial}$-problem; distributional boundary values; extensible currents; regularizing operators

1. Introduction

Solving $\partial\bar{\partial}$ follows from the pluripotential theory, which can be traced back to the 1940s [1,2] and still has a lot of attention. There are many interesting contributions concerning the $\partial\bar{\partial}$-problem, among which are [3–7]. More precisely, Nikitina [5] considered the $\partial\bar{\partial}$-equation on positive $(1,1)$-closed currents on complex manifolds. To build functional calculus for forms f on a positive current T, it requires an auxiliary $(1,1)$-Kähler form $\omega > 0$. With respect to this form, one can equip a metric on T and hence obtain the induced norm $\|f\|_{\omega,T}$ of f on T. The differential operators ∂ and $\bar{\partial}$ also act on positive currents. A current T is closed if $dT = 0$. For a closed current T, we say that a form $u \in L^2_{r,s}(T)$ is a solution to the induced equation

$$\bar{\partial}\partial_\omega u = f \quad \text{on } T \tag{1.1}$$

if

$$\bar{\partial}\partial_\omega(u \wedge T) = f \wedge T, \quad f \in L^2_{r+1,s+1}(T)$$

in the sense of currents (see Definition 4 in [5]), where the subscript ω indicates that the exterior calculus is done w. r. t. the ω-metric. For simplicity, the subscript ω may be omitted from the notations when there is no danger of confusion. The main result in [5] reads as follows: if T is a positive $(1,1)$-closed current in a pseudoconvex domain in $\mathbb{C}^n$, then there is a solution $u \in L^2_{n-r-1,s-1}(T)$ to Eq (1.1) for every $f \in L^2_{n-r,s}(T) \cap \ker(\overline{\partial})$, $s-r-1 \geq 1$. The case of currents of higher bidegree is also discussed. It is noteworthy that the $\partial\overline{\partial}$-approach adopted in [5] is totally different from this one applied in the current paper, where we are concerned with the $\partial\overline{\partial}$-problem for classes of differential forms having boundary traces in the currents sense. The ingredients of our approach include regularity results for both d-and $\overline{\partial}$-equations in the $W^{1,p}_{\text{loc}}$-Sobolev spaces.

Let us now recall those results that are more related to ours. In [8], Lojasiewicz and Tomassini proved that if f is a differential form on a bounded domain in $\mathbb{C}^n$ and has a boundary value, in the sense of currents, then f is an extensible current. This result helped Sambou et al, to study the $\partial\overline{\partial}$-equation for extensible currents and for differential forms with boundary values, in the sense of currents, in a series of papers. To be more precise, Sambou proved in [9] that if T is a $\overline{\partial}$-closed extensible current of bidegree $(n, n-s)$ on a completely strictly q-convex domain with C^∞-boundary in an n-dimensional complex manifold, $0 \leq q \leq n-1$, $1 \leq n-q \leq s \leq n$, then there is an extensible current S of bidegree $(n, n-s+1)$ such that $\overline{\partial}S = T$. As a corollary, he proved also that if f is a $\overline{\partial}$-closed $(0,1)$-form of class C^∞ on a completely strictly pseudoconvex domain and has a boundary value, in the sense of currents, then there is a function u of class C^∞, having a boundary value, in the sense of distributions, and solving the equation $\overline{\partial}u = f$.

In [10], Sambou and Sané generalized the corollary by Sambou [9] to the case of $(0,s)$-forms, where they proved that if f is a $\overline{\partial}$-closed $(0,s)$-form, $1 \leq s \leq n$, of class C^∞ on a smooth, strictly pseudoconvex domain, and admitting a boundary value, in the sense of currents, then there exists a $(0, s-1)$-form g of class C^∞ with boundary value, in the sense of currents, such that $\overline{\partial}g = f$.

Let D be a pseudoconvex domain with C^∞-boundary ∂D in $\mathbb{C}^n$ such that $H^i(D) = 0$, $i \geq 1$, and $H^j(\partial D) = 0$, $1 \leq j \leq 2n-2$, where $H^k(D)$ (respectively, $H^k(\partial D)$) is the de Rham cohomology group of smooth k-forms on D (respectively, on ∂D). Then, by using the $\overline{\partial}$-solving result from [10], Souhaibou et al. proved in [6] that for every d-closed (r,s)-form f of class $C^\infty(D)$ $(1 \leq r, s \leq n)$ with a boundary trace, in the currents sense, there is a $(r-1, s-1)$-form g of class $C^\infty(D)$ with a boundary trace, in the sense of currents, such that $\partial\overline{\partial}g = f$.

For the case of unbounded domains, Bodian et al. showed in [7] that the $\partial\overline{\partial}$-problem is solvable for extensible currents on a half-space in $\mathbb{C}^n$. This allowed Souhaibou et al. [11] to extend the result of [6] to the half-space case for the same class of differential forms. Their proof is achieved by inspiring some results from Brinkschulte [12].

Motivated by the aforementioned results, the following question was raised: If f is a d-closed (r,s)-form with L^p_{loc}-coefficients, does there exist a $(r-1, s-1)$-form u with L^p_{loc}-coefficients and satisfies the equation $\partial\overline{\partial}u = f$?

Positive answer to this question is introduced in Section 3 for L^p_{loc}-forms on the half-complex space

$$\Omega = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : \text{Im}(z_n) > 0\}$$

which is an example of an unbounded pseudoconvex domain as well as its complement.

We notice that there is an essential difference between $L^p_{\text{loc}}(\Omega)$ and $W^{1,p}_{\text{loc}}(\Omega)$: Roughly speaking, functions in $L^p_{\text{loc}}(\Omega)$ do not admit traces on $\partial\Omega$, while functions in $W^{1,p}_{\text{loc}}(\Omega)$ have boundary traces belonging to $W^{1-\frac{1}{p},p}_{\text{loc}}(\partial\Omega)$ (cf. [13, Theorem 1.4.46] or [14, pp. 315]). Luckily, this viewpoint allows us to address the $\partial\overline{\partial}$-problem for differential forms with $W^{1,p}_{\text{loc}}(\Omega)$-coefficients and having boundary traces in the currents sense; see Section 4 below for more details.

Now, we briefly return to the $\partial\overline{\partial}$-cohomologies. For compact Kähler manifolds Z one has that the Bott–Chern cohomology $H^{\bullet,\bullet}_{BC}(Z)$ is naturally isomorphic to the Dolbeault cohomology $H^{\bullet,\bullet}(Z)$; see [15, Lemma 5.15, Remark 5.16, 5.21, Lemma 5.11]. Furthermore, the Hodge$\star$-operator associated with any Hermitian metric on X induces an isomorphism between Bott–Chern and Aeppli cohomologies, i.e.,

$$H^{r,s}_{BC}(Z) \simeq H^{n-s,n-r}_{A}(Z), \quad \forall\ r, s \in \mathbb{N}.$$

In general, for compact non-Kähler manifolds, the natural maps

$$H^{\bullet,\bullet}_{BC}(Z) \to H^{\bullet,\bullet}(Z) \quad \text{and} \quad H^{\bullet,\bullet}_{BC}(Z) \to H^{\bullet}(Z, \mathbb{C})$$

induced by the identity are neither injective nor surjective; see the example given in [16, Section 1.c]. We refer to the monograph [17] by Angella for results concerning the characterization of compact complex manifolds by means of their Bott–Chern and Aeppli cohomologies. Certain isomorphisms and regularity results related to de Rham, Bott–Chern, and Aeppli cohomologies are introduced in Section 5.

Let us now present the main $L^p_{\text{loc}}(\Omega)$-existence theorem.

Theorem 1.1. *Let* $\Omega = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \text{Im}(z_n) > 0\}$*. For all* $r, s \in [1, n]$*, we have the following assertions.*

(i) *If* $f \in L^p_{r,s}(\Omega, \text{loc}) \cap \ker(d)$*,* $1 \leq p \leq \infty$*, then there is a form* $u \in L^p_{r-1,s-1}(\Omega, \text{loc})$ *satisfying* $\partial\overline{\partial}u = f$*.*

(ii) *If* $f \in W^{1,p}_{r,s}(\Omega, \text{loc}) \cap \ker(d)$*,* $1 \leq p < \infty$*, is a form admitting a boundary value, in the sense of currents, then there exists a form* $g \in W^{1,p}_{r-1,s-1}(\Omega, \text{loc})$ *admitting a boundary value, in the sense of currents, such that* $\partial\overline{\partial}g = f$*.*

The proof of assertion (i) depends on pushing out a bumping technique, while the proof of (ii) is twofold, namely, we solve respectively the equations $du = f$ and $\overline{\partial}u = f$ with regularity in the Sobolev spaces $W^{1,p}_{*}(\Omega, \text{loc})$, hence the $\partial\overline{\partial}$-solution becomes a combination of the resulting d- and $\overline{\partial}$-solutions. The key issue to prove (ii) is to construct suitable L^p_{loc}-regularizing operators for d- and $\overline{\partial}$-complexes, respectively.

2. Function spaces

We list here the basic spaces of functions and distributions that will be used throughout the paper. Let M be an open set in a differentiable manifold X of dimension N. For de Rham calculus we recall the needed basic function spaces (cf. [13]).

$C^\infty(M)$: the space of C^∞-smooth functions on M with its classical Fréchet topology.

$C^\infty(\overline{M})$: the subspace of C^∞-smooth functions up to the boundary of M; this is the space of the restrictions to M of functions in $C^\infty(X)$. We endow $C^\infty(\overline{M})$ with the Fréchet topology induced by $C^\infty(X)$.

$\mathcal{D}(M)$: the space of smooth; compactly supported functions on M, which is a topological vector space with the standard inductive limit topology.

$\mathcal{E}^k(M)$: the Fréchet space of k-forms of class C^∞ on M, where k is a finite integer ≥ 0.

$\mathcal{D}^k(M)$: the space of forms in $\mathcal{E}^k(M)$ with compact supports in M.

$\mathcal{D}'^k(M)$: the space of k-currents on M, the topological dual of the space $\mathcal{D}^{N-k}(M)$, endowed with the topology of uniform convergence on bounded subsets of $\mathcal{D}^{N-k}(M)$. In particular, a distribution is a 0-current.

$\check{\mathcal{D}}'^k(M)$: the space of all extensible k-currents T on M. Such currents T are defined as restrictions to M of currents $\tilde{T}$ on X. The associated de Rham cohomology group is denoted by $\check{H}^k(M)$.

$L^p(M)$: the Banach space of measurable functions such that

$$\|f\|_{L^p(M)} := \Big(\int_M |f|^p d\mu\Big)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty,$$

where $d\mu$ is the Lebesgue measure on X. If $p = \infty$, we set

$$\|f\|_{L^\infty(M)} = \operatorname{ess.}\sup_M |f| < \infty.$$

$L^p_k(M)$: the class of k-forms whose coefficients are in $L^p(M)$, $1 \leq p \leq \infty$.

$L^p_{\text{loc}}(M)$: the Fréchet space of p-locally integrable functions on M endowed with the topology of L^p-convergence on compact subsets of M.

$L^p_{k,\text{loc}}(M)$: the space of k-forms on M with coefficients in $L^p_{\text{loc}}(M)$.

The formula

$$\langle f, \phi \rangle = \int_M f \wedge \phi, \quad f \in L^p_{k,\text{loc}}(M), \quad \phi \in \mathcal{D}^k(M)$$

gives an embedding $L^p_{k,\text{loc}}(M) \subset \mathcal{D}'^k(M)$.

A differentiable form $f \in L^p_{k,\text{loc}}(M)$ is called the weak d-exterior derivative (the d-derivative in the sense of currents) of a form $\theta \in L^p_{k-1,\text{loc}}(M)$, and we write $d\theta = f$ if, for each $\phi \in \mathcal{D}^{N-k}(M)$, we have

$$\int_M f \wedge \phi = (-1)^k \int_M \theta \wedge d\phi.$$

$L^p_{k,c}(M)$: the subspace of $L^p_{k,\text{loc}}(M)$ consisting of forms with compact supports in M. This subspace is provided with the inductive limit topology.

$W^{m,p}_{\text{loc}}(M), m \in \mathbb{N} \cup \{0\}, p \in [1, \infty)$: the Sobolev space of functions f defined on M such that f and its distributional derivatives $\partial^\alpha f$ of order $|\alpha| \leq m$ are in $L^p_{\text{loc}}(M)$. The topology on $W^{m,p}_{\text{loc}}(M)$ is defined by the semi-norms:

$$|f|_{W^{m,p}_{\text{loc}}(M)} = \sum_{|\alpha| \leq m} \Big(\int_M |\partial^\alpha f|^p d\mu\Big)^{\frac{1}{p}}. \tag{2.1}$$

Topologized in this way, $W^{m,p}_{\text{loc}}(M)$ is a Fréchet space. For $p \in [1, \infty)$, we denote by p' the conjugate exponent to p, i.e., $p^{-1} + p'^{-1} = 1$. The space $W^{-m,p'}_{\text{loc}}(M)$ is defined as the topological dual of the completion of $\mathcal{D}(M)$ under the semi-norm (2.1); see e.g., [18, Theorem 3.9]. $W^{m,p}_{k,\text{loc}}(M)$ stands for the Sobolev space of k-forms whose coefficients belong to $W^{m,p}_{\text{loc}}(M)$.

$\check{W}^{m,p}_{k,\mathrm{loc}}(M)$: the $W^{m,p}_{\mathrm{loc}}$-Sobolev space of extensible k-currents on M. The corresponding de Rham cohomology group is denoted by $\check{H}^k_{W^{m,p}_{\mathrm{loc}}}(M)$.

We turn now to the complex case. If M is a domain in a complex manifold X of complex dimension n. Let $0 \le r \le n$ and $1 \le s \le n$. As in [19], we denote by:

$\mathcal{E}^{r,s}(M)$: the Fréchet space of (r,s)-forms of class C^∞ on M endowed with the topology of uniform convergence of the forms and all their derivatives on compact subsets of M. For every $k \in \{0,1,\dots,2n\}$, we have

$$\mathcal{E}^k(M) = \bigoplus_{r+s=k} \mathcal{E}^{r,s}(M).$$

The complex structure of M splits the exterior differential operator

$$d : \mathcal{E}^k(M) \to \mathcal{E}^{k+1}(M)$$

uniquely into

$$d = \partial + \overline{\partial}$$

and the $\partial\bar{\partial}$-operator is defined as

$$\partial\bar{\partial} : \mathcal{E}^{r,s}(M) \to \mathcal{E}^{r+1,s+1}(M).$$

$\mathcal{D}^{r,s}(M)$: the space of (r,s)-forms of class C^∞ and compactly supported in M.

$\mathcal{D}'^{r,s}(M)$: the space of currents of bidegree (r,s) on M. $\mathcal{D}'^{r,s}(X)$ is, by definition, the topological dual space to the space $\mathcal{D}^{n-r,n-s}(X)$ with the C^∞-topology.

$\check{\mathcal{D}}'^{r,s}(M)$: the space of extensible currents of bidegree (r,s) on M. The associated Dolbeault cohomology group is denoted by $\check{H}^{r,s}(M)$.

$L^p_{r,s}(M, \mathrm{loc})$: the space of (r,s)-forms on M whose coefficients belong to $L^p_{\mathrm{loc}}(M)$.

$W^{m,p}_{r,s}(M, \mathrm{loc})$: the Sobolev space of (r,s)-forms with $W^{m,p}_{loc}(M)$-coefficients.

$\check{W}^{m,p}_{r,s}(M, \mathrm{loc})$: the $W^{m,p}_{r,s}(M, \mathrm{loc})$-Sobolev space of extensible (r,s)-currents on M. The corresponding Dolbeault cohomology group is denoted by $\check{H}^{r,s}_{W^{m,p}_{\mathrm{loc}}}(M)$.

Taking the restriction of the $\bar{\partial}$-operator to $L^p_{r,s}(M, \mathrm{loc})$, in the sense of currents, we get an unbounded operator whose domain of definition is the set of forms f with $L^p_{\mathrm{loc}}(M)$-coefficients such that $\bar{\partial} f$ has also $L^p_{\mathrm{loc}}(M)$-coefficients; moreover, since $\bar{\partial}^2 = 0$, we get a complex of unbounded operators $(L^p_{r,s}(M, \mathrm{loc}), \bar{\partial})$; see e.g., [20].

3. L^p-existence theorem for $\partial\overline{\partial}$ on a half-space

In [21], Tarkhanov adapted the Norguet's integral formulas (see [22]) for solving the d-equation in L^p-scales on q-convex domains in $\mathbb{R}^n$. By using the L^p-solutions to the d-equation and pushing out the bumping technique by Kerzman [23], we conclude the following theorem.

Theorem 3.1. *Let* $\Omega = \{x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} | \, x_{n+1} > 0\}$ *be the upper half-space in* $\mathbb{R}^{n+1}$. *For every* $f \in L^p_{k,\mathrm{loc}}(\Omega) \cap \ker(d)$, *there exists a form* $u \in L^p_{k-1,\mathrm{loc}}(\Omega)$ *that satisfies* $du = f$.

Proof. Denote by $B_r := \{x \in \mathbb{R}^{n+1} : \|x\| < r\}$ the Euclidean ball of center 0 and radius r in $\mathbb{R}^{n+1}$ and set $B^- = B_r \cap \Omega$ and $B^+ = B_r \cap \Omega^c$. Define

$$\widetilde{f} = \begin{cases} f, & B^-; \\ 0, & \overline{B^+}. \end{cases}$$

Then $\widetilde{f} \in L^p_k(B_r) \cap \ker(d)$; see e.g., [24, Section 2.1]. Since B_r is convex, there exists a form $g \in L^p_{k-1}(B_r)$ such that $dg = \widetilde{f}$ in B_r (see [21]) and $dg = 0$ in $\overline{B^+}$. Put

$$\widetilde{g} = \begin{cases} g, & B^+; \\ 0, & \overline{B^-}. \end{cases}$$

It is clear that $d\widetilde{g} = 0$ in $\overline{B}_r$ and $\widetilde{g} \in L^p_{k-1}(B_r)$. If $k = 1$, we can take $g \equiv 0$ on B^+, so g has a support in B^+. If $k > 1$, $\exists \, h \in L^p_{k-2}(B_r)$ such that $dh = \widetilde{g}$.

Set

$$\widehat{g} = g - dh.$$

Then $\widehat{g} \in L^p_{k-1}(B_r)$, $\widehat{g} = 0$ on B^+, and $d\widehat{g} = f$ in B^-. Exhausting $\mathbb{R}^{n+1}$ by a sequence of open balls $\{B_\delta\}_{\delta \in \mathbb{N} \cup \{0\}}$ each of radius δ and center 0. On each B_δ, we can find $g_\delta \in L^p_{k-1}(B_\delta)$ such that

$$\begin{aligned} dg_\delta &= f \quad \text{in } B^-_\delta, \\ g_\delta &= 0 \quad \text{in } B^+_\delta. \end{aligned}$$

Indeed, since $dg_{\delta+2} = f$ in $B_{\delta+2}$, $dg_{\delta+1} = f$ in $B_{\delta+1}$, and $B_\delta \subset\subset B_{\delta+1}$, then $d(g_{\delta+2} - g_{\delta+1}) = 0$ in $B_{\delta+1}$, $(g_{\delta+2} - g_{\delta+1}) \in L^p_{k-1}(B_{\delta+1})$, and $g_{\delta+2} - g_{\delta+1} = 0$ in $\overline{B^+_{\delta+1}}$. Thus, there exists $u_{\delta+1} \in L^p_{k-2}(B_{\delta+1})$ satisfying $du_{\delta+1} = (g_{\delta+2} - g_{\delta+1})$ in $B^-_{\delta+1}$ and $u_{\delta+1} \equiv 0$ in $\overline{B^+_{\delta+1}}$. Choose a cut-off function $\chi \in \mathcal{D}(B_{\delta+2})$ such that $0 \le \chi(x) \le 1$ and $\chi \equiv 1$ in $\overline{B}_{\delta+1}$. Therefore,

$$g_{\delta+2} - d(1-\chi)u_{\delta+1} = g_{\delta+1} + d(\chi u_{\delta+1}) \quad \text{on} \quad B_{\delta+1}.$$

Setting

$$\psi_{\delta+2} = g_{\delta+2} - d(1-\chi)u_{\delta+1},$$

We have $d\psi_{\delta+2} = f$ in $B_{\delta+2}$, $\psi_{\delta+2} = g_{\delta+1}$ in $B_{\delta+1}$, and $\psi_{\delta+2} \equiv 0$ in $B^+_{\delta+2}$. Thus, we can find a sequence $\{v_\delta\}_\delta$, $v_\delta \in L^p_{k-1,\mathrm{loc}}(B_\delta)$, satisfying $dv_\delta = f$ in B_δ, $v_{\delta+1} = v_\delta$ in B_δ, $v_\delta \equiv 0$ in B^+_δ. Setting $v = \lim\limits_{\delta \to \infty} v_\delta$, then $v \in L^p_{k-1,\mathrm{loc}}(\mathbb{R}^{n+1})$, $v \equiv 0$ in $\overline{\Omega^c}$, and solving $dv = f$ in Ω. Hence $u = v|_\Omega \in L^p_{k-1,\mathrm{loc}}(\Omega)$ is the desired form.

Solving the $\overline{\partial}$-equation is an important question in the theory of several complex variables. For L^p-solutions to $\overline{\partial}u = f$ on q-convex domains in $\mathbb{C}^n$, we refer to [25] and the references therein. Despite of a great deal of the material for $\overline{\partial}$ is strictly analogous to corresponding material for d, the formalism above works in the complex case, where

$$\Omega = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n; \mathrm{Im}(z_n) > 0\}$$

and f is a $\overline{\partial}$-closed (r, s)-form. Therefore, we can immediately obtain:

Theorem 3.2. *Let* $\Omega = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \mathrm{Im}(z_n) > 0\}$. *Let* $\alpha \in L^p_{r,s}(\Omega, loc)$, $\overline{\partial}\alpha = 0$. *Then, there exists a form* $\beta \in L^p_{r,s-1}(\Omega, loc)$ *such that* $\overline{\partial}\beta = \alpha$.

Theorems 3.1 and 3.2 enable us to prove assertion (i) of Theorem 1.1 as follows.

Proof of Theorem 1.1 (i). Let $f \in L^p_{r,s}(\Omega, \mathrm{loc}) \cap \ker(d)$. Due to Theorem 3.1, there is a $(r+s-1)$-form g with coefficients belonging to $L^p_{\mathrm{loc}}(\Omega)$ and solving the equation $dg = f$. Without loss of generality, we can decompose g into a $(r-1, s)$-form g_1 and a $(r, s-1)$-form g_2 whose coefficients are in $L^p_{\mathrm{loc}}(\Omega)$. We then have

$$dg = d(g_1 + g_2) = dg_1 + dg_2 = f.$$

As $d = \partial + \bar{\partial}$, by the bidegree reasons, we have

$$\bar{\partial}g_1 = 0 \quad \text{and} \quad \partial g_2 = 0.$$

Then

$$\partial g_1 + \bar{\partial}g_2 = f. \tag{3.1}$$

By Theorem 3.2, there are two forms $h_1, h_2 \in L^p_{r-1,s-1}(\Omega, \mathrm{loc})$ such that

$$\bar{\partial}h_1 = g_1 \quad \text{and} \quad \partial h_2 = g_2.$$

Equation (3.1) then becomes

$$\partial\bar{\partial}h_1 + \bar{\partial}\partial h_2 = f,$$

but $\partial\bar{\partial} = -\bar{\partial}\partial$, and hence

$$\partial\bar{\partial}h_1 - \partial\bar{\partial}h_2 = \partial\bar{\partial}(h_1 - h_2) = f.$$

Setting $u = h_1 - h_2$. It is obvious that $u \in L^p_{r-1,s-1}(\Omega, \mathrm{loc})$ with $\partial\overline{\partial}u = f$.

4. $W^{1,p}_{\mathrm{loc}}$-regularity to $\partial\overline{\partial}$ for forms with distributional boundary values

Now we are in a position to prove part (ii) of Theorem 1.1. To this end, we need to prove $W^{1,p}_{\mathrm{loc}}$-regularity results for both d- and $\overline{\partial}$-equations. Let us begin with the real case.

4.1. The d-equation

There are many books on distribution theory each of them contains the basic definitions and properties of distributions; see, e.g., [13], [24], and [26]. Following [13, Chapter 9], we recall the following definitions.

Definition 4.1. *Let* X *be a differentiable manifold and* $\Omega \subset X$ *be a* C^∞*-smooth domain of defining function* ρ. *Let* $\Omega_\varepsilon = \{x \in \Omega \mid \rho(x) < -\varepsilon\}$. *A function* $f \in C^\infty(\Omega)$ *is said to have a distributional boundary value, if there is a distribution* T_b *defined on* $\partial\Omega$ *such that for any function* $\varphi \in \mathcal{D}(\partial\Omega)$, *we have:*

$$\langle T_b, \varphi \rangle = \lim_{\varepsilon \to 0} \int_{\partial\Omega_\varepsilon} f\varphi_\varepsilon d\sigma \tag{4.1}$$

where $\varphi_\varepsilon = i^*_\varepsilon\tilde{\varphi}$ *with* $\tilde{\varphi}$ *being an extension of* φ *to* Ω, $i_\varepsilon : \partial\Omega_\varepsilon \to X$ *being the canonical injection, and* $d\sigma$ *denotes the volume element.*

A differential form of class C^∞ *on* Ω *is said to have a boundary value in the sense of currents if its coefficients have distributional boundary values.*

As an example, it follows from Eq (4.1) that any holomorphic function f that can be extended continuously to $\partial\Omega$ admits a distributional boundary trace $T_b = f \wedge [\partial\Omega]^{0,1}$, where $[\partial\Omega]^{0,1}$ is the bidegree $(0,1)$-part of the integration current on $\partial\Omega$. Recently, it was shown in [19] that extensible smooth functions on bounded smooth domains admit distributional boundary traces.

Definition 4.2. *A function $f \in C^\infty(\Omega)$ is said to have polynomial growth of finite order $\gamma > 0$, if there is a constant $C > 0$ such that for each $x \in \Omega$ we have*

$$|f(x)| \leq C\mathrm{dist}(x, \partial\Omega)^{-\gamma}$$

where $\mathrm{dist}(x, \partial\Omega) := \inf\{|x - y|;\ y \in \partial\Omega\}$.

To demonstrate this phenomenon, we point out that smooth functions with polynomial growth on piecewise smooth domains have distributional boundary values; see [10]. It is interesting to mention also that harmonic functions defined on bounded smooth domains admit distributional boundary values if and only if they have polynomial growth of finite order near the boundary; see [27].

Proposition 4.3. *Let D be an open set in a C^∞-differentiable manifold X of dimension n. Then, the natural mapping*

$$\imath : \check{H}^k_{W^{1,p}_{\mathrm{loc}}}(D) \longrightarrow \check{H}^k(D), \quad k \geq 1,\ p \geq 1,$$

is an isomorphism.

Proof. We start with recalling the L^p-adapted properties of the de Rham operators presented in [28]. Namely, it was proved that there are linear regularizing operators R_ε and homotopy operators A_ε depending on a parameter $\varepsilon > 0$ such that

$$R_\varepsilon : \mathcal{D}'^k(X) \longrightarrow \mathcal{E}^k(X), \quad A_\varepsilon : \mathcal{D}'^k(X) \longrightarrow \mathcal{D}'^{k-1}(X)$$

and enjoying the following properties:

(i) For all $T \in \mathcal{D}'^k(X)$,
$$T - R_\varepsilon T = dA_\varepsilon T + A_\varepsilon dT \tag{4.2}$$
(ii) If $T \in W^{1,p}_{k,\mathrm{loc}}(X)$, then $R_\varepsilon T \in W^{1,p}_{k,\mathrm{loc}}(X)$, $A_\varepsilon T \in W^{1,p}_{k-1,\mathrm{loc}}(X)$,
(iii) $R_\varepsilon dT = dR_\varepsilon T$,
(iv) If $T \in W^{1,p}_{k,\mathrm{loc}}(X)$, then $R_\varepsilon T \to T$, $R_\varepsilon dT \to dT$, and $A_\varepsilon T \to 0$ as $\varepsilon \to 0$ in $W^{1,p}_{k,\mathrm{loc}}(X)$.
(v) The supports of $R_\varepsilon T$ and $A_\varepsilon T$ are contained in the ε-neighborhood of the support of T.
(vi) A_ε does not increase the singular support of T.
(vii) The regularity of $A_\varepsilon T$ is $1 - 0$ better than that of T in an ε-neighborhood of any open set U in X.

To finish the proof, we show that the mapping $\imath$ is bijective as follows.

Injectivity: Let $T \in \check{H}^k_{W^{1,p}_{\mathrm{loc}}}(D)$ such that $\imath([T]) = 0$ in $\check{H}^k(D)$. Then there exists a current $S \in \check{D}'^{k-1}(D)$ so that $dS = T$. Suppose that $\widetilde{T}$ and $\widetilde{S}$ are extensions of T and S with supports in $\overline{D}$, so that $d\widetilde{S} = \widetilde{T}$. Applying (4.2) to $\widetilde{S}$, we get

$$d\widetilde{S} = d(R_\varepsilon\widetilde{S} + A_\varepsilon\widetilde{T}) = \widetilde{T},$$

and hence

$$T = \widetilde{T}|_D = d(R_\varepsilon\widetilde{S} + A_\varepsilon\widetilde{T})|_D.$$

Since $A_\varepsilon \widetilde{T}$ has regularity better than that of $\widetilde{T}$ in an ε-neighborhood of D and each of R_ε and A_ε is continuous on $W^{1,p}_{*,\mathrm{loc}}(D)$, then $(R_\varepsilon \widetilde{S} + A_\varepsilon \widetilde{T})|_D \in W^{1,p}_{k-1,\mathrm{loc}}(D)$. This means that $[T] = 0$ in $\check{H}^k_{W^{1,p}_{\mathrm{loc}}}(D)$. The map ι is injective.

Surjectivity: Let $T \in \check{\mathcal{D}}'^k(D) \cap \ker(d)$. Let $\widetilde{T}$ be an extension of T to X with support in $\overline{D}$. Then

$$\widetilde{T} = R_\varepsilon \widetilde{T} + A_\varepsilon d\widetilde{T} + dA_\varepsilon \widetilde{T}, \quad d\widetilde{T}|_D = 0.$$

Thus

$$T = \widetilde{T}|_D = (R_\varepsilon \widetilde{T} + A_\varepsilon d\widetilde{T})|_D + dA_\varepsilon \widetilde{T}|_D.$$

The supports of $R_\varepsilon \widetilde{T}|_D$ and $A_\varepsilon d\widetilde{T}|_D$ are contained in some ε-neighborhood of $\overline{D}$; then they are extensible currents. As the regularity of $A_\varepsilon d\widetilde{T}|_D$ is better than the regularity of $d\widetilde{T}|_D$, and A_ε is continuous on $W^{1,p}_{*,\mathrm{loc}}(D)$, then $A_\varepsilon d\widetilde{T}|_D \in \check{W}^{1,p}_{k,\mathrm{loc}}(D)$. In addition, $R_\varepsilon \widetilde{T}|_D \in \mathcal{E}^k(D)$. Hence $[T] = [(R_\varepsilon \widetilde{T} + A_\varepsilon d\widetilde{T})|_D]$ in $\check{H}^k(D)$. The map ι is surjective.

Theorem 4.4. *Let* Ω *be as in Theorem* 3.1. *Then, we have* $\check{H}^k_{W^{1,p}_{\mathrm{loc}}}(\Omega) = 0$.

Proof. Due to [7], we have $\check{H}^k(\Omega) = 0$. By Proposition 4.3, we immediately get $\check{H}^k_{W^{1,p}_{\mathrm{loc}}}(\Omega) = 0$.

Theorem 4.5. *Let* Ω *be as in Theorem* 3.1. *For* $1 \le p < \infty$, $1 \le k \le n$, *let* $f \in W^{1,p}_{k,\mathrm{loc}}(\Omega)$ *be a d-closed form with a boundary value in the sense of currents. Then there exists a form u in* $W^{1,p}_{k-1,\mathrm{loc}}(\Omega)$ *with boundary value, in the currents sense, such that* $du = f$.

Proof. Since $f \in W^{1,p}_{k,\mathrm{loc}}(\Omega) \cap \ker(d)$, then $[f]$ is an extensible current and hence, by Theorem 4.4, there is a current $\Phi \in \check{W}^{1,p}_{k-1,\mathrm{loc}}(\Omega)$ such that

$$d\Phi = f \tag{4.3}$$

Let S be an extension of Φ to $\mathbb{R}^{n+1}$ with support in $\overline{\Omega}$. Consider a current F defined by $F = dS$ which is an extension of f to $\mathbb{R}^{n+1}$. Applying Eq (4.2) to S, we see that

$$dS = d(R_\varepsilon S + A_\varepsilon F) = F.$$

This shows that $(R_\varepsilon S + A_\varepsilon F)|_\Omega$ is another solution to Eq (4.3). Since $R_\varepsilon S \in \mathcal{E}^{k-1}(\overline{\Omega})$, it has a distributional boundary value on $\partial\Omega$. However, the operator A_ε does not increase the singular support; continuous on $W^{1,p}_{k,\mathrm{loc}}(\Omega)$, and $F|_\Omega \in W^{1,p}_{k,\mathrm{loc}}(\Omega)$, then $A_\varepsilon F|_\Omega \in W^{1,p}_{k-1,\mathrm{loc}}(\Omega)$. Therefore,

$$(R_\varepsilon S + A_\varepsilon F)|_\Omega \in W^{1,p}_{k-1,\mathrm{loc}}(\Omega).$$

We claim now that $A_\varepsilon F|_\Omega$ admits a distributional boundary value on $\partial\Omega$. Since $\overline{\Omega}$ is unbounded, take a closed ball $\overline{B}_r \subset \mathbb{R}^{n+1}$ of center 0 and radius r such that $\overline{B}_r \cap \overline{\Omega} \neq \emptyset$. Hence $F|_{\overline{B}_r \cap \overline{\Omega}}$ is an extensible current of finite order. Now, $A_\varepsilon F|_{\overline{B}_r \cap \overline{\Omega}}$ behaves like $\langle F, \mathcal{N}(x-y)\rangle$, where $\mathcal{N}(x) = c_m |x|^{2-m}$, $m \ge 3$, is the Newtonian potential or the fundamental solution of convolution type for the Laplacian Δ in $\mathbb{R}^m \setminus \{0\}$ (see e.g., [29, Section 2.4]). Set

$$\omega(x) = \langle F, \mathcal{N}(x-y)\rangle, \quad x \in \overline{B}_r \cap \Omega.$$

As $\mathcal{N}(x)$ is locally integrable, we introduce a suitable cut-off function. Let $x \in \overline{B}_r \cap \Omega$ be fixed and denote by d_x the distance of x to $\partial(\overline{B}_r \cap \Omega)$. Choose a cut-off function $\rho \in \mathcal{D}(B_{d_x/2}(x))$ such that $0 \leq \rho(x) \leq 1$ and $\rho(x)|_{B_{d_x/4}(x)} = 1$. Then, decompose the kernel $\mathcal{N}(x)$ into two kernels

$$\mathcal{N}_1(x) = c_n \frac{\rho(x)}{|x|^{n-1}}, \quad \mathcal{N}_2(x) = c_n \frac{(1-\rho(x))}{|x|^{n-1}},$$

and hence $\omega(x)$ can be written as

$$\omega(x) = \omega_1(x) + \omega_2(x),$$

with

$$\omega_1(x) = \langle F, \mathcal{N}_1(x-y)\rangle = \int_{\overline{B}_r\cap\Omega} f(y)\mathcal{N}_1(x-y)dy$$

and

$$\begin{aligned}\omega_2(x) = \langle F, \mathcal{N}_2(x-y)\rangle &= \int_{\overline{B}_r\cap\Omega} f(y)\mathcal{N}_2(x-y)dy \\ &= \int_{(\overline{B}_r\cap\Omega)\setminus B_{d_x/4}(x)} f(y)\mathcal{N}_2(x-y)dy.\end{aligned}$$

Since $\mathcal{N}_1(x)$ is compactly supported and f is locally integrable, then $\omega_1(x)$ is a C^∞-differential form on $\overline{B}_r \cap \overline{\Omega}$, cf. [24, Lemma 2.9], and hence it admits a boundary trace in the sense of currents on $\overline{B}_r \cap \partial\Omega$. Going further, observe that $|\mathcal{N}_2(x-y)| = O(|x-y|^{1-n})$, i.e., the kernels decay like $|x-y|^{1-n}$ for large $|x-y|$, and note also that $|x-y| \geq d_x/4$ for points $y \in (\overline{B}_r \cap \Omega) \setminus B_{d_x/4}(x)$ near to the boundary. Based on these observations, we estimate $|\omega_2|$ as follows:

$$\begin{aligned}\left|\omega_2(x)\right| &= \Big|\int_{(\overline{B}_r\cap\Omega)\setminus B_{d_x/4}(x)} f(y)\mathcal{N}_2(x-y)dy\Big| \\ &\leq \int_{(\overline{B}_r\cap\Omega)\setminus B_{d_x/4}(x)} \frac{|(1-\rho(y))f(y)|}{|x-y|^{n-1}}|dy| \\ &\leq \frac{4}{d_x^{n-1}} \int_{(\overline{B}_r\cap\Omega)\setminus B_{d_x/4}(x)} |f(y)||dy| \\ &\leq c(n) d_x^{1-n} \|f\|_{L^1((\overline{B}_r\cap\Omega)\setminus B_{d_x/4}(x))} \leq C(n) d_x^{1-n}.\end{aligned}$$

Thus, ω_2 has polynomial growth of finite order; it follows then from [10, Proposition 3.1] that ω_2 admits a distributional boundary trace on $\overline{B}_r \cap \partial\Omega$.

Pick a family of balls $\{\overline{B}_\ell\}_{\ell\in\mathbb{N}}$ in $\mathbb{R}^{n+1}$ such that $\overline{B}_\ell \cap \partial\Omega \neq \emptyset$ and $\partial\Omega \subset \bigcup_{\ell\in\mathbb{N}} \overline{B}_\ell$. On each $\overline{B}_\ell \cap \Omega$, $R_\varepsilon S + A_\varepsilon F$ admits a distributional boundary trace V_ℓ on $\overline{B}_\ell \cap \partial\Omega$. Further, on $\overline{B}_{\ell+1} \cap \partial\Omega$, it admits a distributional boundary trace $V_{\ell+1}$. Therefore, $d(V_{\ell+1} - V_\ell) = 0$ on $\overline{B}_\ell \cap \partial\Omega$. Since $\overline{B}_\ell \cap \partial\Omega$ is a convex domain in $\mathbb{R}^n$, for each ℓ, there exists a $(k-2)$-current h_ℓ on $\overline{B}_\ell \cap \partial\Omega$ such that

$$dh_\ell = V_{\ell+1} - V_\ell. \tag{4.4}$$

Let χ be a C^∞-function on $\partial\Omega$ such that $\chi(x) = 1$ for $x \in \overline{B}_{\ell-1} \cap \partial\Omega$ and has a compact support in $\overline{B}_{\ell+1} \cap \partial\Omega$. Rewrite Eq (4.4) as

$$V_{\ell+1} - d(1-\chi)h_\ell = V_\ell + d(\chi h_\ell) \qquad \text{on} \quad \overline{B}_\ell \cap \partial\Omega,$$

and set

$$T_{\ell+1} = V_{\ell+1} - d(1-\chi)h_\ell, \quad T_\ell = V_\ell + d(\chi h_\ell).$$

Then

$$T = \lim_{\ell\to\infty} T_\ell$$

is a distributional boundary value of $(R_\varepsilon S + A_\varepsilon F)|_\Omega$ on $\partial\Omega$. The form u defined by

$$u := (R_\varepsilon S + A_\varepsilon F)|_\Omega$$

belongs to $W^{1,p}_{k-1,\mathrm{loc}}(\Omega)$, admits a boundary trace, in the sense of currents, on $\partial\Omega$, and solves the equation $du = f$ in Ω.

4.2. The $\overline{\partial}$-version

Let X be an n-dimensional complex manifold. Following [30], for each $\varepsilon > 0$, $0 \le r \le n$, $1 \le s \le n$, there exist linear operators

$$\widetilde{R}_\varepsilon : \mathcal{D}'^{r,s}(X) \to \mathcal{E}^{r,s}(X), \quad \widetilde{A}_\varepsilon : \mathcal{D}'^{r,s}(X) \to \mathcal{D}'^{r,s-1}(X)$$

such that the operator $\widetilde{A}_\varepsilon$, modulo a smooth term, is the Martinelli–Bochner operator, and hence continuous from $L^p_{r,s}(X, \mathrm{loc})$ to $L^p_{r,s-1}(X, \mathrm{loc})$, $p \ge 1$ and from $\mathcal{E}^{r,s}(X)$ to $\mathcal{E}^{r,s-1}(X)$. Moreover, for any $T \in \mathcal{D}'^{r,s}(X)$, we have the $\overline{\partial}$-homotopy relation

$$T = \widetilde{R}_\varepsilon T + \widetilde{A}_\varepsilon \bar{\partial} T + \bar{\partial}\widetilde{A}_\varepsilon T. \tag{4.5}$$

Using Eq (4.5) and proceeding as in the proof of Proposition 4.3, we obtain a version for $\overline{\partial}$-cohomologies.

Proposition 4.6. *Let $D \subset X$ be an open set. Then, the natural mapping*

$$\jmath : \check{H}^{r,s}_{W^{1,p}_{\mathrm{loc}}}(D) \longrightarrow \check{H}^{r,s}(D)$$

is an isomorphism.

Theorem 4.7. *Let $\Omega = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \mathrm{Im}(z_n) > 0\}$. Let $f \in W^{1,p}_{r,s}(\Omega, \mathrm{loc})$ be a $\overline{\partial}$-closed form with a boundary value in the sense of currents. Then there exists a form $\alpha \in W^{1,p}_{r,s-1}(\Omega, \mathrm{loc})$ having a boundary value in the sense of currents such that $\overline{\partial}\alpha = f$.*

Proof. Let $f \in W^{1,p}_{r,s}(\Omega, \mathrm{loc}) \cap \ker(\overline{\partial})$ with boundary value in the sense of currents. According to [8], $f \in \check{\mathcal{D}}^{r,s}(\Omega)$. Therefore, by Proposition 4.6 and Theorem 43 in [7], we have

$$\check{H}^{r,s}_{W^{1,p}_{\mathrm{loc}}}(\Omega) = 0.$$

Then there exists an extensible $(r, s-1)$-current u with coefficients in $W^{1,p}_{\mathrm{loc}}(\Omega)$ such that $\bar{\partial}u = f$. Let ψ be a $W^{1,p}_{\mathrm{loc}}$-extension with support in $\overline{\Omega}$ of u to $\mathbb{C}^n$, and consider a current Γ defined by $\Gamma = \bar{\partial}\psi$ which is an extension of f to $\mathbb{C}^n$. Thanks to Eq (4.5), we obtain

$$\psi = \widetilde{R}_\varepsilon\psi + \widetilde{A}_\varepsilon\bar{\partial}\psi + \bar{\partial}A_\varepsilon\psi,$$

i.e.,

$$\psi = \widetilde{R}_\varepsilon \psi + \widetilde{A}_\varepsilon \Gamma + \bar{\partial} \widetilde{A}_\varepsilon \psi.$$

Apply $\overline{\partial}$ to both sides, we obtain

$$\bar{\partial}\psi = \bar{\partial}(\widetilde{R}_\varepsilon \psi + \widetilde{A}_\varepsilon \Gamma) = \Gamma.$$

Thus $(\widetilde{R}_\varepsilon \psi + \widetilde{A}_\varepsilon \Gamma)|_\Omega$ is also a solution to the equation $\bar{\partial} u = f$. Since $\widetilde{R}_\varepsilon \psi \in \mathcal{E}^{r,s-1}(\overline{\Omega})$, it has a boundary value in the sense of currents on $\partial\Omega$. As the operator $\widetilde{A}_\varepsilon$ does not increase the singular support and is continuous on $L^p_*(\Omega, \mathrm{loc}) \supset W^{1,p}_*(\Omega, \mathrm{loc})$, and $\Gamma|_\Omega \in W^{1,p}_{r,s}(\Omega, \mathrm{loc})$, then $\widetilde{A}_\varepsilon \Gamma|_\Omega \in W^{1,p}_{r,s-1}(\Omega, \mathrm{loc})$. This shows that

$$(\widetilde{R}_\varepsilon S + \widetilde{A}_\varepsilon \Gamma)|_\Omega \in W^{1,p}_{r,s-1}(\Omega, \mathrm{loc}).$$

The next step is to show that $\widetilde{A}_\varepsilon \Gamma|_\Omega$ has a distributional boundary trace. Recall first that the $\overline{\partial}$-Laplacian is defined by $\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^* \bar{\partial}$ and maps $\mathcal{E}^{r,s}(\mathbb{C}^n)$ into itself as an elliptic differential operator of order 2, where $\bar{\partial}^*$ is the formal adjoint of $\overline{\partial}$. As the Euclidean metric is Kähler, then $\Box = \frac{1}{2}\Delta$, so the application of $\Box$ to forms is equivalent to the application of Δ in $\mathbb{R}^{2n}$ to each of their coefficients with accuracy up to a nonessential factor $-1/4$. Therefore, $\mathcal{K}(x) = c_{2n}|z|^{2-2n}$, $n \geq 2$, is the Newtonian potential on $\mathbb{C}^n$ or the elementary solution of convolution type for the complex Laplacian $\Box$. Since $\overline{\Omega}$ is unbounded; consider a compact set K in $\mathbb{C}^n$ such that $\overset{\circ}{K} \neq \emptyset$ and $\overset{\circ}{K} \cap \partial\Omega \neq \emptyset$, then $\Gamma|_{K\cap\overline{\Omega}}$ is an extensible current of finite order, and $\widetilde{A}_\varepsilon \Gamma|_{K\cap\overline{\Omega}}$ has the same nature as $\langle \Gamma, \mathcal{K}(z-\zeta)\rangle$. Thus, proceeding as in the real case, we can show that $\widetilde{A}_\varepsilon \Gamma|_{K\cap\overline{\Omega}}$ admits a distributional boundary vale on $\overset{\circ}{K} \cap \partial\Omega$.

Exhaust $\mathbb{C}^n$ by a sequence of compact sets $\{K_j\}_{j\in\mathbb{N}}$, i.e.,

$$\mathbb{C}^n = \bigcup_{j\in\mathbb{N}} K_j, \quad K_j \subset \overset{\circ}{K}_{j+1},$$

so that $\overset{\circ}{K}_j \cap \partial\Omega \neq \emptyset$ and $\partial\Omega \subset \bigcup_{j\in\mathbb{N}} K_j$. On each $\overset{\circ}{K}_j \cap \Omega$, $\widetilde{R}_\varepsilon \psi + \widetilde{A}_\varepsilon \Gamma$ admits a distributional boundary trace U_j on $\overset{\circ}{K}_j \cap \partial\Omega$. On $\overset{\circ}{K}_{j+1} \cap \partial\Omega$, it has a distributional boundary trace U_{j+1}. Thus, $\overline{\partial}_b(U_{j+1} - U_j) = 0$ on $\overset{\circ}{K}_j \cap \partial\Omega$. Since the boundary is Levi flat, then there exists a $(r, s-2)$-current h_j on $\overset{\circ}{K}_j \cap \partial\Omega$ such that

$$\overline{\partial}_b h_j = U_{j+1} - U_j.$$

Let χ be a function of class C^∞ on $\partial\Omega$ such that $\chi \equiv 1$ on $\overset{\circ}{K}_{j-1} \cap \partial\Omega$ and with compact support in $\overset{\circ}{K}_{j+1} \cap \partial\Omega$. We see that

$$U_{j+1} - \overline{\partial}_b(1-\chi)h_j = U_j + \overline{\partial}_b(\chi h_j) \qquad \text{on} \quad \overset{\circ}{K}_j \cap \partial\Omega.$$

Set

$$T_{j+1} = V_{j+1} - \overline{\partial}_b(1-\chi)h_j, \quad T_j = U_j + \overline{\partial}_b(\chi h_j).$$

Then

$$T = \lim_{j\to\infty} T_j$$

represents a boundary trace of $(\widetilde{R}_\varepsilon\psi + \widetilde{A}_\varepsilon\Gamma)|_\Omega$ on $\partial\Omega$ in the sense of currents. Then the form

$$\alpha := (\widetilde{R}_\varepsilon\psi + \widetilde{A}_\varepsilon\Gamma)|_\Omega$$

belongs to $W^{1,p}_{r,s-1}(\Omega, \mathrm{loc})$, admits a boundary trace, in the sense of currents, on $\partial\Omega$, and solves the equation $\overline{\partial}\alpha = f$ in Ω.

4.3. Proof of Theorem 1.1 (ii)

Let $f \in W^{1,p}_{r,s}(\Omega, \mathrm{loc})$ be a d-closed form admitting boundary value in the sense of currents. According to Theorem 4.5, there is a form $u \in W^{1,p}_{r+s-1}(\Omega, \mathrm{loc})$ with boundary value, in the sense of currents, such that

$$du = f.$$

Without loss of generality, we can split u into two forms $u_1 \in W^{1,p}_{r-1,s}(\Omega, \mathrm{loc})$ and $u_2 \in W^{1,p}_{r,s-1}(\Omega, \mathrm{loc})$ such that each of u_1 and u_2 admits a boundary value in the sense of currents. We then have

$$du = du_1 + du_2 = f.$$

Since $d = \partial + \overline{\partial}$, by the bidegree reasons, one has

$$\overline{\partial}u_1 = 0 \quad \text{and} \quad \partial u_2 = 0.$$

From Theorem 4.7, we can find two forms $h_1, h_2 \in (W^{1,p}_{r-1,s-1}(\Omega, \mathrm{loc})$ with boundary values, in the sense of currents, such that

$$\overline{\partial}h_1 = u_1 \quad \text{and} \quad \partial h_2 = u_2.$$

Therefore, we have

$$\begin{aligned} f &= \partial u_1 + \overline{\partial}u_2 \\ &= \partial\overline{\partial}h_1 + \overline{\partial}\partial h_2 \\ &= \partial\overline{\partial}(h_1 - h_2). \end{aligned}$$

The form $g := (h_1 - h_2) \in W^{1,p}_{r-1,s-1}(\Omega, \mathrm{loc})$, has a boundary value in the sense of currents, and satisfies the equation $\partial\overline{\partial}g = f$. This proves assertion (ii) in Theorem 1.1.

5. Isomorphisms and regularity results

In this section, we introduce some isomorphisms and regularity results in relation to de Rham cohomology groups and the $\partial\overline{\partial}$-cohomology groups. Let X be a C^∞-differentiable manifold. As usual, the spaces $H^k_{L^p}(X)$ and $H^k_{\mathrm{curr}}(X)$ denote the de Rham cohomology groups for k-forms with L^p-coefficients and for k-currents, respectively. Corresponding cohomologies for compactly supported datum are denoted respectively by $H^k_{c,L^p}(X)$ and $H^k_{c,\mathrm{curr}}(X)$.

Lemma 5.1. *Keeping the notations as above, the natural mappings*

$$j : H^k_{c,L^p}(X) \to H^k_{c,\mathrm{curr}}(X) \quad \text{and} \quad i : H^k_c(X) \to H^k_{c,L^p}(X)$$

are isomorphisms.

Proof. Let us prove the first isomorphism. To this end, as in [31] or [32], we show that the natural map j is bijective as follows:

Injectivity: As in the proof of Proposition 4.3, there are L^p-regularizing operators R_ε and homotopy operators A_ε, $\varepsilon > 0$, with properties similar to (i)–(vii); see [28] or [33].

Consider a class $[f]$ in $H^k_{c,L^p}(X)$ such that $i[f] = [0]$ in $H^k_{c,\mathrm{cur}}(X)$. This means that there is a $(k-1)$-current S with compact support such that $dS = f$ in X. By Eq (4.2), we obtain $f = d(R_\varepsilon S + A_\varepsilon f)$. Since $A_\varepsilon f$ has regularity better than that of f, and since the operators R_ε and A_ε are continuous on $L^p_*(X)$ and have, by property (v), compact supports contained in some ε-neighborhood of the support of f, therefore $(R_\varepsilon S + A_\varepsilon f) \in L^p_{k-1,c}(X)$. Thus $[f] = [0]$ in $H^k_{c,L^p}(X)$. This shows the injectivity of j.

Surjectivity: Let $[f] \in H^k_{c,\mathrm{cur}}(X)$ such that $df = 0$. From Eq (4.2), we have $f = R_\varepsilon f + dA_\varepsilon f$. Thanks to the properties (v) and (vii), we have $R_\varepsilon f \in \mathcal{D}^{k-1}(X) \subset L^p_{k-1}(X)$, $A_\varepsilon f \in L^p_{k-1,c}(X)$. Then $[f] = [R_\varepsilon f]$ in $H^k_{c,\mathrm{cur}}(X)$. Thus the mapping j is surjective. The second isomorphism is proved by proceeding with the same arguments.

Using the L^p_{loc}-de Rham regularizing operators, we can moreover show that the natural mappings

$$\mathcal{I} : H^k(X) \longrightarrow H^k_{L^p_{\mathrm{loc}}}(X), \quad \Psi : H^k_{L^p_{\mathrm{loc}}}(X) \longrightarrow H^k_{\mathrm{curr}}(X)$$

are isomorphisms.

Corollary 5.2. *Let X be a differentiable manifold of class C^∞. For every $f \in L^p_{k,\mathrm{loc}}(X) \cap \ker(d)$ and any neighborhood U of the support of f, there is a form $g \in L^p_{k-1,\mathrm{loc}}(X)$ with support in U such that $f - dg \in \mathcal{E}^k(X)$.*

Proof. Choose a neighborhood V of the support of f such that $\overline{V} \subset U$ and let $\chi_0, \chi_1 \in C^\infty(X)$ such that $\chi_0 = 1$ on a neighborhood of $X \setminus V$ and vanishes in a neighborhood of the support of f, $\chi_1 = 1$ in a neighborhood of $X \setminus U$ and vanishes on a neighborhood of $\overline{V}$. Since f is d-closed and the map $\mathcal{I}$ is surjective, then there is a form $g_0 \in L^p_{k-1,\mathrm{loc}}(X)$ such that $u = f - dg_0 \in \mathcal{E}^k(X)$. So, $u = -dg_0$ on $X \setminus \operatorname{supp} f$. Since $\mathcal{I}$ is injective, there exists a form v of class C^∞ on $X \setminus \operatorname{supp} f$ such that $u = dv$ on $X \setminus \operatorname{supp} f$. We thus have $d(g_0 + \chi_0 v) = 0$ on $X \setminus \overline{V}$ and then there exists a form $g_1 \in L^p_{k-1,\mathrm{loc}}(X \setminus \overline{V})$ such that $g_0 + \chi_0 v - dg_1 = w \in \mathcal{E}^k(X \setminus \overline{V})$. The $(k-1)$-form $g = g_0 + \chi_0 v - \chi_1 w - d(\chi_1 g_1)$ with L^p_{loc}-coefficients and a support in U, moreover,

$$f - dg = f - d(g_0 + \chi_0 v - \chi_1 w - d(\chi_1 g_1)) = u - d(\chi_0 v - \chi_1 w)$$

is of class $C^\infty(X)$.

Let X be a complex manifold of complex dimension n. For every $p, q \in \{1, \ldots, n\}$, the Bott–Chern cohomology group of smooth (p,q)-forms on X is defined in [34] as

$$H^{p,q}_{BC}(X) = \frac{\ker(\partial : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p+1,q}(X)) \cap \ker(\bar\partial : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p,q+1}(X))}{\mathrm{Im}(\partial\bar\partial : \mathcal{E}^{p-1,q-1}(X) \to \mathcal{E}^{p,q}(X))}$$

Case of either $p = 0$ or $q = 0$. For example, if $q = 0$, then the $(p,0)$-Bott–Chern cohomology group is given, from definition, by

$$H^{p,0}_{BC}(M) = \{f \in \Gamma(M, \Omega^p_M) |\ \partial f = 0\},$$

where Ω_M^p is the sheaf of holomorphic p-forms on M. Thanks to the symmetric property of Bott–Chern cohomology, we have $H_{BC}^{0,q}(M) = \overline{H_{BC}^{q,0}(M)}$. The Bott–Chern cohomology group of smooth (p,q)-forms with compact support in X is defined similarly and is denoted by $H_{BC,c}^{p,q}(X)$. We recall also that the Aeppli cohomology group is defined in [35] by

$$H_A^{p,q}(X) = \frac{\ker(\partial\overline{\partial} : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p+1,q+1}(X))}{\mathrm{Im}(\partial : \mathcal{E}^{p-1,q}(X) \to \mathcal{E}^{p,q}(X)) + \mathrm{Im}\,(\overline{\partial} : \mathcal{E}^{p,q-1}(X) \to \mathcal{E}^{p,q}(X))}.$$

In particular, if $p = q = 0$, then

$$H_A^{0,0}(X) = \frac{\ker(\partial\overline{\partial} : \mathcal{E}^{0,0}(X)) \to \mathcal{E}^{1,1}(X)}{\Omega_X^0 + \overline{\Omega}_X^0},$$

where Ω_X^0 (resp. $\overline{\Omega}_X^0$) is the sheaf of holomorphic (resp. anti-holomorphic) functions on X. The Aeppli cohomology group of smooth (p,q)-forms with compact support in X is defined analogously and is denoted by $H_{A,c}^{p,q}(X)$. Finally, $\tilde{H}_{BC}^{p,q}(X)$ and $\tilde{H}_A^{p,q}(X)$ refer, respectively, to the Bott–Chern and Aeppli cohomology groups of currents of bidegree (p,q).

For compact Hermitian manifolds X, by using certain resolutions of the sheaf of germs of $\partial\overline{\partial}$-closed forms, Bigolin proved in [36] that the algebraic isomorphisms:

$$H_{BC}^{p,q}(X) \simeq \tilde{H}_{BC}^{p,q}(X), \quad H_A^{p,q}(X) \simeq \tilde{H}_A^{p,q}(X),$$
$$H_{BC,c}^{p,q}(X) \simeq \tilde{H}_{BC,c}^{p,q}(X), \quad H_{A,c}^{p,q}(X) \simeq \tilde{H}_{A,c}^{p,q}(X)$$

hold true for all integers $p,q \in \{1,\cdots,n\}$. We introduce below an alternative proof depending on the $\partial\overline{\partial}$-Hodge decomposition formulas for (p,q)-forms on compact complex manifolds.

Lemma 5.3. *Let X be a compact Hermitian complex manifold of complex dimension n. Then, the natural map*

$$j : H_{BC}^{p,q}(X) \to \tilde{H}_{BC}^{p,q}(X)$$

is isomorphism.

Proof. The Hodge theory for elliptic complexes (see [16, Proposition 2.1]) asserts the existence of linear operators

$$H : \mathcal{D}'^{p,q}(X) \to \mathcal{E}^{p,q}(X), \quad J : \mathcal{D}'^{p,q}(X) \to \mathcal{D}^{p-1,q-1}(X),$$
$$M : \mathcal{D}'^{p+1,q}(X) \to \mathcal{D}'^{p,q}(X), \quad N : \mathcal{D}'^{p,q+1}(X) \to \mathcal{D}'^{p,q}(X)$$

such that any (p,q)-current T admits the $\partial\overline{\partial}$-homotopy formula:

$$T = HT + \partial\overline{\partial}JT + M\partial T - N\overline{\partial}T, \tag{5.1}$$

where

$$\begin{cases} H = H^* = H^2, \\ \partial H = \overline{\partial}H = H\partial\overline{\partial} = 0, \\ \ker(I - H) = \mathrm{Im}(H) = \{f \in \mathcal{E}^{p,q}(X) \,|\, \partial f = \overline{\partial} f = (\partial\overline{\partial})^* f = 0\}. \end{cases}$$

Injectivity: Let $[T] \in H^{p,q}_{BC}(X)$ so that $[T] = 0$ in $\tilde{H}^{p,q}_{BC}(X)$, namely $T = \partial\bar{\partial}S$ for some $(p-1, q-1)$-current S on X. Then, in view of Eq (5.1), we have $S = JT$. Thus, if $T = f$ is a C^∞ (p,q)-form on X, then $f = \partial\bar{\partial}Jf$ with Jf a C^∞ $(p-1, q-1)$-form on X. Hence $[T] = 0$ in $H^{p,q}_{BC}(X)$. Thus the map j is injective.

Surjectivity: Let $T \in \mathcal{D}'^{p,q}(X) \cap \ker(d)$. As $\partial T = 0$, $\bar{\partial}T = 0$, hence (5.1) becomes

$$T = HT + \partial\bar{\partial}JT.$$

Since HT is a C^∞ d-closed (p,q)-form and $JT \in \mathrm{Im}\partial\bar{\partial}$, therefore we deduce that the map j is surjective.

Lemma 5.4. *Let X be a compact Hermitian complex manifold of complex dimension n. Then, the natural map*

$$i : H^{p,q}_A(X) \to \tilde{H}^{p,q}_A(X)$$

is surjective. If in addition X is regular in the sense of [37], *i.e., it satisfies the condition*

$$\ker(\partial\bar{\partial}) = \ker(\partial) + \mathrm{Im}(\bar{\partial})$$

then i is injective.

Proof. We first prove that i is surjective. According to Aeppli decomposition; see [16, p. 10], any (p,q)-current α can be decomposed as

$$\alpha = h\alpha + \partial^*\bar{\partial}^*\eta + \partial\mu + \bar{\partial}\lambda,$$

where $h\alpha \in \mathcal{E}^{p,q}(X)$ so that $\partial^* h\alpha = \bar{\partial}^* h\alpha = \partial\bar{\partial}h\alpha = 0$, $\eta \in \mathcal{D}'^{p+1,q+1}(X)$, $\mu \in \mathcal{D}'^{p,q-1}(X)$, and $\lambda \in \mathcal{D}'^{p,q-1}(X)$. Let $\alpha \in \mathcal{D}'^{p,q}(X) \cap \ker(\partial\bar{\partial})$. Since $\partial\bar{\partial}\alpha = 0$, we must have $\partial\bar{\partial}(\partial^*\bar{\partial}^*\eta) = 0$. We claim that $\partial^*\bar{\partial}^*\eta = 0$. We have

$$\|\partial^*\bar{\partial}^*\eta\|^2 = \langle\partial^*\bar{\partial}^*\eta, \partial^*\bar{\partial}^*\eta\rangle = \langle\bar{\partial}^*\eta, \partial\partial^*\bar{\partial}^*\eta\rangle = \langle\eta, \bar{\partial}\partial(\partial^*\bar{\partial}^*\eta)\rangle = 0.$$

Thus $\partial^*\bar{\partial}^*\eta = 0$. Hence α has the representation:

$$\alpha = h\alpha + \partial\mu + \bar{\partial}\lambda,$$

with $h\alpha$ is a C^∞ $\partial\bar{\partial}$-closed (p,q)-form and $\partial\mu + \bar{\partial}\lambda \in \mathrm{Im}\partial + \mathrm{Im}\bar{\partial}$. The map i is then surjective.

Injectivity: Let $[\alpha] \in H^{p,q}_A(X)$ such that $\partial\bar{\partial}\alpha = 0$ and $i([\alpha]) = 0$ in $\tilde{H}^{p,q}_A(X)$. This means that there exist two currents $\beta_1 \in \mathcal{D}'^{p-1,q}(X)$ and $\beta_2 \in \mathcal{D}'^{p,q-1}(X)$ such that $\alpha = \partial\beta_1 + \bar{\partial}\beta_2$. It was shown in [38, Proposition 3.1] that compact manifolds are regular if and only if they satisfy the $\partial\bar{\partial}$-Lemma, hence, by the regularity assumption, α is also $\partial\bar{\partial}$-exact.

Therefore, as shown above, if $\alpha = f \in \mathcal{E}^{p,q}(X)$, then $f = \partial\bar{\partial}Jf = \partial(\bar{\partial}\frac{Jf}{2}) + \bar{\partial}(-\partial\frac{Jf}{2})$. This amounts to $\beta_1 = \bar{\partial}\frac{Jf}{2}$ and $\beta_2 = -\partial\frac{Jf}{2}$, which would be C^∞-forms on X. This proves that $[\alpha] = 0$ in $H^{p,q}_A(X)$, and hence i is injective.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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