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Research article

The existence of periodic solutions for nonconservative superlinear second order ODEs: a rotation number and spiral analysis approach

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Abstract: We investigate the existence of periodic solutions for nonconservative superlinear second-order differential equations in the sense of rotation numbers. Specifically, we focus on equations whose solutions at infinity behave comparably to a suitable linear system. By employing a rotation number approach, spiral analysis, and fixed-point theorems, we establish the existence of periodic solutions for nonconservative superlinear second-order differential equations. Among the equations we consider, a notable subclass is partially superlinear second-order differential equations, which provide a concrete illustration of our results. Our results extend several recent results, thereby advancing to a more comprehensive understanding of periodic behavior in nonconservative systems.

Keywords: periodic solutions; nonconservative; superlinear; rotation numbers; spiral analysis

1. Introduction

We investigate the existence of periodic solutions within the framework of nonconservative second-order differential equations

$$x'' + f(t, x, x') = 0. \tag{1.1}$$

The function $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is presumed to be a Carathéodory function that is T-periodic in the time variable and locally Lipschitz continuous with respect to (x, x'). The growth of f with respect to its second variable is characterized by superlinear properties, interpreted through rotation numbers.

Researching periodic solutions in nonconservative superlinear second-order differential equations is a complex and challenging area, intertwining mathematical theory with practical applications. As discussed by Gidoni in [1], the question of periodic solutions is closely related to the broader problem regarding the boundedness, or unboundedness, of solutions for (1.1). An important result in this area was established by Struwe [2] and subsequently extended by Capietto et al. [3], as well as Gidoni [1].

Recently, Wang et al. [4] further advanced the study of the existence of periodic solutions in nonconservative equations

$$x'' + f(t,x) + p(t,x,x') = 0. \tag{1.2}$$

By employing topological degree theory, they established a fixed-point theorem with angular descriptions, which serves as a complement to the Poincaré–Birkhoff twist theorem. Their results extend the results of [3], [1], and [2] by relaxing the assumption of global existence of solutions and providing a broader understanding of periodic solutions in nonconservative contexts.

In [4], the function f is required to satisfy the following growth condition:

(A_1) $\lim\limits_{|x|\to+\infty} \dfrac{f(t,x)}{x} = +\infty$ for a.e. $t \in [0,T]$ uniformly.

We note that the superlinear condition (A_1) is required to hold for a.e. $t \in [0,T]$. However, consider the case where

$$f(t,x) = \frac{s(t)}{8}x^3 - \frac{1}{8}x\cos 2\pi t, \tag{1.3}$$

where $s(t)$ is a 1-periodic function defined as

$$s(t) = \begin{cases} \sin 2\pi t, & t \in (0,1/2), \\ 0, & t \in [0,1]\setminus(0,1/2). \end{cases}$$

It is evident that $f(t,x)$ does not satisfy condition (A_1). However,

$$\lim_{|x|\to+\infty} \frac{f(t,x)}{x} = \lim_{|x|\to+\infty} \frac{\sin 2\pi t}{8}x^2 - \frac{1}{8}\cos 2\pi t = +\infty, \tag{1.4}$$

for $t \in (0,1/2)$. This indicates that $f(t,x)$ exhibits superlinear growth only over the subinterval $(0,1/2)$. This leads to a natural question: under these circumstances, can it still guarantee the existence of a periodic solution to Eq (1.2)?

To address this issue, the aim of this paper is to further extend these existing results to the case where superlinearity is considered in the sense of rotation numbers. Specifically, we aim to investigate the existence of periodic solutions for the following generalization of nonconservative superlinear second-order equations (1.2), where we replace the traditional growth conditions with those involving rotation numbers

(A_1') There exist functions $a_n \in L^1([0,T])$, $n \in \mathbb{N}$ such that

$$\liminf_{|x|\to+\infty} \frac{f(t,x)}{x} \geq a_n(t) \quad \text{uniformly a.e. in } t \in [0,T],$$

and $\rho(a_n) \to +\infty$ as $n \to +\infty$.

Here, $\rho(a_n)$ is the rotation number of system $x' = \frac{1}{2}y$ and $-y' = a_n(t)x$, which is defined in Section 2. Papers related to rotation numbers can be found in [5–13], as well as the references cited therein. Additionally, we assume that

(H_1) The function $f : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ is continuous, while $p : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function; both functions are T-periodic with respect to the first variable.

(H_2) There exists a nonempty open bounded set E that contains the origin O, and if $(x(0), x'(0)) \in \partial E$, then the solution $x(\cdot)$ of (1.2) exists on the interval $[0, T]$, with the condition that $(x(t), x'(t)) \neq (0, 0)$ for all $t \in [0, T]$.

(H_3) There exist two positive constants $C_p,\ D_p$, and a positive function $\gamma_p \in L^1([0, T])$ such that $|p(t, x, y)| < \gamma_p(t) + C_p|x| + D_p|y|$, for every $(x, y) \in \mathbb{R}^2$ and a.e. $t \in [0, T]$.

The primary result of this paper is summarized as follows.

Theorem 1.1. *Assume that* (1.2) *satisfies conditions* (H_1), (A_1'), (H_2), *and* (H_3). *Then, Eq* (1.2) *has at least one T-periodic solution.*

Remark 1.1. *In Theorem 1.1, the only distinction lies in condition* (A_1'), *which replaces* (A_1) *from Theorem 1.1 in [4]. Notably, the assumption* (A_1) *implies the hypothesis* (A_1'). *As a result, Theorem 1.1 broadens the scope of the recent findings in [4]. A more detailed discussion is presented in Section 4.*

To prove Theorem 1.1, we apply Theorem 2.1 established by Wang et al. [4], a fixed-point theorem with angular descriptions. The key to applying Theorem 2.1 from [4] lies in identifying a continuous map that is well-defined and establishing the twist condition on an annulus. However, the solutions to Eq (1.2) may not exist globally, and as a result, the corresponding Poincaré map might not be well-defined, which does not meet the requirements of Theorem 2.1 in [4]. To address this issue, we modify the system using the spiral behavior of large-amplitude solutions from the original equation (see Lemma 3.1 below). Additionally, by defining the rotation number of the linear system and estimating the relationship between the rotation numbers of the linear and nonlinear systems, we identify the rapid rotation properties of large-amplitude solutions under the superlinear condition in the sense of rotation numbers (see Lemma 2.3 below). This leads us to find an appropriate annulus that satisfies the twist condition. Other methods for studying second-order ordinary differential equations can be found in [14–16].

The remaining sections of the paper are organized as follows. Section 2 presents the definition and the relationship of the rotation number in planar systems. In Section 3, we analyze the behavior of large-amplitude solutions to system (2.6). In Section 4, we modify system (2.6) and prove the existence of periodic solutions for non-conservative superlinear equations in the context of rotation numbers. Furthermore, we present a partially superlinear example in this section to illustrate the importance of our results.

2. Rotation numbers and properties

Assuming that $(x(t), y(t)) \in \mathbb{R}^2$ does not reach the origin during the interval $[0, t]$, one can convert it to polar coordinates as follows:

$$x(t) = r(t)\cos\theta(t), \quad y(t) = r(t)\sin\theta(t).$$

The t-rotation number of $(x(t), y(t))$ over this interval is defined by

$$\mathrm{Rot}((x(t), y(t)); [0, t]) = \frac{1}{2\pi}(\theta(0) - \theta(t)) = \frac{1}{2\pi}\int_0^t \frac{y(t)x'(t) - x(t)y'(t)}{x(t)^2 + y(t)^2}dt.$$

Here, $\mathrm{Rot}((x(t), y(t)); [0, t])$ refers to the total algebraic number of clockwise rotations made by the trajectory of $(x(t), y(t))$ around the origin during the time interval $[0, t]$.

Next, we consider the planar system

$$\begin{cases} x' = \frac{1}{2}y, \\ -y' = a_n(t)x. \end{cases} \tag{L_a}$$

In this system, the angular function $\theta(t)$ satisfies

$$-\theta'(t) = a_n(t)\cos^2(\theta(t)) + \frac{1}{2}\sin^2(\theta(t)). \tag{2.1}$$

Here, $\theta(t)$ depends solely on the initial time and the angular function $\theta(0) \in \mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$. We denote the t-rotation number of the solution (x, y) of (L_a) over the interval $[0, t]$ as $\mathrm{Rot}^{a_n}(t; v)$, where $v = (1, \theta_0) \in \Gamma_0 = \{\xi = (r, \theta) : r = 1,\ \theta \in \mathbb{R}\}$, θ_0 is the polar angle of $z(0)$ in the polar coordinates. When the interval $[0, t]$ coincides with $[0, T]$, we use the notation $\mathrm{Rot}^{a_n}(v)$ for simplicity.

Additionally, for a given $n \in \mathbb{N}$, the function

$$a_n(t)\cos^2(\theta(t)) + \frac{1}{2}\sin^2(\theta(t))$$

is 2π-periodic in θ and T-periodic in t. Therefore, Eq (2.1) describes a differential equation on a torus. Thus, the rotation number of (2.1)

$$\rho(a_n) = \lim_{t\to+\infty} \frac{\theta_0 - \theta(t)}{t}$$

exists and is independent of θ_0, as stated in Theorem 2.1 of Chapter 2 in Hale [17].

In the following discussion, we define

$$b_n(t) = a_n(t) - C_p - 2D_p^2 - |\gamma_p(t)|, \tag{2.2}$$

where C_p, D_p, and $\gamma_p(t)$ are introduced in (H_3). Using similar arguments, we denote by $\mathrm{Rot}^{b_n}(t; v)$ the t-rotation number of the solution for the system

$$\begin{cases} x' = \frac{1}{2}y, \\ -y' = b_n(t)x, \end{cases} \tag{L_b}$$

over the interval $[0, t]$. When the interval is $[0, T]$, we use the notation $\mathrm{Rot}^{b_n}(v)$ for convenience. We now establish the following relation between the rotation number $\rho(a_n)$ and the t-rotation number $\mathrm{Rot}^{b_n}(v)$.

Lemma 2.1. *If $\rho(a_n) \to +\infty$ as $n \to +\infty$, then both $\mathrm{Rot}^{a_n}(v) \to +\infty$ and $\mathrm{Rot}^{b_n}(v) \to +\infty$ as $n \to +\infty$.*

Proof. The system (L_a) is now written as $Jz' = \nabla L_n(t, z)$ (following the notation in [6]), where

$$L_n(t, z) = \frac{a_n(t)x^2}{2} + \frac{y^2}{4},$$

and J is defined as $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Based on the definitions of $\rho(a_n)$ and $\mathrm{Rot}^{a_n}(v)$, we have

$$\rho(L_n) = \rho(a_n) \quad \text{and} \quad \mathrm{Rot}^{L_n}(T; v) = \mathrm{Rot}^{a_n}(v),$$

where the definitions of $\rho(L_n)$ and $\mathrm{Rot}^{L_n}(T;v)$ can be found in [6]. Applying Lemma 2.2 from [6], it follows that $\rho(a_n) > K_0$ if and only if $\mathrm{Rot}^{a_n}(v) > K_0$ for all $v \in \mathbb{S}^1$ and $K_0 \in \mathbb{Z}^+$. Thus, if $\rho(a_n) \to +\infty$ as $n \to +\infty$, then $\mathrm{Rot}^{a_n}(v) \to +\infty$ as $n \to +\infty$.

Consider the system (L_b) in general polar coordinates

$$x(t) = \frac{r(t)\cos(\varphi(t))}{n}, \quad y(t) = r(t)\sin(\varphi(t)). \tag{2.3}$$

Then, we have

$$\begin{aligned} -\varphi'(t) &= \frac{n(x'y - xy')}{n^2x^2 + y^2} = \frac{n[(a_n(t) - C_p - 2D_p^2 - |\gamma_p(t)|)x^2 + \frac{1}{2}y^2]}{n^2x^2 + y^2} \\ &\geq \frac{n(a_n(t)x^2 + \frac{1}{2}y^2)}{n^2x^2 + y^2} - \frac{(C_p + 2D_p^2 + |\gamma_p(t)|)(n^2x^2 + y^2)}{n(n^2x^2 + y^2)} \\ &= \frac{n(a_n(t)x^2 + \frac{1}{2}y^2)}{n^2x^2 + y^2} - \frac{C_p + 2D_p^2 + |\gamma_p(t)|}{n}. \end{aligned} \tag{2.4}$$

Note that the generalized polar coordinate (2.3) essentially represents a form of elliptic coordinates. Consequently, according to the definition $\mathrm{Rot}^{a_n}(v)$, we obtain

$$\mathrm{Rot}^{a_n}(v) = \frac{1}{2\pi}\int_T^0 \bar{\varphi}'(t)dt = \frac{1}{2\pi}\int_0^T \frac{n(a_n(t)x^2 + \frac{1}{2}y^2)}{n^2x^2 + y^2}dt \to +\infty \ (n \to +\infty) \tag{2.5}$$

where $\bar{\varphi}(t)$ denotes the argument function of (x, y) with respect to system (L_a) in the generalized polar coordinates (2.3). Combining (2.4) with (2.5), we have

$$\begin{aligned} \mathrm{Rot}^{b_n}(v) &= \frac{1}{2\pi}[\varphi(0) - \varphi(T)] \\ &\geq \frac{1}{2\pi}\int_0^T \frac{n(a_n(t)x^2 + \frac{1}{2}y^2)}{n^2x^2 + y^2}dt - \frac{(C_p + 2D_p^2)T}{2n\pi} - \frac{1}{2n\pi}\int_0^T |\gamma_p(t)|dt, \end{aligned}$$

which implies that

$$\mathrm{Rot}^{b_n}(v) \to +\infty \quad \text{as } n \to \infty.$$

We can express Eq (1.2) as the planar system

$$\begin{cases} x' = y, \\ -y' = f(t, x) + p(t, x, y). \end{cases} \tag{2.6}$$

Let $z(t; z_0) = (x(t; x_0), y(t; y_0))$ denote the solution to (2.6), with $z(0; z_0) = z_0$. For brevity, we write $\mathrm{Rot}^f(t; z) = \mathrm{Rot}(z(t); [0, t])$. When the interval is $[0, T]$, we use the notation $\mathrm{Rot}^f(z)$ for convenience. We now examine the relationship between the t-rotation number of the solution to system (2.6) and that of the linear system (L_b).

Lemma 2.2. *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function, T-periodic in the first variable, such that*

$$\liminf_{|x|\to+\infty} \frac{f(t, x)}{x} \geqslant a_n(t) \quad \textit{uniformly a.e. in } t \in [0, T],$$

for a certain $n \in \mathbb{N}$. Then, for every $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that for any solution of (2.6) *with $|z(t)| \geqslant R_\varepsilon$ for all $t \in [0, T]$, it holds that*

$$\mathrm{Rot}^f(t; z) \geqslant \mathrm{Rot}^{b_n}(t; v) - \varepsilon, \quad \textit{for all } t \in [0, T], \ \textit{with } v = (1, \theta_0). \tag{2.7}$$

Proof. Let $\varepsilon > 0$ be fixed. In Lemma 2.2 of [6], we define

$$L_1(t,z) = \frac{b_n(t)}{2}x^2 + \frac{y^2}{4}, \quad L_2(t,z) = \frac{b_n(t)-\delta}{2}x^2 + \frac{y^2}{4} \quad \text{and} \quad V(z) = x^2 + y^2,$$

where δ is a sufficiently small positive number. Then, it follows that

$$\mathrm{Rot}^{L_1}(t;v) - \mathrm{Rot}^{L_2}(t;v) \le \frac{\varepsilon}{2}, \quad \forall\, t \in [0,T],\ v \in \Gamma_0.$$

Let $\mathrm{Rot}^{b_n-\delta}(t;v)$ denote the t-rotation number of the solution to the system

$$\begin{cases} x' = \frac{1}{2}y, \\ -y' = (b_n(t)-\delta)x. \end{cases}$$

From the definitions of $\mathrm{Rot}^{b_n}(t;v)$ and $\mathrm{Rot}^{b_n-\delta}(t;v)$, we have

$$\mathrm{Rot}^{b_n}(t;v) = \mathrm{Rot}^{L_1}(t;v)$$

and

$$\mathrm{Rot}^{b_n-\delta}(t;v) = \mathrm{Rot}^{L_2}(t;v)$$

for all $t \in [0,T]$ and $v \in \Gamma_0$. Thus, it follows that

$$\mathrm{Rot}^{b_n}(t;v) - \mathrm{Rot}^{b_n-\delta}(t;v) \le \frac{\varepsilon}{2}, \quad \forall\, t \in [0,T],\ v \in \Gamma_0. \tag{2.8}$$

Assume, for contradiction, that the assertion of (2.7) does not hold. This implies that for every $m \in \mathbb{N}$, there exists a solution $z_m(t)$ of (2.6) defined on $[0,T]$ with $|z_m(t)| \geqslant m$ for all $t \in [0,T]$ such that, for some $t_m \in [0,T]$,

$$\mathrm{Rot}^f(t_m;z_m) < \mathrm{Rot}^{b_n}(t_m;v_m) - \varepsilon, \tag{2.9}$$

where $v_m = (1,\alpha_m)$ and α_m is the polar angle of $z_m(0)$ in the polar coordinates. Without loss of generality, assume $t_m \to \tau \in [0,T]$ and $v_m \to \bar{v} = (1,\alpha) \in \Gamma_0$ as $m \to \infty$ with $\alpha_m \to \alpha$. Note that $\mathrm{Rot}^{b_n}(\cdot;\cdot)$ is continuous on $[0,T] \times \Gamma_0$. Taking the upper limit as $m \to \infty$ on both sides of (2.9), we obtain

$$\limsup_{m\to\infty} \mathrm{Rot}^f(t_m;z_m) \le \mathrm{Rot}^{b_n}(\tau;\bar{v}) - \varepsilon. \tag{2.10}$$

By (A_1') and the continuity of f, there exists $l := l_\delta \in L^1([0,T],\mathbb{R}_+)$ such that

$$f(t,x)x \geqslant (a_n(t)-\delta)x^2 - l(t), \quad \forall\, x \in \mathbb{R} \text{ and a.e. } t \in [0,T]. \tag{2.11}$$

Consider that $(r_m(t),\theta_m(t))$ represents the polar coordinates of $z_m(t) = (x_m(t),y_m(t))$ with $r_m(t) \geqslant m$. From (H_3), (2.6), and (2.11), we have

$$\begin{aligned} -\theta_m'(t) &= \frac{y_m x_m' - x_m y_m'}{x_m^2 + y_m^2} \\ &= \frac{y_m^2 + x_m\left(f\left(t,x_m\right) + p\left(t,x_m,y_m\right)\right)}{x_m^2 + y_m^2} \\ &\geqslant \frac{y_m^2 + x_m f\left(t,x_m\right) - \left|p\left(t,x_m,y_m\right)\right|\left|x_m\right|}{x_m^2 + y_m^2} \\ &\geqslant \frac{y_m^2 + \left(a_n(t)-\delta\right)x_m^2 - l(t) - \gamma_p(t)\left|x_m\right| - C_p x_m^2 - D_p\left|x_m\right|\left|y_m\right|}{x_m^2 + y_m^2}, \end{aligned}$$

which holds for a.e. $t\in[0,T]$. Note that $2D_p^2x_m^2 - D_p\,|x_m|\,|y_m| + \frac{1}{2}y_m^2 > 0$, so

$$-\theta_m'(t) \geqslant \frac{\left(a_n(t) - C_p - 2D_p^2 - \delta\right)x_m^2 + \frac{1}{2}y_m^2 - \gamma_p(t)\,|x_m| - l(t)}{x_m^2 + y_m^2},$$

which holds for a.e. $t\in[0,T]$. Further, we have $\gamma_p(t)(x_m^2 - |x_m| + 1) > 0$, which implies that

$$-\gamma_p(t)\,|x_m| > -\gamma_p(t)x_m^2 - \gamma_p(t), \quad \text{for a.e. } t\in[0,T].$$

Therefore, we obtain

$$\begin{aligned}-\theta_m'(t) &\geqslant \frac{\left(a_n(t) - C_p - 2D_p^2 - \gamma_p(t) - \delta\right)x_m^2 + \frac{1}{2}y_m^2 - \gamma_p(t) - l(t)}{x_m^2 + y_m^2}\\ &\geqslant (b_n(t) - \delta)\cos^2(\theta_m(t)) + \frac{1}{2}\sin^2(\theta_m(t)) - \frac{\gamma_p(t) + l(t)}{m}\end{aligned}$$

for $x_m \in \mathbb{R}$ and a.e. $t\in[0,T]$, where $b_n(t)$ is introduced in (2.2).

Based on a result concerning differential inequalities [18], we can conclude that

$$\mathrm{Rot}^f(t_m; z_m) = \frac{\theta_m(0) - \theta_m(t_m)}{2\pi} = \frac{\alpha_m - \theta_m(t_m)}{2\pi} \geqslant \frac{\alpha_m - \vartheta_m(t_m)}{2\pi}, \tag{2.12}$$

where $\vartheta_m(t_m)$ is the solution of

$$\begin{cases}-\theta'(t) = (b_n(t) - \delta)\cos^2(\theta(t)) + \frac{1}{2}\sin^2(\theta(t)) - \frac{\gamma_p(t) + l(t)}{m},\\ \theta(0) = \alpha_m.\end{cases}$$

By applying the principle of continuous dependence on initial conditions, it can be shown that as $m \to \infty$, the function $\vartheta_m(t)$ converges uniformly to $\bar{\theta}(t)$ on the interval $[0,T]$, where $\bar{\theta}(t)$ is the solution to the differential equation

$$\begin{cases}-\theta'(t) = (b_n(t) - \delta)\cos^2(\theta(t)) + \frac{1}{2}\sin^2(\theta(t))\\ \theta(0) = \alpha.\end{cases}$$

In particular, the uniform convergence implies that $\vartheta_m(t_m) \to \bar{\theta}(\tau)$ $(m \to \infty)$. Consider the expression

$$\begin{aligned}\mathrm{Rot}^{b_n-\delta}(\tau; \bar{v}) &= \frac{1}{2\pi}\int_0^\tau \frac{(b_n(t) - \delta)x^2 + \frac{1}{2}y^2}{x^2 + y^2}dt\\ &= -\frac{1}{2\pi}\int_0^\tau (\bar{\theta}(t))'dt = \frac{\bar{\theta}(0) - \bar{\theta}(\tau)}{2\pi}.\end{aligned}$$

Thus, from (2.12), it follows that

$$\liminf_{m\to\infty} \mathrm{Rot}^f(t_m; z_m) \geqslant \lim_{m\to\infty} \frac{\alpha_m - \vartheta_m(t_m)}{2\pi} = \frac{\alpha - \bar{\theta}(\tau)}{2\pi} = \mathrm{Rot}^{b_n-\delta}(\tau; \bar{v}),$$

which combined with (2.10) gives

$$\mathrm{Rot}^{b_n}(\tau; \bar{v}) - \mathrm{Rot}^{b_n-\delta}(\tau; \bar{v}) \geqslant \varepsilon.$$

This result contradicts the inequality (2.8), thereby confirming the validity of (2.7).

Based on Lemma 2.2, we can obtain the rapid rotational property of large amplitude solutions for the superlinear equations in the sense of rotation number.

Lemma 2.3. *Let* (H_1), (A_1'), *and* (H_3) *be satisfied. Then, for any integer* j, *there exists a sufficiently large radius* R_j *such that every solution* $z(t)$ *of* (2.6) *with* $|z(t)| \geq R_j$ *for a.e.* $t \in [0, T]$ *satisfies the condition* $\mathrm{Rot}^f(z) > j$.

Proof. According to (A_1') and Lemma 2.1, for any integer j, there exists a sufficiently large integer N such that

$$\mathrm{Rot}^{b_n}(v) > j, \quad \text{for } n > N.$$

We can select an appropriate $\varepsilon > 0$ such that

$$\mathrm{Rot}^{b_n}(v) - \varepsilon > j, \quad \text{for } n > N.$$

By Lemma 2.2, for this choice of $\varepsilon > 0$, there exists a sufficiently large R_j such that, for every solution $z(t)$ to system (2.6) with $|z(t)| \geqslant R_j$ for a.e. $t \in [0, T]$, it holds that

$$\mathrm{Rot}^f(z) \geqslant \mathrm{Rot}^{b_n}(v) - \varepsilon > j.$$

This completes the proof.

3. Spiral analysis for large-amplitude solutions

In this section, we primarily analyze the behavior of large-amplitude solutions to system (2.6). The following lemma establishes that solutions of this system exhibit spiral behavior when the amplitude is large. By utilizing these spiral properties, we can estimate the relationship between changes in the polar radius of the solution and variations in the rotational angle. Additionally, this allows us to determine an appropriate bound for adjusting our system. For simplicity, let $(r(t), \theta(t))$ represent the polar coordinates of $z(t; z_0)$, with $(r(0), \theta(0)) = (r_0, \theta_0)$.

Lemma 3.1. *Assume the conditions* (H_1), (A_1'), *and* (H_3) *are satisfied. For any positive integer* N_0 *and sufficiently large* r_*, *there exist two strictly increasing functions* $\xi^{\pm}_{N_0}(r) : [r_*, +\infty) \to \mathbb{R}$ *such that*

$$\xi^{\pm}_{N_0}(r) \to +\infty \quad \Leftrightarrow \quad r \to +\infty.$$

If a solution $z(t)$ *of system* (2.6) *satisfies* $r_0 \geqslant r_*$, *then either*

$$\xi^{-}_{N_0}(r_0) \leq r(t) \leq \xi^{+}_{N_0}(r_0), \quad \textit{for } t \in [0, T];$$

or there exists a time $\hat{t}_{N_0} \in (0, T)$ *such that*

$$\theta(\hat{t}_{N_0}) - \theta(0) = -2N_0\pi,$$

and

$$\xi^{-}_{N_0}(r_0) \leq r(t) \leq \xi^{+}_{N_0}(r_0), \quad \textit{for } t \in [0, \hat{t}_{N_0}].$$

Proof. We partition $\mathbb{R}^2$ into four regions as follows:

$$\mathcal{D}_1 = \{(x,y)|x \geqslant 0, y > 0\}; \qquad \mathcal{D}_2 = \{(x,y)|x > 0, y \leq 0\};$$
$$\mathcal{D}_3 = \{(x,y)|x \leq 0, y < 0\}; \qquad \mathcal{D}_4 = \{(x,y)|x < 0, y \geqslant 0\}.$$

Consider the function

$$S(x,y) = \frac{x^2}{2} + \frac{y^2}{2}. \tag{3.1}$$

According to condition (A_1'), for any fixed $n \in \mathbb{N}$, there exist $\varepsilon_0 \leq 1$ and M_{ε_0} such that

$$f(t,x) \geqslant (a_n(t) - \varepsilon_0)x, \quad \text{for } x > M_{\varepsilon_0} \text{ and a.e. } t \in [0,T],$$

and

$$f(t,x) \leqslant (a_n(t) - \varepsilon_0)x, \quad \text{for } x < -M_{\varepsilon_0} \text{ and a.e. } t \in [0,T].$$

Thus, we have

$$yf(t,x) \geqslant (a_n(t) - \varepsilon_0)xy, \quad \text{for } |x| > M_{\varepsilon_0} \text{ and } (x,y) \in \mathcal{D}_1 \cup \mathcal{D}_3. \tag{3.2}$$

Define $\bar{K} = \max\{|f(t,x)| : t \in [0,T], |x| \leq M_{\varepsilon_0}\}$. For sufficiently large r, if $|x| \leq M_{\varepsilon_0}$, then $|y|$ is sufficiently large and $|y| \leq y^2$.

Consider first the case where $(x,y) \in \mathcal{D}_1 \cup \mathcal{D}_3$. When $|x| \leq M_{\varepsilon_0}$, using (2.6) and (3.1), we obtain

$$\begin{aligned}
\frac{dS(x,y)}{dt} &= xx' + yy' = xy - y(f(t,x) + p(t,x,y)) \\
&\leq xy + |y|(|f(t,x)| + |p(t,x,y)|) \\
&\leq xy + |y|\left(\bar{K} + \gamma_p(t) + C_p|x| + D_p|y|\right) \\
&\leq \left(1 + C_p\right)xy + \left(\bar{K} + \gamma_p(t) + D_p\right)y^2 \\
&\leq 2(\bar{K} + \gamma_p(t) + D_p + 1 + C_p)S(x,y).
\end{aligned} \tag{3.3}$$

For the case where $|x| > M_{\varepsilon_0}$, by (2.6), (3.1), and (3.2), we have

$$\begin{aligned}
\frac{dS(x,y)}{dt} &= xx' + yy' = xy - yf(t,x) - yp(t,x,y) \\
&\leq xy - (a_n(t) - \varepsilon_0)xy + |y||p(t,x,y)| \\
&\leq (1 - a_n(t) + \varepsilon_0)\,xy + |y|\left(\gamma_p(t) + C_p|x| + D_p|y|\right).
\end{aligned} \tag{3.4}$$

For sufficiently large r, if $|x| > M_{\varepsilon_0}$, then either $|y| \leq M_{\varepsilon_0} < x^2$ or $|y| > M_{\varepsilon_0}$. If $|y| < x^2$, then applying (3.4), we obtain

$$\begin{aligned}
\frac{dS(x,y)}{dt} &\leq \left(2 + |a_n(t)| + C_p\right)xy + \gamma_p(t)x^2 + D_py^2 \\
&\leq 2\left(\gamma_p(t) + D_p + 2 + |a_n(t)| + C_p\right)S(x,y).
\end{aligned} \tag{3.5}$$

If $|y| > M_{\varepsilon_0}$, then $|y| \le y^2$. By (3.4), we have

$$\begin{aligned}\frac{dS(x,y)}{dt} &\le \left(2 + |a_n(t)| + C_p\right)xy + \left(\gamma_p(t) + D_p\right)y^2 \\ &\le 2\left(\gamma_p(t) + D_p + 2 + |a_n(t)| + C_p\right)S(x,y). \end{aligned} \tag{3.6}$$

By combining (3.3), (3.5), and (3.6), we deduce the existence of a positive function $c_1(t) \in L^1([0,T])$ such that

$$\frac{dS(x,y)}{dt} \le c_1(t)S(x,y), \tag{3.7}$$

where $c_1(t) = 2\left(\gamma_p(t) + D_p + 2 + |a_n(t)| + \bar{K} + C_p\right)$.

Next, consider the case when $(x,y) \in \mathcal{D}_2 \cup \mathcal{D}_4$. Here, for $|x| \le M_{\varepsilon_0}$, we obtain

$$\begin{aligned}\frac{dS(x,y)}{dt} &= xx' + yy' = xy - y(f(t,x) + p(t,x,y)) \\ &\geqslant -\Big(|xy| + |y|(|f(t,x)| + |p(t,x,y)|)\Big) \\ &\geqslant -\Big(|xy| + |y|\left(\bar{K} + \gamma_p(t) + C_p|x| + D_p|y|\right)\Big).\end{aligned}$$

For $|x| > M_{\varepsilon_0}$, we have

$$\begin{aligned}\frac{dS(x,y)}{dt} &= xx' + yy' = xy - y(f(t,x) + p(t,x,y)) \\ &\geqslant xy - (a_n(t) - \varepsilon_0)xy - yp(t,x,y) \\ &\geqslant -\Big((2 + |a_n(t)|)|xy| + |y|(\gamma_p(t) + C_p|x| + D_p|y|)\Big).\end{aligned}$$

By similar arguments as for $(x,y) \in \mathcal{D}_1 \cup \mathcal{D}_3$, we find that

$$\frac{dS(x,y)}{dt} \geqslant -c_1(t)S(x,y).$$

Consider now the function

$$T(x,y) = \frac{y^2}{2} + \tilde{F}_+(x) + \frac{x^2}{2},$$

where $\tilde{F}_+(x) = \int_0^x \tilde{f}_+(s)ds$ and $\tilde{f}_+(x) = \operatorname{sgn} x \max_{0\le t\le T}\{|f(t,x)|\}$. As $\tilde{F}_+(x) \to +\infty$ when $|x| \to +\infty$, it follows that

$$T(x,y) \to +\infty \iff x^2 + y^2 \to +\infty.$$

For sufficiently large r, if $0 < |y| < 1$, then $|x|$ is large enough, and $|y| \le x^2$. If $|y| \geqslant 1$, then $|y| \le y^2$.

For $(x,y) \in \mathcal{D}_1 \cup \mathcal{D}_3$, we can obtain

$$\begin{aligned}\frac{dT(x,y)}{dt} &= yy' + \tilde{f}_+(x)y + xx' \\ &= \tilde{f}_+(x)y - y(f(t,x) + p(t,x,y)) + xy \\ &\geqslant y(\tilde{f}_+(x) - f(t,x)) - |y|\,|\,p(t,x,y)| - |xy|\end{aligned}$$

$$\geqslant -|y|\left(\gamma_p(t) + C_p|x| + D_p|y|\right) - |xy|$$
$$\geqslant -\left(\gamma_p(t)|y| + (C_p+1)|xy| + D_p y^2\right). \tag{3.8}$$

When $0 < |y| < 1$, applying (3.8) gives us

$$\frac{dT(x,y)}{dt} \geqslant -\left(\gamma_p(t)x^2 + \frac{C_p+1}{2}\left(x^2+y^2\right) + D_p y^2\right)$$
$$\geqslant -2\left(\gamma_p(t) + C_p + 1 + D_p\right)T(x,y).$$

If $|y| \geqslant 1$, applying (3.8) leads to

$$\frac{dT(x,y))}{dt} \geqslant -\left(\gamma_p(t)y^2 + \frac{C_p+1}{2}\left(x^2+y^2\right) + D_p y^2\right)$$
$$\geqslant -2\left(\gamma_p(t) + C_p + 1 + D_p\right)T(x,y).$$

Thus, we find that there exists a positive function $c_2(t) \in L^1([0,T])$ such that

$$\frac{dT(x,y)}{dt} \geqslant -c_2(t)T(x,y),$$

where $c_2(t) = 2(\gamma_p(t) + C_p + 1 + D_p)$.

For $(x,y) \in \mathcal{D}_2 \cup \mathcal{D}_4$, we have

$$\begin{aligned}\frac{dT(x,y)}{dt} &= yy' + \tilde{f}_+(x)y + xx' \\ &= \tilde{f}_+(x)y - y(f(t,x) + p(t,x,y)) + xy \\ &\leq y(\tilde{f}_+(x) - f(t,x)) + |y||p(t,x,y)| + |xy| \\ &\leq \gamma_p(t)|y| + \left(C_p+1\right)|xy| + D_p|y|^2.\end{aligned}$$

By similar arguments as for $(x,y) \in \mathcal{D}_1 \cup \mathcal{D}_3$, we obtain

$$\frac{dT(x,y)}{dt} \leqslant c_2(t)T(x,y). \tag{3.9}$$

In [6, Lemma 4.1], with $\varphi(x) = 0$ and $\frac{\partial H}{\partial y}(t,x,y) = y$, the functions are defined as follows:

$$V(x,y) = \begin{cases} S(x,y), & \text{for } (x,y) \in \mathcal{D}_1 \cup \mathcal{D}_3, \\ T(x,y), & \text{for } (x,y) \in \mathcal{D}_2 \cup \mathcal{D}_4, \end{cases}$$

and

$$U(x,y) = \begin{cases} T(x,y), & \text{for } (x,y) \in \mathcal{D}_1 \cup \mathcal{D}_3, \\ S(x,y), & \text{for } (x,y) \in \mathcal{D}_2 \cup \mathcal{D}_4. \end{cases}$$

By (3.7) and (3.9), we obtain

$$\frac{dV(x,y)}{dt} \leqslant c_1(t)V(x,y), \quad \text{for } (x,y) \in \mathcal{D}_1 \cup \mathcal{D}_3$$

and

$$\frac{dV(x,y)}{dt} \leqslant c_2(t)V(x,y), \quad \text{for } (x,y) \in \mathcal{D}_2 \cup \mathcal{D}_4.$$

It is evident that $V(x,y) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^+$ is piecewise continuously differentiable. Additionally, $V(x,y) \to +\infty$ if and only if $x^2 + y^2 \to +\infty$, and $V(x,\varphi(x))$ is monotonic with respect to x for $(x,y) \in \mathcal{D}_i$, where $i = 1,2,3,4$. The function $U(x,y)$ exhibits similar properties to $V(x,y)$. According to Lemmas 4.1 and 3.3, along with the definitions of the upper and lower spiral functions $\xi^{\pm}(\cdot)$ as described in [6], we can conclude the existence of two strictly increasing functions $\xi^{\pm}_{N_0}(r)$ for system (2.6). This concludes the proof.

Remark 3.1. *Although Lemmas 4.1 and 3.3 in [6] emphasize the characteristics of solutions in conservative Hamiltonian systems, these findings are also relevant to non-conservative systems. The demonstration of Lemma 3.1 uses the results obtained for nonconservative systems.*

4. Modified planar systems and the existence of periodic solutions

Next, we will use Theorem 2.1 from Wang et al. [4], which provides a fixed-point theorem with angular descriptions, to prove Theorem 1.1. It is evident that solutions to system (2.6) may not be globally defined, which implies that its corresponding Poincaré map might not be well-defined. Theorem 2.1 in [4] requires the mapping to be continuous, so we cannot directly apply it to (2.6).

To address this issue, we will modify system (2.6) by utilizing the spiral properties of large-amplitude solutions established in Lemma 3.1. This modification guarantees the global existence of solutions, allowing us to apply Theorem 2.1 from [4] to establish the existence of periodic solutions. Furthermore, we will use the angular description results from Theorem 2.1 in [4] to demonstrate that the obtained periodic solutions are in fact solutions to the original system (2.6). We modify system (2.6) as follows:

$$\begin{cases} x' = y, \\ -y' = \lambda(r^2)(f(t,x) + p(t,x,y)) + (1 - \lambda(r^2))x, \end{cases} \tag{4.1}$$

where $\lambda(r^2) = \lambda(x^2 + y^2) \in C^\infty(\mathbb{R}_+, \mathbb{R})$ is a defined truncated function given as

$$\lambda(r^2) = \begin{cases} 1, & r \leq r_1, \\ \text{smooth connection}, & r_1 < r < r_2, \\ 0, & r \geqslant r_2. \end{cases}$$

The values of r_1 and r_2 will be determined based on the spiral properties as outlined in Lemma 3.1; for further details, refer to the proof of Theorem 1.1. Let $\tilde{z}(t;\tilde{z}_0)$ denote the solution to equation (4.1) with the initial condition $\tilde{z}(0;\tilde{z}_0) = \tilde{z}_0$. The T-rotation number of $\tilde{z}(t;\tilde{z}_0)$ is denoted by $\mathrm{Rot}^{f_\lambda}(\tilde{z})$. We use $(\tilde{r}(t), \tilde{\theta}(t))$ to represent the polar coordinates of $\tilde{z}(t;\tilde{z}_0)$. The following properties can be easily verified:

(i) The initial value problem for (4.1) has a unique solution.

(ii) Every solution $\tilde{z}(t;\tilde{z}_0)$ is defined for all $t \in \mathbb{R}$. Moreover, if $\tilde{z}_0 \neq (0,0)$, then $\tilde{z}(t;\tilde{z}_0) \neq (0,0)$ for every $t \in \mathbb{R}$.

Next, consider the Poincaré map defined by

$$\Phi : \mathbb{R}^2 \to \mathbb{R}^2, \quad \tilde{z}_0 \mapsto \tilde{z}(T;\tilde{z}_0).$$

Thus, the Poincaré map Φ associated with system (4.1) is well-defined and continuous.

Next, we will apply the fixed-point theorem with angular descriptions recently established in [4] to prove Theorem 1.1 of this paper.

Proof of Theorem 1.1. We will divide the proof into four steps.

Step 1. Under condition (H_2), if $z_0 \in \partial E$, then the solution $z(t; z_0)$ to (2.6) exists on the interval $t \in [0, T]$ and $z(t; z_0) \neq (0, 0)$ for all $t \in [0, T]$. This demonstrates the so-called "elastic" property of the solutions, which indicates that

$$r^- \leq |z(t)| \leq r^+, \quad \text{for } t \in [0, T] \text{ and } z_0 \in \partial E,$$

where $r^\pm$ are positive real numbers. Owing to the uniqueness of solutions for the corresponding Cauchy problems, the continuous dependence of the solutions of (2.6) on the initial conditions is guaranteed. As a result, $\mathrm{Rot}^f(z)$ is a continuous function. Thus, there exists an integer m such that

$$\mathrm{Rot}^f(z) < m, \quad \text{for } z_0 \in \partial E,$$

which leads to

$$\theta(T) - \theta(0) > -2m\pi, \quad \text{for } z_0 \in \partial E. \tag{4.2}$$

Step 2. For any integer $j \geqslant m + 1$, according to Lemma 2.3, there exists $R_j > r^+$ such that every solution $z(t)$ of (2.6) with $|z(t)| \geqslant R_j$ for a.e. $t \in [0, T]$ satisfies

$$\mathrm{Rot}^f(z) > j,$$

which leads to

$$\theta(T) - \theta(0) < -2j\pi. \tag{4.3}$$

Let us choose

$$r_1 = \xi_j^+(R_\infty), \quad R_\infty \geqslant (\xi_j^-)^{-1}(R_j) \quad \text{and} \quad r_2 > r_1,$$

where $\xi_j^\pm(\cdot)$ are functions defined in Lemma 3.1. According to (4.1), if $r \leq r_1$, the system (4.1) is equivalent to (2.6). Therefore, using (4.2), we can deduce that

$$\tilde{\theta}(T) - \tilde{\theta}(0) > -2m\pi, \quad \text{for } \tilde{z}_0 \in \partial E. \tag{4.4}$$

Next, consider the solution of (4.1) starting from $\tilde{z}_0 \in \Gamma =: \{(x, y) : r = R_\infty\}$. If $R_j \leq |\tilde{z}(t)| \leq r_1$ for a.e. $t \in [0, T]$, then (4.1) is equivalent to (2.6). In this case, applying (4.3) yields

$$\tilde{\theta}(T) - \tilde{\theta}(0) < -2j\pi < -2m\pi, \quad \text{for } \tilde{z}_0 \in \Gamma. \tag{4.5}$$

If there exists $t_1 \in (0, T)$ such that $|\tilde{z}(t_1)| > r_1$, then there exists a time $t_1' \in (0, t_1)$ such that $|\tilde{z}(t_1')| = r_1$ and $|\tilde{z}(t)| < r_1$ for a.e. $t \in [0, t_1')$. By Lemma 3.1, we have

$$\tilde{\theta}(t_1') - \tilde{\theta}(0) = \theta(t_1') - \theta(0) = -2j\pi. \tag{4.6}$$

Alternatively, from (4.1), we observe that $x'y = y^2 > 0$ when $x = 0$ and $y \neq 0$. Thus, a nonzero solution of (4.1) performs clockwise rotations around the y-axis. If the solution transitions from the positive (negative) y-axis to the negative (positive) y-axis, the angular function $\tilde{\theta}(t)$ increases by $-\pi$. On the other hand, if the trajectory oscillates within the left (right) half-plane, the angle function $\tilde{\theta}(t)$ increases by less than π. Consequently,

$$\tilde{\theta}(t) - \tilde{\theta}(t_1') < \pi, \quad \text{for all } t \in (t_1', T]. \tag{4.7}$$

Combining (4.7) with (4.6), we obtain

$$\begin{aligned} \tilde{\theta}(T) - \tilde{\theta}(0) &= \tilde{\theta}(T) - \tilde{\theta}(t_1') + \tilde{\theta}(t_1') - \tilde{\theta}(0) \\ &< \pi + (-2j\pi) < -2m\pi, \quad \text{for } \tilde{z}_0 \in \Gamma. \end{aligned} \tag{4.8}$$

Finally, if there exists a time $t_2 \in (0, T)$ such that $|\tilde{z}(t_2)| < R_j$, the validity of (4.8) can be proven using similar arguments as those presented above.

Step 3. By applying Theorem 2.1 in [4], we establish that there exists at least one fixed point z_0^* for Φ. Consequently, system (4.1) has at least one T-periodic solution $\tilde{z}(t)$ that starts from z_0^*.

Step 4. We will further show that $\tilde{z}(t)$ remains in the circle $r = r_1$, which implies that $\tilde{z}(t)$ is in fact a T-periodic solution of system (2.6). If $z_0^* \in E$, the choice of r_1 ensures that $|\tilde{z}(t)| \le r_1$. If $z_0^* \in \overline{I(\Gamma)} \setminus E$, then by Theorem 2.1 in [4], we have

$$\tilde{\theta}(T) - \tilde{\theta}(0) = -2k\pi, \quad \text{for } k \le m. \tag{4.9}$$

Assume there exists $t_0' \in (0, T]$ such that $|\tilde{z}(t_0')| > r_1$. Using an argument similar to that in (4.6), we can find $t_0'' \in (0, t_0']$ such that

$$\tilde{\theta}(t_0'') - \tilde{\theta}(0) = -2j\pi.$$

Furthermore, using an argument analogous to (4.8), we obtain

$$\begin{aligned} \tilde{\theta}(T) - \tilde{\theta}(0) &= \tilde{\theta}(T) - \tilde{\theta}(t_0'') + \tilde{\theta}(t_0'') - \tilde{\theta}(0) \\ &< \pi + (-2j\pi) < -2m\pi, \end{aligned}$$

which contradicts (4.9). Therefore, system (2.6) has at least one T-periodic solution, and the same conclusion applies to Eq (1.2). This completes the proof.

To illustrate the applicability of Theorem 1.1, we will discuss the problem mentioned in the introduction, which satisfies the superlinear condition only on a partial interval, thereby concretely demonstrating the theorem's utility.

Example 4.1. *Consider the non-conservative superlinear equation*

$$x'' + f(t, x) + \frac{1}{8}x' - \frac{1}{10}\cos 2\pi t = 0, \tag{4.10}$$

where $f(t, x)$ is given by (1.3). *We can demonstrate that* (4.10) *has at least one* 1*-periodic solution.*

Proof. To demonstrate this result within the context of Theorem 1.1, we consider

$$p(t, x, x') = \frac{x'}{8} - \frac{1}{10}\cos 2\pi t.$$

The conditions (H_1) and (H_3) can be easily verified, so we only need to verify (A_1') and (H_2).

Let us first verify condition (A_1'). From (1.3), we have

$$\frac{f(t,x)}{x} \geqslant -\frac{1}{8}\cos 2\pi t, \quad \text{for } t \in [0,1].$$

Therefore, for any fixed $n \in \mathbb{N}$, it holds that

$$\lim_{|x|\to+\infty} \frac{f(t,x)}{x} \geqslant a_n(t), \quad \text{for a.e. } t \in [0,1] \text{ uniformly,}$$

where

$$a_n(t) = \begin{cases} n^2, & t \in (0,1/2); \\ -\dfrac{1}{8}\cos 2\pi t, & t \in [0,1]\setminus(0,1/2). \end{cases}$$

Next, we prove that $\rho(a_n) \to +\infty$ as $n \to +\infty$. We start by using generalized polar coordinates

$$x(t) = \frac{r(t)}{n}\cos\psi(t), \quad y(t) = \sqrt{2}r(t)\sin\psi(t) \tag{4.11}$$

and consider the following system

$$\begin{cases} x' = \frac{1}{2}y, \\ -y' = a_n(t)x, \end{cases}$$

which leads to

$$-\psi'(t) = \frac{\sqrt{2}n(x'y - xy')}{2(n^2x^2 + \frac{1}{2}y^2)} = \frac{\sqrt{2}n(a_n(t)x^2 + \frac{1}{2}y^2)}{2(n^2x^2 + \frac{1}{2}y^2)}.$$

Thus, $-\psi'(t) = \sqrt{2}n/2$ for $t \in (0,1/2)$, and

$$\begin{aligned} -\psi'(t) &= \frac{\sqrt{2}n(-\frac{1}{8}x^2\cos 2\pi t + \frac{1}{2}y^2)}{2(n^2x^2 + \frac{1}{2}y^2)} \geqslant -\frac{\frac{1}{8}\sqrt{2}nx^2}{2(n^2x^2 + \frac{1}{2}y^2)} \\ &\geqslant -\frac{\frac{1}{8}\sqrt{2}(n^2x^2 + \frac{1}{2}y^2)}{2n(n^2x^2 + \frac{1}{2}y^2)} = -\frac{\sqrt{2}}{16n}, \end{aligned} \tag{4.12}$$

for $t \in [0,1]\setminus(0,1/2)$. Note that the generalized polar coordinates in (4.11) effectively correspond to a variant of elliptic coordinates. Consequently, using the definition of $\mathrm{Rot}^{a_n}(v)$, we can deduce that

$$\mathrm{Rot}^{a_n}(v) = -\frac{1}{2\pi}\int_0^1 \psi'(t)dt \geqslant \frac{1}{2\pi}\left(\frac{\sqrt{2}}{4}n - \frac{\sqrt{2}}{16n}\right) \to +\infty \ (n \to +\infty).$$

Using a method similar to that in the proof of Lemma 2.1, it can be shown that $\rho(a_n) \to +\infty$ as $n \to +\infty$.

Next, we verify condition (H_2). The Eq (4.10) can be rewritten as

$$\begin{cases} x' = y, \\ -y' = f(t,x) + \dfrac{y}{8} - \dfrac{1}{10}\cos 2\pi t, \end{cases} \tag{4.13}$$

If $(x, y) \neq (0, 0)$ for $t \in [0, 1]$, it can be represented in polar coordinates as

$$x = r\cos\theta, \quad y = r\sin\theta.$$

For sufficiently small values of $r \leq 1$, the following result can be obtained using (4.13) and (1.3):

$$\begin{aligned}|r'| = \left|\frac{xx' + yy'}{r}\right| &\leq \frac{\frac{9}{8}|xy| + \frac{1}{8}y^2 + \frac{1}{8}\left|x^3y\right| + \frac{1}{10}|y|}{r} \\ &\leq \frac{(\frac{1}{4} + 1)|xy| + \frac{1}{8}r^2 + \frac{1}{10}r}{r} \leq \frac{3}{4}r + \frac{1}{10}.\end{aligned}$$

This implies that

$$0 < r(0)e^{-\frac{3}{4}} + \frac{2}{15}\left(e^{-\frac{3}{4}} - 1\right) \leq r(t) \leq r(0)e^{\frac{3}{4}} + \frac{2}{15}\left(e^{\frac{3}{4}} - 1\right) < 1$$

for $t \in [0, 1]$, provided that $\frac{2}{15}(e^{\frac{3}{4}} - 1) < r(0) \leq \frac{2}{15}e^{\frac{3}{4}}$. Therefore, condition (H_2) is satisfied with the nonempty open bounded set

$$E = \left\{(x, y) : \sqrt{x^2 + y^2} < \frac{2}{15}e^{\frac{3}{4}}\right\}.$$

According to Theorem 1.1, Eq (4.10) has at least one 1-periodic solution.

Remark 4.1. *Assumption (A_1) implies that (A_1') is satisfied. Indeed, from (A_1), for any fixed $n \in \mathbb{N}$, it holds that*

$$\lim_{|x|\to+\infty} \frac{f(t, x)}{x} \geqslant a_n(t), \quad \text{for a.e.} t \in [0, T] \text{ uniformly},$$

where $a_n(t) = n^2$ for $t \in [0, T]$. Using a method similar to that in Example 4.1, we can show that $\rho(a_n) \to +\infty$ as $n \to +\infty$.

By applying Theorem 1.1 and Remark 4.1, we can derive the following corollaries, which provide further insight into the implications of the theorem. Notably, Corollary 4.1 is Theorem 1.1 in [4]. Therefore, our Theorem 1.1 extends the corresponding result from [4].

Corollary 4.1. *Suppose that (1.2) satisfies (H_1), (A_1), (H_2), and (H_3). Then Eq (1.2) has at least one T-periodic solution.*

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. P. Gidoni, Existence of a periodic solution for superlinear second order ODEs, *J. Differ. Equations*, **345** (2023), 401–417. https://doi.org/10.1016/j.jde.2022.11.054
2. M. Struwe, Multiple solutions of anticoercive boundary value problems for a class of ordinary differential equations of second order, *J. Differ. Equations*, **37** (1980), 285–295. https://doi.org/10.1016/0022-0396(80)90099-6
3. A. Capietto, J. Mawhin, F. Zanolin, A continuation approach to superlinear periodic boundary value problems, *J. Differ. Equations*, **88** (1990), 347–395. https://doi.org/10.1016/0022-0396(90)90102-U
4. S. Wang, F. Chen, D. Qian, The existence of periodic solution for superlinear second order ODEs by a new fixed point approach, *J. Differ. Equations*, **410** (2024), 481–512. https://doi.org/10.1016/j.jde.2024.07.031
5. D. Qian, P. J. Torres, P. Wang, Periodic solutions of second order equations via rotation numbers, *J. Differ. Equations*, **266** (2019), 4746–4768. https://doi.org/10.1016/j.jde.2018.10.010
6. S. Wang, D. Qian, Subharmonic solutions of indefinite Hamiltonian systems via rotation numbers, *Adv. Nonlinear Stud.*, **21** (2021), 557–578. https://doi.org/10.1515/ans-2021-2134
7. S. Wang, C. Liu, Periodic solutions of superlinear planar Hamiltonian systems with indefinite term, *J. Appl. Anal. Comput.*, **13** (2023), 2542–2554. http://dx.doi.org/10.11948/20220426
8. S. Wang, Periodic solutions of weakly coupled superlinear systems with indefinite terms, *Nonlinear Differ. Equations Appl. NoDEA*, **29** (2022), 1–22. https://doi.org/10.1007/s00030-022-00768-1
9. C. Liu, D. Qian, P. J. Torres, Non-resonance and double resonance for a planar system via rotation numbers, *Results Math.*, **76** (2021), 1–23. https://doi.org/10.1007/s00025-021-01401-w
10. M. Zhang, The rotation number approach to eigenvalues of the one-dimensional p-Laplacian with periodic potentials, *J. London Math. Soc.*, **64** (2001), 125–143. https://doi.org/10.1017/S0024610701002277
11. J. Chu, G. Meng, Z. Zhang, Continuous dependence and estimates of eigenvalues for periodic generalized Camassa-Holm equations, *J. Differ. Equations*, **269** (2020), 6343–6358. https://doi.org/10.1016/j.jde.2020.04.042
12. P. Yan, M. Zhang, Rotation number, periodic Fučik spectrum and multiple periodic solutions, *Commun. Contemp. Math.*, **12** (2010), 437–455. https://doi.org/10.1142/S0219199710003877
13. J. Chu, M. Zhang, Rotation numbers and Lyapunov stability of elliptic periodic solutions, *Discrete Contin. Dyn. Syst.*, **21** (2008), 1071–1094. https://doi.org/10.3934/dcds.2008.21.1071
14. J. Li, Y. Cheng, Barycentric rational interpolation method for solving time-dependent fractional convection-diffusion equation, *Electron. Res. Arch.*, **31** (2023), 4034–4056. https://doi.org/10.3934/era.2023205

15. J. Li, Barycentric rational collocation method for fractional reaction-difusion equation, *AIMS Math.*, **8** (2023), 9009–9026. https://doi.org/10.3934/math.2023451

16. J. Li, X. Su, K. Zhao, Barycentric interpolation collocation algorithm to solve fractional differential equations, *Math. Comput. Simul.*, **205** (2023), 340–367. https://doi.org/10.1016/j.matcom.2022.10.005

17. J. K. Hale, *Ordinary Differential equations*, (eds. R. E. Krieger and P. Co. Huntington), New York, 1980.

18. V. Lakshmikantham, S. Leela, *Differential and Integral Inequalities: Theory and Applications*, Academic Press, New York and London, 1969.

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