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Research article

Stability analysis and security control of nonlinear singular semi-Markov jump systems

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Abstract: This paper mainly studied the stochastic stability and the design of state feedback controllers for nonlinear singular continuous semi-Markov jump systems under false data injection attacks. Based on the Lyapunov function and the implicit function theorem, a basic stochastic stability condition of the system was given to ensure that the nonlinear singular semi-Markov jump system under attack was regular, impulse-free, unique, and stochastically stable. On this basis, the stochastic admissible linear matrix inequality conditions of the system were obtained by using the singular value decomposition of the matrix and Schur's complement lemma. To design the state feedback controller, based on the upper and lower bounds of the time-varying transition probability of the semi-Markov jump system and the singular value decomposition method, the stochastic stable linear matrix inequality condition of the closed-loop system under the false data injection attack was established. Finally, the validity and feasibility of the results were verified by numerical examples.

Keywords: semi-Markov jump systems; singular system; false data injection attacks; state feedback controller

1. Introduction

A cyber-physical system is a kind of complex feedback control system, which is the product of deep integration of the physical system and information world. However, with the continuous popularization of cyber-physical systems in practical applications, the security threats faced by cyber-physical systems are also on the rise. Network attacks occur frequently, which greatly reduce the performance of the system and even make the system unstable. Generally speaking, there are three main categories of network attacks, namely denial of service attacks (DoS Attacks), false data injection (FDI) attacks, and replay attacks. In [1], the filter and controller design of the event-triggered Markov transition system under DoS attacks were studied. Shen et al. [2] studied the secure synchronization control of Markov jump neural networks under DoS attacks. FDI attacks interfere with system operations by injecting false data into the system, thereby destroying system performance and reducing its stability. Cao et al. [3] studied the finite-time sliding mode control problem of Markov jumping systems attacked by FDI.Xiao et al. [4] designed a new adaptive event-triggered scheme, and studied the adaptive event-triggered state estimation problem of discrete large-scale systems under FDI attacks. Replay attack is when an attacker intercepts and resends a client's data request, thereby destroying the integrity and security of the data. Guo et al. [5] proposed a coding detection scheme based on state estimation, and discussed the impact of replay attacks on the security of remote state estimation in cyber-physical systems. They also introduced a detection scheme based on output coding, and explored the detection issue of replay attacks on data transmitted through feedback channels in cyber-physical systems [6]. In [7], the authors studied the composite $\mathcal{H}_\infty$ control of hidden Markov jump systems under replay attacks.

A Markov jump system, as a special kind of stochastic hybrid system, can usually be used to represent unpredictable structural changes in the system, such as component failures, parameter changes, and environmental mutations in many practical systems. In [8–16], authors took the Markov jump system as the research object. For example, Shen et al. [17] studied the mismatched quantized output feedback control of fuzzy Markov jump systems via a dynamic guaranteed cost triggering scheme. In [18], they studied fuzzy fault-tolerant tracking control of Markov jump systems with unknown mismatch faults. However, the Markov jump system also has many limitations in application. Markov jump systems are difficult to apply in practical systems due to the limitation of residence time. Different from the Markov jump system, the residence time of the semi-Markov jump system can not only obey the exponential distribution but also obey some other distributions such as the Weber distribution, Gaussian distribution, etc. In practical application, since the semi-Markov jump system can describe a more general actual system, the application of the model is also more extensive. For example, the semi-Markov jump system model is used to model the epidemic contagion system [19], and the semi-Markov jump system is used to describe and model the sociological network [20]. In theoretical research, Mu et al. [21] studied the stability of semi-Markov jump systems with stochastic pulse jumping. In [22], the stochastic stability and the design of state feedback controllers for a class of nonlinear continuous-time semi-Markov jump systems with time-varying transition rates were studied. [23–25] studied the stability analysis of discrete-time semi-Markov jump linear systems. Zhu et al. [26] studied sufficient conditions for the stochastic stability of a semi-Markov switching stochastic system without bounded transition rates. In [27], they studied the $\mathcal{H}_\infty$ filtering problem for a class of semi-Markov jump linear systems with residence time-dependent transition rates. In the above work on the processing of semi-Markov jump systems, there are two methods for dealing with time-varying transition probabilities in semi-Markov jump systems. The first method uses the mathematical expectation of the transition probability to design the sliding mode control rate [28]. The second is to assume that the time-varying transition probability is bounded. The lower and upper bounds of the transition probability are used to construct a sliding mode controller for a nonlinear time-delay semi-Markov jump system [29]. It is worth noting that the authors considered semi-Markov jump systems and used the Lyapunov function to analyze the stability of the system, but all of them did not consider the possibility of attacks on the systems.

The singular semi-Markov jump system is an extension of the semi-Markov jump system, which can describe the state of the system with different probability distributions in different time periods, and there may be delays in state transitions. Shen et al. [30] studied the l2-l∞ control of singular perturbation semi-Markov jump systems. Based on this, they studied the optimal control problem of fast sampling singular perturbation systems [31]. Sivaranjani et al. [32] used singular integral inequalities to study the synchronization problem of semi-Markov jump complex dynamical networks with time-varying coupling delays for actuator faults. Kaviarasan et al. [33] used the state decomposition method to study singular semi-Markov jump systems based on the control design of dissipative constraints. Ding et al. [34] studied the extended dissipative disturbance rejection control problem for a switched singular semi-Markov jump system with multiple disturbances and time delays. The nonlinear singular semi-Markov jump system is a stochastic process model, which combines the characteristics of the nonlinear system and the semi-Markov jump system, and is used to describe the complex system with randomness and uncertainty.

Inspired by the above work, this paper focuses on analyzing the stochastic stability and the design of the state feedback controller for impulse-free nonlinear singular continuous semi-Markov jump systems under FDI attacks and gives the sufficient linear matrix inequality condition for the stochastic stability of the system, proving that the system can still achieve stability when it is attacked.

The organization of this paper is as follows: Section 2 mainly gives the preliminaries. Section 3 mainly studies the stochastic stability analysis of singular nonlinear semi-Markov jump systems under random network FDI attacks. Section 4 mainly studies the state feedback controller design of singular nonlinear semi-Markov transfer systems under random network FDI attacks. In Section 5, we give three numerical examples to verify the results.

2. Preliminaries

The main mathematical symbols in this paper are listed in Table 1.

Lemma 1. *[35](Lipschitz Condition) For a function $f(t,x)$, if any two points (t,x) and (t,y) in a neighborhood of point (t_0,x_0) satisfy the inequality $\|f(t,x)-f(t,y)\| \leq L\|x-y\|$, where L is a constant, then this inequality is called the Lipschitz condition. L is the Lipschitz constant. $f(t,x)$ is the Lipschitz function about x.*

Definition 1. *(The Second Equivalent Form of a Singular System) From the theory of linear algebra, there must be two invertible matrices Q and P so that $QEP = diag(I_r,0)$ holds true for any given matrix E, where $r = rank(E)$. If $x = P\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then the system $E\dot{x} = Ax + Bu$ is restricted and equivalent to $\begin{cases} \dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1u \\ 0 = A_{21}x_1 + A_{22}x_2 + B_2u \end{cases}$, where $x_1 \in R^r$ and $x_2 \in R^{n-r}$, and $QAP = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $QB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ are two block matrices of appropriate dimension.*

Remark 1. *The second equivalent form of the singular system has strong physical and practical significance. Its first equation is the dynamic equation, which can be regarded as the dynamic terms of many subsystems of the singular system. The second equation is an algebraic equation, which can be regarded as the interconnection relationship between the subsystems of the singular system.*

Table 1. Nomenclature.

Symbol	Representation
X^{-1}	The inverse matrix of matrix X
X^T	The transpose matrix of matrix X
X^{-T}	The transpose matrix of the inverse matrix X
I	The identity matrix of suitable dimension
0	The zero matrix of suitable dimension
$diag\{\cdots\}$	The diagonal matrix
$rank(X)$	The rank of matrix X
$det(X)$	The determinant of the square matrix X
$deg\{P(x)\}$	The highest degree of a polynomial $P(x)$
$X > 0$	The matrix X is a positive definite matrix
$*$	Similar parts of the matrix
$\|\|x\|\|$	The Euclidean norm of the vector x
R^{n*n}	An $n * n$ dimensional real matrix
$E\{\cdot\}$	The mathematical expectation
$p\{A\}$	The probability of event A happening

Definition 2. *For a singular system:*

1). If $det(sE - A_i) \not\equiv 0$, the singular system is regular.

2). If $deg\{det(sE - A_i)\} = rank(E)$, then the singular system is impulse-free.

3). A system is stochastically stable if for any initial condition, $(x(0), r_0)$, and there exists a constant $T(x(0), r_0) > 0$ such that $E\left\{\int_0^\infty \|x(t)\|^2 dt \,|(x(0), r_0)\right\} \le T(x(0), r_0)$.

4). If the singular system satisfies the above three conditions at the same time, the system is said to be stochastically admissible.

Definition 3. *For a semi-Markov jump system $x(t) = f(x(t), r_t)$, if the Lyapunov function of the system is $V(x(t), r_t)$, then the weak infinitesimal operator is*

$$\varphi V(x(t), r_t) = \lim_{\Delta\to 0} \frac{E\{V(x(t+\Delta), r_{t+\Delta})|x(t), r_t\} - V(x(t), r_t)}{\Delta},$$

where Δ is a small positive number.

Lemma 2. *[36](Dynkin Formula) If there is a random variable $\psi(t)$ satisfying $\psi(0) = y$, $g(x)$ is a first-order differentiable function of x, then there is $E\{g(\psi(t))\} = g(y) + E\left\{\int_0^t Lg(\psi(\rho))d\rho\right\}$, where L represents the rate of change of $g(\psi(t))$ over time t.*

Definition 4. *(Linear Matrix Inequalities) The general mathematical expression of the linear matrix inequality is $F(x) = F_0 + \sum_{i=1}^{l} x_i F_i < 0$, where $x_1, x_2, \cdots, x_l$ is a set of real decision variables, and F_i is the given symmetric matrix. It can be seen from the above formula that $F(x)$ is a negative definite matrix. That is to say, for any non-zero vector $\varsigma \in R^n$, $\varsigma^T F(x)\varsigma < 0$ holds true. This is the same construction form $V(x) = x^T Px$ of the Lyapunov function commonly used in Lyapunov's second method.*

3. Stability analysis of singular nonlinear semi-Markov jump systems under FDI attacks

3.1. System model

This paper mainly discusses the nonlinear singular semi-Markov jump system with a Lipschitz nonlinear term, and the model is as follows:

$$E\dot{x} = A(r_t)x + B(r_t)u_b + f_{r_t}(t, x), \tag{3.1}$$

where $x \in R^n$ represents the state variable of the system and $u_b \in R^p$ refers to the control input of the system. $A(r_t) \in R^{n*n}$ and $B(r_t) \in R^{n*n}$ are known constant matrices. For convenience, let $r_t = i$, and we define $A(r_t) = A_i, B(r_t) = B_i$ and $f_{r_t}(t, x) = f_i(t, x)$. Matrix $E \in R^{n*n}$ is an irreversible (singular) matrix and satisfies $rank(E) = r \leq n$. $r_t, t \geq 0$ means the mode of the system, which is a continuous-time semi-Markov process taking values on a finite set $N = \{1, 2, \cdots, N\}$, and its transition probability is as follows:

$$\Pr\{r_{t+\Lambda} = j | r_t = i\} = \begin{cases} \lambda_{ij}(h)\Delta + o(\Delta) & \text{if } i \neq j \\ 1 + \lambda_{ij}(h)\Delta + o(\Delta) & \text{if } i = j \end{cases},$$

where $\lambda_{ij}(h) \geq 0$ is the transition probability of mode i jumping to mode j when $i \neq j$, while satisfying $\lambda_{\text{ii}}(h) = -\sum_{j=1}^{N} \lambda_{ij}(h), h > 0$, and $\lim_{\Delta \to 0} \frac{o(\Delta)}{\Delta} = 0$. In fact, the transition probability $\lambda_{ij}(h)$ is bounded, and $\underline{\lambda}_{ij} < \lambda_{ij}(h) < \bar{\lambda}_{ij}$. For any $i, j \in N$, $\underline{\lambda}_{ij}$ and $\bar{\lambda}_{ij}$ represent the lower bound and upper bound of the transition probability $\lambda_{ij}(h)$, respectively.

$f_i(t, x)$ is a nonlinear function and satisfies $f_i(t, 0) = 0$ and the Lipschitz nonlinear condition, that is

$$\|f_i(t, x) - f_i(t, \tilde{x})\| \leq \|L_i(x - \tilde{x})\|. \tag{3.2}$$

In the process of signal transmission, it is unavoidable to be affected by network attacks (as shown in Figure 1). In this paper, the random FDI attack model is mainly considered, and the model is as follows:

$$u_b = u + \delta(t)\psi_i(t, x), \tag{3.3}$$

where $\psi_i(t, x)$ represents the false system information specifically crafted by the attacker to manipulate the system, which satisfies $\psi_i(t, x) \leq \Psi_i x$. Ψ_i is a known constant matrix. $\delta(t)$ is a stochastic variable that obeys the Bernoulli distribution and satisfies the following formula. In (3.3), $\delta(t) = 1$ means that there is an FDI attack, and otherwise, no attack occurs.

$$\Pr\{\delta(t) = 1\} = E\{\delta(t)\} = \rho$$

$$\Pr\{\delta(t) = 0\} = 1 - E\{\delta(t)\} = 1 - \rho, \rho \in [0, 1]$$

Adding the FDI attack model into the system, the system model of the nonlinear singular semi-Markov jump system under FDI attacks is obtained by (3.1) and (3.3) as shown in (3.4).

$$E\dot{x} = A_i x + B_i u + \delta(t)B_i\psi_i(t, x) + f_i(t, x) \tag{3.4}$$

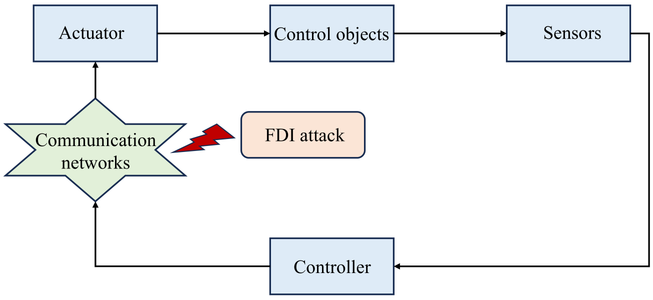


Figure 1. Cyber-physical system under FDI attacks.

3.2. Stochastic stability analysis

Theorem 1. *If the system (3.4) has an invertible matrix P_i and a scalar $\alpha > 0$ for each mode $i \in N$ when $u = 0$ is the input, and the following conditions are satisfied, then the system (3.4) is stochastically admissible and has a unique solution when $u = 0$.*

$$E^T P_i = P_i^T E \geq 0 \tag{3.5}$$

$$\begin{bmatrix} \sum_{j=1}^{N} \lambda_{ij}(h) E^T P_j + A_i^T P_i + P_i^T A_i + (L_i + \delta B_i \Psi_i)^T (L_i + \delta B_i \Psi_i) + \alpha I & P_i^T \\ * & -I \end{bmatrix} < 0 \tag{3.6}$$

Proof. We first prove that the system (3.4) is regular and impulse-free when $u = 0$. According to (3.6), we can get $\sum_{j=1}^{N} \lambda_{ij}(h) E^T P_j + A_i^T P_i + P_i^T A_i + (L_i + \delta B_i \Psi_i)^T (L_i + \delta B_i \Psi_i) + \alpha I < 0$. Since $\alpha > 0$ and Lipschitz constant $L_i > 0$, we have $\sum_{j=1}^{N} \lambda_{ij}(h) E^T P_j + A_i^T P_i + P_i^T A_i < 0$. Because $rank(E) = r$, there exists two invertible matrices $M, N \in R^{n*n}$ such that

$$MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, MA_i N = \begin{bmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{bmatrix}, M^{-T} P_i N = \begin{bmatrix} P_{i1} & P_{i2} \\ P_{i3} & P_{i4} \end{bmatrix}, \tag{3.7}$$

where $A_{i1}, P_{i1} \in R^{r*r}$ and $A_{i4}, P_{i4} \in R^{(n-r)*(n-r)}$. Then we have

$$N^T P_i^T EN = N^T P_i^T M^{-1} MEN = \begin{bmatrix} P_{i1} & P_{i2} \\ P_{i3} & P_{i4} \end{bmatrix}^T \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} P_{i1}^T I_r & 0 \\ P_{i2}^T I_r & 0 \end{bmatrix},$$

$$N^T E^T P_i N = N^T E^T M^T M^{-T} P_i N = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_{i1} & P_{i2} \\ P_{i3} & P_{i4} \end{bmatrix} = \begin{bmatrix} P_{i1} I_r & P_{i2} I_r \\ 0 & 0 \end{bmatrix}.$$

According to (3.5), we have

$$P_{i1} \geq 0, P_{i2} = 0. \tag{3.8}$$

According to (3.7) and (3.8), we can get

$$N^T \left(\sum_{j=1}^{N} \lambda_{ij}(h) E^T P_j + A_i^T P_i + P_i^T A_i\right) N < 0. \tag{3.9}$$

Then, we have

$$N^T A_i^T P_i N = N^T A_i^T M^{-1} M P_i N = \begin{bmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{bmatrix}^T \begin{bmatrix} P_{i1} & 0 \\ P_{i3} & P_{i4} \end{bmatrix} = \begin{bmatrix} \bigstar & \bigstar \\ \bigstar & A_{i4}^T P_{i4} \end{bmatrix},$$

$$N^T P_i^T A_i N = N^T P_i^T M^T M^{-T} A_i N = \begin{bmatrix} P_{i1} & 0 \\ P_{i3} & P_{i4} \end{bmatrix}^T \begin{bmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{bmatrix} = \begin{bmatrix} \bigstar & \bigstar \\ \bigstar & P_{i4}^T A_{i4} \end{bmatrix},$$

where $\bigstar$ represents the matrix that is not used in the proof process. According to (3.9), there is $\begin{bmatrix} \bigstar & \bigstar \\ \bigstar & A_{i4}^T P_{i4} + P_{i4}^T A_{i4} \end{bmatrix} < 0$, namely $A_{i4}^T P_{i4} + P_{i4}^T A_{i4} < 0$. From this, it can be obtained that A_{i4}^T is an invertible matrix. That is, A_i is an invertible matrix, and then we have $\det(sE - A_i) \neq 0$ and $\deg\{\det(sE - A_i)\} = rank(E)$. According to Definition 2, it can be seen that the system (3.4) is regular and impulse-free when $u = 0$.

Next we prove that the system (3.4) has a unique solution when $u = 0$. First let $\tilde{L}_i = \frac{\partial f_i}{\partial x}|_{x=\tilde{x}}$. Then in the neighborhood of $\tilde{x}$, $f_i(t, x)$ can be written in the following form:

$$f_i(t, x) = f_i(t, \tilde{x}) + \tilde{L}_i(x - \tilde{x}) + o(\|(x - \tilde{x})\|). \tag{3.10}$$

According to (3.4) and (3.10), when $u = 0$, system (3.4) can be rewritten in the following form:

$$E\dot{x} = (A_i + \tilde{L}_i)x + f_i(t, \tilde{x}) - \tilde{L}_i\tilde{x} + \delta(t)B_i\psi_{\rm i}(t, x) + o(\|(x - \tilde{x})\|). \tag{3.11}$$

Let $\tilde{A}_i = A_i + \tilde{L}_i$, the system can be described as

$$E\dot{x} = \tilde{A}_i x + f_i(t, \tilde{x}) - \tilde{L}_i\tilde{x} + \delta(t)B_i\psi_i(t, x) + o(\|(x - \tilde{x})\|). \tag{3.12}$$

According to Lipschitz condition $\|f_{\rm i}(t, x) - f_i(t, \tilde{x})\| \leq \|L_i(x - \tilde{x})\|$, we can get $\|f_i(t, x) - f_i(t, \tilde{x})\|^2 \leq (x - \tilde{x})^T L_i^T L_i(x - \tilde{x})$, combined with the following formula:

$$\|f_i(t, x) - f_i(t, \tilde{x})\|^2 = (x - \tilde{x})^T \tilde{L}_i^T \tilde{L}_i(x - \tilde{x}) + 2(x - \tilde{x})^T \tilde{L}_i^T o(\|(x - \tilde{x})\|) + o(\|(x - \tilde{x})\|)^T o(\|(x - \tilde{x})\|).$$

Let $\|x - \tilde{x}\| \to 0$, and we have

$$\tilde{L}_i^T \tilde{L}_i \leq L_i^T L_i. \tag{3.13}$$

Let $T_i = \begin{bmatrix} I & 0 \\ \tilde{L}_i & I \end{bmatrix}$. Multiply T_i^T on the left side of (3.6) and multiply T_i on the right side, we can get $\begin{bmatrix} \Phi & P_i^T - \tilde{L}_i^T \\ P_i - L_i & -I \end{bmatrix} < 0$, and

$$\begin{aligned} \Phi &= \sum_{j=1}^{N} \lambda_{ij}(h)E^T P_j + (A_i + \tilde{L}_i)^T P_i + P_i^T(A_i + \tilde{L}_i) + (L_i + \delta B_i\Psi_i)^T(L_i + \delta B_i\Psi_i) + \alpha I - \tilde{L}_i^T \tilde{L}_i \\ &= \sum_{j=1}^{N} \lambda_{ij}(h)E^T P_j + \tilde{A}_i^T P_i + P_i^T \tilde{A}_i + L_i^T L_i - \tilde{L}_i^T \tilde{L}_i + \delta\Psi_i^T B_i^T L_i + \delta L_i^T B_i\Psi_i + \delta^2\Psi_i^T B_i^T B_i\Psi_i + \alpha I. \end{aligned}$$

According to (3.13) and $\alpha > 0$, we have $\sum_{j=1}^{N} \lambda_{ij}(h)E^T P_j + \tilde{A}_i^T P_i + P_i^T \tilde{A}_i < 0$.

Similar to the proof that system (E, A_i) is regular and impulse-free, it can be obtained that system $(E, \tilde{A}_i)$ is also regular and impulse-free. According to the second equivalent form of the singular system, let $MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, MA_iN = \begin{bmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{bmatrix}, M\tilde{L}_iN = \begin{bmatrix} \tilde{L}_{i1} & \tilde{L}_{i2} \\ \tilde{L}_{i3} & \tilde{L}_{i4} \end{bmatrix}, N^{-1}x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, Mf_i(t, x) = \begin{bmatrix} f_{i1}(t, x_1, x_2) \\ f_{i2}(t, x_1, x_2) \end{bmatrix}$, and $\delta(t)MB_i\psi_i(t, x) = \begin{bmatrix} \delta(t)B_{i1}\psi_{i1}(t, x_1, x_2) \\ \delta(t)B_{i2}\psi_{i2}(t, x_1, x_2) \end{bmatrix}$, where $x_1, f_{i1} \in R^r, x_2, f_{i2} \in R^{n-r}, \tilde{L}_{i1} \in R^{r*r}$, and $\tilde{L}_{i4} \in R^{(n-r)*(n-r)}$. When $u = 0$, the system is equivalent to being limited by the following system:

$$\dot{x}_1 = A_{i1}x_1 + A_{i2}x_2 + \delta(t)B_{i1}\psi_{i1}(t, x_1, x_2) + f_{i1}(t, x_1, x_2), \tag{3.14}$$

$$0 = A_{i3}x_1 + A_{i4}x_2 + \delta(t)B_{i2}\psi_{i2}(t, x_1, x_2) + f_{i2}(t, x_1, x_2). \tag{3.15}$$

Since the system $(E, \tilde{A}_i)$ is regular and impulse-free, we have $\frac{\partial(A_{i4}x_2 + f_{i2}(t, x_1, x_2))}{\partial x}|_{x_1=\tilde{x}_1, x_2=\tilde{x}_2} = A_{i4} + \tilde{L}_{i4}$, where $A_{i4} + \tilde{L}_{i4}$ is the invertible matrix. According to the implicit function theorem, in the neighborhood of $x_1 = \tilde{x}_1, x_2 = \tilde{x}_2$, there is a unique continuous function $x_2 = \frown{f}_{i2}(t, x_1)$ that satisfies $\tilde{x}_2 = \frown{f}_{i2}(t, \tilde{x}_1)$ and makes the algebraic equation of the system (3.14) valid.

$$\dot{x}_1 = A_{i1}x_1 + A_{i2}\frown{f}_{i2}(t, x_1) + \delta(t)B_{i1}\psi_{i1}(t, x_1, \frown{f}_{i2}(t, x_1)) + f_{i1}(t, x_1, \frown{f}_{i2}(t, x_1))$$

Substituting the above solution into (3.15), we have

$$0 = A_{i3}x_1 + A_{i4}\frown{f}_{i2}(t, x_1) + \delta(t)B_{i2}\psi_{i2}(t, x_1, \frown{f}_{i2}(t, x_1)) + f_{i2}(t, x_1, \frown{f}_{i2}(t, x_1)).$$

It can be obtained that the system has a unique solution when $u = 0$.

Finally, it is proved that the system is stochastically stable when $u = 0$. Consider the stochastic Lyapunov functional of the following system $V(x(t), r_t) = x^T(t)E^T P_{r_t}x(t)$. Then the infinitesimal operator of the Lyapunov function is

$$\begin{aligned} \varphi V(x(t), r_t) &= \lim_{\Delta \to 0} \frac{E\{V(x(t + \Delta), r_{t+\Delta})|x(t), r_t\} - V(x(t), r_t)}{\Delta} \\ &= \lim_{\Delta \to 0} \frac{1}{\Delta}\{ \sum_{j=1, j\neq i}^{N} \Pr\{r_{t+\Delta} = j|r_t = i\}x^T(t + \Delta)E^T P_j x(t + \Delta) \\ &\quad + \Pr\{r_{t+\Delta} = j|r_t = i\}x^T(t + \Delta)E^T P_i x(t + \Delta) - x^T(t)E^T P_i x(t)\}. \end{aligned}$$

Since the distribution function of the residence time of the semi-Markov jump system is not memoryless in mode i, and according to the nature of the conditional probability, we have

$$\begin{aligned}
\varphi V(x(t), r_t) =& \lim_{\Delta\to 0}\frac{1}{\Delta}\{\sum_{j=1,j\neq i}^{N}\frac{q_{ij}(F_i(h+\Delta)-F_i(h))}{1-F_i(h)}x^T(t+\Delta)E^TP_jx(t+\Delta)\\
&+\frac{1-F_i(h+\Delta)}{1-F_i(h)}x^T(t+\Delta)E^TP_ix(t+\Delta)-x^T(t)E^TP_ix(t)\}\\
=& \lim_{\Delta\to 0}\frac{1}{\Delta}\{\sum_{j=1,j\neq i}^{N}\frac{q_{ij}(F_i(h+\Delta)-F_i(h))}{1-F_i(h)}x^T(t+\Delta)E^TP_jx(t+\Delta)\\
&+\frac{1-F_i(h+\Delta)}{1-F_i(h)}x^T(t+\Delta)E^TP_ix(t+\Delta)-\frac{1-F_i(h+\Delta)}{1-F_i(h)}x^T(t)E^TP_ix(t+\Delta)\\
&+\frac{1-F_i(h+\Delta)}{1-F_i(h)}x^T(t)E^TP_ix(t+\Delta)-\frac{1-F_i(h+\Delta)}{1-F_i(h)}x^T(t)E^TP_ix(t)\\
&-\frac{F_i(h+\Delta)-F_i(h)}{1-F_i(h)}x^T(t)E^TP_ix(t)\},
\end{aligned}$$

where h is the residence time of the nonlinear singular semi-Markov jump system. $F_i(t)$ is the cumulative distribution function of the residence time when the system is in mode i. q_{ij} is the probability density function from mode i to mode j. Based on the properties of the cumulative distribution function, we can get

$$\lim_{\Delta\to 0}\frac{F_i(h+\Delta)-F_i(h)}{1-F_i(h)}=0,\ \lim_{\Delta\to 0}\frac{1}{\Delta}\frac{F_i(h+\Delta)-F_i(h)}{1-F_i(h)}=\lambda_i(h).$$

Furthermore, the above formula can be simplified to

$$\varphi V(x(t), r_t) = x^T(t)\sum_{j=1,j\neq i}^{N}q_{ij}\lambda_i(h)E^TP_jx(t)+\dot{x}^T(t)E^TP_ix(t)+x^T(t)P_i^TE\dot{x}(t)-\lambda_i(h)x^T(t)E^TP_ix(t).$$

When $i\neq j$, we have $\lambda_{ij}(h)=q_{ij}\lambda_i(h)$ and $\lambda_{ii}(h)=-\sum_{j=1,j\neq i}^{N}\lambda_{ij}(h)$, which can be further extended to

$$\varphi V(x(t), r_t) = x^T(t)\sum_{j=1}^{N}\lambda_{ij}(h)E^TP_jx(t)+\dot{x}^T(t)E^TP_ix(t)+x^T(t)P_i{}^TE\dot{x}(t).$$

Let $\varsigma(t)=\begin{bmatrix}x(t)\\ f_i+\delta B_i\psi_i\end{bmatrix}$. Multiply the left side of (3.6) by $\varsigma^T(t)$ and the right side by $\varsigma(t)$, and we have

$$\begin{aligned}
&\varsigma^T(t)\begin{bmatrix}\sum_{j=1}^{N}\lambda_{ij}(h)E^TP_j+A_i^TP_i+P_i^TA_i+(L_i+\delta B_i\Psi_i)^T(L_i+\delta B_i\Psi_i)+\alpha I & P_i^T\\ * & -I\end{bmatrix}\varsigma(t)\\
=& x^T(t)\sum_{j=1}^{N}\lambda_{ij}(h)E^TP_jx(t)+x^T(t)A_i^TP_ix(t)+x^T(t)P_i^TA_ix(t)\\
&+x^T(t)(L_i+\delta B_i\Psi_i)^T(L_i+\delta B_i\Psi_i)x(t)+x^T(t)\alpha x(t)+f_i^TP_ix(t)\\
&+\delta\psi_i^TB_i^TP_ix(t)+x^T(t)P_i^Tf_i+x^T(t)P_i^T\delta B_i\psi_i-(f_i+\delta B_i\psi_i)^T(f_i+\delta B_i\psi_i).
\end{aligned}$$

Bringing system $E\dot{x}(t) = A_i x(t) + \delta(t)B_i\psi_i(t,x) + f_i(t,x)$ into $\varphi V(x(t), r_t)$, we have

$$\begin{aligned}\varphi V(x(t), r_t) = x^T(t)\sum_{j=1}^{N}\lambda_{ij}(h)E^T P_j x(t) + x^T(t)A_i^T P_i x(t) + x^T(t)P_i^T A_i x(t)\\ + f_i^T P_i x(t) + x^T P_i{}^T f_i + x^T(t)P_i^T\delta(t)B_i\psi_i + \delta(t)\psi_i^T B_i^T P_i x(t).\end{aligned}$$

Since the nonlinear function $f_i(t,x)$ satisfies $f_i(t,0) = 0$, there is $f_i \le L_i x$, and according to $\psi_i(t,x) \le \Psi_i x$, we can get $x^T(t)(L_i + \delta B_i\Psi_i)^T(L_i + \delta B_i\Psi_i)x(t) - (f_i + \delta B_i\psi_i)^T(f_i + \delta B_i\psi_i) \ge 0$. Therefore we have

$$\varphi V(x(t), r_t) \le \varsigma^T(t)\begin{bmatrix}\sum\limits_{j=1}^{N}\lambda_{ij}(h)E^T P_j + A_i^T P_i + P_i^T A_i + (L_i + \delta B_i\Psi_i)^T(L_i + \delta B_i\Psi_i) & P_i^T\\ * & -I\end{bmatrix}\varsigma(t).$$

According to (3.6), we can get

$$\begin{aligned}&\varphi V(x(t), r_t) + \alpha x^T(t)x(t) \le\\ &\varsigma^T(t)\begin{bmatrix}\sum\limits_{j=1}^{N}\lambda_{ij}(h)E^T P_j + A_i^T P_i + P_i^T A_i + (L_i + \delta B_i\Psi_i)^T(L_i + \delta B_i\Psi_i) + \alpha I & P_i^T\\ * & -I\end{bmatrix}\varsigma(t) \le 0.\end{aligned}$$

In other words, we have

$$\varphi V(x(t), r_t) \le -\alpha x^T(t)x(t). \tag{3.16}$$

Applying Lemma 2 to (3.16), for each $i = r_t, i \in N, t > 0$, we have

$$\begin{aligned}E\{V(x(t), r_t)\} - V(x_0, r_0) &= E\{\int_0^t \varphi V(x(s), r_s)ds|(x_0, r_0)\}\\ &\le -\alpha E\{\int_0^t x^T(s)x(s)ds|(x_0, r_0)\}.\end{aligned} \tag{3.17}$$

For any $t > 0$, we have $E\{\int_0^t x^T(s)x(s)ds|(x_0, r_0)\} \le \dfrac{1}{\alpha}V(x_0, r_0)$. According to Definition 2, it can be seen that the system (3.4) is stochastically stable at that time. In summary, the system is stochastically admissible and has a unique solution, and the theorem is proved. □

Remark 2. *In a jump system, the dwell time h is a stochastic variable following a continuous-time probability distribution function F. For example, when F obeys the Weibull distribution, its cumulative distribution function* $F(h) = \begin{cases}1 - \exp(-\frac{h}{\gamma})^\beta, h \ge 0\\ 0, h < 0\end{cases}$. *The probability distribution function* $f(h) = \begin{cases}\frac{\beta}{\gamma^\beta}h^{\beta-1}\exp(-(\frac{h}{\gamma})^\beta), h \ge 0\\ 0, h < 0\end{cases}$. *At this time, the expression of its probability function* $\lambda(h)$ *is*

$$\lambda(h) = \frac{f(h)}{1 - F(h)} = \frac{\beta}{\lambda^\beta}h^{\beta-1}.$$

When the parameter $\beta = 1$, the residence time h obeys the exponential distribution

$$f(h) = \begin{cases} \frac{1}{\gamma}\exp(-\frac{h}{\gamma}), h \geq 0 \\ 0, h < 0 \end{cases}.$$

At this time, the system changes from a singular semi-Markovian jump system to a singular Markovian jump system, and the transition probability is constant, that is

$$\lambda_{ij}(h) = \frac{f_{ij}(h)}{1 - F_{ij}(h)} = \frac{\lambda_{ij} e^{-\lambda_{ij} h}}{1 - (1 - e^{-\lambda_{ij} h})} = \lambda_{ij}.$$

The semi-Markov jump system is the generalization of the Markov jump system, and the Markov jump system is a special case of the semi-Markov jump system.

Remark 3. *If the nonlinear term $f_{r_t}(t, x)$ of the system is equal to zero, the system transforms from a singular nonlinear semi-Markov jump system to a singular linear semi-Markov jump system. At the same time, the stochastic stability condition of the singular linear semi-Markov jump system under FDI attacks when $u = 0$ is obtained.*

Remark 4. *If for each mode $i \in N$, there exists a matrix $P_i > 0$ and a scalar $\alpha > 0$ such that $E^T P_i = P_i^T E \geq 0$, and $\sum_{j=1}^{N} \lambda_{ij}(h) E^T P_j + A_i^T P_i + P_i^T A_i + \delta^2 \Psi_i^T B_i^T B_i \Psi_i + \alpha I < 0$. Then the singular linear semi-Markov jump system is stochastically stable when input $u = 0$. Furthermore, if $E = I$, the system changes from a singular system to a non-singular system. At the same time, the stochastic stability condition of the linear non-singular semi-Markov jump system under FDI attacks when $u = 0$ is obtained.*

Remark 5. *If there exists a matrix $P_i > 0$ and a scalar $\alpha > 0$ for each mode $i \in N$, the formula $\sum_{j=1}^{N} \lambda_{ij}(h) P_j + A_i^T P_i + P_i^T A_i + \delta^2 \Psi_i^T B_i^T B_i \Psi_i + \alpha I < 0$ holds, and the system $\dot{x} = A(r_t)x + B(r_t)u_b$ is stochastically stable with input $u = 0$.*

Lemma 3. *[37](Schur's Complementary Lemma) For a given symmetric matrix $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$, where $S_{11} \in R^{n*n}$, then the following three conditions are equivalent.*

1). $S < 0$.
2). $S_{11} < 0, S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$.
3). $S_{22} < 0, S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$.

Theorem 2. *If the system (3.4) has two matrices $M_i > 0, Q_i$ and a scalar $\alpha > 0$ for each mode $i \in N$ such that the following linear matrix inequality holds, then the system (3.4) is stochastically admissible when $u = 0$ and there is a unique solution.*

$$\begin{bmatrix} \underline{\Lambda}_i & I & (SQ_i + M_i E^T)^T (L_i + \delta B_i \Psi_i)^T & (SQ_i + M_i E^T)^T & \underline{Z}_i \\ * & -I & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -\frac{1}{\alpha} I & 0 \\ * & * & * & * & -\Theta_i \end{bmatrix} < 0 \tag{3.18}$$

$$\begin{bmatrix} \bar{\Lambda}_i & I & (SQ_i + M_iE^T)^T(L_i + \delta B_i\Psi_i)^T & (SQ_i + M_iE^T)^T & \bar{Z}_i \\ * & -I & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -\frac{1}{\alpha}I & 0 \\ * & * & * & * & -\Theta_i \end{bmatrix} < 0 \tag{3.19}$$

where

$$\begin{aligned}
&\underline{\Lambda}_i = \underline{\lambda}_{ii}EM_iE^T + A_i(SQ_i + M_iE^T) + (SQ_i + M_iE^T)^T A_i^T, \\
&\bar{\Lambda}_i = \bar{\lambda}_{ii}EM_iE^T + A_i(SQ_i + M_iE^T) + (SQ_i + M_iE^T)^T A_i^T, \\
&\underline{Z}_i = [\sqrt{\underline{\lambda}_{i1}}EM_i\bar{V}, \cdots, \sqrt{\underline{\lambda}_{i(i-1)}}EM_i\bar{V}, \sqrt{\underline{\lambda}_{i(i+1)}}EM_i\bar{V}, \cdots, \sqrt{\underline{\lambda}_{iN}}EM_i\bar{V}], \\
&\bar{Z}_i = [\sqrt{\bar{\lambda}_{i1}}EM_i\bar{V}, \cdots, \sqrt{\bar{\lambda}_{i(i-1)}}EM_i\bar{V}, \sqrt{\bar{\lambda}_{i(i+1)}}EM_i\bar{V}, \cdots, \sqrt{\bar{\lambda}_{iN}}EM_i\bar{V}], \\
&\Theta_i = diag\{\bar{V}^TM_1\bar{V}, \cdots, \bar{V}^TM_{i-1}\bar{V}, \bar{V}^TM_{i+1}\bar{V}, \cdots, \bar{V}^TM_N\bar{V}\}, \\
&\bar{V} = V[I_r \quad 0]^T.
\end{aligned}$$

Here U and V are orthogonal matrices. After singular value decomposition, we get

$$E = U\begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}V^T. \tag{3.20}$$

$\Sigma_r = diag\{\sigma_1 \cdots \sigma_r\}$ *is non-singular and* $\sigma_1 \cdots \sigma_r$ *is the singular value of matrix E.* $S \in R^{n*(n-r)}$ *is any matrix satisfying* $ES = 0$ *and* $rank(S) = n - r$.

Proof. First, let $U^TA_iV = \begin{bmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{bmatrix}, V^TM_iV = \begin{bmatrix} M_{i1} & M_{i2} \\ M_{i2}^T & M_{i3} \end{bmatrix}$, and $Q_iU = [Q_{i1} \quad Q_{i2}]$, where $A_{i1} \in R^{r*r}, A_{i4} \in R^{(n-r)*(n-r)}, M_{i1} \in R^{r*r}, M_{i3} \in R^{(n-r)*(n-r)}, Q_{i1} \in R^{(n-r)*r}$, and $Q_{i2} \in R^{(n-r)*(n-r)}$. Furthermore, we have

$$\begin{aligned}
&\underline{Z}_i = [\sqrt{\underline{\lambda}_{i1}}U\bar{M}_i, \cdots, \sqrt{\underline{\lambda}_{i(i-1)}}U\bar{M}_i, \sqrt{\underline{\lambda}_{i(i+1)}}U\bar{M}_i, \cdots, \sqrt{\underline{\lambda}_{iN}}U\bar{M}_i] \\
&\Theta_i = diag\{M_{11}, \cdots, M_{(i-1)1}, M_{(i+1)1}, \cdots, M_{N1}\},
\end{aligned} \tag{3.21}$$

and

$$\bar{M}_i = [\Sigma_rM_{i1} \quad 0]^T. \tag{3.22}$$

Let $V^TS = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$, and we know that $S_1 = 0$ from $ES = U\begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}V^TS = 0$. That is to say,

$$V^TS = \begin{bmatrix} 0 \\ S_2 \end{bmatrix}, \tag{3.23}$$

where $S_2 \in R^{(n-r)*(n-r)}, andrank(S_2) = n - r$. According to (3.18), we can get

$$U^T\underline{\Lambda}_iU = U^T(\underline{\lambda}_{ii}EM_iE^T + A_i(SQ_i + M_iE^T) + (SQ_i + M_iE^T)^TA_i^T)U < 0. \tag{3.24}$$

Due to

$$U^T A_i S Q_i U = U^T A_i V V^T S Q_i U = \begin{bmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{bmatrix} \begin{bmatrix} 0 \\ S_2 \end{bmatrix} \begin{bmatrix} Q_{i1} & Q_{i2} \end{bmatrix} = \begin{bmatrix} \bigstar & \bigstar \\ \bigstar & A_{i4} S_2 Q_{i2} \end{bmatrix},$$

$$U^T Q_i^T S^T A_i^T U = (Q_i U)^T (V^T S)^T (U^T A_i V)^T = \begin{bmatrix} Q_{i1} & Q_{i2} \end{bmatrix}^T \begin{bmatrix} 0 \\ S_2 \end{bmatrix}^T \begin{bmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{bmatrix}^T = \begin{bmatrix} \bigstar & \bigstar \\ \bigstar & Q_{i2}^T S_2^T A_{i4}^T \end{bmatrix}.$$

Then (3.24) is equivalent to $\begin{bmatrix} \bigstar & \bigstar \\ \bigstar & A_{i4} S_2 Q_{i2} + Q_{i2}^T S_2^T A_{i4}^T \end{bmatrix} < 0$, where $\bigstar$ means matrices that are not used in the proof. From this, $A_{i4} S_2 Q_{i2} + Q_{i2}^T S_2^T A_{i4}^T < 0$ can be obtained, so the matrix Q_{i2} is an invertible matrix. Let $X_i = S Q_i + M_i E^T$, and we have

$$X_i = S Q_i + M_i E^T = V \begin{bmatrix} M_{i1} \Sigma_r & 0 \\ S_2 Q_{i1} + M_{i2}^T \Sigma_r & S_2 Q_{i2} \end{bmatrix} U^T. \tag{3.25}$$

From this, it can be obtained that the matrix X_i is an invertible matrix, further combined with (3.20), and we have

$$\sum_{j=1, j \neq i}^{N} \underline{\lambda}_{ij} X_i^T E^T X_j^{-1} X_i = \sum_{j=1, j \neq i}^{N} \underline{\lambda}_{ij} U \begin{bmatrix} \Sigma_r M_{i1} \\ 0 \end{bmatrix} M_{j1}^{-1} \begin{bmatrix} \Sigma_r M_{i1} \\ 0 \end{bmatrix}^T U^T.$$

According to (3.21), (3.22), and (3.25), and using Lemma 2 and (3.18), we can get

$$\begin{bmatrix} \sum_{j=1}^{N} \underline{\lambda}_{ij} X_i^T E^T X_j^{-1} X_i + X_i^T A_i^T + A_i X_i + X_i^T (L_i + \delta B_i \Psi_i)^T (L_i + \delta B_i \Psi_i) X_i + \alpha X_i^T X_i & I \\ * & -I \end{bmatrix} < 0.$$

Multiply the above formula on the left by $\begin{bmatrix} X_i^{-T} & 0 \\ 0 & I \end{bmatrix}$ and on the right by $\begin{bmatrix} X_i^{-1} & 0 \\ 0 & I \end{bmatrix}$. Then we have

$$\begin{bmatrix} \sum_{j=1}^{N} \underline{\lambda}_{ij} E^T X_j^{-1} + A_i^T X_i^{-1} + X_i^{-T} A_i + (L_i + \delta B_i \Psi_i)^T (L_i + \delta B_i \Psi_i) + \alpha I & X_i^{-T} \\ * & -I \end{bmatrix} < 0. \tag{3.26}$$

Similarly, according to (3.19), after the same above-mentioned processing, we can get

$$\begin{bmatrix} \sum_{j=1}^{N} \bar{\lambda}_{ij} E^T X_j^{-1} + A_i^T X_i^{-1} + X_i^{-T} A_i + (L_i + \delta B_i \Psi_i)^T (L_i + \delta B_i \Psi_i) + \alpha I & X_i^{-T} \\ * & -I \end{bmatrix} < 0. \tag{3.27}$$

For a particular h, the transition probability $\lambda_{ij}(h)$ can be written as $\lambda_{ij}(h) = \theta_1 \underline{\lambda}_{ij} + \theta_2 \bar{\lambda}_{ij}$, where $\theta_1 + \theta_2 = 1, \theta_1 > 0, \theta_2 > 0$. We have

$$\begin{aligned} &\theta_1 \begin{bmatrix} \sum_{j=1}^{N} \underline{\lambda}_{ij} E^T X_j^{-1} + A_i^T X_i^{-1} + X_i^{-T} A_i + (L_i + \delta B_i \Psi_i)^T (L_i + \delta B_i \Psi_i) + \alpha I & X_i^{-T} \\ * & -I \end{bmatrix} + \\ &\theta_2 \begin{bmatrix} \sum_{j=1}^{N} \bar{\lambda}_{ij} E^T X_j^{-1} + A_i^T X_i^{-1} + X_i^{-T} A_i + (L_i + \delta B_i \Psi_i)^T (L_i + \delta B_i \Psi_i) + \alpha I & X_i^{-T} \\ * & -I \end{bmatrix} \\ = &\begin{bmatrix} \sum_{j=1}^{N} \lambda_{ij}(h) E^T X_j^{-1} + A_i^T X_i^{-1} + X_i^{-T} A_i + (L_i + \delta B_i \Psi_i)^T (L_i + \delta B_i \Psi_i) + \alpha I & X_i^{-T} \\ * & -I \end{bmatrix} < 0. \end{aligned}$$

According to (3.25), we can get

$$EX_i = X_i^T E^T = EM_iE^T \geq 0. \tag{3.28}$$

Furthermore, we have

$$X_i^{-T}(X_i^T E^T)X_i^{-1} = X_i^{-T}(EX_i)X_i^{-1} = E^T X_i^{-1} = X_i^{-T} E \geq 0. \tag{3.29}$$

Let $P_i = X_i^{-1} = (SQ_i + M_iE)^{-T}$, and it can be concluded that

$$E^T P_i = P_i^T E \geq 0,$$

$$\begin{bmatrix} \sum_{j=1}^{N} \lambda_{ij}(h)E^T P_j + A_i^T P_i + P_i^T A_i + (L_i + \delta B_i\Psi_i)^T(L_i + \delta B_i\Psi_i) + \alpha I & P_i^T \\ * & -I \end{bmatrix} < 0.$$

According to Theorem 1, it can be obtained that the system (3.4) is stochastically admissible and has a unique solution when $u = 0$, and Theorem 2 is proved. □

Remark 6. *When the transition probability $\lambda_{ij}(h)$ is constant, that is, $\bar{\lambda}_{ij} = \underline{\lambda}_{ij} = \lambda_{ij}$, the system transforms from a nonlinear singular semi-Markov jump system to a nonlinear singular Markov jump system. Theorem 2 becomes the stochastic stability condition of the nonlinear singular Markov jump system under FDI attacks.*

Theorem 3. *If for each mode i, there exists a matrix $M_i > 0$, Q_i and a scalar $\alpha > 0$ such that the following linear matrix inequality holds, then the system is stochastically admissible and has a unique solution.*

$$\begin{bmatrix} \Lambda_i & I & (SQ_i + M_iE^T)^T(L_i + \delta B_i\Psi_i)^T & (SQ_i + M_iE^T)^T & Z_i \\ * & -I & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -\frac{1}{\alpha}I & 0 \\ * & * & * & * & -\Theta_i \end{bmatrix} < 0$$

In addition,

$$\Lambda_i = \lambda_{ii}EM_iE^T + A_i(SQ_i + M_iE^T) + (SQ_i + M_iE^T)^T A_i^T,$$
$$Z_i = [\sqrt{\lambda_{i1}}EM_i\bar{V}, \cdots, \sqrt{\lambda_{i(i-1)}}EM_i\bar{V}, \sqrt{\lambda_{i(i+1)}}EM_i\bar{V}, \cdots, \sqrt{\lambda_{iN}}EM_i\bar{V}].$$

Proof. The proof is similar to Theorem 2. □

Remark 7. *Furthermore, if the nonlinear term of the system $f_{r_t}(t, x) = 0$, the system is further transformed into a linear continuous-time Markov jump system. Theorem 2 becomes the random stability condition of the linear continuous-time Markov jump system under FDI attacks.*

Theorem 4. *If for each mode i, there exists a matrix $M_i > 0$, Q_i and a scalar $\alpha > 0$ such that the following linear matrix inequality holds, then the system is stochastically admissible and has a unique solution.*

$$\begin{bmatrix} \Lambda_i & I & (SQ_i + M_iE^T)^T(\delta B_i\Psi_i)^T & (SQ_i + M_iE^T)^T & Z_i \\ * & -I & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -\frac{1}{\alpha}I & 0 \\ * & * & * & * & -\Theta_i \end{bmatrix} < 0$$

Proof. The proof is similar to Theorem 2. □

4. Design of state feedback controller under FDI attacks

4.1. System model

The nonlinear singular semi-Markov jump system model is the same as in Section 3. Let $u = K_ix$, and we can get the nonlinear singular semi-Markov jump system model with a state feedback controller under random FDI attacks as follows:

$$E\dot{x} = (A_i + B_iK_i)x + B_i\delta(t)\psi_i(t,x) + f_i(t,x). \tag{4.1}$$

For the convenience of expression, let $A_{ci} = A_i + B_iK_i$, and then we have the closed-loop system:

$$E\dot{x} = A_{ci}x + B_i\delta(t)\psi_i(t,x) + f_i(t,x). \tag{4.2}$$

4.2. State feedback controller design

Theorem 5. *If for each mode $i \in N$, there are two matrices $M_i > 0, Q_i$, $E^{\perp}Q_i^T$ is an invertible matrix, and scalar $\alpha > 0$, so that the following formula holds, then the closed-loop system (4.2) is stochastically stable and has a unique solution. The controller gain matrix of the system is as follows:*

$$K_i = R_i(SQ_i + M_iE^T)^{-1},$$

$$\begin{bmatrix} \underline{\Xi}_i & I & (SQ_i + M_iE^T)^T(L_i + \delta B_i\Psi_i)^T & (SQ_i + M_iE^T)^T & \underline{Z}_i \\ * & -I & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -\frac{1}{\alpha}I & 0 \\ * & * & * & * & -\Lambda_i \end{bmatrix} < 0, \tag{4.3}$$

$$\begin{bmatrix} \bar{\Xi}_i & I & (SQ_i + M_iE^T)^T(L_i + \delta B_i\Psi_i)^T & (SQ_i + M_iE^T)^T & \bar{Z}_i \\ * & -I & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -\frac{1}{\alpha}I & 0 \\ * & * & * & * & -\Lambda_i \end{bmatrix} < 0. \tag{4.4}$$

In addition,

$$\begin{aligned}
&\underline{\Xi}_i = \underline{\lambda}_{ii} E M_i E^T + A_i(SQ_i + M_i E^T) + B_i R_i + (SQ_i + M_i E^T)^T A_i^T + R_i^T B_i^T,\\
&\bar{\Xi}_i = \underline{\lambda}_{ii} E M_i E^T + A_i(SQ_i + M_i E^T) + B_i R_i + (SQ_i + M_i E^T)^T A_i^T + R_i^T B_i^T,\\
&\underline{Z}_i = [\sqrt{\underline{\lambda}_{i1}} E M_i \bar{V}, \cdots, \sqrt{\underline{\lambda}_{i(i-1)}} E M_i \bar{V}, \sqrt{\underline{\lambda}_{i(i+1)}} E M_i \bar{V}, \cdots, \sqrt{\underline{\lambda}_{iN}} E M_i \bar{V}],\\
&\bar{Z}_i = [\sqrt{\bar{\lambda}_{i1}} E M_i \bar{V}, \cdots, \sqrt{\bar{\lambda}_{i(i-1)}} E M_i \bar{V}, \sqrt{\bar{\lambda}_{i(i+1)}} E M_i \bar{V}, \cdots, \sqrt{\bar{\lambda}_{iN}} E M_i \bar{V}],\\
&\Theta_i = diag\{\bar{V}^T M_1 \bar{V}, \cdots, \bar{V}^T M_{i-1} \bar{V}, \bar{V}^T M_{i+1} \bar{V}, \cdots, \bar{V}^T M_N \bar{V}\},\\
&\bar{V} = V[I_r \quad 0]^T,
\end{aligned}$$

where U and V are orthogonal matrices. After singular value decomposition, we get

$$E = U \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} V^T.$$

$\Sigma_r = diag\{\sigma_1 \cdots \sigma_r\}$ *is non-singular and* $\sigma_1 \cdots \sigma_r$ *is the singular value of matrix E.* $S \in R^{n*(n-r)}$ *is any matrix satisfying* $ES = 0$ *and* $rank(S) = n - r$. $E^\perp \in R^{(n-r)*n}$ *is any matrix satisfying* $E^\perp E = 0$ *and* $rank(E^\perp) = n - r$.

Proof. First, let

$$V^T M_i V = \begin{bmatrix} M_{i1} & M_{i2} \\ M_{i2}^T & M_{i3} \end{bmatrix}, Q_i U = \begin{bmatrix} Q_{i1} & Q_{i2} \end{bmatrix}, E^\perp U = \begin{bmatrix} E_1^\perp & E_2^\perp \end{bmatrix}.$$

Since $E^\perp E = 0$, we can get $E^\perp U = \begin{bmatrix} 0 & E_2^\perp \end{bmatrix}$, where $E_1^\perp = 0, E_2^\perp \in R^{(n-r)*(n-r)}$ are two invertible matrices, and since $E^\perp Q_i^T = E^\perp U U^T Q_i^T = \begin{bmatrix} 0 & E_2^\perp \end{bmatrix} \begin{bmatrix} Q_{i1} & Q_{i2} \end{bmatrix}^T = E_2^\perp Q_{i2}^T$ is an invertible matrix, Q_{i2} is an invertible matrix. Further from $SQ_i + M_i E^T = V \begin{bmatrix} M_{i1}\Sigma_r & 0 \\ S_2 Q_{i1} + M_{i2}^T \Sigma_r & S_2 Q_{i2} \end{bmatrix} U^T$, it can be seen that $SQ_i + M_i E^T$ is an invertible matrix.

According to Theorem 2, the closed-loop system (4.2) is stochastically stable if the following conditions are satisfied.

$$\begin{bmatrix} \underline{\Lambda}_{ci} & I & (SQ_i + M_i E^T)^T (L_i + \delta B_i \Psi_i)^T & (SQ_i + M_i E^T)^T & \underline{Z}_i \\ * & -I & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -\frac{1}{\alpha} I & 0 \\ * & * & * & * & -\Theta_i \end{bmatrix} < 0 \tag{4.5}$$

$$\begin{bmatrix} \bar{\Lambda}_{ci} & I & (SQ_i + M_i E^T)^T (L_i + \delta B_i \Psi_i)^T & (SQ_i + M_i E^T)^T & \bar{Z}_i \\ * & -I & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -\frac{1}{\alpha} I & 0 \\ * & * & * & * & -\Theta_i \end{bmatrix} < 0 \tag{4.6}$$

In addition, we have

$$\underline{\Lambda}_{ci} = \underline{\lambda}_{ii}EM_iE^T + A_{ci}(SQ_i + M_iE^T) + (SQ_i + M_iE^T)^T {A_{ci}}^T,$$
$$\bar{\Lambda}_{ci} = \bar{\lambda}_{ii}EM_iE^T + A_{ci}(SQ_i + M_iE^T) + (SQ_i + M_iE^T)^T {A_{ci}}^T.$$

Let $R_i = K_i(SQ_i + M_iE^T)$, that is to say, $K_i = R_i(SQ_i + M_iE^T)^{-1}$. In addition, because (4.5) and (4.6) are equivalent to (4.3) and (4.4), Theorem 5 is proved. □

Remark 8. When the transition probability $\lambda_{ij}(h)$ is constant, that is, $\bar{\lambda}_{ij} = \underline{\lambda}_{ij} = \lambda_{ij}$ and the nonlinear term $f_{r_t}(t, x) = 0$ of the system, according to Theorem 5, the state feedback stabilization condition for the system to become a continuous-time singular linear Markov jump system can be obtained, namely

$$\begin{bmatrix} \Xi_i & I & (SQ_i + M_iE^T)^T(\delta B_i\Psi_i)^T & (SQ_i + M_iE^T)^T & Z_i \\ * & -I & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -\frac{1}{\alpha}I & 0 \\ * & * & * & * & -\Lambda_i \end{bmatrix} < 0.$$

In addition, we have

$$\Xi_i = \lambda_{ii}EM_iE^T + A_i(SQ_i + M_iE^T) + B_iR_i + (SQ_i + M_iE^T)^T A_i^T + R_i^T B_i^T,$$
$$Z_i = [\sqrt{\lambda_{i1}}EM_i\bar{V}, \cdots, \sqrt{\lambda_{i(i-1)}}EM_i\bar{V}, \sqrt{\lambda_{i(i+1)}}EM_i\bar{V}, \cdots, \sqrt{\lambda_{iN}}EM_i\bar{V}].$$

5. Numerical simulation

In this section, three numerical examples are given to illustrate the validity of the stochastic stability condition of system (3.4) in Theorem 3 and the state feedback stabilization condition of closed-loop system (4.2) in Theorem 5.

Example 1. Consider the stochastic stability of a nonlinear singular continuous-time semi-Markov jump system with the following parameters.

$$E = \begin{bmatrix} 2.35 & 0.86 & 0 \\ 0.6 & 0.72 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} -4.2 & -1.12 & -0.35 \\ 0 & 0 & 0.84 \\ -2.2 & -2.64 & -0.77 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -8.65 & -2.54 & -2.03 \\ -0.6 & -0.72 & 0.84 \\ -3.3 & -3.96 & -0.77 \end{bmatrix}, L_1 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, L_2 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T, B_2 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T, \Psi_1 = \begin{bmatrix} 0.1 & 0.1 & 0.1 \end{bmatrix}, \Psi_2 = \begin{bmatrix} 0.2 & 0.2 & 0.2 \end{bmatrix}$$

Assuming that the residence time h of each mode follows the Weibull distribution of the scale parameter $\gamma = 1$ and the shape parameter $\beta = 2$, the cumulative distribution function is as follows:

$$F(h) = \begin{cases} 1 - \exp(-h^2), h \geq 0 \\ 0, h < 0 \end{cases}.$$

Its probability distribution function is as follows:

$$f(h) = \begin{cases} 2h\exp(-h^2), h \geq 0 \\ 0, h < 0 \end{cases}.$$

Therefore, the transition probability function is

$$\lambda(h) = \frac{f(h)}{1 - F(h)} = 2h.$$

According to the probability distribution function of the Weibull distribution with scale parameter $\gamma = 1$ and shape parameter $\beta = 2$, when the transition probability h is in the interval $[0.1000, 4.6000]$, the probability of system transition is greater than 0.99. So it can be assumed that the lower and upper bounds of the transition probability are

$$\left[\underline{\lambda}_{ij}\right]_{2*2} = \begin{bmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{bmatrix}, \left[\bar{\lambda}_{ij}\right]_{2*2} = \begin{bmatrix} -4.6 & 4.6 \\ 4.6 & -4.6 \end{bmatrix}.$$

Let $S = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$, that is, the conditions $ES = 0$ and $rank(S) = n - r$ are satisfied, and the linear matrix inequality (3.18) and (3.19) can be obtained by using the linear matrix inequalities toolbox in MATLAB.

$$M_1 = \begin{bmatrix} 28.6551 & -26.4673 & 0.0000 \\ -26.4673 & 28.0566 & 0.0000 \\ 0.0000 & 0.0000 & 222.1177 \end{bmatrix}, M_2 = \begin{bmatrix} 25.9485 & -23.7717 & 0.0000 \\ -23.7717 & 23.6364 & 0.0000 \\ 0.0000 & 0.0000 & 222.1177 \end{bmatrix}$$

$$\alpha = 0.0041, Q_1 = \begin{bmatrix} -6.1074 & -23.0682 & 19.2619 \end{bmatrix}, Q_2 = \begin{bmatrix} 4.7655 & -4.7633 & 3.6497 \end{bmatrix}$$

For simulation, the nonlinear function is taken as

$$f_1(t, x_1, x_2, x_3) = \begin{bmatrix} \sin(0.01x_1 + 0.01x_2) \\ \sin(0.05x_2) \\ \sin(0.01x_2 - 0.01x_3) \end{bmatrix}, f_2(t, x_1, x_2, x_3) = \begin{bmatrix} \sin(0.02x_1 - 0.02x_2) \\ \sin(0.1x_2) \\ \sin(0.01x_2 + 0.01x_3) \end{bmatrix}.$$

The network attack function is

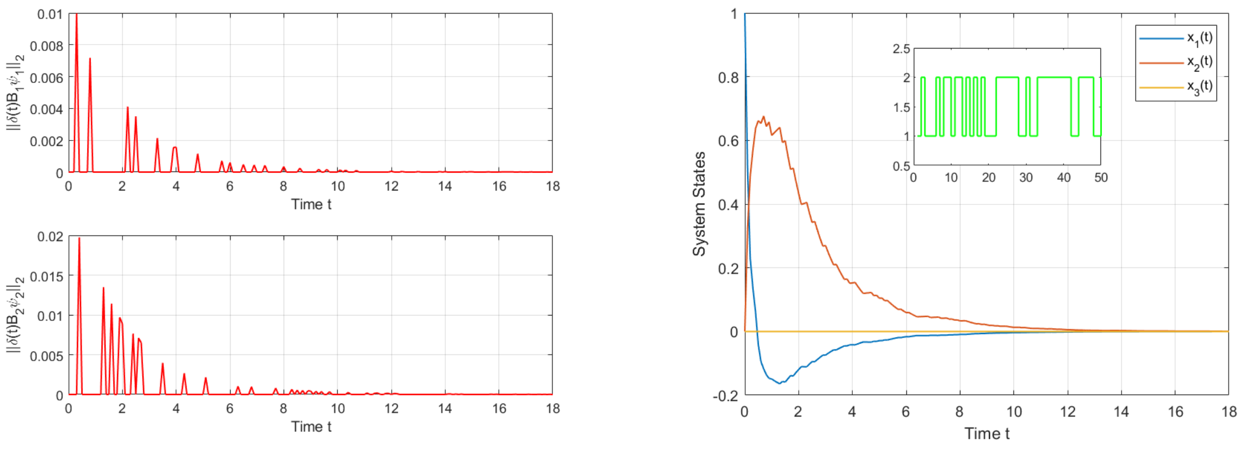
$$\psi_1(t, x_1, x_2, x_3) = \sin(0.01x_1 + 0.01x_2 + 0.01x_3),$$

$$\psi_2(t, x_1, x_2, x_3) = \sin(0.02x_1 + 0.02x_2 + 0.02x_3),$$

which satisfies the conditions $f_i(t, 0) = 0$, $\|f_i(t, x) - f_i(t, \tilde{x})\| \leq \|L_i(x - \tilde{x})\|$, and $\psi_i(t, x) \leq \Psi_i x$. Variable $\delta(t)$ obeys the Bernoulli distribution, generating stochastic numbers from 0 to 1. If the stochastic number is in the range $\left[0, \bar{\delta}\right]$, then $\delta(t) = 1$. If the stochastic number is in $\left[\bar{\delta}, 1\right]$, then $\delta(t) = 0$. Let $\bar{\delta} = 0.49$, and we have

$$\Pr\{\delta(t) = 1\} = E\{\delta(t)\} = 0.49, \Pr\{\delta(t) = 0\} = 1 - E\{\delta(t)\} = 1 - \rho = 0.51.$$

Taking the initial value as $Ex_0 = \begin{bmatrix} 2.35 & 0.6 & 0 \end{bmatrix}^T$, the stochastic false data injected by the attacker into the nonlinear singular semi-Markov jump system is shown in Figure 2(a). The simulation results of the state trajectory of the system and the stochastic mode of evolution are shown in Figure 2(b), and it can be seen that the system is stochastically stable.



(**a**) Stochastic false data injected by the FDI attacker

(**b**) State responses of the system

Figure 2. The stochastic stability of a nonlinear singular continuous-time semi-Markov jump system.

Example 2. Consider the problem of state feedback stabilization of a nonlinear singular continuous-time semi-Markov jump system with the following parameters. It should be noted that, except for the following parameters, other parameters are consistent with Example 1.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 7 & -1 \\ 0 & 2 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -14 & -12 & 1 \\ 0 & 10 & -1 \\ 0 & 15 & -7 \end{bmatrix}$$

Let $S = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}^T$. Solving linear matrix inequalities (4.3) and (4.4), we get

$$M_1 = \begin{bmatrix} 2.3505 & 0.0000 & -1.3271 \\ 0.0000 & 10.4026 & 0.0000 \\ -1.3271 & 0.0000 & 3.0335 \end{bmatrix}, M_2 = \begin{bmatrix} 2.1030 & 0.0000 & -1.3029 \\ 0.0000 & 10.4026 & 0.0000 \\ -1.3029 & 0.0000 & 3.5680 \end{bmatrix},$$

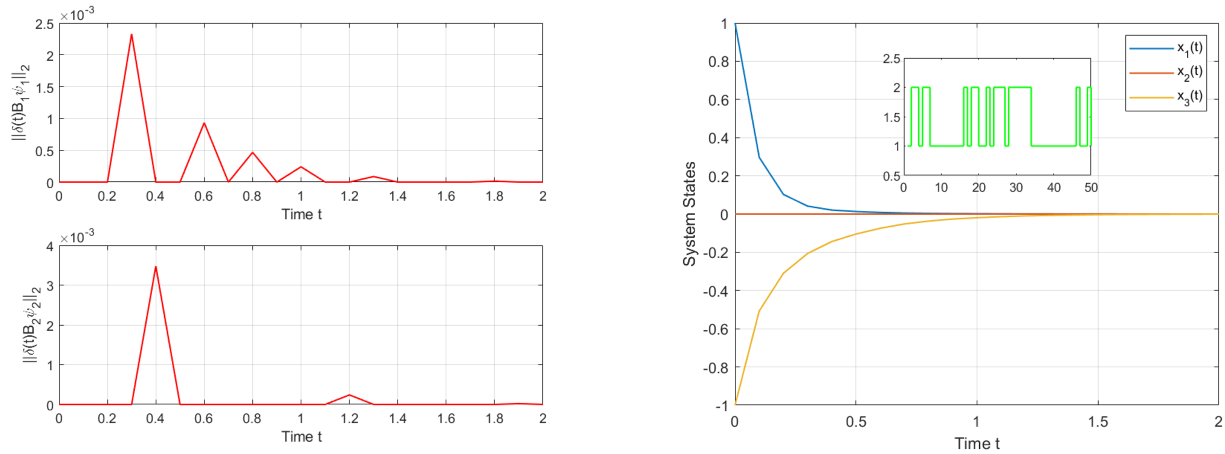
$$\alpha = 0.0869, Q_1 = \begin{bmatrix} -0.5049 & 0.9227 & -0.5341 \end{bmatrix}, Q_2 = \begin{bmatrix} -1.5057 & 0.5769 & 0.8349 \end{bmatrix},$$

$$R_1 = \begin{bmatrix} -6.8064 & 1.2457 & -1.1721 \end{bmatrix}, R_2 = \begin{bmatrix} -40.3798 & 21.8775 & 27.6501 \end{bmatrix}.$$

According to $K_i = R_i(SQ_i + M_iE^T)^{-1}$, the state feedback gain matrix can be obtained as

$$K_1 = \begin{bmatrix} -3.5721 & -1.3501 & -1.7114 \end{bmatrix}, K_2 = \begin{bmatrix} 9.3714 & -37.9196 & 2.2982 \end{bmatrix}.$$

Taking the initial value of $Ex_0 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T$, stochastic false data injected by the attacker is shown in Figure 3(a), and the simulation results of the state trajectory and the stochastic mode of evolution are shown in Figure 3(b). It can be seen that the closed-loop system is stochastically stable.



(a) Stochastic false data injected by the FDI attacker **(b)** State responses of the closed-loop system

Figure 3. Nonlinear singular semi-Markov systems with a state feedback controller.

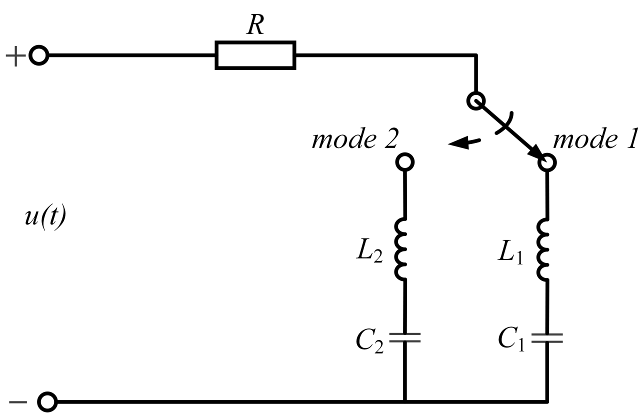


Figure 4. An RLC series circuit.

Example 3. Contemplate an RLC series circuit as depicted in Figure 4, sourced from reference [38, 39]. We define $x_1(t) = u_C(t)$ and $x_2(t) = i_{\mathcal{L}}(t)$ as the state vectors, where $u_C(t)$ is the voltage across the capacitor and $i_{\mathcal{L}}(t)$ is the current flowing through the inductor at that moment. Two modes are considered to describe the switching behavior regulated by a semi-Markov chain. According to Kirchhoff's law, we get $\mathcal{L}_i \frac{di_{\mathcal{L}}(t)}{dt} + u_C(t) + Ri_{\mathcal{L}}(t) = u(t)$ and $C_i \frac{du_C(t)}{dt} = i_{\mathcal{L}}(t)$. We then derive the following state space $\dot{x}(t) = \begin{bmatrix} 0 & \frac{1}{C_i} \\ -\frac{1}{\mathcal{L}_i} & -\frac{R}{\mathcal{L}_i} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{\mathcal{L}_i} \end{bmatrix} u(t)$. Considering the content of this paper, we add FDI attack and nonlinear terms and use them as interference terms, so we get $\dot{x}(t) = \begin{bmatrix} 0 & \frac{1}{C_i} \\ -\frac{1}{\mathcal{L}_i} & -\frac{R}{\mathcal{L}_i} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{\mathcal{L}_i} \end{bmatrix} u_b(t) + f_i(t, x)$, where the resistance $R = 50\Omega$, $\mathcal{L}_1 = 4H$, $\mathcal{L}_2 = 8H$, $C_1 = 0.2F$, $C_2 = 0.8F$, and the other parameters are as follows:

$$\psi_1(t, x_1, x_2) = \sin(0.01x_1 + 0.01x_2), \psi_2(t, x_1, x_2) = \sin(0.02x_1 + 0.02x_2),$$

$$f_1(t, x_1, x_2) = \begin{bmatrix} \sin(0.01x_1 + 0.01x_2) \\ \sin(0.05x_2)) \end{bmatrix}, f_2(t, x_1, x_2) = \begin{bmatrix} \sin(0.02x_1 - 0.02x_2) \\ \sin(0.1x_2)) \end{bmatrix}.$$

Solving the linear matrix inequalities, we get

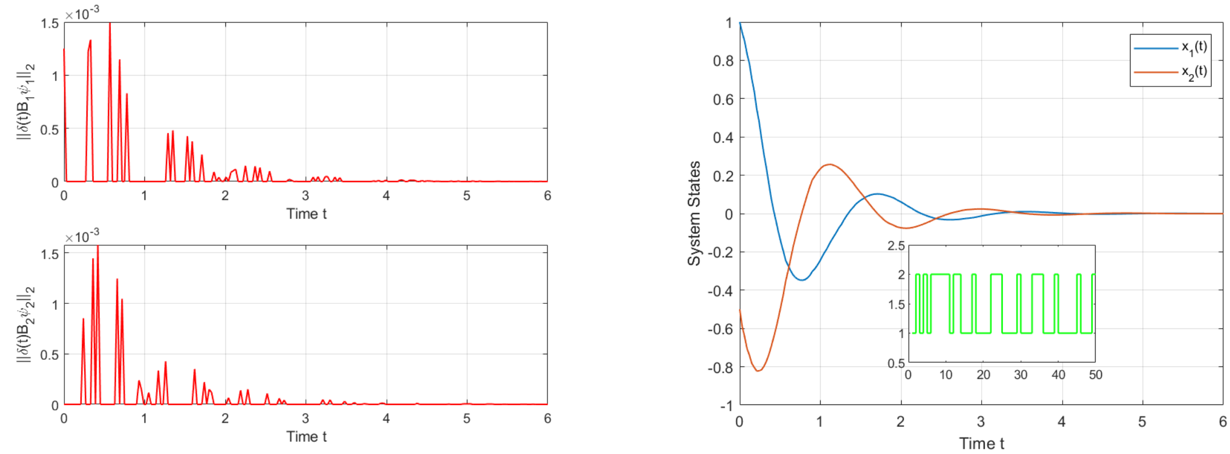
$$M_1 = \begin{bmatrix} 1.0522 & -0.3044 \\ -0.3044 & 1.1077 \end{bmatrix}, M_2 = \begin{bmatrix} 2.0297 & -1.0834 \\ -1.0834 & 3.0059 \end{bmatrix},$$

$$\alpha = 0.2082, R_1 = \begin{bmatrix} -36.9955 & 47.9857 \end{bmatrix}, R_2 = \begin{bmatrix} -69.9046 & 119.9846 \end{bmatrix}.$$

According to $K_i = R_i(SQ_i + M_iE^T)^{-1}$, the state feedback gain matrix can be obtained as

$$K_1 = \begin{bmatrix} -24.5829 & 36.5645 \end{bmatrix}, K_2 = \begin{bmatrix} -16.2626 & 34.0544 \end{bmatrix}.$$

Stochastic false data injected by the attacker is shown in Figure 5(a), and the simulation results of the state trajectory and the stochastic mode of evolution are shown in Figure 5(b).



(a) Stochastic false data injected by the FDI attacker

(b) State responses of the closed-loop system

Figure 5. Nonlinear non-singular semi-Markov systems with a state feedback controller.

6. Conclusions

This paper mainly studies the stability analysis and stabilization of nonlinear singular semi-Markov jump systems under FDI attacks. First, based on the Lyapunov functional and implicit function theorem, the basic stochastic stability conditions of nonlinear singular semi-Markov jump systems under FDI attacks are obtained. Then, the solvable stochastic stability conditions of linear matrix inequalities are given by means of matrix singular value decomposition and Schur's complement lemma. Under the stochastic stability condition of the linear matrix inequality, the state feedback controller of the system is designed. Finally, three numerical examples are employed to demonstrate the effectiveness of the results. The main innovation of this paper is that the FDI attack is introduced in the nonlinear singular semi-Markov jump system, and the stochastic stability analysis and the design of the state feedback controller of the system under network attack are studied. In the follow-up, the stability analysis and stabilization of nonlinear singular semi-Markov jump systems under other forms of network attacks can be considered in various ways. At the same time, the method adopted in this paper can be extended to the stability analysis and stabilization problems of singular semi-Markov jump systems with time delay, and the finite-time control problems of such systems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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