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Research article

Almost periodic positive solutions of two generalized Nicholson's blowflies equations with iterative term

Youqian Bai and Yongkun Li*

Department of Mathematics, Yunnan University, Kunming 650091, China

* **Correspondence:** Email: yklie@ynu.edu.cn.

Abstract: This article considered two generalized Nicholson's blowflies equations with iteration term and time delay, as well as with immigration, and Nicholson's blowflies equation with iteration term and time delay, as well as harvesting term, respectively. Under appropriate conditions, the existence and uniqueness of almost periodic positive solutions for these two equations were established, respectively, by employing Banach's fixed point theorem. These results were brand new.

Keywords: Nicholson's blowflies models; iterative terms; time delays; Banach fixed point theorem; almost periodic solutions

1. Introduction

The Australian sheep fly has caused serious damage to sheep herds in many parts of the world, including Australia, New Zealand, South Africa, and North America. It causes 90% of skin myiasis, resulting in millions of sheep deaths and billions of dollars in annual losses [1].

Female flies lay 300 eggs in open wounds or dirty areas of wool, and after no more than 24 hours, the larvae appear from the eggs, causing sheep to exhibit many symptoms such as restlessness, tangled wool, and an imperceptible unique odor, attracting other flies to lay more eggs. Therefore, if this flyworm disease is not treated immediately, it can lead to sheep death.

To address the issues mentioned above, in 1954, A. J. Nicholson conducted a series of experiments on sheep flies and discovered characteristic periodic oscillations or periods of approximately 35 to 40 days, corresponding to delays of 9 to 15 days. Based on the experimental data obtained by Nicholson [2], Gurney et al. [3] proposed the following model, which can well match the experimental data:

$$y'(t) = \alpha y(t-\sigma)\exp\left[-\frac{y(t-\sigma)}{K}\right] - \zeta y(t),$$

in which y indicates the population of mature adults, α stands for the maximum per capita daily egg production rate, ζ signifies per capita daily death rate, σ indicates the approximate time of the lifecycle, and $1/K$ represents the size at which the fly population reproduces at its maximum rate.

Since then, various dynamics of this model have been extensively studied, and various generalizations of this model have been proposed, such as models with disparate mature delay and feedback delay [4], models with multiple delays [5, 6], models with harvesting terms [7], models with immigration [8], models with random disturbances [9, 10], models with two distinct distributed delays [11], models represented by iterative differential equations [12, 13], and so on. Meanwhile, the various qualitative properties of these models have also been extensively studied. It should be pointed out that numerous models in life sciences are described by iterative differential equations [12–16]. Nevertheless, due to the fact that iterative differential equations are equations with deviating arguments that not only depend on time variables, but also on state variables, it is difficult to study the dynamics of the models represented by these differential equations.

Recently, with the help of the Schauder fixed point theorem, Bouakkaz and Khemis [12] studied the periodic positive solutions for a generalized Nicholson's blowflies equation

$$u'(t) = -a(t)u(t) + b(t)u(t-\sigma)e^{-\gamma(t)u(t-\sigma)} - cu(t-\sigma)E(t, u(t), u^{[2]}(t), \ldots, u^{[n]}(t)),$$

in which the last term indicates the harvesting term and $u^{[2]}(t) = u(u(t))$, $u^{[3]}(t) = u(u(u(t)))$, $\ldots$, $u^{[n]}(t) = u(u^{[n-1]}(t))$.

Very recently, by applying the Banach contraction principle, Khemis [13] established the existence and uniqueness of periodic positive solutions to the following Nicholson's fly equation involving time-varying delays and iteration terms:

$$y'(t) = -a(t)y(t) + b(t)y(t-\sigma(t))e^{-c(t)y^{[2]}(t)}, \tag{1.1}$$

in which $y^{[2]}(t) = y(y(t))$.

As is well known, even if all the parameters (a, b, c, σ) in (1.1) are periodic functions, if their periods are noncommensurable, (1.1) will become an almost periodic equation rather than a periodic equation. In this situation, (1.1) generally does not have a periodic solution, but it may have almost periodic solutions. Therefore, to search for the almost periodic solutions of Nicholson's blowflies equations is more reasonable than to search for their periodic solutions [17–22]. However, there have been no published papers on the almost periodic solutions for Eq (1.1) so far. To fill this gap, this article will establish the existence and uniqueness of almost periodic positive solutions for two generalized models of (1.1).

The first equation is the Nicholson's blowflies equation with iterative terms and immigration:

$$u'(t) = -A(t)u(t) + B(t)u(t-\sigma(t))e^{-C(t)u^{[n]}(t)} + I(t), \tag{1.2}$$

where $A, B, C \in C(\mathbb{R}, (0, +\infty))$, $I, \sigma \in C(\mathbb{R}, \mathbb{R}^+)$, and $u^{[n]}(t) = u(u^{n-1]}(t)), n \geq 1$, $I(t)$ represents immigration.

Obviously, when $n = 2$ and $I(t) \equiv 0$, Eq (1.2) reduces to Eq (1.1).

The second equation is the following Nicholson's blowflies equation with iterative terms and harvesting term:

$$u'(t) = -a(t)u(t) + b(t)u(t-\sigma(t))e^{-c(t)u^{[n]}(t)} - H(t)u(t-\sigma(t)), \tag{1.3}$$

where $a, b, c \in C(\mathbb{R}, (0, +\infty))$, $H, \sigma \in C(\mathbb{R}, \mathbb{R}^+)$, and $u^{[n]}(t) = u(u^{n-1]}(t)), n \geq 1$, $H(t)$ represents harvesting rate.

Clearly, when $n = 2$ and $H(t) \equiv 0$, Eq (1.3) also reduces to Eq (1.1).

The rest of this article is arranged as follows: In the second section, we review the definition and some basic properties of almost periodic functions, and introduce three useful lemmas. In the third section, we will state and prove the main results of this paper. Finally, in the fourth section, we present two examples to demonstrate the effectiveness of our results.

2. Preliminaries

Let us denote by $CB(\mathbb{R}, \mathbb{R})$ the space of all bounded and continuous functions $f : \mathbb{R} \to \mathbb{R}$.

Definition 2.1. *[23] A function $u \in CB(\mathbb{R}, \mathbb{R})$ is called almost periodic, if for each $\varepsilon > 0$, one can find a positive number $l = l(\varepsilon)$ such that every interval of length $l(\varepsilon)$ contains a point $\delta = \delta(\varepsilon)$ satisfying*

$$|u(t + \delta) - u(t)| < \varepsilon$$

for all $t \in \mathbb{R}$. The family of such functions will be signified as $AP(\mathbb{R}, \mathbb{R})$.

Lemma 2.1. *[23] The space $AP(\mathbb{R}, \mathbb{R})$ endowed with the supremum norm is a Banach algebra.*

Lemma 2.2. *[24, 25] Let $u, v \in AP(\mathbb{R}, \mathbb{R})$. Then, $u(\cdot - v(\cdot)) \in AP(\mathbb{R}, \mathbb{R})$.*

Lemma 2.3. *[23] If $B \in AP(\mathbb{R}, \mathbb{R})$ and $A \in AP(\mathbb{R}, \mathbb{R}^+)$ satisfying $\inf_{t\in\mathbb{R}} A(t) > 0$, then the nonhomogeneous linear differential equation*

$$u'(t) = -A(t)u(t) + B(t)$$

admits a unique almost periodic solution that can be expressed as

$$u(t) = \int_{-\infty}^{t} e^{-\int_s^t A(\tau)\mathrm{d}\tau} B(s)\mathrm{d}s.$$

3. Main results

In this section, we will present and demonstrate the main results of this article.

For convenience, for $u \in BC(\mathbb{R}, \mathbb{R})$, we will use the following notations:

$$u^+ = \sup_{s\in\mathbb{R}} |u(s)| \text{ and } u^- = \inf_{s\in\mathbb{R}} |u(s)|.$$

Throughout this paper, as to Eq (1.2), we use the following assumptions:

(G_1) Functions $A, B, C, I, \sigma \in AP(\mathbb{R}, \mathbb{R}^+)$ with $A^- > 0$ and $I^- > 0$, and $n \geq 1$ is an integer.
(G_2) There is a positive constant K such that $\frac{B^+K+I^+}{A^-} \leq K$.
(G_3) There is a constant $M > 0$ such that $(B^+K + I^+)\left(1 + \frac{A^+}{A^-}\right) \leq M$, in which K is mentioned in (G_2).
(G_4) $\frac{B^+}{A^-}\left[1 + KC^+\frac{1-M^n}{1-M}\right] < 1$, in which K, M is mentioned in (G_3).

Also, as to Eq (1.3), we use the following hypotheses:

(H_1) Functions $H, a, b, c, \sigma \in AP(\mathbb{R}, \mathbb{R}^+)$ with $a^- > 0$ and the integer number $n \geq 1$.
(H_2) There is a positive number $\mathcal{K}$ such that $b^- e^{-c^+\mathcal{K}} - H^+ > 0$.
(H_3) There is a positive number $\mathcal{M}$ satisfying that $(b^+ + H^+)\mathcal{K}\Big(1 + \frac{a^+}{a^-}\Big) \leq \mathcal{M}$, where $\mathcal{K}$ is mentioned in (H_2).
(H_4) $\frac{1}{a^-}\Big(H^+ + b^+ + b^+\mathcal{K}c^+\frac{1-\mathcal{M}^n}{1-\mathcal{M}}\Big) < 1$, where $\mathcal{K}, \mathcal{M}$ is mentioned in (H_3).

Lemma 3.1. *Let* (G_1) *hold. If u is a bounded solution to Eq* (1.2)*, then u solves the following integral equation:*

$$u(t) = \int_{-\infty}^{t} e^{-\int_s^t A(\tau)d\tau}\Big[B(s)u\left(s - \sigma(s)\right)e^{-C(s)u^{[n]}(s)} + I(s)\Big]\,\mathrm{ds}, \tag{3.1}$$

and vice versa.

Proof. Let u be a bounded solution to Eq (1.2). Multiplying both sides of (1.2) by $e^{\int_{t_0}^t A(\tau)d\tau}$ and integrating from t_0 to t, we have

$$u(t)e^{\int_{t_0}^t A(\tau)d\tau} = u(t_0) + \int_{t_0}^{t} e^{\int_{t_0}^s A(\tau)d\tau}[B(s)u(s - \sigma(s))e^{-C(s)u^{[n]}(s)} + I(s)]ds,$$

that is,

$$u(t) = u(t_0)e^{-\int_{t_0}^t A(\tau)d\tau} + \int_{t_0}^{t} e^{-\int_s^t A(\tau)d\tau}[B(s)u(s - \sigma(s))e^{-C(s)u^{[n]}(s)} + I(s)]ds.$$

Letting $t_0 \to -\infty$, we see that u satisfies (3.1).

If u solves (3.1), differentiating both sides of (3.1) with respect to t, one finds that u also solves (1.2). The proof is finished.

Similarly, one can prove:

Lemma 3.2. *Let* (H_1) *be fulfilled. If u is a bounded solution to Eq* (1.3)*, then u solves the following integral equation:*

$$u(t) = \int_{-\infty}^{t} e^{-\int_s^t a(\tau)\mathrm{d}\tau}\Big[b(s)u\left(s - \sigma(s)\right)e^{-c(s)u^{[n]}(s)} - H(s)u\left(s - \sigma(s)\right)\Big]\,\mathrm{ds},$$

and vice versa.

Theorem 3.1. *Let* (G_1)*–*(G_4) *be fulfilled. Then, Eq* (1.2) *possesses one unique almost periodic positive solution belonging to* $\mathbb{Q}$*, where*

$$\mathbb{Q} = \left\{u \in AP(\mathbb{R}, \mathbb{R}^+), 0 \leq u(t) \leq K, |u(t_2) - u(t_1)| \leq M\,|t_2 - t_1|\,, \forall t_1, t_2 \in \mathbb{R}\right\}.$$

Proof. In view of Lemma 3.1, let us consider a mapping $\mathcal{T} : \mathbb{Q} \to \mathbb{Q}$ defined as follows:

$$(\mathcal{T}u)(t) = \int_{-\infty}^{t} e^{-\int_s^t A(\tau)d\tau}\Big[B(s)u\left(s - \sigma(s)\right)e^{-C(s)u^{[n]}(s)} + I(s)\Big]\,\mathrm{ds}.$$

First, we will demonstrate that $0 \leq (\mathcal{T}u)(t) \leq K$ for all $u \in \mathbb{Q}$ and $t \in \mathbb{R}$. Indeed, one finds that

$$\begin{aligned}(\mathcal{T}u)(t) &= \int_{-\infty}^{t} e^{-\int_s^t A(\tau)d\tau}\Big[B(s)u\,(s-\sigma(s))\,e^{-C(s)u^{[n]}(s)} + I(s)\Big]\,\mathrm{ds}\\ &\leq \int_{-\infty}^{t} e^{-A^-(t-s)}\,(B^+K + I^+)\,\mathrm{ds}\\ &= \frac{B^+K + I^+}{A^-}.\end{aligned} \tag{3.2}$$

Thus, from conditions (G_2), (G_2), and (3.2), we arrive at

$$0 \leq (\mathcal{T}u) \leq K.$$

Second, we will verify that $|(\mathcal{T}u)(t_2) - (\mathcal{T}u)(t_1)| \leq M\,|u(t_2) - u(t_1)|$. In fact, for any $t_1, t_2 \in \mathbb{R}$ with $t_2 > t_1$, we deduce that

$$\begin{aligned}&|(\mathcal{T}u)(t_2) - (\mathcal{T}u)(t_1)|\\ =&\left|\int_{-\infty}^{t_2} e^{-\int_s^{t_2} A(\tau)d\tau}\Big[B(s)u\,(s-\sigma(s))\,e^{-C(s)u^{[n]}(s)} + I(s)\Big]\,\mathrm{ds}\right.\\ &\left.-\int_{-\infty}^{t_1} e^{-\int_s^{t_1} A(\tau)d\tau}\Big[B(s)u\,(s-\sigma(s))\,e^{-C(s)u^{[n]}(s)} + I(s)\Big]\,\mathrm{ds}\right|\\ \leq&\int_{t_1}^{t_2}\left|e^{-\int_s^{t_2} A(\tau)d\tau}\Big[B(s)u\,(s-\sigma(s))\,e^{-C(s)u^{[n]}(s)} + I(s)\Big]\right|\,\mathrm{ds}\\ &+\left|\int_{-\infty}^{t_1}(e^{-\int_s^{t_2} A(\tau)d\tau} - e^{-\int_s^{t_1} A(\tau)d\tau})\left|B(s)u\,(s-\sigma(s))\,e^{-C(s)u^{[n]}(s)} + I(s)\right|\,\mathrm{ds}\right|\\ \leq&\int_{t_1}^{t_2}(B^+K + I^+)\mathrm{ds} + \int_{-\infty}^{t_1}(B^+K + I^+)\left|e^{-\int_s^{t_2} A(\tau)d\tau} - e^{-\int_s^{t_1} A(\tau)d\tau}\right|\,\mathrm{ds}\\ =&(B^+K + I^+)\,|t_2 - t_1| + \int_{-\infty}^{t_1}(B^+K + I^+)\left|e^{-\int_s^{t_2} A(\tau)d\tau} - e^{-\int_s^{t_1} A(\tau)d\tau}\right|\,\mathrm{ds}.\end{aligned} \tag{3.3}$$

By the mean value theorem, we infer that

$$\begin{aligned}\left|e^{-\int_s^{t_2} A(\tau)d\tau} - e^{-\int_s^{t_1} A(\tau)d\tau}\right| &\leq \left|e^{-\int_s^{t_1} A(\tau)d\tau}\int_{t_1}^{t_2} A(\tau)d\tau\right|\\ &\leq A^+ e^{-A^-(t_1-s)}|t_2 - t_1|.\end{aligned} \tag{3.4}$$

Thus, based on (G_3), from (3.3) and (3.4), we infer that

$$\begin{aligned}&|(\mathcal{T}u)(t_2) - (\mathcal{T}u)(t_1)|\\ \leq&(B^+K + I^+)\,|t_2 - t_1| + \int_{-\infty}^{t_1}(B^+K + I^+)\,A^+ e^{-A^-(t_1-s)}|t_2 - t_1|\mathrm{ds}\\ \leq&(B^+K + I^+)\left(1 + \frac{A^+}{A^-}\right)|t_2 - t_1|\\ \leq&M|t_2 - t_1|.\end{aligned}$$

Third, we will prove that $\mathcal{T}u \in AP(\mathbb{R}, \mathbb{R}^+)$ for every $u \in \mathbb{Q}(\subset AP(\mathbb{R}, \mathbb{R}^+))$. According to Lemma 2.3, we only need to show that $h(s) =: B(s)u\left(s - \sigma(s)\right) e^{-C(s)u^{[n]}(s)} + I(s)$ is almost periodic. In order to prove $h \in AP(\mathbb{R}, \mathbb{R})$ by Lemmas 2.1 and 2.2, it suffices to prove that $e^{-C(s)u^{[n]}(s)}$ is almost periodic. Notice that

$$\begin{aligned}
&\left|e^{-C(s+\sigma)u^{[n]}(s+\sigma)} - e^{-C(s)u^{[n]}(s)}\right| \\
\leq &\left|C(s+\sigma)u^{[n]}(s+\sigma) - C(s)u^{[n]}(s)\right| \\
\leq &C(s+\sigma)\left|u^{[n]}(s+\sigma) - u^{[n]}(s)\right| + u^{[n]}(s)\left|C(s+\sigma) - C(s)\right| \\
\leq &C^+\left|u^{[n]}(s+\sigma) - u^{[n]}(s)\right| + K\left|C(s+\sigma) - C(s)\right| \\
\leq &C^+M^{n-1}\left|u(s+\sigma) - u(s)\right| + K\left|C(s+\sigma) - C(s)\right|.
\end{aligned}$$

By Definition 2.1 and the fact that $u, C \in AP(\mathbb{R}, \mathbb{R})$, we deduce that $h \in AP(\mathbb{R}, \mathbb{R})$. Consequently, we can gain that $\mathcal{T}u \in AP(\mathbb{R}, \mathbb{R}^+)$.

Finally, we will prove that the mapping $\mathcal{T}$ is a contraction one from $\mathbb{Q}$ to itself. Actually, for any $u, v \in \mathbb{Q}, t \in \mathbb{R}$, we get

$$\begin{aligned}
\left|(\mathcal{T}u)(t) - (\mathcal{T}v)(t)\right| = &\left|\int_{-\infty}^{t} e^{-\int_s^t A(\tau)d\tau}\left[B(s)u\left(s - \sigma(s)\right) e^{-C(s)u^{[n]}(s)} + I(s)\right] \mathrm{ds}\right. \\
&\left. - \int_{-\infty}^{t} e^{-\int_s^t A(\tau)d\tau}\left[B(s)v\left(s - \sigma(s)\right) e^{-C(s)v^{[n]}(s)} + I(s)\right] \mathrm{ds}\right| \\
\leq &\left|\int_{-\infty}^{t} e^{-\int_s^t A(\tau)d\tau}B(s)\left|u\left(s - \sigma(s)\right) - v\left(s - \sigma(s)\right)\right| e^{-C(s)u^{[n]}(s)}\mathrm{ds}\right| \\
&+ \left|\int_{-\infty}^{t} e^{-\int_s^t A(\tau)d\tau}B(s)v\left(s - \sigma(s)\right)\left|e^{-C(s)u^{[n]}(s)} - e^{-C(s)v^{[n]}(s)}\right| \mathrm{ds}\right|.
\end{aligned}$$

Note that

$$\begin{aligned}
&\left|e^{-C(s)u^{[n]}(s)} - e^{-C(s)v^{[n]}(s)}\right| \\
\leq &C(s)\left|u^{[n]}(s) - v^{[n]}(s)\right| \\
\leq &C^+\left[\left|u^{[n-1]}(u(s)) - u^{[n-1]}(v(s))\right| + \left|u^{[n-1]}(v(s)) - u^{[n-2]}\left(v^{[2]}(s)\right)\right|\right. \\
&+ \left|u^{[n-2]}\left(v^{[2]}(s)\right) - u^{[n-3]}\left(v^{[3]}(s)\right)\right| + \ldots \\
&\left. + \left|u\left(v^{[n-1]}(s)\right) - v\left(v^{[n-1]}(s)\right)\right|\right] \\
\leq &C^+\left[M^{n-1} + M^{n-2} + \ldots + M + 1\right]\|u - v\| \\
\leq &C^+\frac{1 - M^n}{1 - M}\|u - v\|.
\end{aligned}$$

We can deduce that

$$\begin{aligned}
&\left|(\mathcal{T}u)(t) - (\mathcal{T}v)(t)\right| \\
\leq &\left|\int_{-\infty}^{t} e^{-A^-(t-s)}B^+\left|u\left(s - \sigma(s)\right) - v\left(s - \sigma(s)\right)\right| \mathrm{ds}\right|
\end{aligned}$$

$$+\left|\int_{-\infty}^{t} e^{-A^{-}(t-s)}B^{+}KC^{+}\frac{1-M^{n}}{1-M}\|u-v\|\,\mathrm{ds}\right|$$
$$\leq\frac{B^{+}}{A^{-}}\left(1+KC^{+}\frac{1-M^{n}}{1-M}\right)\|u-v\|,$$

which yields

$$\|\mathcal{T}u-\mathcal{T}v\|\leq\frac{B^{+}}{A^{-}}\left(1+KC^{+}\frac{1-M^{n}}{1-M}\right)\|u-v\|.$$

In view of hypothesis (G_4), we see that $\mathcal{T}$ is a contraction. As a result, according to Banach fixed point theorem, $\mathcal{T}$ admits one unique fixed point in $\mathbb{Q}$, namely, Eq (1.2) admits a unique solution in $\mathbb{Q}$. This completes the proof.

Theorem 3.2. *Assume that* (H_1)–(H_4) *hold. Then, Eq* (1.3) *has a unique almost periodic positive solution in* Ω*, where*

$$\Omega=\left\{u\in AP(\mathbb{R},\mathbb{R}^{+}),0\leq u(t)\leq\mathcal{K},|u(t_2)-u(t_1)|\leq\mathcal{M}|t_2-t_1|,\forall t_1,t_2\in\mathbb{R}\right\}.$$

Proof. Based on Lemma 3.2, let us consider an operator $\Phi:\Omega\to\Omega$ defined as follows:

$$(\Phi u)(t)=\int_{-\infty}^{t} e^{-\int_{s}^{t}a(u)\mathrm{d}u}\Big[b(s)u(s-\sigma(s))e^{-c(s)u^{[n]}(s)}-H(s)u(s-\sigma(s))\Big]\mathrm{ds},$$

where $u\in\Omega, t\in\mathbb{R}$.

First of all, for any $u\in\Omega, t\in\mathbb{R}$, by (H_2), we infer that

$$(\Phi u)(t)=\int_{-\infty}^{t} e^{-\int_{s}^{t}a(u)\mathrm{d}u}u(s-\sigma(s))\left[b(s)e^{-c(s)u^{[n]}(s)}-H(s)\right]\mathrm{ds}$$
$$\geq\int_{-\infty}^{t} e^{-a^{+}(t-s)}u(s-\sigma(s))\left(b^{-}e^{-c^{+}\mathcal{K}}-H^{+}\right)\mathrm{ds}\geq 0.$$

Second, for any $t_1,t_2\in\mathbb{R}$ with $t_2>t_1$, by (H_3), we deduce that

$$\begin{aligned}
&|(\Phi u)(t_2)-(\Phi u)(t_1)|\\
\leq&\int_{t_1}^{t_2}\left|e^{-\int_{s}^{t_2}a(u)\mathrm{d}u}\left[b(s)u(s-\sigma(s))e^{-c(s)u^{[n]}(s)}-H(s)u(s-\sigma(s))\right]\right|\mathrm{ds}\\
&+\int_{-\infty}^{t_1}\left|e^{-\int_{s}^{t_2}a(u)\mathrm{d}u}-e^{-\int_{s}^{t_1}a(u)\mathrm{d}u}\right|\\
&\times\left|b(s)u(s-\sigma(s))e^{-c(s)u^{[n]}(s)}-H(s)u(s-\sigma(s))\right|\mathrm{ds}\\
\leq&(b^{+}+H^{+})\mathcal{K}|t_2-t_1|+\int_{-\infty}^{t_1}(b^{+}+H^{+})\mathcal{K}a^{+}e^{-a^{-}(t_1-s)}|t_2-t_1|\mathrm{ds}\\
\leq&(b^{+}+H^{+})\mathcal{K}\left(1+\frac{a^{+}}{a^{-}}\right)|t_2-t_1|\\
\leq&\mathcal{M}|t_2-t_1|.
\end{aligned}$$

Then, similar to the corresponding proof arguments used in Theorem 3.1, one can easily get that $\Phi x\in AP(\mathbb{R},\mathbb{R}^{+})$ for each $u\in\Omega$. Consequently, the operator Φ is a self-mapping.

Finally, for any $u, v \in \Omega, t \in \mathbb{R}$, we have

$$\begin{aligned}
&|(\Phi u)(t) - (\Phi v)(t)| \\
\leq &\int_{-\infty}^{t} e^{-\int_s^t a(u)\mathrm{d}u} b(s) \left| u(s-\sigma(s)) e^{-c(s)u^{[n]}(s)} - v(s-\sigma(s)) e^{-c(s)v^{[n]}(s)} \right| \mathrm{ds} \\
&+ \int_{-\infty}^{t} e^{-\int_s^t a(u)\mathrm{d}u} H(s) |u(s-\sigma(s)) - v(s-\sigma(s))| \mathrm{ds} \\
\leq &\frac{b^+}{a^-}\Big(1 + \mathcal{K} c^+ \frac{1-\mathcal{M}^n}{1-\mathcal{M}}\Big)\|u - v\| \\
&+ \left| \int_{-\infty}^{t} e^{-a^-(t-s)} H^+ |u(s-\sigma(s)) - v(s-\sigma(s))| \mathrm{ds} \right| \\
\leq &\frac{1}{a^-}\Big(H^+ + b^+ + b^+ \mathcal{K} c^+ \frac{1-\mathcal{M}^n}{1-\mathcal{M}}\Big)\|u - v\|,
\end{aligned}$$

which combined with (H_4) indicates that Φ is a contraction. This ends the proof.

4. Examples

This section presents two examples to show the effectiveness of the results obtained in this article.

Example 4.1. *Let us consider the Nicholson's Blowflies model with iterative term and immigration as follows:*

$$\begin{aligned}
u'(t) = &-(e + 0.01\cos^2 t)u(t) + (1 + ((e-2.1)|\sin t|)u(t - 2|\cos\sqrt{2}t|)e^{-0.005\sin^2 t u^{[2]}(t)} \\
&+ 0.01\sin^2\sqrt{3}t.
\end{aligned} \tag{4.1}$$

Obviously, Eq (4.1) *satisfies* (G_1)*, and by simple calculation, we have*

$$A^+ = e + 0.01, \quad A^- = e, \quad B^+ = e - 1.1, \quad C^+ = 0.005, \quad I^+ = 0.01.$$

Take $M = 10$ *and* $K = 1$*. Then, we have*

$$\begin{aligned}
&\frac{B^+K + I^+}{A^-} = \frac{(e-1.1) + 0.01}{e} < 1, \\
&(B^+K + I^+)\Big(1 + \frac{A^+}{A^-}\Big) = (e - 1.1 + 0.01) \times \Big(1 + \frac{e + 0.01}{e}\Big) < 10, \\
&\frac{B^+}{A^-}\Big[1 + KC^+(M+1)\Big] = \frac{e - 1.1}{e}(1 + 0.055) < 1.
\end{aligned}$$

Hence, conditions (G_1)*–*(G_3) *are also satisfied. As a consequence, from Theorems 3.1,* (4.1) *has a unique almost periodic positive solution.*

Example 4.2. *Let us consider the Nicholson's Blowflies model with iterative term and harvesting term as follows:*

$$u'(t) = -(4 - \cos^2 t)u(t) + (0.11 + 0.1|\sin t|)u(t - 6|\cos 2t|)e^{-\sin^2 t u^{[3]}(t)} - 0.01\sin^2(\sqrt{7}t)u(t - 6|\cos 2t|). \tag{4.2}$$

Obviously, Eq (4.2) *satisfies* (H_1)*, and by simple calculation, we have*

$$a^+ = 4, \quad a^- = 3, \quad b^+ = 0.21, \quad b^- = 0.11, \quad c^+ = 1, \quad H^+ = 0.01.$$

Take $\mathcal{M} = 3$ *and* $\mathcal{K} = 1$*. Then, we have*

$$b^- e^{-c^+\mathcal{K}} - H^+ \doteq 0.03 > 0,$$
$$(b^+ + H^+)\mathcal{K}\Big(1 + \frac{a^+}{a^-}\Big) \doteq 0.513 < 3$$
$$\frac{1}{a^-}\Big(H^+ + b^+ + b^+\mathcal{K}c^+\frac{1-\mathcal{M}^n}{1-\mathcal{M}}\Big) \doteq 0.983 < 1.$$

Hence, conditions (H_2)*–*(H_4) *are also satisfied. As a consequence, from Theorem 3.2,* (4.2) *has a unique almost periodic positive solution.*

Remark 4.1. *Examples 4.1 and 4.2 manifest that although the various parameters in the environment in which the sheep fly live, namely, the parameters in Eqs (1.2) and (1.3) are periodic, the density of the sheep fly population represented by Eqs (1.2) and (1.3) exhibits almost periodic oscillations rather than periodic oscillations due to the noncommensurability of periodicity of these parameters.*

5. Conclusions

This article has established the existence and uniqueness of almost periodic positive solutions for two generalized equations to Eq (1.1). The results gained in this article are new. The method presented in this article provides an example for further studying the existence of almost periodic solutions of models described by iterative differential equations in other types of life sciences. Our future efforts will focus on studying the dynamics of diffusive Nicholson's Blowflies equations with iterative terms.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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