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Research article

Nonradial singular solutions for elliptic equations with exponential nonlinearity

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Abstract: For any $R > 0$, infinitely many nonradial singular solutions can be constructed for the following equation:

$$-\Delta u = e^u \quad \text{in } B_R\backslash\{0\}, \tag{0.1}$$

where $B_R = \{x \in \mathbb{R}^N\ (N \geq 3) : \ |x| < R\}$. To construct nonradial singular solutions, we need to consider asymptotic expansion at the isolated singular point $x = 0$ of a prescribed solution of (0.1). Then, nonradial singular solutions of (0.1) can be constructed by using the asymptotic expansion and introducing suitable weighted Hölder spaces.

Keywords: nonradial singular solutions; asymptotic expansions; exponential nonlinearity; weighted spaces

1. Introduction

We are interested in singular solutions of the following equation with exponential nonlinearity:

$$\Delta u + e^u = 0 \quad \text{in } B_R\backslash\{0\}, \tag{1.1}$$

where $R > 0$ and $B_R = \{x \in \mathbb{R}^N\ (N \geq 3) : \ |x| < R\}$ is a ball.

By a singular solution of (1.1) we mean that $u \in C^2(B_R\backslash\{0\})$ and 0 is a nonremovable singular point of u.

It is easily known that (1.1) admits a (trivial) radial singular solution:

$$U_s(x) = U_s(|x|) := -2\ln|x| + \ln[2(N-2)]. \tag{1.2}$$

We are mainly concerned with nonradial singular solutions of (1.1) in this paper.

When $N = 2$, by using the moving plane method, the authors of [1] proved that every solution of

$$\begin{cases} \Delta u + e^u = 0 \quad \text{in} \quad \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^u dx < \infty \end{cases} \tag{1.3}$$

has the form

$$u(x) = \ln \frac{32\lambda^2}{(4 + \lambda^2|x - x_0|^2)^2}, \quad \lambda > 0, \; x_0 \in \mathbb{R}^2.$$

For $n > 0$, symmetry and uniqueness results were obtained in [2] for the solutions of the following problem:

$$\begin{cases} \Delta u + |x|^{2(n-1)} e^u = 0 \quad \text{in} \quad \mathbb{R}^2, \\ \int_{\mathbb{R}^2} |x|^{2(n-1)} e^u dx < \infty. \end{cases} \tag{1.4}$$

If $n = 1$, problem (1.4) reduces to (1.3); also, classification of solutions of (1.4) can be found in [1]. Under the condition that $n \geq 2$ is an integer, the authors of [2] showed that problem (1.4) admits radial and nonradial solutions, but, when $n > 0$ is not an integer, problem (1.4) only has radial solutions. Note that, for each $n > 0$, if $u(x)$ is a solution of (1.4), we can perform the following transformation:

$$v(x) = 2(n - 1) \ln |x| + u(x),$$

we see that $v(x)$ satisfies the following equation

$$\begin{cases} \Delta v + e^v = 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \{0\}, \\ \int_{\mathbb{R}^2 \backslash \{0\}} e^v dx < \infty. \end{cases} \tag{1.5}$$

The results in [2] imply that (1.5) admits a family of radial and nonradial singular solutions.

The asymptotic behavior of singular solutions of the problem given by

$$\begin{cases} \Delta u + e^u = 0 \quad \text{in } D_1 \backslash \{0\}, \\ \int_{D_1 \backslash \{0\}} e^u dx < \infty \end{cases} \tag{1.6}$$

where $D_1 \subset \mathbb{R}^2$ is the unit disc, was studied in [3]. The authors of [3] obtained that if $u \in C^2(D_1 \backslash \{0\})$ is a singular solution of (1.6), then there is $\alpha > -2$ such that

$$u(x) = \alpha \ln |x| + O(1) \quad \text{as } |x| \to 0.$$

In a recent paper [4], the authors continued the study in [3] and obtained asymptotic expansions up to arbitrary orders for $u(x)$ as $|x| \to 0$.

Under the condition that $N \geq 2$, the structure of finite Morse index solutions of the equation

$$\Delta u + e^u = 0 \quad \text{in } \mathbb{R}^N \tag{1.7}$$

was studied in [5–7]. In particular, under the condition that $N = 3$, the asymptotic behavior at $x = 0$ of solutions u with $|x|^2 e^u \in L^\infty(\mathbb{R}^3)$ of the equation

$$\Delta u + e^u = 0 \quad \text{in } \mathbb{R}^3 \backslash \{0\}, \tag{1.8}$$

was classified in [8]. For the case that $N = 3$, if we write

$$u(x) = -2\ln|x| + \Theta(\theta)$$

where $(|x|, \theta) \in (0, \infty) \times S^2$ denotes the spherical coordinates in $\mathbb{R}^3\backslash\{0\}$, we find that $\Theta(\theta)$ must satisfy

$$\Delta_{\mathbb{S}^2}\Theta - 2 + e^{\Theta} = 0 \tag{1.9}$$

on S^2 where $\Delta_{\mathbb{S}^2}$ is the Laplace-Beltrami operator on (S^2, g_0) and g_0 is the standard round metric. It means that the Gaussian curvature of the metric $g = e^{\Theta}g_0$ on S^2 is $\frac{1}{2}$. This and related equations have been studied for more than three decades. Chang and Yang [9] and Onofri [10] described all regular solutions of (1.9). Specifically, axially symmetric solutions of (1.9) can be written explicitly as $\Theta(\theta) = \log 2 - 2\log(\sqrt{c^2+1} - c\cos\theta)$, where $c \in \mathbb{R}$ is constant and $\theta \in [0, \pi]$ is the geodesic distance from the north pole of S^2. Hence,

$$u(x) = -2\ln|x| + \log 2 - 2\log(\sqrt{c^2+1} - c\cos\theta)$$

is a one-parameter family of non-radial singular solutions of (1.8).

Recently, singular solutions in different settings have also been studied in [11] and [12]. The authors of [12] obtained the existence and asymptotic behavior of singular solutions to quasilinear elliptic inequalities with nonlocal terms. Moreover, by using mini-max and asymptotic approximation methods, the existence of positive singular solutions to the planar logarithmic Choquard equation with exponential nonlinearity was established in [11].

In this paper, we study singular solutions of (1.1) in $B_R \subset \mathbb{R}^N$ ($N \geq 3$). We are interested in not only the asymptotic behavior of singular solutions of (1.1) at $x = 0$, but also the existence of nonradial singular solutions of (1.1). The structure of nonradial singular solutions of the equation

$$\Delta u + e^u = 0 \quad \text{in } \mathbb{R}^N\backslash\{0\} \text{ with } N \geq 3 \tag{1.10}$$

remains largely open. Motivated by the main ideas in [13], the authors of [14, 15] obtained infinitely many nonradial singular solutions of (1.10) of the following form given $4 \leq N \leq 10$:

$$u(x) = -2\ln|x| + \Theta(\theta), \tag{1.11}$$

where $\Theta(\theta)$ is a non-constant solution of the equation

$$\Delta_{\mathbb{S}^{N-1}}\Theta - 2(N-2) + e^{\Theta} = 0 \tag{1.12}$$

on S^{N-1}, where $\Delta_{\mathbb{S}^{N-1}}$ is the Laplace-Beltrami operator on (S^{N-1}, g_0). They constructed infinitely many axially symmetric non-constant classical solutions of (1.12). The only singular solutions to (1.10) known so far are the (trivial) radial singular solution $U_s(x)$ and the solutions given in (1.11). It is clear that they are also the singular solutions to (1.1).

We will construct a new type of singular solutions of (1.1) in the following form:

$$u(x) - U_s(x) = O(|x|^{\epsilon}) \quad \text{as } |x| \to 0, \tag{1.13}$$

for some $\epsilon > 0$.

Our main result is as follows.

Theorem 1.1. *For any* $R > 0$, *Eq* (1.1) *admits infinitely many nonradial singular solutions* $u(x)$ *of the following form:*

$$u(x) = U_s(x) + O(|x|^{\sigma_+^{(2)}}) \quad as\ |x| \to 0, \tag{1.14}$$

where

$$\sigma_+^{(2)} = -\frac{1}{2}(N-2) + \frac{1}{2}\sqrt{N^2 - 4N + 20} > 0.$$

It is known from Theorem 1.1 that the parameter ϵ in (1.13) is $\sigma_+^{(2)}$.

To prove Theorem 1.1, we firstly study the detailed asymptotic behavior at $x = 0$ of a prescribed singular solution u of

$$\Delta u + e^u = 0 \quad \text{in } B\backslash\{0\} \tag{1.15}$$

with the form (1.13), where $B := B_1 = \{x \in \mathbb{R}^N\ (N \geq 3) : \ |x| < 1\}$ is the unit ball. Then, the infinitely many nonradial singular solutions of the form (1.14) can be constructed by using the asymptotic expansion and introducing suitable Hölder spaces.

This paper is organized as follows. In Section 2, we obtain asymptotic expansions near the isolated singular point $x = 0$ of solutions of (1.15). In Section 3, we establish weighted Hölder spaces and invertible operators related to our Equation (1.15). In Section 4, we construct infinitely many singular solutions of (1.1) and show that the singular solutions that we have constructed are non-radial singular solutions.

2. Asymptotic expansion of a prescribed singular solution $u \in C^2(B\backslash\{0\})$ of (1.15) that satisfies (1.13)

We will establish asymptotic expansions of the singular solution $u \in C^2(B\backslash\{0\})$ of (1.15) such that (1.13) is satisfied.

Let $v(x) = u(x) - U_s(x)$. Then $v(x)$ satisfies

$$-\Delta v = 2(N-2)|x|^{-2}(e^v - 1) \quad \text{in } B\backslash\{0\}. \tag{2.1}$$

Making the following transformations:

$$t = \ln r, \quad w(t,\theta) = v(r,\theta),$$

we see from (2.1) that $w(t,\theta)$ satisfies

$$w_{tt} + (N-2)w_t + \Delta_{\mathbb{S}^{N-1}} w + 2(N-2)(e^w - 1) = 0 \quad \text{in } (-\infty, 0) \times \mathbb{S}^{N-1}. \tag{2.2}$$

We write (2.2) in the following forms

$$w_{tt} + (N-2)w_t + \Delta_{\mathbb{S}^{N-1}} w + 2(N-2)w + 2(N-2)(e^w - 1 - w) = 0 \quad \text{in } (-\infty, 0) \times \mathbb{S}^{N-1} \tag{2.3}$$

and

$$\mathcal{L}w + \mathcal{F}(w) = 0 \quad \text{in } (-\infty, 0) \times \mathbb{S}^{N-1}, \tag{2.4}$$

where

$$\mathcal{L}w = w_{tt} + (N-2)w_t + \Delta_{\mathbb{S}^{N-1}} w + 2(N-2)w, \quad \mathcal{F}(w) = 2(N-2)(e^w - 1 - w).$$

Moreover, (1.13) implies that

$$w(t,\theta) = O(e^{\epsilon t}) \quad \text{uniformly for } \theta \in \mathbb{S}^{N-1} \text{ as } t \to -\infty. \tag{2.5}$$

Define a linearized operator

$$\mathcal{L} = \frac{\partial^2}{\partial t^2} + (N-2)\frac{\partial}{\partial t} + \Delta_{\mathbb{S}^{N-1}} + 2(N-2). \tag{2.6}$$

Obviously, $\mathcal{L}$ can decouple into infinitely many ordinary differential operators, i.e.,

$$\mathcal{L}_k = \frac{d^2}{dt^2} + (N-2)\frac{d}{dt} - \lambda_k + 2(N-2) \tag{2.7}$$

for $k = 0, 1, 2, \ldots$, where λ_k is the k-th eigenvalue of

$$-\Delta_{S^{N-1}} Q = \lambda Q \tag{2.8}$$

and $\lambda_k = k(N-2+k)$ with the following multiplicity:

$$m_k = \frac{(N-2+2k)(N-3+k)!}{k!(N-2)!}.$$

The $\{Q_1^k(\theta), \ldots, Q_{m_k}^k(\theta)\}$ with $\|Q_j^k\|_{L^2(S^{N-1})} = 1$ denotes the basis of the eigenspace $H_k(S^{N-1}) \subset L^2(S^{N-1})$ corresponding to λ_k. Then two roots of characteristic polynomial of (2.7) are as follows:

$$\sigma_\pm^{(k)} = -\frac{1}{2}(N-2) \pm \frac{1}{2}\sqrt{(N-2)(N-10) + 4k(N-2+k)}. \tag{2.9}$$

For $k = 0$, we have from (2.9) that

$$\sigma_\pm^{(0)} = \begin{cases} -\frac{1}{2}(N-2) \pm \frac{i}{2}\sqrt{(N-2)(10-N)}, & \text{for } 3 \le N \le 9, \\ -\frac{1}{2}(N-2) < 0, & \text{for } N = 10, \\ -\frac{1}{2}(N-2) \pm \frac{1}{2}\sqrt{(N-2)(N-10)} < 0, & \text{for } N \ge 11. \end{cases} \tag{2.10}$$

For $k = 1$,

$$\sigma_\pm^{(1)} = -\frac{(N-2)}{2} \pm \frac{|N-4|}{2}. \tag{2.11}$$

Then,

$$\sigma_+^{(1)} = \begin{cases} 0, & \text{for } N = 3, \\ -\frac{(N-2)}{2} + \frac{|N-4|}{2} < 0, & \text{for } N \ge 4 \end{cases} \tag{2.12}$$

and

$$\sigma_-^{(1)} = -\frac{(N-2)}{2} - \frac{|N-4|}{2} < 0, \quad \text{for } N \ge 3. \tag{2.13}$$

For $k \ge 2$, the fact that $k(N-2+k) > 2(N-2)$ implies that

$$(N-2)(N-10) + 4k(N-2+k) > (N-2)^2,$$

we see from (2.9) that

$$\sigma_+^{(k)} > 0, \quad \sigma_-^{(k)} < 0. \tag{2.14}$$

It is clear that

$$\sigma_+^{(k+1)} > \sigma_+^{(k)} > 0, \quad \sigma_-^{(k+1)} < \sigma_-^{(k)} < 0 \quad \text{for any } k \ge 2.$$

Proposition 2.1. *Assume that $N \geq 3$ and $u = u(r)$ is a radial solution of* (1.15) *that satisfies*

$$u(r) - U_s(r) = O(r^\epsilon) \text{ for } r \text{ near } 0 \text{ and some } \epsilon > 0.$$

Then

$$u(r) \equiv U_s(r) \quad \text{for } r \in (0, 1].$$

Proof. Bt applying the following transformations:

$$v(r) = u(r) - U_s(r), \quad w(t) = v(r), \quad t = \log r,$$

by (2.3), $w(t)$ satisfies the following ordinary diifferential equation (ODE):

$$w_{tt} + (N-2)w_t + 2(N-2)w + f(w) = 0 \quad \text{in } (-\infty, 0), \tag{2.15}$$

where

$$f(w) = 2(N-2)(e^w - 1 - w) = O(w^2) = O(e^{2\epsilon t}).$$

Note that $w(t) = O(e^{\epsilon t})$ for t near $-\infty$. Therefore, for $3 \leq N \leq 9$,

$$\begin{aligned} w(t) &= A_1 e^{\tau t}\cos\gamma t + A_2 e^{\tau t}\sin\gamma t - B_1 e^{\tau t}\cos\gamma t \int_{-\infty}^{t} e^{-\tau s}[-f(w(s))]\sin\gamma s ds \\ &\quad - B_2 e^{\tau t}\sin\gamma t \int_{-\infty}^{t} e^{-\tau s}[-f(w(s))]\cos\gamma s ds, \end{aligned} \tag{2.16}$$

where $|B_1| = |B_2| = 1/\gamma$, $|f(w(t))| = O(e^{2\epsilon t})$,

$$\tau = -\frac{1}{2}(N-2), \quad \gamma = \frac{1}{2}\sqrt{(N-2)(10-N)}.$$

Since $w(t) \to 0$ as $t \to -\infty$, we obtain from (2.16) that $A_1 = A_2 = 0$ and

$$w(t) = B_1 e^{\tau t}\cos\gamma t \int_{-\infty}^{t} e^{-\tau s} O(w^2(s))\sin\gamma s ds + B_2 e^{\tau t}\sin\gamma t \int_{-\infty}^{t} e^{-\tau s} O(w^2(s))\cos\gamma s ds, \tag{2.17}$$

and

$$|w(t)| \leq C e^{2\epsilon t} := e^{\beta + 2\epsilon t} \quad \text{for some fixed } \beta > 0 \text{ with } C = e^\beta \text{ and } t \text{ near } -\infty. \tag{2.18}$$

Substituting (2.18) into (2.17), we see that

$$|w(t)| \leq e^{3\beta + 4\epsilon t} \quad \text{for } t \text{ near } -\infty. \tag{2.19}$$

We can do the same process to obtain that $w(t) \equiv 0$ for $t \leq -\frac{4\beta}{\epsilon}$. Since $w(t)$ satisfies the ODE in (2.15), then $w(t) \equiv 0$ for $t \in (-\infty, 0)$.

For $N = 10$, we see that

$$\begin{aligned} w(t) &= A_1 e^{\tau t} + A_2 t e^{\tau t} \\ &\quad - B_1 e^{\tau t} \int_{-\infty}^{t} s e^{-\tau s}[-f(w(s))] ds - B_2 t e^{\tau t} \int_{-\infty}^{t} e^{-\tau s}[-f(w(s))] ds, \end{aligned} \tag{2.20}$$

where $|B_1| = |B_2| = 1$. Note that $\tau = -\frac{(N-2)}{2} < 0$. Since $w(t) \to 0$ as $t \to -\infty$, we see that $A_1 = A_2 = 0$. Arguments similar to those in the proof for the case of $3 \le N \le 9$ imply that $w(t) \equiv 0$ for $t \in (-\infty, 0)$.

For $N \ge 11$, we see that

$$\begin{aligned} w(t) &= A_1 e^{\sigma_+^{(0)} t} + A_2 e^{\sigma_-^{(0)} t} \\ &\quad - B_1 e^{\sigma_+^{(0)} t} \int_{-\infty}^{t} e^{-\sigma_+^{(0)} s}[-f(w(s))]ds - B_2 e^{\sigma_-^{(0)} t} \int_{-\infty}^{t} e^{-\sigma_-^{(0)} s}[-f(w(s))]ds, \end{aligned} \tag{2.21}$$

where $|B_1| = |B_2| = \left|\frac{1}{\sigma_-^{(0)} - \sigma_+^{(0)}}\right|$. Note that $\sigma_+^{(0)} < 0$ and $\sigma_-^{(0)} < 0$. Since $w(t) \to 0$ as $t \to -\infty$, we see that $A_1 = A_2 = 0$. Arguments similar to those in the proof for the case of $3 \le N \le 9$ imply that $w(t) \equiv 0$ for $t \in (-\infty, 0)$. This completes the proof of this proposition.

Lemma 2.2. *Assume that $N \ge 3$ and $u \in C^2(B\backslash\{0\})$ is a singular solution of (1.15) that satisfies (1.13). Defining $w(t, \theta) = u(x) - U_s(x)$ and $t = \ln r$, it follows that $w(t, \theta) = O(e^{\epsilon t})$ for $t \in (-\infty, -1]$, and*

$$\max_{S^{N-1}} |w(t, \theta)| \le C e^{\sigma_+^{(2)} t} \quad \textit{for } t \in (-\infty, -1]. \tag{2.22}$$

Proof. Let

$$w(t, \theta) = \sum_{k=0}^{\infty} \sum_{j=1}^{m_k} w_j^k(t) Q_j^k(\theta).$$

Then $w_j^k(t)$ satisfies the following equation:

$$(w_j^k)''(t) + (N-2)(w_j^k)'(t) + [2(N-2) - \lambda_k] w_j^k(t) = -g_j^k(t), \tag{2.23}$$

where

$$g_j^k(t) = \int_{S^{N-1}} \mathcal{F}(w(t, \theta)) Q_j^k(\theta) d\theta.$$

Note that

$$\|w\|_{L^2(S^{N-1})}^2 = \sum_{k=0}^{\infty} \sum_{j=1}^{m_k} (w_j^k(t))^2, \quad \|\mathcal{F}(w)\|_{L^2(S^{N-1})}^2 = \sum_{k=0}^{\infty} \sum_{j=1}^{m_k} (g_j^k(t))^2.$$

Since $\mathcal{F}(w) = O(w^2)$ and $w(t, \theta) = O(e^{\epsilon t})$, we see that

$$\|\mathcal{F}(w)\|_{L^2(S^{N-1})} = O(e^{\epsilon t} \|w\|_{L^2(S^{N-1})}). \tag{2.24}$$

On the other hand, it follows from (2.23) that for $k \ge 2$, $T \ll -1$ and $t < T$,

$$\begin{aligned} w_j^k(t) &= A_{j,1}^k e^{\sigma_+^{(k)} t} + A_{j,2}^k e^{\sigma_-^{(k)} t} \\ &\quad + B_{j,1}^k \int_{t}^{T} e^{\sigma_+^{(k)}(t-s)}[-g_j^k(s)]ds - B_{j,2}^k \int_{-\infty}^{t} e^{\sigma_-^{(k)}(t-s)}[-g_j^k(s)]ds, \end{aligned}$$

where

$$|B_{j,1}^k| = |B_{j,2}^k| = \left|\frac{1}{\sigma_-^{(k)} - \sigma_+^{(k)}}\right|.$$

Since $w_j^k(t) \to 0$ as $t \to -\infty$, we have that $A_{j,2}^k = 0$. Moreover,

$$w_j^k(T) = A_{j,1}^k e^{\sigma_+^{(k)}T} - B_{j,2}^k \int_{-\infty}^{T} e^{\sigma_-^{(k)}(T-s)}[-g_j^k(s)]ds$$

and

$$A_{j,1}^k = O(e^{-\sigma_+^{(k)}T}).$$

Then,

$$\begin{aligned} w_j^k(t) &= O(e^{\sigma_+^{(k)}(t-T)}) \\ &\quad + B_{j,1}^k \int_t^T e^{\sigma_+^{(k)}(t-s)}[-g_j^k(s)]ds - B_{j,2}^k \int_{-\infty}^{t} e^{\sigma_-^{(k)}(t-s)}[-g_j^k(s)]ds \end{aligned} \tag{2.25}$$

and for small enough $\delta > 0$,

$$\begin{aligned} [w_j^k(t)]^2 &\le O(e^{2\sigma_+^{(k)}(t-T)}) + 4(B_{j,1}^k)^2\Big(\int_t^T e^{\delta(t-s)}ds\Big)\Big(\int_t^T e^{(2\sigma_+^{(k)}-\delta)(t-s)}(g_j^k(s))^2 ds\Big) \\ &\quad + 4(B_{j,2}^k)^2\Big(\int_{-\infty}^{t} e^{-\delta(t-s)}ds\Big)\Big(\int_{-\infty}^{t} e^{(2\sigma_-^{(k)}+\delta)(t-s)}(g_j^k(s))^2 ds\Big) \\ &\le Ce^{2\sigma_+^{(k)}(t-T)} + C_\delta \int_t^T e^{(2\sigma_+^{(2)}-\delta)(t-s)}(g_j^k(s))^2 ds \\ &\quad + C_\delta \int_{-\infty}^{t} e^{(2\sigma_-^{(2)}+\delta)(t-s)}(g_j^k(s))^2 ds, \end{aligned}$$

with constants $C > 0$ and $C_\delta > 0$ being dependent on δ and independent of (j, k).

For $k = 1$, $T \ll -1$ and $t < T$,

$$\begin{aligned} w_j^1(t) &= A_{j,1}^1 e^{\sigma_+^{(1)}t} + A_{j,2}^1 e^{\sigma_-^{(1)}t} \\ &\quad - B_{j,1}^1 \int_{-\infty}^{t} e^{\sigma_+^{(1)}(t-s)}[-g_j^1(s)]ds - B_{j,2}^1 \int_{-\infty}^{t} e^{\sigma_-^{(1)}(t-s)}[-g_j^1(s)]ds, \end{aligned}$$

where

$$|B_{j,1}^1| = |B_{j,2}^1| = \Big|\frac{1}{\sigma_-^{(1)} - \sigma_+^{(1)}}\Big|.$$

Note that $\sigma_+^{(1)} \le 0$ and $\sigma_-^{(1)} < 0$. Since $w_j^1(t) \to 0$ as $t \to -\infty$, we have that $A_{j,1}^1 = A_{j,2}^1 = 0$,

$$w_j^1(t) = -B_{j,1}^1 \int_{-\infty}^{t} e^{\sigma_+^{(1)}(t-s)}[-g_j^1(s)]ds - B_{j,2}^1 \int_{-\infty}^{t} e^{\sigma_-^{(1)}(t-s)}[-g_j^1(s)]ds \tag{2.26}$$

and

$$(w_j^1(t))^2 = O(e^{4\epsilon t}). \tag{2.27}$$

Note that

$$(g_j^1(t))^2 \le C\|\mathcal{F}(w)\|_{L^2(S^{N-1})}^2 \le Ce^{4\epsilon t}.$$

Similarly, we have

$$(g_1^0(t))^2 = O(e^{4\epsilon t}), \quad (w_1^0(t))^2 = O(e^{4\epsilon t}). \tag{2.28}$$

Then,

$$\begin{aligned}\sum_{k=2}^{\infty}\sum_{j=1}^{m_k}(w_j^k(t))^2 &\le C\sum_{k=2}^{\infty}\sum_{j=1}^{m_k}e^{2\sigma_+^{(k)}(t-T)}+C_\delta\int_t^T e^{(2\sigma_+^{(2)}-\delta)(t-s)}\sum_{k=2}^{\infty}\sum_{j=1}^{m_k}(g_j^k(s))^2ds\\ &\quad +C_\delta\int_{-\infty}^t e^{(2\sigma_-^{(2)}+\delta)(t-s)}\sum_{k=2}^{\infty}\sum_{j=1}^{m_k}(g_j^k(s))^2ds\\ &\le C\sum_{k=2}^{\infty}\sum_{j=1}^{m_k}e^{2\sigma_+^{(k)}(t-T)}\\ &\quad +C\int_t^T e^{(2\sigma_+^{(2)}-\delta)(t-s)}e^{4\epsilon s}ds+C\int_t^T e^{(2\sigma_+^{(2)}-\delta)(t-s)}e^{2\epsilon s}\sum_{k=2}^{\infty}\sum_{j=1}^{m_k}(w_j^k(s))^2ds\\ &\quad +C\int_{-\infty}^t e^{(2\sigma_-^{(2)}+\delta)(t-s)}e^{4\epsilon s}ds+C\int_{-\infty}^t e^{(2\sigma_-^{(2)}+\delta)(t-s)}e^{2\epsilon s}\sum_{k=2}^{\infty}\sum_{j=1}^{m_k}(w_j^k(s))^2ds.\end{aligned}$$

Notice that

$$\sum_{k=2}^{\infty}\sum_{j=1}^{m_k}(g_j^k(t))^2=\|\mathcal{F}(w)\|^2_{L^2(S^{N-1})}-\Big[(g_1^0(t))^2+\sum_{j=1}^{N}(g_j^1(t))^2\Big]$$

and

$$\|\mathcal{F}(w)\|^2_{L^2(S^{N-1})}=O(e^{2\epsilon t})\|w\|^2_{L^2(S^{N-1})}=O(e^{2\epsilon t})\Big[\sum_{k=2}^{\infty}\sum_{j=1}^{m_k}(w_j^k(t))^2+\Big((w_1^0(t))^2+\sum_{j=1}^{N}(w_j^1(t))^2\Big)\Big],$$

since $\mathcal{F}(w)=O(w^2)$. We also know that

$$\sum_{k=2}^{\infty}\sum_{j=1}^{m_k}e^{2\sigma_+^{(k)}(t-T)}=\sum_{k=2}^{\infty}m_ke^{2\sigma_+^{(k)}(t-T)}=O(e^{2\sigma_+^{(2)}(t-T)}),$$

since

$$\lim_{k\to\infty}\frac{m_{k+1}e^{2(\sigma_+^{(k+1)}-\sigma_+^{(2)})(t-T)}}{m_ke^{2(\sigma_+^{(k)}-\sigma_+^{(2)})(t-T)}}=\lim_{k\to\infty}\Big[\frac{m_{k+1}}{m_k}e^{2(\sigma_+^{(k+1)}-\sigma_+^{(k)})(t-T)}\Big]=e^{2(t-T)}<\frac{1}{2}.$$

Let $[W(t)]^2=\sum_{k=2}^{\infty}\sum_{j=1}^{m_k}(w_j^k(t))^2$. We have that, if $4\epsilon\neq 2\sigma_+^{(2)}-\delta$,

$$\begin{aligned}[W(t)]^2 &\le Ce^{2\sigma_+^{(2)}(t-T)}+Ce^{4\epsilon t}+C\int_t^T e^{(2\sigma_+^{(2)}-\delta)(t-s)}e^{4\epsilon s}ds\\ &\quad +C\int_t^T e^{(2\sigma_+^{(2)}-\delta)(t-s)}e^{2\epsilon s}[W(s)]^2ds+C\int_{-\infty}^t e^{(2\sigma_-^{(2)}+\delta)(t-s)}e^{2\epsilon s}[W(s)]^2ds.\end{aligned}$$

Now we show that, for $t<T\ll -1$,

$$\|w\|_{L^2(S^{N-1})}=\Big[\sum_{k=0}^{\infty}\sum_{j=1}^{m_k}(w_j^k(t))^2\Big]^{\frac{1}{2}}=O(e^{\sigma_+^{(2)}t}). \tag{2.29}$$

In fact, two cases can occur: (i) $4\epsilon \geq [2\sigma_+^{(2)} - \delta]$ and (ii) $4\epsilon < [2\sigma_+^{(2)} - \delta]$.

It suffices to consider that $4\epsilon > [2\sigma_+^{(2)} - \delta]$ in the case (i). So

$$\begin{aligned}[W(t)]^2 &\leq Ce^{(2\sigma_+^{(2)}-\delta)(t-T)} + C\int_t^T e^{(2\sigma_+^{(2)}-\delta)(t-s)}e^{2\epsilon s}[W(s)]^2 ds \\ &\quad + C\int_{-\infty}^t e^{(2\sigma_-^{(2)}+\delta)(t-s)}e^{2\epsilon s}[W(s)]^2 ds. \qquad (2.30)\end{aligned}$$

Set

$$K_1(t) = \int_t^T e^{(2\sigma_+^{(2)}-\delta)(t-s)}[W(s)]^2 ds, \quad K_2(t) = \int_{-\infty}^t e^{(2\sigma_-^{(2)}+\delta)(t-s)}[W(s)]^2 ds.$$

Then, if $|T|$ is large,

$$\begin{aligned}(K_2 - K_1)'(t) &= (2\sigma_-^{(2)} + \delta)K_2(t) - (2\sigma_+^{(2)} - \delta)K_1(t) + 2[W(t)]^2 \\ &\leq (2\sigma_-^{(2)} + \delta)K_2(t) - (2\sigma_+^{(2)} - \delta)K_1(t) + Ce^{2\epsilon T}(K_1(t) + K_2(t)) + Ce^{(2\sigma_+^{(2)}-\delta)(t-T)} \\ &\leq Ce^{(2\sigma_+^{(2)}-\delta)(t-T)}\end{aligned}$$

where $\sigma_-^{(2)} < 0$, $\sigma_+^{(2)} > 0$. Since $K_1(t) \to 0$ and $K_2(t) \to 0$ as $t \to -\infty$, then for $t < T$,

$$K_2(t) \leq K_1(t) + Ce^{(2\sigma_+^{(2)}-\delta)t}. \qquad (2.31)$$

Substituting (2.31) into (2.30), we have

$$[W(t)]^2 \leq Ce^{(2\sigma_+^{(2)}-\delta)t} + Ce^{2\epsilon T}\int_t^T e^{(2\sigma_+^{(2)}-\delta)(t-s)}[W(s)]^2 ds. \qquad (2.32)$$

It follows from arguments similar to those in [16, 17] that, for $t < T$ (enlarge $|T|$ if necessary),

$$[W(t)]^2 \leq C\epsilon_T e^{(2\sigma_+^{(2)}-\delta-\epsilon_T)t}, \qquad (2.33)$$

where $\epsilon_T = Ce^{2\epsilon T}$ (i.e., C is independent of ϵ). On the other hand, (2.27) and (2.28) imply that

$$(w_j^1(t))^2 = O(e^{4\epsilon t}) = O(e^{(2\sigma_+^{(2)}-\delta)t}), \quad j = 1, 2, \ldots, m_1,$$

$$(w_1^0(t))^2 = O(e^{4\epsilon t}) = O(e^{(2\sigma_+^{(2)}-\delta)t})$$

where $4\epsilon > 2\sigma_+^{(2)} - \delta$. Therefore, for $t < T$ (i.e., T is sufficiently negative),

$$\sum_{k=0}^{\infty}\sum_{j=1}^{m_k}(w_j^k(t))^2 = O(e^{(2\sigma_+^{(2)}-\delta-\epsilon_T)t}) \qquad (2.34)$$

and

$$\|w\|_{L^2(S^{N-1})} = O(e^{(\sigma_+^{(2)}-\frac{\delta}{2}-\frac{\epsilon_T}{2})t}). \qquad (2.35)$$

Using (2.35), we obtain

$$\|\mathcal{F}(w)\|^2_{L^2(S^{N-1})} = O(e^{2\epsilon t})\|w\|^2_{L^2(S^{N-1})} = O(e^{(2\sigma_+^{(2)}-\delta-\epsilon_T+2\epsilon)t}). \qquad (2.36)$$

Choosing δ sufficiently small such that $\delta < 2\epsilon - \epsilon_T$, then

$$\begin{aligned}\sum_{j=1}^{m_2}|w_j^2(t)| &\le Ce^{\sigma_+^{(2)}(t-T)} + C\int_t^T e^{\sigma_+^{(2)}(t-s)}\sum_{j=1}^{m_2}|g_j^2(s)|ds \\ &\quad +C\int_{-\infty}^t e^{\sigma_-^{(2)}(t-s)}\sum_{j=1}^{m_2}|g_j^2(s)|ds \\ &\le Ce^{\sigma_+^{(2)}t} + C\int_t^T e^{\sigma_+^{(2)}(t-s)}e^{(\sigma_+^{(2)}-\frac{\delta}{2}-\frac{\epsilon_T}{2}+\epsilon)s}ds \\ &\quad +C\int_{-\infty}^t e^{\sigma_-^{(2)}(t-s)}e^{(\sigma_+^{(2)}-\frac{\delta}{2}-\frac{\epsilon_T}{2}+\epsilon)s}ds \\ &\le Ce^{\sigma_+^{(2)}t}.\end{aligned}$$

So for $t < T$,

$$\sum_{j=1}^{m_2}|w_j^2(t)|^2 \le \Big[\sum_{j=1}^{m_2}|w_j^2(t)|\Big]^2 \le Ce^{2\sigma_+^{(2)}t}. \tag{2.37}$$

Moreover, we choose $0 < \delta < \min\{2\epsilon - \epsilon_T, 2\sigma_+^{(3)} - 2\sigma_+^{(2)}\}$; then,

$$\begin{aligned}\sum_{k=3}^{\infty}\sum_{j=1}^{m_k}(w_j^k(t))^2 &\le C\sum_{k=3}^{\infty}\sum_{j=1}^{m_k}e^{2\sigma_+^{(k)}(t-T)} + C_\delta\int_t^T e^{(2\sigma_+^{(3)}-\delta)(t-s)}\sum_{k=3}^{\infty}\sum_{j=1}^{m_k}(g_j^k(s))^2ds \\ &\quad +C_\delta\int_{-\infty}^t e^{(2\sigma_-^{(3)}+\delta)(t-s)}\sum_{k=3}^{\infty}\sum_{j=1}^{m_k}(g_j^k(s))^2ds \\ &\le Ce^{2\sigma_+^{(3)}t} + C\int_t^T e^{(2\sigma_+^{(3)}-\delta)(t-s)}e^{(2\sigma_+^{(2)}-\delta-\epsilon_T+2\epsilon)s}ds \\ &\quad +C\int_{-\infty}^t e^{(2\sigma_-^{(3)}+\delta)(t-s)}e^{(2\sigma_+^{(2)}-\delta-\epsilon_T+2\epsilon)s}ds \\ &\le Ce^{2\sigma_+^{(3)}t} + C\max\{e^{(2\sigma_+^{(3)}-\delta)t}, e^{(2\sigma_+^{(2)}-\delta-\epsilon_T+2\epsilon)t}\} + Ce^{(2\sigma_+^{(2)}-\delta-\epsilon_T+2\epsilon)t} \\ &\le Ce^{2\sigma_+^{(2)}t}.\end{aligned} \tag{2.38}$$

It is known from (2.36) and $\delta < 2\epsilon - \epsilon_T$ that

$$\|\mathcal{F}(w)\|^2_{L^2(\mathbb{S}^{N-1})} = O(e^{2\sigma_+^{(2)}t}), \quad \sum_{j=1}^{m_1}(g_j^1)^2 = O(e^{2\sigma_+^{(2)}t}), \quad (g_1^0)^2 = O(e^{2\sigma_+^{(2)}t}).$$

By (2.26) and $g_j^1(t) = O(e^{\sigma_+^{(2)}t})$, we obtain

$$\sum_{j=1}^{m_1}(w_j^1(t))^2 = O(e^{2\sigma_+^{(2)}t}). \tag{2.39}$$

Similarly,

$$(w_1^0(t))^2 = O(e^{2\sigma_+^{(2)}t}). \tag{2.40}$$

We obtain from (2.37)–(2.40) that

$$\|w\|_{L^2(\mathbb{S}^{N-1})} = O(e^{\sigma_+^{(2)} t}). \tag{2.41}$$

For any fixed $(t, \theta) \in (-\infty, T-1) \times \mathbb{S}^{N-1}$, by applying the interior L^∞-estimate to (2.4) with $(t-1, t+1) \times \mathbb{S}^{N-1}$, we obtain from (2.41) and (2.24) that

$$|w(t,\theta)| \le C\{\|w\|_{L^2((t-1,t+1)\times\mathbb{S}^{N-1})} + \|\mathcal{F}(w)\|_{L^2((t-1,t+1)\times\mathbb{S}^{N-1})}\} \le Ce^{\sigma_+^{(2)} t}, \tag{2.42}$$

where $C > 0$ is independent of t. Note that we can also use arguments similar to those in the proof of [18] to obtain

$$\max_{\theta\in\mathbb{S}^{N-1}} |w(t,\theta)| \le Me^{\sigma_+^{(2)} t} \quad \text{for } t \in (-\infty, T-1). \tag{2.43}$$

Defining

$$v(r,\theta) = w(t,\theta), \quad r = e^t,$$

it follows that $v(r,\theta)$ satisfies

$$\Delta v + \frac{2(N-2)v}{r^2} + \frac{\mathcal{F}(v)}{r^2} = 0 \text{ in } B_R\backslash\{0\}, \tag{2.44}$$

where $R = e^{T-1}$. For any $x_0 \in B_R\backslash\{0\}$, denote $r_0 = |x_0| > 0$ and $\Omega = B_{r_0/2}(x_0)$. Consider (2.44) to be a linear equation in Ω as in Lemma 5.1 and Theorem 5.1 of [18] with

$$k = k_1 = 1, \quad h(x) \equiv 0, \quad |c| = \frac{Q}{r_0^2}, \quad k_2 = \frac{Q}{r_0^2}$$

where $Q = Q(v) > 0$. Then, (2.41) implies that there is a positive constant (independent of r_0)

$$M = M(k_1/k, k_2 r_0^2) = M(Q) = M(v)$$

such that

$$\sup_{x\in B_{r_0/4}(x_0)} |v(x)| \le Mr_0^{\sigma_+^{(2)}}.$$

In particular, we have

$$|v(x_0)| \le Mr_0^{\sigma_+^{(2)}},$$

$$\max_{|x|=r} |v(x)| \le Mr^{\sigma_+^{(2)}}.$$

Hence, (2.43) follows for $t \in (-\infty, T-1)$.

For the case of $4\epsilon = 2\sigma_+^{(2)} - \delta$, we may choose δ' a little larger than δ such that $0 < \delta < \delta'$ and $4\epsilon > 2\sigma_+^{(2)} - \delta'$. By similar arguments, we can prove (2.41) and (2.43).

For the case (ii), by $\mathcal{F}(w) = O(e^{2\epsilon t})$, $\sum_{k=2}^{\infty}\sum_{j=1}^{m_k}(g_j^k(s))^2 = O(e^{4\epsilon t})$ and $4\epsilon < 2\sigma_+^{(2)} - \delta < 2\sigma_+^{(2)}$, we can obtain

$$\begin{aligned}\sum_{k=2}^{\infty}\sum_{j=1}^{m_k}(w_j^k(t))^2 \le\ & C\sum_{k=2}^{\infty}\sum_{j=1}^{m_k} e^{2\sigma_+^{(k)}(t-T)} + C\int_t^T e^{(2\sigma_+^{(2)}-\delta)(t-s)}\sum_{k=2}^{\infty}\sum_{j=1}^{m_k}(g_j^k(s))^2 ds \\ & + C\int_{-\infty}^t e^{(2\sigma_-^{(2)}+\delta)(t-s)}\sum_{k=2}^{\infty}\sum_{j=1}^{m_k}(g_j^k(s))^2 ds\end{aligned}$$

$$\leq \quad Ce^{4\epsilon t}.$$

Then

$$[W(t)]^2 \leq Ce^{4\epsilon t} \quad \text{for } t < T.$$

Together with (2.27) and (2.28), we know that

$$\|w\|_{L^2(\mathbb{S}^{N-1})} = O(e^{2\epsilon t}).$$

Arguments similar to those in the proof of (2.43) imply that

$$\max_{\theta \in \mathbb{S}^{N-1}} |w(t,\theta)| \leq Me^{2\epsilon t} \quad \text{for } t \in (-\infty, -1], \tag{2.45}$$

where $M := M(w) > 0$. As a consequence,

$$\max_{\mathbb{S}^{N-1}} |\mathcal{F}(w)| \leq Ce^{4\epsilon t} \quad \text{for } t \in (-\infty, -1]. \tag{2.46}$$

Then (2.26) implies that

$$w_j^1(t) = O(e^{4\epsilon t}), \quad j = 1, 2, \dots m_1. \tag{2.47}$$

Similarly,

$$w_1^0(t) = O(e^{4\epsilon t}). \tag{2.48}$$

Therefore,

$$\begin{aligned} [W(t)]^2 \quad \leq \quad & Ce^{2\sigma_+^{(2)}(t-T)} + C\int_t^T e^{(2\sigma_+^{(2)}-\delta)(t-s)} e^{8\epsilon s} ds + C\int_t^T e^{(2\sigma_+^{(2)}-\delta)(t-s)} e^{2\epsilon s} [W(s)]^2 ds \\ & + C\int_{-\infty}^t e^{(2\sigma_-^{(2)}+\delta)(t-s)} e^{8\epsilon s} ds + C\int_{-\infty}^t e^{(2\sigma_-^{(2)}+\delta)(t-s)} e^{2\epsilon s} [W(s)]^2 ds. \end{aligned} \tag{2.49}$$

Note that

$$\sum_{k=2}^{\infty} \sum_{j=1}^{m_k} (g_j^k(t))^2 \leq Ce^{2\epsilon t}([W(t)]^2 + e^{8\epsilon t}).$$

In what follows, there are also two cases: (a) $8\epsilon \geq [2\sigma_+^{(2)} - \delta]$ and (b) $8\epsilon < [2\sigma_+^{(2)} - \delta]$.

For the case (a), using (2.49) and arguments similar to those in the proof of (i), we can obtain (2.43).

The case (b) implies that $\mathcal{F}(w) = O(e^{4\epsilon t})$. Then

$$[W(t)]^2 \leq Ce^{8\epsilon t} \quad \text{for } t < T.$$

This, (2.47) and (2.48) imply that

$$\|w\|_{L^2(\mathbb{S}^{N-1})} = O(e^{4\epsilon t}) \tag{2.50}$$

and

$$\|\mathcal{F}(w)\|_{L^2(\mathbb{S}^{N-1})} = O(e^{5\epsilon t}). \tag{2.51}$$

Therefore,

$$g_j^1(t) = O(e^{5\epsilon t}), \quad w_j^1(t) = O(e^{5\epsilon t}), \quad w_1^0(t) = O(e^{5\epsilon t}). \tag{2.52}$$

Then we have

$$\sum_{k=2}^{\infty}\sum_{j=1}^{m_k}(g_j^k(t))^2 \le Ce^{2\epsilon t}([W(t)]^2 + e^{10\epsilon t})$$

and

$$\begin{aligned}[W(t)]^2 \le\ & Ce^{2\sigma_+^{(2)}(t-T)} + C\int_t^T e^{(2\sigma_+^{(2)}-\delta)(t-s)}e^{10\epsilon s}ds + C\int_t^T e^{(2\sigma_+^{(2)}-\delta)(t-s)}e^{2\epsilon s}[W(s)]^2 ds \\ & +C\int_{-\infty}^t e^{(2\sigma_-^{(2)}+\delta)(t-s)}e^{10\epsilon s}ds + C\int_{-\infty}^t e^{(2\sigma_-^{(2)}+\delta)(t-s)}e^{2\epsilon s}[W(s)]^2 ds. \qquad (2.53)\end{aligned}$$

Similarly, we still consider two cases: $10\epsilon \ge [2\sigma_+^{(2)} - \delta]$ and $10\epsilon < [2\sigma_+^{(2)} - \delta]$; then, we obtain (2.43).

The proof of this lemma is complete.

Theorem 2.3. *Assume that* $N \ge 3$ *and* $u \in C^2(B\backslash\{0\})$ *is a singular solution of* (1.15) *that satisfies* (1.13). *Defining* $w(t,\theta) = u(x) - U_s(x)$ *and* $t = \ln r$, *there is a positive number sequence* $\{\mu_k\}_{k\ge1}$, *strictly increasing and converging to* ∞ *with*

$$\mu_1 = \sigma_+^{(2)} \qquad (2.54)$$

such that for any positive integer $n \gg 1$ *and any* $(t,\theta) \in (-\infty,-1)\times S^{N-1}$,

$$w(t,\theta) = \sum_{k=1}^{n}\sum_{\ell=0}^{k-1} c_{k\ell}(\theta)t^\ell e^{\mu_k t} + O\Big(|t|^n e^{\mu_{n+1}t}\Big), \qquad (2.55)$$

where

$$c_{k\ell}(\theta) = \sum_{i=0}^{M_{k\ell}} a_{k\ell i}Q_i(\theta) \qquad (2.56)$$

and $M_{k\ell}$ *is a nonnegative integer depending on* N, k, ℓ; $a_{k\ell i}$ *is constant and* $Q_i(\theta)$ *is a linear combination of* $\{Q_1^i(\theta), Q_2^i(\theta), \dots, Q_{m_i}^i(\theta)\}$. *Especially, for* $k = 1$,

$$c_{10}(\theta) = a_{102}Q_2(\theta),$$

where a_{102} *is a constant.*

Proof. By using the starting estimate (2.22), constructing the index set $\mathcal{I}$, and examining the equation of w, the expansion of $w(t,\theta)$ can be established via similar arguments to those in Theorem 1.1 of [19].

Let $\{\rho_k\}_{k\ge1}$ be positive strictly increasing and converging to ∞:

$$\rho_1 = \sigma_+^{(2)}, \rho_2 = \sigma_+^{(3)}, \dots, \rho_k = \sigma_+^{(k+1)}, \dots.$$

Also, let $\mathbb{Z}_+$ be the collection of nonnegative integers. Define the index set $\mathcal{I}$ by

$$\mathcal{I} = \Big\{\sum_{k\ge1} n_k\rho_k :\ n_k \in \mathbb{Z}_+ \ \text{ with finitely many } n_k > 0\Big\}. \qquad (2.57)$$

Set

$$\mathcal{I}_\rho = \{\rho_k :\ k \ge 1\} \qquad (2.58)$$

and

$$\mathcal{I}_{\tilde{\rho}} = \Big\{ \sum_{k=1}^{i} n_k\rho_k : \; n_k \in \mathbb{Z}_+, \quad \sum_{k=1}^{i} n_k \geq 2 \Big\}. \tag{2.59}$$

Assume that $\mathcal{I}_{\tilde{\rho}}$ is given by a strictly increasing sequence $\{\tilde{\rho}_k\}_{k\geq 1}$ with $\tilde{\rho}_1 = 2\rho_1$. There may be identical elements in $\mathcal{I}_\rho$ and $\mathcal{I}_{\tilde{\rho}}$.

For $\tilde{\rho}_k \in \mathcal{I}_{\tilde{\rho}}$, there are nonnegative integers $n_1, \ldots, n_{i_1}$ such that

$$n_1 + \ldots + n_{i_1} \geq 2, \quad n_1\rho_1 + \ldots + n_{i_1}\rho_{i_1} = \tilde{\rho}_k. \tag{2.60}$$

The collections of nonnegative integers $n_1, \ldots, n_{i_1}$ that satisfy (2.60) are finite. Set

$$\tilde{M}_k = \max\{2n_1 + 3n_2 + \ldots + (i_1+1)n_{i_1} : \; n_1, \ldots, n_{i_1} \text{ are nonnegative integers satisfying (2.60)}\}. \tag{2.61}$$

Arrange $\mathcal{I}$ as follows:

$$\rho_1 < \ldots < \rho_{i_1} \leq \tilde{\rho}_1 < \ldots < \tilde{\rho}_{l_1} \leq \rho_{i_1+1} < \ldots < \rho_{i_2} \leq \tilde{\rho}_{l_1+1} < \ldots < \tilde{\rho}_{l_2} \leq \rho_{i_2+1} < \ldots. \tag{2.62}$$

Note that if $\rho_1 < \tilde{\rho}_1 < \rho_2$, we choose $i_1 = 1$ and $l_1 = 1$ and the arrangement of (2.62) becomes $\rho_1 < \tilde{\rho}_1 < \rho_2 < \ldots$. Similarly, if $\rho_{i_k+1} \leq \tilde{\rho}_{l_k+1} < \rho_{i_k+2}$ for some $k \geq 1$, define $i_k + 1 = i_{k+1}$ and $l_k + 1 = l_{k+1}$. We do not consider the multiplicity of ρ_k here, since all terms containing $e^{\rho_k t}$ in the expansions of $w(t,\theta)$ can be combined as one term. We know

$$\mathcal{L}(w) = -\mathcal{F}(w),$$

where $\mathcal{F}(w) = \sum_{k=2}^{\infty} b_k w^k$ for $|w| < \hat{\epsilon}$ with some sufficiently small $\hat{\epsilon} > 0$; also, the expansion of $\mathcal{F}(w)$ consists of terms including $\sum_{\ell=0}^{I(k)} \Big(\sum_{i=0}^{\tilde{M}_k} c_{k\ell i} Q_i(\theta) \Big) t^\ell e^{\tilde{\rho}_k t}$ for $\tilde{\rho}_k \in \mathcal{I}_{\tilde{\rho}}$ in (2.59).

Define

$$\mu_1 = \rho_1, \mu_2 = \rho_2, \mu_3 = \rho_3, \ldots, \mu_{i_1} = \rho_{i_1}, \mu_{i_1+1} = \tilde{\rho}_1, \ldots \tag{2.63}$$

according to the arrangement in (2.62). To ensure that $\{\mu_k\}_{k\geq 1}$ is a strictly increasing sequence of positive constants, when $\rho_{i_1} = \tilde{\rho}_1$, define $\mu_{i_1} = \rho_{i_1}$ and $\mu_{i_1+1} = \tilde{\rho}_2$. In this case, an extra power of t term corresponding to μ_{i_1} may appear in the expansion of $w(t,\theta)$. Similarly, make the same choices of μ_k for the cases $\tilde{\rho}_{l_1} = \rho_{i_1+1}, \rho_{i_2} = \tilde{\rho}_{l_1+1}, \tilde{\rho}_{l_2} = \rho_{i_2+1}$, etc. As a consequence, for any positive integer $n \gg 1$ and any $(t,\theta) \in (-\infty, -1) \times S^{N-1}$,

$$w(t,\theta) = \sum_{k=1}^{n} \sum_{\ell=0}^{k-1} c_{k\ell}(\theta) t^\ell e^{\mu_k t} + O\Big(|t|^n e^{\mu_{n+1} t}\Big), \tag{2.64}$$

where

$$c_{k\ell}(\theta) = \sum_{i=0}^{M_{k\ell}} a_{k\ell i} Q_i(\theta) \tag{2.65}$$

and $M_{k\ell}$ is a nonnegative integer that is dependent on N, k, ℓ, $a_{k\ell i}$ is constant and $Q_i(\theta)$ is in the span of $Q_1^i(\theta), Q_2^i(\theta), \ldots, Q_{m_i}^i(\theta)$. Especially, for $k = 1$,

$$c_{10}(\theta) = a_{102} Q_2(\theta).$$

The proof of this theorem is complete.

3. Linearized equations and inverse of the operator

We will introduce the appropriate weighted Hölder spaces and obtain the inverse of the operator $\mathcal{L}$ on those spaces, where $\mathcal{L}$ is given in (2.4). We use some ideas from [20], where the authors derived singular solutions to the following equation:

$$\Delta u + \frac{N(N-2)}{4} u^{\frac{N+2}{N-2}} = 0 \quad \text{in } B\backslash\{0\}.$$

Fix a $t_0 < 0$. For a nonnegative integer i, $\alpha \in (0,1)$, and $\mu \in \mathbb{R}$, define

$$\|v\|_{C^i_\mu((-\infty,t_0]\times\mathbb{S}^{N-1})} = \sum_{j=0}^{i} \sup_{(t,\theta)\in(-\infty,t_0]\times\mathbb{S}^{N-1}} e^{-\mu t}|\nabla^j v(t,\theta)|,$$

and

$$\|v\|_{C^{i,\alpha}_\mu((-\infty,t_0]\times\mathbb{S}^{N-1})} = \|v\|_{C^i_\mu((-\infty,t_0]\times\mathbb{S}^{N-1})} + \sup_{t\le t_0-1} e^{-\mu t}[\nabla^i v]_{C^\alpha([t-1,t+1]\times\mathbb{S}^{N-1})},$$

where $[\cdot]_{C^\alpha}$ is the usual Hölder semi-norm.

Definition 3.1. *The collection of functions v in $C^i((-\infty,t_0]\times\mathbb{S}^{N-1})$ with a finite norm $\|v\|_{C^{i,\alpha}_\mu((-\infty,t_0]\times\mathbb{S}^{N-1})}$ is the weighted Hölder space $C^{i,\alpha}_\mu((-\infty,t_0]\times\mathbb{S}^{N-1})$.*

For $\mu > 0$ and some $g \in C^{0,\alpha}_\mu((-\infty,t_0]\times\mathbb{S}^{N-1})$, to consider the linear equation given by

$$\mathcal{L}v = g \ \text{ in } \ (-\infty,t_0)\times\mathbb{S}^{N-1}, \tag{3.1}$$

we introduce a boundary condition on $t = t_0$ such that

$$\mathcal{L}: \ C^{2,\alpha}_\mu((-\infty,t_0]\times\mathbb{S}^{N-1}) \to C^{0,\alpha}_\mu((-\infty,t_0]\times\mathbb{S}^{N-1})$$

has a bounded inverse. However, since signs of coefficients of zero order terms are inappropriate, we cannot directly apply the maximum principle to the following Dirichlet boundary-value problem

$$\begin{cases} \mathcal{L}v = g & \text{in } (-\infty,t_0)\times\mathbb{S}^{N-1}, \\ v = \varphi & \text{on } \{t_0\}\times\mathbb{S}^{N-1}. \end{cases} \tag{3.2}$$

Lemma 3.1. *Let $\mu > 0$, $g \in C^0_\mu((-\infty,t_0]\times\mathbb{S}^{N-1})$, and $\varphi \in C^0(\mathbb{S}^{N-1})$. Then, there is at most one solution $v \in C^2_\mu((-\infty,t_0]\times\mathbb{S}^{N-1})$ of* (3.2).

Proof. Assume that $g = 0$, $\varphi = 0$ and $v \in C^2_\mu((-\infty,t_0]\times\mathbb{S}^{N-1})$ is a solution of (3.2). For each $k \ge 0$, define

$$v_k(t) = \int_{\mathbb{S}^{N-1}} v(t,\theta)Q_k(\theta)d\theta.$$

So $\mathcal{L}_k(v_k) = 0$ on $(-\infty,t_0)$ and $v_k(t_0) = 0$. This implies that v_k is a linear combinations of the basis of $\text{Ker}(\mathcal{L}_k)$. In particular, for $k = 0$,

$$v_0(t) = \begin{cases} c_0^1 e^{\Re(\sigma_+^{(0)})t}\cos\gamma t + c_0^2 e^{\Re(\sigma_+^{(0)})t}\sin\gamma t, & \text{for } 3 \le N \le 9, \\ c_0^1 e^{\sigma_+^{(0)}t} + c_0^2 t e^{\sigma_+^{(0)}t}, & \text{for } N = 10, \\ c_0^1 e^{\sigma_+^{(0)}t} + c_0^2 e^{\sigma_-^{(0)}t}, & \text{for } N \ge 11, \end{cases}$$

and for $k \geq 1$,

$$v_k(t) = c_k^1 e^{\sigma_+^{(k)} t} + c_k^2 e^{\sigma_-^{(k)} t},$$

where c_k^1 and c_k^2 are constants for $k = 0, 1, 2, \ldots$. By the assumption, we have the following for any $t \in (-\infty, t_0)$:

$$|e^{-\mu t} v_k(t)| \leq C. \tag{3.3}$$

Hence, $v_k = 0$ for $k = 0$ and $k = 1$. Note that

$$\begin{cases} \Re(\sigma_+^{(0)}) < 0, \ \Re(\sigma_-^{(0)}) < 0, & \text{for } 3 \leq N \leq 9, \\ \sigma_+^{(0)} < 0, \ \sigma_-^{(0)} < 0, & \text{for } N \geq 10, \end{cases}$$

$\sigma_+^{(1)} \leq 0$ and $\sigma_-^{(1)} < 0$ for $N \geq 3$. Moreover, $v_k(t) = c_k^1 e^{\sigma_+^{(k)} t}$ for $k \geq 2$, which decays exponentially as $t \to -\infty$ (note that $c_k^2 = 0$ since $\sigma_-^{(k)} < 0$ for $k \geq 2$). Since $v_k(t_0) = 0$, we can directly obtain $c_k^1 = 0$ and $v_k(t) \equiv 0$ for $k \geq 2$. In conclusion, $v_k = 0$ for all $k \geq 0$, i.e., $v \equiv 0$.

Lemma 3.2. *Let $\alpha \in (0, 1)$, $\mu > 0$, $g \in C_\mu^{0,\alpha}((-\infty, t_0] \times \mathbb{S}^{N-1})$, and $\varphi \in C^{2,\alpha}(\mathbb{S}^{N-1})$. Suppose that $v \in C_\mu^{2,\alpha}((-\infty, t_0] \times \mathbb{S}^{N-1})$ is a solution of (3.2). Then*

$$\|v\|_{C_\mu^{2,\alpha}((-\infty,t_0]\times\mathbb{S}^{N-1})} \leq C\Big[\|v\|_{C_\mu^0((-\infty,t_0]\times\mathbb{S}^{N-1})} + \|g\|_{C_\mu^{0,\alpha}((-\infty,t_0]\times\mathbb{S}^{N-1})} + e^{-\mu t_0}\|\varphi\|_{C^{2,\alpha}(\mathbb{S}^{N-1})}\Big], \tag{3.4}$$

where C is a positive constant that is only dependent on N, α, μ and is independent of t_0.

Proof. Using similar arguments to that of Lemma 2.5 of [20], consider two cases:
(i) $t < t_0 - 2$. We have

$$\begin{aligned} &\sum_{j=0}^{2} \sup_{\mathbb{S}^{N-1}} |\nabla^j v(t, \cdot)| + [\nabla^2 v]_{C^\alpha([t-1,t+1]\times\mathbb{S}^{N-1})} \\ &\leq C\Big[\|v\|_{L^\infty([t-2,t+2]\times\mathbb{S}^{N-1})} + \|g\|_{L^\infty([t-2,t+2]\times\mathbb{S}^{N-1})} + [g]_{C^\alpha([t-2,t+2]\times\mathbb{S}^{N-1})}\Big], \end{aligned}$$

where C is a positive constant that is independent of t. We estimate $[g]_{C^\alpha([t-2,t+2]\times\mathbb{S}^{N-1})}$, by setting (t_1, θ_1), $(t_2, \theta_2) \in [t-2, t+2] \times \mathbb{S}^{N-1}$ with $(t_1, \theta_1) \neq (t_2, \theta_2)$. There are two cases: $|t_1 - t_2| \leq 2$ and $|t_1 - t_2| > 2$.

When $|t_1 - t_2| \leq 2$, choose $t' \in [t-1, t+1]$ such that $t_1, t_2 \in [t'-1, t'+1]$ is satisfied. Then,

$$[g]_{C^\alpha([t-2,t+2]\times\mathbb{S}^{N-1})} \leq \max\Big\{ \sup_{t'\in[t-1,t+1]} [g]_{C^\alpha([t'-1,t'+1]\times\mathbb{S}^{N-1})}, \|g\|_{L^\infty([t-2,t+2]\times\mathbb{S}^{N-1})}\Big\}.$$

So,

$$\begin{aligned} &\sum_{j=0}^{2} \sup_{\mathbb{S}^{N-1}} |\nabla^j v(t, \cdot)| + [\nabla^2 v]_{C^\alpha([t-1,t+1]\times\mathbb{S}^{N-1})} \\ &\leq C\Big[\|v\|_{L^\infty([t-2,t+2]\times\mathbb{S}^{N-1})} + \|g\|_{L^\infty([t-2,t+2]\times\mathbb{S}^{N-1})} + \sup_{t'\in[t-1,t+1]} [g]_{C^\alpha([t'-1,t'+1]\times\mathbb{S}^{N-1})}\Big]. \end{aligned}$$

We multiply both sides by $e^{-\mu t}$ and take the supremum over $t \in (-\infty, t_0 - 2)$. The following holds

$$\sum_{j=0}^{2} \sup_{t\in(-\infty,t_0-2)} \sup_{\mathbb{S}^{N-1}} e^{-\mu t}|\nabla^j v(t, \cdot)| + \sup_{t\in(-\infty,t_0-2)} e^{-\mu t}[\nabla^2 v]_{C^\alpha([t-1,t+1]\times\mathbb{S}^{N-1})}$$

$$\leq C\Big[\|v\|_{C^0_\mu((-\infty,t_0]\times\mathbb{S}^{N-1})} + \|g\|_{C^{0,\alpha}_\mu((-\infty,t_0]\times\mathbb{S}^{N-1})}\Big], \tag{3.5}$$

where C is a positive constant that is independent of t_0.

(ii) $t_0 - 2 \leq t \leq t_0$. From the boundary Schauder estimate, we see that

$$\sum_{j=0}^{2} \sup_{\mathbb{S}^{N-1}} |\nabla^j v(t,\cdot)| + [\nabla^2 v]_{C^\alpha([t_0-3,t_0]\times\mathbb{S}^{N-1})}$$
$$\leq C\Big[\|v\|_{L^\infty([t_0-4,t_0]\times\mathbb{S}^{N-1})} + \|g\|_{L^\infty([t_0-4,t_0]\times\mathbb{S}^{N-1})} + [g]_{C^\alpha([t_0-4,t_0]\times\mathbb{S}^{N-1})} + \|\varphi\|_{C^{2,\alpha}(\mathbb{S}^{N-1})}\Big].$$

Similarly,

$$\sum_{j=0}^{2} \sup_{t\in[t_0-2,t_0]} \sup_{\mathbb{S}^{N-1}} e^{-\mu t}|\nabla^j v(t,\cdot)| + \sup_{t\in[t_0-2,t_0-1]} e^{-\mu t}[\nabla^2 v]_{C^\alpha([t-1,t+1]\times\mathbb{S}^{N-1})}$$
$$\leq C\Big[\|v\|_{C^0_\mu((-\infty,t_0]\times\mathbb{S}^{N-1})} + \|g\|_{C^{0,\alpha}_\mu((-\infty,t_0]\times\mathbb{S}^{N-1})} + e^{-\mu t_0}\|\varphi\|_{C^{2,\alpha}(\mathbb{S}^{N-1})}\Big]. \tag{3.6}$$

Combining (3.5) and (3.6), (3.4) holds.

Then, by arguments similar to those in [20–22], we obtain the L^∞ estimates of solutions on finite cylinders to (3.2) with a 0 boundary value.

Lemma 3.3. *Let $\mu > \rho_1$ and $\mu \neq \rho_k$ for $k \geq 1$, T and t_0 be constants with $t_0 \leq 0$ and $T - t_0 \leq -4$, and $g \in C^0([T,t_0]\times\mathbb{S}^{N-1})$. Suppose that $v \in C^2([T,t_0]\times\mathbb{S}^{N-1})$ satisfies the following:*

$$\begin{cases} \mathcal{L}v = g & \text{in } (T,t_0)\times\mathbb{S}^{N-1}, \\ v = 0 & \text{on } (\{T\}\cup\{t_0\})\times\mathbb{S}^{N-1}, \end{cases}$$

and $\int_{\mathbb{S}^{N-1}} v(t,\theta)Q_k(\theta)d\theta = 0$ for $k = 0,1,\cdots,K$, where K is the largest integer satisfying that $\rho_{K-1} < \mu$. Then,

$$\sup_{(t,\theta)\in[T,t_0]\times\mathbb{S}^{N-1}} e^{-\mu t}|v(t,\theta)| \leq C \sup_{(t,\theta)\in[T,t_0]\times\mathbb{S}^{N-1}} e^{-\mu t}|g(t,\theta)|, \tag{3.7}$$

where C is a positive constant dependent only on N,μ and independent of T and t_0.

Proof. Note that $\rho_1 = \sigma_+^{(2)}$.

Let the sequences $\{T_i\}$, $\{t_i\}$, $\{v_i\}$ and $\{g_i\}$ with $t_i \leq 0$ and $T_i - t_i \leq -4$, satisfy the following:

$$\begin{cases} \mathcal{L}v_i = g_i & \text{in } (T_i,t_i)\times\mathbb{S}^{N-1}, \\ v_i = 0 & \text{on } (\{T_i\}\cup\{t_i\})\times\mathbb{S}^{N-1}, \end{cases}$$

and

$$\sup_{(t,\theta)\in[T_i,t_i]\times\mathbb{S}^{N-1}} e^{-\mu t}|g_i(t,\theta)| = 1,$$
$$\sup_{(t,\theta)\in[T_i,t_i]\times\mathbb{S}^{N-1}} e^{-\mu t}|v_i(t,\theta)| \to \infty \quad \text{as } i\to\infty.$$

There exists $t_i^* \in (T_i,t_i)$ that satisfies

$$M_i = \sup_{\mathbb{S}^{N-1}} e^{-\mu t_i^*}|v_i(t_i^*,\cdot)| = \sup_{(t,\theta)\in[T_i,t_i]\times\mathbb{S}^{N-1}} e^{-\mu t}|v_i(t,\theta)| \to \infty \quad \text{as } i\to\infty.$$

Set

$$\tilde{v}_i(t,\theta) = M_i^{-1} e^{-\mu t_i^*} v_i(t + t_i^*, \theta), \tag{3.8}$$

$$\tilde{g}_i(t,\theta) = M_i^{-1} e^{-\mu t_i^*} g_i(t + t_i^*, \theta). \tag{3.9}$$

The following holds:

$$\sup_{\mathbb{S}^{N-1}} |\tilde{v}_i(0,\cdot)| = 1,$$

for any $(t,\theta) \in [T_i - t_i^*, t_i - t_i^*] \times \mathbb{S}^{N-1}$,

$$|e^{-\mu t}\tilde{v}_i(t,\theta)| \leq 1 \tag{3.10}$$

and

$$\mathcal{L}\tilde{v}_i = \tilde{g}_i \quad \text{for } t \in (T_i - t_i^*, t_i - t_i^*) \times \mathbb{S}^{N-1}.$$

Assume the following for some $\tau_- \in \mathbb{R}^- \cup \{-\infty\}$ and $\tau^+ \in \mathbb{R}^+ \cup \{\infty\}$:

$$T_i - t_i^* \to \tau_-, \quad t_i - t_i^* \to \tau_+. \tag{3.11}$$

From (3.10) we have

$$|\tilde{v}_i| \leq Ce^{\mu(t_i^* - T_i)} \quad \text{on } (T_i - t_i^*, T_i - t_i^* + 2) \times \mathbb{S}^{N-1},$$

and hence

$$\left|\frac{d^2\tilde{v}_i}{dt^2} + (N-2)\frac{d\tilde{v}_i}{dt} + \Delta_\theta \tilde{v}_i\right| \leq Ce^{\mu(t_i^* - T_i)} \quad \text{on } (T_i - t_i^*, T_i - t_i^* + 2) \times \mathbb{S}^{N-1}.$$

Since $\tilde{v}_i = 0$ on $\{T_i - t_i^*\} \times \mathbb{S}^{N-1}$, we have

$$|\nabla \tilde{v}_i| \leq Ce^{\mu(t_i^* - T_i)} \quad \text{on } (T_i - t_i^*, T_i - t_i^* + 1) \times \mathbb{S}^{N-1}.$$

This implies that $T_i - t_i^*$ remains bounded away from zero. Similar arguments imply that $t_i - t_i^*$ is bounded away from zero. As a consequence, $0 \in (\tau_-, \tau_+)$. Let

$$\tilde{v}_i \to \hat{v} \quad \text{in the compact set of } (\tau_-, \tau_+). \tag{3.12}$$

Moreover, $\tilde{g}_i \to 0$ in every compact set of (τ_-, τ_+). So the following holds:

$$\hat{v} \neq 0,$$

$$|e^{-\mu t}\hat{v}(t,\theta)| \leq 1 \quad \text{for any } (t,\theta) \in (\tau_-, \tau_+) \times \mathbb{S}^{N-1}, \tag{3.13}$$

$$\mathcal{L}\hat{v} = 0 \quad \text{on } (\tau_-, \tau_+) \times \mathbb{S}^{N-1},$$

and

$$\lim_{t \to \tau_*} \hat{v}(t,\theta) = 0 \tag{3.14}$$

where $\tau_* = \tau_-$ or τ_+ if it is finite.

Let

$$\hat{v}_k(t) = \int_{\mathbb{S}^{N-1}} \hat{v}(t,\theta) Q_k(\theta) d\theta. \tag{3.15}$$

Then $\mathcal{L}_k(\hat{v}_k) = 0$; hence, $\hat{v}_k$ is the linear combination of the basis of $\mathrm{Ker}(\mathcal{L}_k)$. We now take $k \geq 2$ with $\rho_k > \mu$. Then

$$\hat{v}_{k+1}(t) = c_{k+1}^1 e^{\rho_k t} + c_{k+1}^2 e^{\sigma_-^{(k+1)} t},$$

where c_{k+1}^1 and c_{k+1}^2 are constants. From (3.13), we obtain the following for any $t \in (\tau_-, \tau_+)$:

$$|e^{-\mu t}\hat{v}_{k+1}(t)| \le C.$$

When $\tau_+ = \infty$, $c_{k+1}^1 = 0$ and hence $\hat{v}_{k+1}(t) = c_{k+1}^2 e^{\sigma_-^{(k+1)} t}$. When τ_+ is finite, $\lim_{t\to\tau_+} \hat{v}_{k+1}(t) = 0$ by (3.14). Similarly, when $\tau_- = -\infty$, $\hat{v}_{k+1}(t) = c_{k+1}^1 e^{\rho_k t} = c_{k+1}^1 e^{\sigma_+^{(k+1)} t}$. When τ_- is finite, $\lim_{t\to\tau_-} \hat{v}_{k+1}(t) = 0$ by (3.14). Thus,

$$\int_{\tau_-}^{\tau_+} \Big[(\partial_t \hat{v}_{k+1})^2 + [\lambda_{k+1} - 2(N-2)\big]\hat{v}_{k+1}^2\Big]dt = 0.$$

Since $\rho_k > \mu > 0$, it follows that $\lambda_{k+1} > 2N$ for each k. This implies that $\lambda_{k+1} - 2(N-2) > 0$ for each k. Therefore, $\hat{v}_{k+1} = 0$ for each k. By the assumption, we have

$$\hat{v}_0 = \hat{v}_1 = \ldots = \hat{v}_k = 0$$

provided that $\rho_{k-1} < \mu$. In conclusion, $\hat{v}_k = 0$ for any $k \ge 0$; hence, $\hat{v} \equiv 0$. This is a contradiction.

Lemma 3.4. *Let $\alpha \in (0,1)$, $\mu > \rho_K$ for some $K \ge 1$, and $g \in C_\mu^{0,\alpha}((-\infty, t_0] \times \mathbb{S}^{N-1})$ with $g(t,\cdot) \in span\{Q_0, Q_1, \ldots, Q_{K+1}\}$ for $t \le t_0$. Then, there is a unique solution $v \in C_\mu^{2,\alpha}((-\infty, t_0] \times \mathbb{S}^{N-1})$ of* (3.1) *with $v(t,\cdot) \in span\{Q_0, Q_1, \ldots, Q_{K+1}\}$ for $t \le t_0$. Furthermore, $g \mapsto v$ is linear, and*

$$\|v\|_{C_\mu^{2,\alpha}((-\infty,t_0]\times\mathbb{S}^{N-1})} \le C\|g\|_{C_\mu^{0,\alpha}((-\infty,t_0]\times\mathbb{S}^{N-1})},$$

where C is a positive constant that is only dependent on N, α, μ and independent of t_0.

Proof. For $k = 0, 1, \ldots, K+1$, define

$$g_k(t) = \int_{\mathbb{S}^{N-1}} g(t,\theta) Q_k(\theta) d\theta.$$

Then,

$$\|g_k\|_{C_\mu^{0,\alpha}((-\infty,t_0])} \le C\|g\|_{C_\mu^{0,\alpha}((-\infty,t_0]\times\mathbb{S}^{N-1})},$$

and

$$g(t,\theta) = \sum_{k=0}^{K+1} g_k(t) Q_k(\theta). \tag{3.16}$$

Consider the following ODE:

$$\mathcal{L}_k v_k = g_k. \tag{3.17}$$

Suppose that there is a solution $v_k \in C_\mu^{2,\alpha}((-\infty, t_0])$ of (3.17) and

$$\|v_k\|_{C_\mu^{2,\alpha}((-\infty,t_0])} \le C\|g_k\|_{C_\mu^{0,\alpha}((-\infty,t_0])}, \tag{3.18}$$

where C is a constant that depends only on N, α, μ and independent of t_0.

If $k = 0$, it is known from Section 2 that $\mathrm{Ker}(\mathcal{L}_0)$ encompasses $e^{\tau t}\cos\gamma t$ and $e^{\tau t}\sin\gamma t$ with

$$\tau = -\frac{1}{2}(N-2) < 0, \quad \gamma = \frac{1}{2}\sqrt{(N-2)(10-N)}$$

provided that $3 \leq N \leq 9$. Ker($\mathcal{L}_0$) encompasses $e^{\sigma_+^{(0)}t}$ and $te^{\sigma_+^{(0)}t}$ with $\sigma_+^{(0)} = \sigma_-^{(0)} < 0$ provided that $N = 10$. Ker($\mathcal{L}_0$) encompasses $e^{\sigma_+^{(0)}t}$ and $e^{\sigma_-^{(0)}t}$ with $\sigma_+^{(0)} < 0$, $\sigma_-^{(0)} < 0$ provided that $N \geq 11$. If $k = 1$, Ker($\mathcal{L}_1$) encompasses $e^{\sigma_+^{(1)}t}$ and $e^{\sigma_-^{(1)}t}$ with $\sigma_+^{(1)} \leq 0$, $\sigma_-^{(1)} < 0$ provided that $N \geq 3$.

We now consider $k = 0$ and $k = 1$. For $k = 0$, we see that

$$v_0(t) = \begin{cases} B_0^1 \int_{-\infty}^t e^{\tau(t-s)} \sin\gamma(t-s) g_0(s)ds, & \text{for } 3 \leq N \leq 9, \\ \int_{-\infty}^t s e^{\sigma_+^{(0)}(t-s)} g_0(s)ds - t\int_{-\infty}^t e^{\sigma_+^{(0)}(t-s)} g_0(s)ds, & \text{for } N = 10, \\ B_0^1 \int_{-\infty}^t e^{\sigma_+^{(0)}(t-s)} g_0(s)ds - B_0^1 \int_{-\infty}^t e^{\sigma_-^{(0)}(t-s)} g_0(s)ds, & \text{for } N \geq 11, \end{cases} \tag{3.19}$$

where

$$|B_0^1| = \begin{cases} \frac{1}{\gamma}, & \text{for } 3 \leq N \leq 9, \\ \left|\frac{1}{\sigma_+^{(0)} - \sigma_-^{(0)}}\right|, & \text{for } N \geq 11. \end{cases}$$

We only consider $v_0(t)$ for $N \geq 11$ and $v_1(t)$. The other cases for $v_0(t)$ can be demonstrated similarly since $\tau < 0$ and $\sigma_+^{(0)} < 0$ in these cases. Let

$$v_k(t) = B_k^1 \int_{-\infty}^t e^{\sigma_+^{(k)}(t-s)} g_k(s)ds - B_k^1 \int_{-\infty}^t e^{\sigma_-^{(k)}(t-s)} g_k(s)ds, \tag{3.20}$$

where

$$|B_k^1| = \left|\frac{1}{\sigma_-^{(k)} - \sigma_+^{(k)}}\right|.$$

Direct calculation implies the following for $t \leq t_0$:

$$e^{-\mu t}|v_k(t)| \leq C \sup_{t \leq t_0} e^{-\mu t}|g_k(t)| = C\|g_k\|_{C_\mu^0((-\infty,t_0])}, \tag{3.21}$$

$$e^{-\mu t}(|v_k'(t)| + |v_k''(t)|) \leq C\|g_k\|_{C_\mu^0((-\infty,t_0])}. \tag{3.22}$$

Set

$$v_k''(t) = R_1(t) + R_2(t),$$

where

$$R_1(t) = B_k^1(e^{\sigma_+^{(k)}t})'' \int_{-\infty}^t e^{-\sigma_+^{(k)}s} g_k(s)ds - B_k^1(e^{\sigma_-^{(k)}t})'' \int_{-\infty}^t e^{-\sigma_-^{(k)}s} g_k(s)ds,$$

and

$$R_2(t) = B_k^1(e^{\sigma_+^{(k)}t})' e^{-\sigma_+^{(k)}t} g_k(t) - B_k^1(e^{\sigma_-^{(k)}t})' e^{-\sigma_-^{(k)}t} g_k(t).$$

Then,

$$\begin{aligned} R_1'(t) \quad = \quad & B_k^1(e^{\sigma_+^{(k)}t})''' \int_{-\infty}^t e^{-\sigma_+^{(k)}s} g_k(s)ds - B_k^1(e^{\sigma_-^{(k)}t})''' \int_{-\infty}^t e^{-\sigma_-^{(k)}s} g_k(s)ds \\ & + B_k^1(e^{\sigma_+^{(k)}t})'' e^{-\sigma_+^{(k)}t} g_k(t) - B_k^1(e^{\sigma_-^{(k)}t})'' e^{-\sigma_-^{(k)}t} g_k(t). \end{aligned}$$

Then we have

$$e^{-\mu t}|R_1'(t)| \leq C\|g_k\|_{C_\mu^0((-\infty,t_0])},$$

and hence, for $t \leq t_0 - 1$,

$$e^{-\mu t}[R_1]_{C^\alpha([t-1,t+1])} \leq C\|g_k\|_{C_\mu^0((-\infty,t_0])},$$

$$e^{-\mu t}[R_2]_{C^{\alpha}([t-1,t+1])} \leq C\|g_k\|_{C^{0,\alpha}_{\mu}((-\infty,t_0])}.$$

Therefore, for $t \leq t_0 - 1$,

$$e^{-\mu t}[v_k'']_{C^{\alpha}([t-1,t+1])} \leq C\|g_k\|_{C^{0,\alpha}_{\mu}((-\infty,t_0])}. \tag{3.23}$$

Combining (3.21), (3.22) and (3.23), (3.18) holds for $k = 0$ with $N \geq 11$ and $k = 1$.

For $2 \leq k \leq K + 1$, since $e^{-\mu t}|g_k(t)| \leq C$ and $\mu > \rho_K$, we can also define

$$v_k(t) = B_k^1 \int_{-\infty}^{t} e^{\sigma_+^{(k)}(t-s)} g_k(s)ds - B_k^1 \int_{-\infty}^{t} e^{\sigma_-^{(k)}(t-s)} g_k(s)ds. \tag{3.24}$$

Note that, at this time, $\sigma_+^{(k)} > 0$ and $\sigma_-^{(k)} < 0$ for $k = 2, \dots, K + 1$ and $\rho_k = \sigma_+^{(k+1)}$. By arguments similar to those in the proof of (3.18) for $k = 0$ with $N \geq 11$ and $k = 1$, we obtain (3.18) for $2 \leq k \leq K + 1$.

With the solution v_k of (3.17) for $k = 0, 1, \dots, K + 1$, we set

$$v(t, \theta) = \sum_{k=0}^{K+1} v_k(t)Q_k(\theta).$$

Then, $\mathcal{L}v = g$ and, by (3.18), we have

$$\begin{aligned}
\|v\|_{C^{2,\alpha}_{\mu}((-\infty,t_0]\times\mathbb{S}^{N-1})} &\leq C \sum_{k=0}^{K+1} \|v_k\|_{C^{2,\alpha}_{\mu}((-\infty,t_0])} \\
&\leq C \sum_{k=0}^{K+1} \|g_k\|_{C^{0,\alpha}_{\mu}((-\infty,t_0])} \\
&\leq C\|g\|_{C^{0,\alpha}_{\mu}((-\infty,t_0]\times\mathbb{S}^{N-1})}.
\end{aligned}$$

Then the extra requirement $v(t, \cdot) \in \text{span}\{Q_0, Q_1, \dots, Q_{K+1}\}$ implies the uniqueness of v.

Lemma 3.5. *Let $\alpha \in (0, 1)$, $\mu > \rho_1$ and $\mu \neq \rho_k$ for any $k \geq 1$; also, $g \in C^{0,\alpha}_{\mu}((-\infty, t_0] \times \mathbb{S}^{N-1})$, with $\int_{\mathbb{S}^{N-1}} g(t, \cdot)Q_k(\theta)d\theta = 0$ for any $k = 0, 1, \dots, K, K + 1$, where K is the largest integer such that $\rho_K < \mu$, and $t \leq t_0$. Then, there exists a unique solution $v \in C^{2,\alpha}_{\mu}((-\infty, t_0] \times \mathbb{S}^{N-1})$ of (3.1) with $v = 0$ on $\{t_0\} \times \mathbb{S}^{N-1}$. Moreover,*

$$\|v\|_{C^{2,\alpha}_{\mu}((-\infty,t_0]\times\mathbb{S}^{N-1})} \leq C\|g\|_{C^{0,\alpha}_{\mu}((-\infty,t_0]\times\mathbb{S}^{N-1})}, \tag{3.25}$$

where C is a positive constant that is dependent on N, α and μ and independent of t_0.

Proof. Assume that $T \leq t_0 - 4$. We claim that there is a solution $v_T \in C^{2,\alpha}([T, t_0] \times \mathbb{S}^{N-1})$ to the following problem:

$$\begin{cases} \mathcal{L}v_T = g & \text{in } (T, t_0) \times \mathbb{S}^{N-1}, \\ v_T = 0 & \text{on } (\{T\} \cup \{t_0\}) \times \mathbb{S}^{N-1}. \end{cases} \tag{3.26}$$

The problem described by (3.26) can be written as follows:

$$\begin{cases} \frac{\partial}{\partial t}(e^{\tau t} \frac{\partial v_T}{\partial t}) + e^{\tau t}\Delta_\theta v_T + 2(N-2)e^{\tau t}v_T = e^{\tau t}g & \text{in } (T, t_0) \times \mathbb{S}^{N-1}, \\ v_T = 0 & \text{on } (\{T\} \cup \{t_0\}) \times \mathbb{S}^{N-1}. \end{cases} \tag{3.27}$$

where $\tau = N - 2$. Consider the energy function

$$\mathcal{G}_T(v) = \int_T^{t_0} \int_{\mathbb{S}^{N-1}} \Big[e^{\tau t}(\partial_t v)^2 + e^{\tau t}|\nabla_\theta v|^2 - 2(N-2)e^{\tau t}v^2 + 2e^{\tau t}gv\Big]dtd\theta.$$

Set

$$\Gamma = \Big\{\phi \in H^1(\mathbb{S}^{N-1}) : \int_{\mathbb{S}^{N-1}} \phi(\theta)Q_k(\theta)d\theta = 0 \text{ for } k = 0, 1, \ldots, K, K+1 \text{ with } \rho_K < \mu\Big\}.$$

Then, for any $\phi \in \Gamma$,

$$\int_{\mathbb{S}^{N-1}} |\nabla_\theta \phi|^2 d\theta \geq 2N \int_{\mathbb{S}^{N-1}} \phi^2 d\theta,$$

since $\lambda_{k+1} \geq \lambda_2 = 2N$ with $\mu > \rho_k \geq \rho_1$. Hence, for any $v \in H_0^1((T, t_0) \times \mathbb{S}^{N-1})$ with $v(t, \cdot) \in \Gamma$ for any $t \in (T, t_0)$, we have

$$\mathcal{G}_T(v) \geq \int_T^{t_0} \int_{\mathbb{S}^{N-1}} \Big[e^{\tau t}(\partial_t v)^2 + [2N - 2(N-2)]e^{\tau t} v^2 + 2e^{\tau t} g v\Big] d\theta dt.$$

The inequality $2N - 2(N-2) > 0$ implies that $\mathcal{G}_T$ is coercive and weakly lower semi-continuous. Then there is a minimizer v_T of $\mathcal{G}_T$ in the following statement:

$$\{v \in H_0^1((T, t_0) \times \mathbb{S}^{N-1}) : \ v(t, \cdot) \in \Gamma \ \text{ for any } t \in (T, t_0)\}.$$

So v_T is a solution of (3.26) that satisfies $v_T(t, \cdot) \in \Gamma$ for any $t \in (T, t_0)$.

We obtain that by Lemma 3.3,

$$\sup_{(t,\theta) \in [T,t_0] \times \mathbb{S}^{N-1}} e^{-\mu t} |v_T(t, \theta)| \leq C \sup_{(t,\theta) \in [T,t_0] \times \mathbb{S}^{N-1}} e^{-\mu t} |g(t, \theta)|,$$

where C is a positive constant that is dependent on N, μ and independent of T and t_0. Fix $T_0 < t_0$. By the interior and boundary Schauder estimates in $[t_0 + T_0, t_0] \times \mathbb{S}^{N-1} \subset [t_0 + T_0 - 1, t_0] \times \mathbb{S}^{N-1}$ and $v_T(t_0, \theta) = 0$, there exists a subsequence v_T that converges to a $C^{2,\alpha}$-solution v of (3.1) in $[t_0 + T_0, t_0] \times \mathbb{S}^{N-1}$ satisfying $v = 0$ on $\{t_0\} \times \mathbb{S}^{N-1}$, as $T \to -\infty$. Then, we can obtain that v_T converges to a $C^{2,\alpha}$-solution v of (3.1) in $(-\infty, t_0] \times \mathbb{S}^{N-1}$ such that the following is satisfied: $v = 0$ on $\{t_0\} \times \mathbb{S}^{N-1}$,

$$\sup_{(t,\theta) \in [T,t_0] \times \mathbb{S}^{N-1}} e^{-\mu t} |v(t, \theta)| \leq C \sup_{(t,\theta) \in [T,t_0] \times \mathbb{S}^{N-1}} e^{-\mu t} |g(t, \theta)|,$$

or

$$\|v\|_{C^0_\mu((-\infty,t_0] \times \mathbb{S}^{N-1})} \leq C \|g\|_{C^0_\mu((-\infty,t_0] \times \mathbb{S}^{N-1})}, \tag{3.28}$$

Combining (3.28) and (3.4) with $\varphi = 0$, (3.25) holds.

Theorem 3.6. *Let $\alpha \in (0, 1)$, $\mu > \rho_1$, $\mu \neq \rho_k$ for any $k \geq 1$ and $g \in C^{0,\alpha}_\mu((-\infty, t_0] \times \mathbb{S}^{N-1})$. Then* (3.1) *admits a solution $v \in C^{2,\alpha}_\mu((-\infty, t_0] \times \mathbb{S}^{N-1})$ and*

$$\|v\|_{C^{2,\alpha}_\mu((-\infty,t_0] \times \mathbb{S}^{N-1})} \leq C \|g\|_{C^{0,\alpha}_\mu((-\infty,t_0] \times \mathbb{S}^{N-1})}, \tag{3.29}$$

where C is a positive constant that is dependent on N, α, μ, and independent of t_0. Also, $g \mapsto v$ is linear.

Proof. Assume that $K \geq 1$ is the largest integer with $\rho_K < \mu$. Define

$$g_k(t) = \int_{\mathbb{S}^{N-1}} g(t, \theta) Q_k(\theta) d\theta \ \text{ for } k = 0, 1, \ldots, K+1.$$

Then $v_1 \in C^{2,\alpha}_{\mu}((-\infty, t_0] \times \mathbb{S}^{N-1})$ is a solution of

$$\mathcal{L}(v_1) = \sum_{k=0}^{K+1} g_k(t)Q_k(\theta) \quad \text{in } (-\infty, t_0] \times \mathbb{S}^{N-1}$$

as in Lemma 3.4. By Lemma 3.5, let $v_2 \in C^{2,\alpha}_{\mu}((-\infty, t_0] \times \mathbb{S}^{N-1})$ be the unique solution of the following problem:

$$\begin{cases} \mathcal{L}v = g - \sum_{k=0}^{K+1} g_k Q_k & \text{in } (-\infty, t_0] \times \mathbb{S}^{N-1}, \\ v = 0 & \text{on } \{t_0\} \times \mathbb{S}^{N-1}. \end{cases}$$

Then $v = v_1 + v_2$ is a solution of (3.1) satisfying that $v(t_0, \theta) = v_1(t_0, \theta) = \sum_{k=0}^{K+1} v_k(t_0)Q_k(\theta)$, where $v_k(t)$ for $k = 0, 1, \ldots, K, K+1$ is given in Lemma 3.4.

Remark 3.7. *Theorem 3.6 implies that the bound of*

$$\mathcal{L}^{-1} : \; C^{0,\alpha}_{\mu}((-\infty, t_0] \times \mathbb{S}^{N-1}) \to C^{2,\alpha}_{\mu}((-\infty, t_0] \times \mathbb{S}^{N-1}) \tag{3.30}$$

is independent of t_0.

4. Nonradial singular solutions of (1.1)

In what follows, singular solutions of (1.1) will be constructed.

We set

$$\mathcal{N}(w) = w_{tt} + (N-2)w_t + \Delta_{\mathbb{S}^{N-1}} w + 2(N-2)(e^w - 1). \tag{4.1}$$

Then w satisfies

$$w_{tt} + (N-2)w_t + \Delta_{\mathbb{S}^{N-1}} w + 2(N-2)(e^w - 1) = 0 \;\text{ in } (-\infty, 0) \times \mathbb{S}^{N-1} \tag{4.2}$$

if $\mathcal{N}(w) = 0$ in $(-\infty, 0) \times \mathbb{S}^{N-1}$. This also implies that $u(x) = U_s(x) + w(\ln|x|, \theta)$ is a solution of (1.15) in $B\backslash\{0\}$.

Theorem 4.1. *Let $U_s(x)$ be given as in* (1.2)*, the index set $\mathcal{I}_\rho$ be given as in* (2.58)*, and $\mu > \rho_1$ with $\mu \notin \mathcal{I}_\rho$. Suppose that $\hat{w} \in C^{2,\alpha}((-\infty, 0] \times \mathbb{S}^{N-1})$ satisfies*

$$|\hat{w}(t,\theta)| + |\nabla \hat{w}(t,\theta)| \to 0 \;\; \textit{as } t \to -\infty \textit{ uniformly in } \theta \in \mathbb{S}^{N-1}, \tag{4.3}$$

and for $(t, \theta) \in (-\infty, 0] \times \mathbb{S}^{N-1}$,

$$|\mathcal{N}(\hat{w})(t,\theta)| + |\nabla(\mathcal{N}(\hat{w}))(t,\theta)| \le Ce^{\mu t}, \tag{4.4}$$

where C is a positive constant. Then, there exist $t_0 < 0$ and a solution $w \in C^{2,\alpha}((-\infty, t_0] \times \mathbb{S}^{N-1})$ of the equation in (4.2) *such that the following is satisfied for* $(t, \theta) \in (-\infty, t_0) \times \mathbb{S}^{N-1}$:

$$|w(t,\theta) - \hat{w}(t,\theta)| \le Ce^{\mu t}, \tag{4.5}$$

where C is a positive constant.

Proof. The proof consists of 4 steps.

Step 1. We claim that there is $z \in C^{2,\alpha}_\mu((-\infty, t_0] \times \mathbb{S}^{N-1})$ such that

$$\mathcal{N}(\hat{w} + z) = 0. \tag{4.6}$$

Rewrite this equation as

$$\mathcal{L}z + \mathcal{N}(\hat{w}) + P(z) = 0, \tag{4.7}$$

and

$$z = \mathcal{L}^{-1}\Big[- \mathcal{N}(\hat{w}) - P(z)\Big], \tag{4.8}$$

where

$$P(z) = 2(N-2)[e^{\hat{w}+z} - e^{\hat{w}} - z]. \tag{4.9}$$

Defining $\mathcal{T}$ by

$$\mathcal{T}(z) = \mathcal{L}^{-1}\Big[- \mathcal{N}(\hat{w}) - P(z)\Big], \tag{4.10}$$

we prove that $\mathcal{T}$ is a contraction on some ball in $C^{2,\alpha}_\mu((-\infty, t_0] \times \mathbb{S}^{N-1})$, for some $t_0 < 0$ with $|t_0|$ large. We set

$$\Gamma_{B,t_0} = \{z \in C^{2,\alpha}_\mu((-\infty, t_0] \times \mathbb{S}^{N-1}) : \ ||z||_{C^{2,\alpha}_\mu((-\infty,t_0]\times\mathbb{S}^{N-1})} \le B\}.$$

Step 2. We claim that $\mathcal{T}$ maps Γ_{B,t_0} to itself, for some fixed B and any t_0 with $|t_0|$ sufficiently large, namely, for any $z \in C^{2,\alpha}_\mu((-\infty, t_0] \times \mathbb{S}^{N-1})$ with $||z||_{C^{2,\alpha}_\mu((-\infty,t_0]\times\mathbb{S}^{N-1})} \le B$, we have that $\mathcal{T}(z) \in C^{2,\alpha}_\mu((-\infty, t_0] \times \mathbb{S}^{N-1})$ and $||\mathcal{T}(z)||_{C^{2,\alpha}_\mu((-\infty,t_0]\times\mathbb{S}^{N-1})} \le B$.

First, it follows from (4.4) that

$$||\mathcal{N}(\hat{w})||_{C^1_\mu((-\infty,t_0]\times\mathbb{S}^{N-1})} \le C_1.$$

Next, set

$$E(z) = 2(N-2)\int_0^1 (e^{\hat{w}+sz} - 1)ds. \tag{4.11}$$

Then, $P(z) = zE(z)$. Take any $z \in C^{2,\alpha}_\mu((-\infty, t_0] \times \mathbb{S}^{N-1})$ with $||z||_{C^{2,\alpha}_\mu((-\infty,t_0]\times\mathbb{S}^{N-1})} \le B$ where B will be determined later. Because

$$|\hat{w}| + |\nabla \hat{w}| \le \epsilon(t),$$

where ϵ is increasing such that $\epsilon(t) \to 0$ as $t \to -\infty$ and

$$|z| + |\nabla z| \le Be^{\mu t}.$$

Then, for $t \le t_0$,

$$|E(z)| + |\nabla E(z)| \le C_2(\epsilon(t) + Be^{\mu t}), \tag{4.12}$$

and hence,

$$\begin{aligned} ||P(z)||_{C^1_\mu((-\infty,t_0]\times\mathbb{S}^{N-1})} &\le C_2(\epsilon(t_0) + Be^{\mu t_0})||z||_{C^1_\mu((-\infty,t_0]\times\mathbb{S}^{N-1})} \\ &\le C_2(\epsilon(t_0) + Be^{\mu t_0})B. \end{aligned}$$

By Theorem 3.6, we have

$$||\mathcal{T}(z)||_{C^{2,\alpha}_\mu((-\infty,t_0]\times\mathbb{S}^{N-1})} \le C|| - \mathcal{N}(\hat{w}) - P(z)||_{C^{0,\alpha}_\mu((-\infty,t_0]\times\mathbb{S}^{N-1})}$$

$$\leq \quad C[C_1 + C_2(\epsilon(t_0) + Be^{\mu t_0})B],$$

where C, C_1 and C_2 are constants that are independent of t_0. Choose $B \geq 2CC_1$ and t_0 with $|t_0|$ sufficiently large such that $CC_2(\epsilon(t_0) + Be^{\mu t_0}) \leq 1/2$. Then,

$$\|\mathcal{T}(z)\|_{C_\mu^{2,\alpha}((-\infty,t_0]\times\mathbb{S}^{N-1})} \leq B.$$

Step 3. We claim that $\mathcal{T} : \Gamma_{B,t_0} \to \Gamma_{B,t_0}$ is a contraction, i.e., for any $z_1, z_2 \in \Gamma_{B,t_0}$,

$$\|\mathcal{T}(z_1) - \mathcal{T}(z_2)\|_{C_\mu^{2,\alpha}((-\infty,t_0]\times\mathbb{S}^{N-1})} \leq \kappa\|z_1 - z_2\|_{C_\mu^{2,\alpha}((-\infty,t_0]\times\mathbb{S}^{N-1})}, \tag{4.13}$$

for some constant $\kappa \in (0, 1)$.

Note that

$$\mathcal{T}(z_1) - \mathcal{T}(z_2) = \mathcal{L}^{-1}[P(z_2) - P(z_1)],$$

and

$$\begin{aligned} P(z_1) - P(z_2) &= z_1E(z_1) - z_2E(z_2) \\ &= (z_1 - z_2)E(z_1) + z_2(E(z_1) - E(z_2)). \end{aligned}$$

By (4.11), we have

$$E(z_1) - E(z_2) = 2(N-2)\int_0^1 [e^{\hat{w}+sz_1} - e^{\hat{w}+sz_2}]ds.$$

Then,

$$|E(z_1) - E(z_2)| + |\nabla(E(z_1) - E(z_2))| \leq C(|z_1 - z_2| + |\nabla(z_1 - z_2)|).$$

By (4.12), we have the following for any $t \leq t_0$,

$$\begin{aligned} &|P(z_1) - P(z_2)| + |\nabla(P(z_1) - P(z_2))| \\ &\quad \leq C(\epsilon(t) + Be^{\mu t})(|z_1 - z_2| + |\nabla(z_1 - z_2)|), \end{aligned}$$

and hence

$$\|P(z_1) - P(z_2)\|_{C_\mu^1((-\infty,t_0]\times\mathbb{S}^{N-1})} \leq C(\epsilon(t_0) + Be^{\mu t_0})\|z_1 - z_2\|_{C_\mu^1((-\infty,t_0]\times\mathbb{S}^{N-1})}.$$

By Theorem 3.6, we obtain

$$\begin{aligned} \|\mathcal{T}(z_1) - \mathcal{T}(z_2)\|_{C_\mu^{2,\alpha}((-\infty,t_0]\times\mathbb{S}^{N-1})} &\leq C\|P(z_1) - P(z_2)\|_{C_\mu^{0,\alpha}((-\infty,t_0]\times\mathbb{S}^{N-1})} \\ &\leq C(\epsilon(t_0) + Be^{\mu t_0})\|z_1 - z_2\|_{C_\mu^{2,\alpha}((-\infty,t_0]\times\mathbb{S}^{N-1})}. \end{aligned}$$

We derive (4.13) by choosing t_0 with t_0 sufficiently negative.

Step 4. The contraction mapping principle implies that there exists $z \in C_\mu^{2,\alpha}((-\infty, t_0] \times \mathbb{S}^{N-1})$ such that $\mathcal{T}(z) = z$. Then $z \in C_\mu^{2,\alpha}((-\infty, t_0] \times \mathbb{S}^{N-1})$ is a solution of (4.6) and hence $w = \hat{w} + z$ is a solution of (4.2).

We call a function $\hat{w}$ that satisfies (4.3) and (4.4) an approximate solution to (4.2) with order μ.

Lemma 4.2. *Assume that $Q_k(\theta)$ is the combination of $Q_1^k(\theta), \ldots, Q_{m_k}^k(\theta)$. Then,*

$$Q_kQ_l = \sum_{i=0}^{k+l} Q_i. \tag{4.14}$$

Proof. The proof is similar to that of Lemma 2.4 in [19]. We omit the details here.

Proposition 4.3. *Let the index sets* $\mathcal{I}_\rho$ *and* $\mathcal{I}_{\tilde\rho}$ *be respectively given as in* (2.58) *and* (2.59) *and* $\mu > \rho_1$ *with* $\mu \notin \mathcal{I}_\rho \cup \mathcal{I}_{\tilde\rho}$. *Let* η *be a solution of* $\mathcal{L}(\eta) = 0$ *on* $\mathbb{R} \times \mathbb{S}^{N-1}$ *such that* $\eta(t,\cdot) \to 0$ *as* $t \to -\infty$ *uniformly on* $\mathbb{S}^{N-1}$. *Therefore for some* $t_0 < 0$, *there exists a smooth function* $\tilde\eta$ *on* $(-\infty, t_0] \times \mathbb{S}^{N-1}$ *such that* $\hat w = \eta + \tilde\eta$ *satisfies* (4.3) *and* (4.4).

Proof. Choose ϕ with $|\phi| < \hat\epsilon$ on $(-\infty, 0) \times \mathbb{S}^{N-1}$, where $\hat\epsilon > 0$ is a sufficiently small number. Then, a simple computation yields

$$\mathcal{N}(\phi) = \mathcal{L}\phi + 2(N-2)(e^\phi - 1 - \phi).$$

Therefore,

$$\mathcal{N}(\phi) = \mathcal{L}(\phi) + \sum_{i=2}^{\infty} a_i \phi^i. \tag{4.15}$$

Assume $K \geq 1$ and $\tilde K$ represent the largest corresponding integer with $\rho_K < \mu$ and $\tilde\rho_{\tilde K} < \mu$, respectively. There is no function that converges to 0 as $t \to -\infty$ in Ker$\mathcal{L}_0$ and Ker$\mathcal{L}_1$; for $k \geq 2$, $\psi_k^+(t) = e^{\sigma_+^{(k)}}$ and $\psi_k^-(t) = e^{\sigma_-^{(k)} t}$ in Ker$\mathcal{L}_k t$. Any term $e^{\rho_k t}$ with $k > K$ in η will produce the term $e^{\tilde\rho_l t}$ with $\tilde\rho_l > \mu$ in $\mathcal{N}(\eta)$; set

$$\eta(t,\theta) = \sum_{k=1}^{K} c_k Q_{k+1}(\theta) e^{\rho_k t}, \tag{4.16}$$

where c_k denotes constants.

For the following case:

$$\mathcal{I}_\rho \cap \mathcal{I}_{\tilde\rho} = \emptyset, \tag{4.17}$$

we will show that there $\tilde\eta_0, \tilde\eta_1, \ldots, \tilde\eta_{\tilde K}$ exists in succession such that, for any $i = 0, 1, \ldots, \tilde K$,

$$\mathcal{N}(\eta + \tilde\eta_0 + \ldots + \tilde\eta_i) = O(e^{\tilde\rho_{i+1} t}). \tag{4.18}$$

Set $\phi = \eta$. From (4.15) and $\mathcal{L}(\eta) = 0$, we obtain

$$\mathcal{N}(\eta) = \sum_{n_1 + \ldots + n_{i_1} \geq 2} a_{n_1 \ldots n_{i_1}} e^{(n_1\rho_1 + \ldots + n_{i_1}\rho_{i_1})t} Q_2^{n_1} \ldots Q_{i_1+1}^{n_{i_1}},$$

where $n_1, \ldots, n_{i_1}$ are nonnegative integers and $a_{n_1 \ldots n_{i_1}}$ is a constant. By the definition of $\mathcal{I}_{\tilde\rho}$, $n_1\rho_1 + \ldots + n_{i_1}\rho_{i_1}$ is some $\tilde\rho_k$. Hence, by Lemma 4.2,

$$\mathcal{N}(\eta) = \sum_{k=1}^{\tilde K} \Big\{ \sum_{m=0}^{\tilde M_k} a_{km} Q_m(\theta) \Big\} e^{\tilde\rho_k t} + O(e^{\tilde\rho_{\tilde K+1} t}), \tag{4.19}$$

where $\tilde M_k$ is defined as in (2.61) and a_{km} is constant. We see that

$$\mathcal{N}(\eta) = O(e^{\tilde\rho_1 t}),$$

where $\tilde\rho_1 = 2\rho_1$. Then $\tilde\eta_0 = 0$.

Assume that (4.18) holds for $0, 1, \ldots, i-1$. Set

$$\tilde{\eta}_i(t,\theta) = \Big(\sum_{m=0}^{\tilde{M}_i} c_{im} Q_m(\theta)\Big) e^{\tilde{\rho}_i t}, \tag{4.20}$$

where c_{im} is constant. This implies that

$$\begin{aligned} \mathcal{N}(\eta + \tilde{\eta}_0 + \ldots + \tilde{\eta}_i) &= \mathcal{L}(\tilde{\eta}_1) + \ldots + \mathcal{L}(\tilde{\eta}_i) \\ &\quad + \sum_{k=1}^{\tilde{K}} \Big\{ \sum_{m=0}^{\tilde{M}_k} a_{km} Q_m(\theta) \Big\} e^{\tilde{\rho}_k t} + O(e^{\tilde{\rho}_{\tilde{K}+1} t}), \end{aligned}$$

where a_{km} is different from that in (4.19). The induction hypothesis implies that

$$\mathcal{N}(\eta + \tilde{\eta}_0 + \ldots + \tilde{\eta}_i) = \mathcal{L}(\tilde{\eta}_i) + \sum_{k=i}^{\tilde{K}} \Big\{ \sum_{m=0}^{\tilde{M}_k} a_{km} Q_m(\theta) \Big\} e^{\tilde{\rho}_k t} + O(e^{\tilde{\rho}_{\tilde{K}+1} t}).$$

Choose $\tilde{\eta}_i$ such that

$$\mathcal{L}(\tilde{\eta}_i) = -\Big\{ \sum_{m=0}^{\tilde{M}_i} a_{im} Q_m(\theta) \Big\} e^{\tilde{\rho}_i t}.$$

So (4.18) holds for i. For $m = 0, 1, \ldots, \tilde{M}_i$, solve

$$\mathcal{L}_m(c_{im} e^{\tilde{\rho}_i t}) = -a_{im} e^{\tilde{\rho}_i t}. \tag{4.21}$$

Similar to that in Lemma 3.4, there is a formula for $c_{im} e^{\tilde{\rho}_i t}$ in terms of $a_{im} e^{\tilde{\rho}_i t}$. For $0 < \rho_m < \tilde{\rho}_i$, the expression is in the form of (3.24). For $\rho_m > \tilde{\rho}_i$, the expression is similar. If $m = 0$ or 1, the expression is (3.19) or (3.20). Therefore, define

$$\tilde{\eta}(t,\theta) = \sum_{k=1}^{\tilde{K}} \Big\{ \sum_{m=0}^{\tilde{M}_k} c_{km} Q_m(\theta) \Big\} e^{\tilde{\rho}_k t}, \tag{4.22}$$

where c_{km} is a constant. We obtain

$$\mathcal{N}(\eta + \tilde{\eta}) = O(e^{\tilde{\rho}_{\tilde{K}+1} t}) = O(e^{\mu t}).$$

The estimate of $\nabla \mathcal{N}(\eta + \tilde{\eta})$ is similar.

Then for the general case, ρ_k can be some $\tilde{\rho}_i$. We only need to modify (4.21). When $\rho_m = \tilde{\rho}_i$, there exist constants c_{i0m} and c_{i1m} such that

$$\mathcal{L}_m((c_{i0m} + t c_{i1m}) e^{\tilde{\rho}_i t}) = -a_{im} e^{\tilde{\rho}_i t}.$$

By iteration, there are more powers of t. Therefore, there exist constants denoted by c_{ijm} with $j = 0, 1, \ldots, J+1$ such that

$$\mathcal{L}_m\Big(\sum_{j=0}^{J+1} c_{ijm} t^j e^{\tilde{\rho}_i t}\Big) = \sum_{j=0}^{J} a_{ijm} t^j e^{\tilde{\rho}_i t}.$$

In conclusion, instead of (4.22), apply

$$\tilde{\eta}(t,\theta) = \sum_{k=1}^{\tilde{K}} \sum_{j=0}^{k} \Big\{ \sum_{m=0}^{\tilde{M}_k} c_{kjm} Q_m(\theta) \Big\} t^j e^{\tilde{\rho}_k t}, \tag{4.23}$$

where c_{kjm} is a constant.

Proof of Theorem 1.1

Theorem 4.1 and Proposition 4.3 imply that we can obtain t_0 and a solution $w \in C^{2,\alpha}((-\infty, t_0] \times \mathbb{S}^{N-1})$ of the equation in (4.2) such that, for $\mu > \rho_1$ with $\mu \notin \mathcal{I}_\rho \cup \mathcal{I}_{\tilde{\rho}}$ and any $(t,\theta) \in (-\infty, t_0) \times \mathbb{S}^{N-1}$,

$$w(t,\theta) = \hat{\eta}(t,\theta) + O(e^{\mu t}), \tag{4.24}$$

$$\hat{\eta}(t,\theta) = \eta(t,\theta) + \tilde{\eta}(t,\theta).$$

Let $r_0 = e^{t_0}$. Then $0 < r_0 < 1$. Since $u(x) = U_s(x) + w(\ln|x|, \theta)$, we add easily see that

$$u(x) = U_s(x) + O(|x|^{\sigma_+^{(2)}}) \quad \text{for } x \in B_{r_0} \backslash \{0\}. \tag{4.25}$$

On the other hand, for any $R > 0$, we see that $\tilde{u}(y) := u(x) + 2\ln(R^{-1} r_0)$, $y = R r_0^{-1} x$ satisfies the following equations:

$$-\Delta_y \tilde{u} = e^{\tilde{u}} \quad \text{in } B_R \backslash \{0\} \tag{4.26}$$

and

$$\tilde{u}(y) = U_s(y) + O(|y|^{\sigma_+^{(2)}}) \quad \text{as } |y| \to 0^+. \tag{4.27}$$

This implies that $\tilde{u}$ is a singular solution of (1.1).

Now, suppose that $\tilde{u}$ is non-radial. It suffices to prove that u is non-radial in $B_{r_0} \backslash \{0\}$. Suppose that $u(x) = u(|x|)$. Then from Proposition 2.1 we have

$$u(x) \equiv U_s(x) \quad \text{for } x \in B_{r_0} \backslash \{0\}. \tag{4.28}$$

Moreover, we can easily see from (4.24) and (4.25) that

$$w(t,\theta) \not\equiv 0.$$

This implies that u and $\tilde{u}$ are non-radial singular solutions of (1.15) in $B_{r_0} \backslash \{0\}$ and (1.1) in $B_R \backslash \{0\}$, respectively.

Since $w(t,\theta)$ depends on the parameter μ, for each $\mu > \rho_1$, different coefficients c_k of $\eta(t,\theta)$ in (4.16) can determine infinitely many $\tilde{u}_\mu$ and t_0 may change. When restricting all coefficients of $\eta(t,\theta)$ in (4.16) in a bounded interval, there is a minimal $t_0 < 0$. Since K of $\eta(t,\theta)$ in (4.16) depends on μ, we can obtain infinitely many $\tilde{u}(y)$ by choosing a sequence of parameters $\mu > \rho_1$ with $\mu \to \infty$. Hence, a family of nonradial singular solutions of (1.1) can be constructed.

5. Conclusions

In this manuscript, infinitely many nonradial singular solutions have been constructed for the equation

$$-\Delta u = e^u \quad \text{in } B_R \backslash \{0\},$$

where $B_R = \{x \in \mathbb{R}^N \ (N \geq 3) : \ |x| < R\}$.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflicts of interest.

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