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https://www.aimspress.com/journal/era

ERA, 32(5): 3092–3112.
DOI: 10.3934/era.2024141
Received: 26 February 2024
Revised: 14 April 2024
Accepted: 17 April 2024
Published: 24 April 2024

Research article

A study for a higher order Riemann-Liouville fractional differential equation with weakly singularity

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Abstract: In this paper, we study an initial value problem with a weakly singular nonlinear fractional differential equation of higher order. First, we establish the existence of global solutions to the problem within the appropriate function space. We then introduce a generalized Riemann-Liouville mean value theorem. Using this theorem, we prove the Nagumo-type uniqueness theorem for the stated problem. Moreover, we give two examples to illustrate the applicability of the existence and uniqueness theorems.

Keywords: weakly singular integral equation; fractional differential equation; initial value problem; global existence

1. Introduction

Recently, weakly singular fractional differential equations have become prominent in the theory of fractional differential equations, drawing extensive attention and leading to significant studies in the literature. The research of Webb [1] is pioneering work, where he identified a critical deficiency in the theory and proposed an appropriate treatment. This deficiency concerns the inability to accurately reduce initial value problems that involve fractional-order differential equations to integral equations. Webb demonstrated how the Riemann-Liouville (R-L) integral operator traverses various function spaces, such as continuous, Holder continuous, absolutely continuous, and L_p function spaces, in a rigorous manner. Using these findings and appropriate mathematical tools, the correct methodology for reducing initial value problems associated with fractional-order differential equations to integral equations was delineated. This refinement has established a more robust foundation for the theory. In addition to this study, Lan [2] made a significant contribution by focusing on reducing boundary-value problems involving R-L fractional differential equations of alpha-order ($\alpha \in (1, 2)$) to integral equations.

Moreover, these integral equations exhibit weak singularities, sometimes single and sometimes double, with respect to the function on the right-hand side of the corresponding differential equation. The study of the uniqueness of solutions to such integral equations has led to the generalization of existing Gronwall-type inequalities in the literature. For further insight, one may refer to [3–5], which are devoted to the newly defined weakly singular Gronwall inequalities. Another determinant of the uniqueness of solutions in weakly singular integral equations is the Nagumo-type uniqueness criterion. Meanwhile, [6] presents two different proof techniques for establishing the uniqueness of solutions of the integral equation corresponding to the classical differential equation under this condition, and they have been used in numerous studies such as [7, 8]. Moreover, Sert and San [9] contributed another proof (the third proof) of the Nagumo-type result for the integral equation with a doubly weak singularity kernel. In this context, we aim to extend their technique to our specific problem.

The studies [10–15] are among the recent works on initial value problems involving singular fractional differential equations. Webb [1] considered the following initial value problem involving multi-index singular nonlinear differential equations in the sense of the Caputo derivative

$$\begin{cases} {}_{C}D^{1+\eta}u(t) = t^{-\gamma}g\left(t, u(t), D_{C}^{\eta}u(t)\right), & t > 0 \\ u(0) = u_0, u'(0) = u_1 \quad u_0, u_1 \in \mathbb{R} \end{cases} \tag{1.1}$$

where $0 \le \gamma < \eta < 1$ and $0 < \beta \le 1$. He proved the local existence and uniqueness of the solutions to the problem when the function g on the right-hand side of the equation is continuous on $[0, T] \times \mathbb{R} \times \mathbb{R}$, i.e., when the right-hand side of the equation has a weakly singularity.

Bilgici and San [16] investigated a similar problem with the R-L fractional derivative (instead of Caputo derivative), as follows:

$$\begin{cases} {}_{RL}D^{\eta}u(t) = g\left(t, u(t), {}_{RL}D^{\eta}u(t)\right), & t > 0 \\ u(0) = 0, \quad {}_{RL}D^{\eta}u(t)|_{t=0} = b, \end{cases} \tag{1.2}$$

where $1 < \eta \le 2$ and the function g in the right-hand side of the equation is continuous on $(0, T] \times \mathbb{R} \times \mathbb{R}$ and has singularity of order $\eta - 1$ at $t = 0$. They established a local existence theorem, along with uniqueness theorems of Nagumo-type, Krasnoselskii-Krein-type, and Osgood-type.

To address the aforementioned issues, we generalize some results of Bilgici and San [16] and investigate the global solutions of the following weakly singular fractional differential equation with initial value conditions:

$$\begin{cases} {}_{RL}D^{\eta}u(t) = g(t, u(t), {}_{RL}D^{\eta-2}u(t), {}_{RL}D^{\eta-1}u(t)), & t > 0 \\ u(0) = 0, \quad {}_{RL}D^{\eta-2}u(t)|_{t=0} = 0, \quad {}_{RL}D^{\eta-1}u(t)|_{t=0} = u_{\eta-1} \in \mathbb{R} \end{cases} \tag{1.3}$$

where $\eta \in (2, 3)$.

Moreover, to the best of our knowledge, the equation in Problem (1.3) has been studied with boundary value conditions and under the condition that the function on the right-hand side is continuous (see, for example [17], [18], and [19]). Here, we extend their ideas for Problem (1.3). First, in view of the research of Webb [1], we try to reduce the problem to the corresponding integral equation in a correct way. Then, we give a global existence theorem for the solution to the integral equation in the appropriate space by proposing some conditions on the function g on the right-hand side of the equation. These two results lead to the following condition:

(K1) Let $g(t, u, v, \omega)$ and $t^{\eta-2}g(t, u, v, \omega)$ be continuous on $(0, \infty) \times \mathbb{R}^3$ and $[0, \infty) \times \mathbb{R}^3$, respectively.

This type of continuity requirement for g alone does not ensure the existence of global solutions to the problem, even if local existence can be established solely based on the continuity of g. This issue relates to the compactness of the operator associated with the problem. To address this, we draw insights from the work of Su and Zhang [17], Jiang [18], and Hao et al. [19], who have studied analogous problems without singularities. In addition, we introduce a Nagumo-type result for the problem by presenting a generalized mean-value theorem for the R-L derivative. Furthermore, two supportive examples are given to demonstrate the applicability of the existence and uniqueness theorems.

2. Preliminaries

2.1. Definitions and some properties of operators in Problem (1.3)

The following definition and some properties of the R-L fractional integral and derivative can be found in [1, 20, 21]. From now on, we will denote the R-L derivative, previously represented by ${}_{RL}D^{\eta}$, solely by D^{η}.

Definition 2.1. The R-L fractional derivative of order $\eta \in (n-1, n)$, $(n \in \mathbb{Z})$ of a function u with the R-L fractional integral $I^{\eta-n}u(t)$ with an n-th order derivative is defined by

$$D^{\eta}u(t) = \frac{d^n}{dt^n} I^{\eta-n}u(t) = \frac{1}{\Gamma(n-\eta)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\eta-1} u(s) ds.$$

In the next section, we use the following properties of the R-L fractional integral and derivative:

- The R-L integral operator has Abelian and semigroup properties, i.e., $I^{\alpha}I^{\beta}u = I^{\alpha+\beta}u = I^{\beta}I^{\alpha}u$ holds for $u \in L[0, T]$, where $L[0, T]$ represents the space of integrable functions on $[0, T]$. Moreover, I^1 means the classical integral operator I.
- $I^{\alpha}u(0) = 0$ is satisfied for $u \in C[0, T]$, where $C[0, T]$ is the space of continuous functions on $[0, T]$.
- Let $m \in \mathbb{Z}$ and $\eta \in \mathbb{R}$ with $0 < m < \eta$. The equality $D^{\eta}u(t) = D^{m}D^{\eta-m}u(t)$ holds for $u \in C^{\eta}[0, T]$, where $C^{\eta}[0, T] = \{u \in C[0, T] : D^{\eta}u(t) \in C[0, T]\}$. Here, $D := \frac{d}{dt}$ is the usual derivative operator and $D^m = \frac{d^m}{dt^m}$.

These properties are generally applied to reduce Problem (1.3) to an integral equation in the next section.

2.2. The definition of the solution space for Problem (1.3)

The continuity condition on the function g in (K1) gives us the insight that it is sufficient for the function u to be of class $C^{\eta-1}[0, \infty)$ (which is the space of functions with a continuous $(\eta-1)$−order derivative) in order to satisfy the equations of Problem (1.3). However, it is known that any continuous function defined on $[0, \infty)$ cannot be bounded; therefore, we cannot speak of a supremum of the functions u, $D^{\eta-2}u$ and $D^{\eta-1}u$ on $[0, \infty)$. To get around this situation, we can bound some classes of these functions by multiplying them by the weight function $\frac{1}{1+t^m}$ $(m \geq 0)$, i.e., the polynomial-weighted function. So, the solution to Problem (1.3) is investigated in a subspace of $C^{\eta-1}[0,\infty)$. This subspace is given by

$$Y = \left\{ u(t) \in C^{\eta-1}[0,\infty) : \sup_{t\in[0,\infty)} \left\{ \frac{|u(t)|}{1+t^{\eta-1}}, \frac{\left|D^{\eta-2}u(t)\right|}{1+t}, \left|D^{\eta-1}u(t)\right| \right\} < +\infty \right\},$$

and the weight functions are determined by the integral representations of $u(t), D^{\eta-2}u(t)$ and $D^{\eta-1}u(t)$, which can be obtained after transforming the problem into an integral equation, as in Lemma 2.1. As described by Su and Zhang [17], Jiang [18], and Hao et al. [19], the space Y is a Banach space when endowed with the following norm

$$||u||_Y = \max\left\{ \sup_{t\in[0,\infty)} \frac{|u(t)|}{1+t^{\eta-1}}, \sup_{t\in[0,\infty)} \frac{\left|D^{\eta-2}u(t)\right|}{1+t}, \sup_{t\in[0,\infty)} \left|D^{\eta-1}u(t)\right| \right\}.$$

2.3. Some tools used for the existence of the solution to Problem (1.3)

One mathematical tool used to establish the existence of solutions to an initial value problem is fixed point theory. Schauder's fixed point theorem is one of the theorems proposed to prove the existence of the fixed point of an operator acting on a Banach space. This theorem guarantees the existence of a fixed point for the operator if it maps a closed convex subset of the Banach space into itself and is a relatively compact operator (see Zeidler [22]). While the continuity condition is often sufficient to show the existence of locally fixed points of the operator, establishing the existence of globally fixed points of the operator requires more conditions and more mathematical effort. In particular, we need a lemma and an extra condition on the function g to prove the relative compactness of the operator acting on space Y. The following lemma, given and proved in [17], should be helpful:

Lemma 2.1. Let be $Z \subset Y$ a bounded set and $\eta \in (2,3)$. Suppose that the following are satisfied:

i) For any $u(t) \in Z$, the functions $\frac{u(t)}{1+t^{\eta-1}}, \frac{D^{\eta-2}u(t)}{1+t}$ and $D^{\eta-1}u(t)$ are equicontinuous on every compact subset of $[0,\infty)$.

ii) For any given $\varepsilon > 0$, there exists a real number $T = T(\varepsilon) > 0$ such that for any $t_1, t_2 > T$ and for any $u(t) \in Z$, the inequalities

$$\left| \frac{u(t_1)}{1+t_1^{\eta-1}} - \frac{u(t_2)}{1+t_2^{\eta-1}} \right| < \varepsilon, \quad \left| \frac{D^{\eta-2}u(t_1)}{1+t_1} - \frac{D^{\eta-2}u(t_2)}{1+t_2} \right| < \varepsilon, \quad \left| D^{\eta-1}u(t_1) - D^{\eta-1}u(t_2) \right| < \varepsilon$$

holds.

Then, the set Z is relatively compact in Y.

3. Main results

3.1. The equivalence of solutions to Problem (1.3) and solutions of the related integral equation

There are several methods in the literature for investigating the existence of solutions to initial value problems involving fractional differential equations. One of these methods entails converting the initial value problem into the corresponding integral equation and investigating the solutions of the integral equation. In what follows, it is shown how the given initial value problem can be reduced to an integral equation.

Lemma 3.1. Let $\eta \in (2,3)$, and let condition (K1) be satisfied. If $u \in C^{\eta-1}[0,T]$, $T > 0$ is a solution of Problem (1.3), then the function u solves the equation

$$u(t) = \frac{u_{\eta-1}}{\Gamma(\eta)}t^{\eta-1} + \frac{1}{\Gamma(\eta)}\int_0^t \frac{g(s,u(s),D^{\eta-2}u(s),D^{\eta-1}u(s))}{(t-s)^{1-\eta}}ds \tag{3.1}$$

and, vice versa.

Proof. $\Longrightarrow$: It is supposed that $u \in C^{\eta-1}[0,T]$ is a solution of Problem (1.3). It is shown that $u \in C^{\eta-1}[0,T]$ solves Eq (3.1). For simplicity, we write the nonlinear function g on the right-hand side of the equation in Problem (1.3) as follows:

$$g(t,\mathcal{D}^{\eta-1}u(t)) = g(t,u(t),D^{\eta-2}u(t),D^{\eta-1}u(t)) \tag{3.2}$$

where

$$\mathcal{D}^{\eta-1} = (D^0, D^{\eta-2}, D^{\eta-1}).$$

Since the condition (K1) is satisfied, $g(t,\mathcal{D}^{\eta-1}u(t)) \in C(0,T]$ and $t^{\eta-2}g(t,\mathcal{D}^{\eta-1}u(t)) \in C[0,T]$. From here, we obtain

$$\int_0^T |g(t,\mathcal{D}^{\eta-1}u(t))|dt = \int_0^T \frac{\left|t^{\eta-2}g(t,\mathcal{D}^{\eta-1}u(t))\right|}{t^{\eta-2}}dt \le M\int_0^T \frac{1}{t^{\eta-2}}dt \le \frac{MT^{3-\eta}}{3-\eta} < \infty.$$

This means that $g(t,\mathcal{D}^{\eta-1}u(t))$ is also integrable, i.e., $g(t,\mathcal{D}^{\eta-1}u(t)) \in C(0,T]\cap L^1[0,T]$. This implies that $D^\eta u(t) \in C(0,T]\cap L^1[0,T]$ since the equation in Problem (1.3) is satisfied. Thus, by integrating both sides of this equation, we obtain

$$D^{\eta-1}u(t) = D^{\eta-1}u(0) + Ig(t,\mathcal{D}^{\eta-1}u(t))) = u_{\eta-1} + Ig(t,\mathcal{D}^{\eta-1}u(t))) \tag{3.3}$$

where $D^{\eta-1}u(0) = u_{\eta-1}$ and the relation $ID^\eta = IDD^{\eta-1}$ was used.

This time, by integrating both sides of the above equation and using the relation $ID^{\eta-1} = IDD^{\eta-2}$ and $D^{\eta-2}u(0) = 0$ one can see that the following equation is yielded

$$D^{\eta-2}u(t) = u_{\eta-1}t + I^2g(t,\mathcal{D}^{\eta-1}u(t))) = u_{\eta-1}t + \int_0^t (t-s)g(s,u(s),D^{\eta-2}u(s),D^{\eta-1}u(s))ds. \tag{3.4}$$

Since $u(0) = 0$, it is known that $I^{3-\eta}u(0) = 0$. Given this and the relation $ID^{\eta-2} = IDI^{3-\eta}$,

$$I^{3-\eta}u(t) = \frac{u_{\eta-1}t^2}{2} + I^3g(t,\mathcal{D}^{\eta-1}u(t))$$

is satisfied for all $t \in [0,T]$. Applying $I^{\eta-2}$ to both sides of the above equation and using the semigroup and Abelian properties of the R-L integral operator, it is seen that

$$Iu(t) = \frac{u_{\eta-1}t^\eta}{\Gamma(\eta+1)} + II^\eta g(t,\mathcal{D}^{\eta-1}u(t))$$

is satisfied for all $t \in [0, T]$. Differentiating both sides, gives Eq (3.1).

$\Longleftarrow$: Now, suppose otherwise and let the integral equation given by Eq (3.1) be solved by $u \in C^{\eta-1}[0, T]$. Applying D^η to both sides of Eq (3.1) and using the relation

$$D^\eta I^\eta u(t) = D^\eta I^{2-\eta} I^\eta u(t) = D^2 I^2 u(t) = u(t)$$

once can deduce that $u \in C^{\eta-1}[0, T]$ is a solution of the equation in Problem (1.3).

On the other hand, let us show that the function u, which we assume as the solution of Eq (3.1), satisfies the initial conditions in Problem (1.3). First, by changing the variable $s = mt$ in the integral of Eq (3.1), it follows that

$$u(0) = \lim_{t\to 0^+} u(t) = \lim_{t\to 0^+} \int_0^t \frac{g(s, \mathcal{D}^{\eta-1}u(s))}{\Gamma(\eta)(t-s)^{1-\eta}} ds = \lim_{t\to 0^+} t^2 \int_0^1 \frac{(mt)^{\eta-2} g(s, \mathcal{D}^{\eta-1}u(s))|_{s=mt}}{\Gamma(\eta) m^{\eta-2}(1-m)^{1-\eta}} dm = 0, \tag{3.5}$$

since the last integral in the above equation is finite.

Second, by applying $D^{\eta-2}$ to both sides of Eq (3.1) and changing the same variable as above, one can see that

$$\begin{aligned} D^{\eta-2}u(0) &= \lim_{t\to 0^+} u_{\eta-1} t + \lim_{t\to 0^+} \int_0^t \frac{(t-s)s^{\eta-2} g(s, \mathcal{D}^{\eta-1}u(s)))}{s^{\eta-2}} ds \\ &= \lim_{t\to 0^+} t^{4-\eta} \int_0^1 \frac{(1-m)(mt)^{\eta-2}}{m^{\eta-2}} g(s, \mathcal{D}^{\eta-1}u(s)))|_{s=mt} dm = 0, \quad (4-\eta > 0) \end{aligned} \tag{3.6}$$

where the relation $D^{\eta-2} I^\eta u(t) = DI^{3-\eta} I^\eta u(t) = DI^3 u(t) = I^2 u(t)$ was used. From this relation, it can easily be found that $D[D^{\eta-2} I^\eta u(t)] = D[DI^{3-\eta} I^\eta u(t)] = D^2 I^3 u(t) = Iu(t)$. Thus, from the integral equation in Eq (3.6), the final result is obtained:

$$D^{\eta-1}u(0) = u_{\eta-1} + \lim_{t\to 0^+} \int_0^t g(s, \mathcal{D}^{\eta-1}u(s))) = u_{\eta-1} + \lim_{t\to 0^+} t^{3-\eta} \int_0^1 \frac{g(s, \mathcal{D}^{\eta-1}u(s)))|_{s=mt}}{m^{\eta-2}} (mt)^{\eta-2} dm = u_{\eta-1}.$$

This completes the proof. □

3.2. The existence of solutions to Problem (1.3)

In the previous subsection, the initial value problem given by Problem (1.3) was converted into the integral equation given by Eq (3.1). There are some ways (i.e., Tonelli approach, fixed poin theory, etc.) to investigate the existence of solutions of Eq (3.1). Here we will use fixed point theory, in particular Schauder's fixed point theorem.

The existence of a global solution to Problem (1.3) is proved by the following theorem.

Theorem 3.1. Let $\eta \in (2, 3)$. Assume that (K1) and the following condition (K2) are satisfied:

(K2) There exist integrable functions denoted by $a_i : [0, \infty) \to [0, \infty)$ $(i = 1, 2, 34)$ such that the following integral inequalities are satisfied:

$$\int_0^\infty \frac{[(1 + s^{\eta-1})a_1(s) + (1 + s)a_2(s) + a_3(s)]}{s^{\eta-2}} ds \le \frac{\Gamma(\eta)}{2}, \quad \int_0^\infty \frac{a_4(s)}{s^{\eta-2}} ds < \infty$$

and for all $t \in [0, \infty)$,

$$|t^{\eta-2} g(t, u, v, w)| \le a_1(t)|u| + a_2(t)|v| + a_3(t)|w| + a_4(t). \tag{3.7}$$

Then, Problem (1.3) has at least one solution $u(t) \in Y$.

Proof. The proof is given in three steps, showing that the conditions of Schauder's fixed point theorem are satisfied. The first step is to show that the operator related to the integral equation in Eq (3.1) maps a closed convex sphere into itself. The second step is devoted to proving the relative compactness of the operator. The last step is devoted to showing the continuity of the operator.

First Step. First, we define an operator that corresponds to the integral equation in Eq (3.1) as follows:

$$\mathcal{L}u(t) = \frac{u_{\eta-1}}{\Gamma(\eta)}t^{\eta-1} + \frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1} g(s, u(s), D^{\eta-2}u(s), D^{\eta-1}u(s))ds \tag{3.8}$$

and, let us show that $\mathcal{L} : U \to U$, where U is the closed, convex ball defined by $U = \{u(t) \in Y : \|u(t)\|_Y \leq R\}$ with

$$R \geq \frac{2\left(u_{\eta-1} + \int_0^\infty \frac{a_4(s)}{s^{\eta-2}}ds\right)}{\Gamma(\eta) - 2\int_0^\infty [(1+s^{\eta-2})a_1(s) + (1+s)a_2(s) + a_3(s)]ds}.$$

For simplicity, in the following integral inequalities let $g(s, \mathcal{D}^{\eta-1}u(s))$ be as in Eq (3.2) and let

$$H(s, \mathcal{D}^{\eta-1}u(s)) = \frac{a_1(s)(1+s^{\eta-1})\frac{|u(s)|}{1+s^{\eta-1}} + a_2(s)(1+s)\frac{|D^{\eta-2}u(s)|}{1+s} + a_3(s)|D^{\eta-1}u(s)|}{s^{\eta-2}}. \tag{3.9}$$

From the inequality $\frac{(t-s)^{\eta-1}}{(1+t)^{\eta-1}} \leq 1$ for all $s \in [0, t]$ and from conditions (K1) and (K2), we have

$$\begin{aligned}
\frac{|\mathcal{L}u(t)|}{1+t^{\eta-1}} &\leq \left|\frac{u_{\eta-1}}{\Gamma(\eta)}\frac{t^{\eta-1}}{1+t^{\eta-1}}\right| + \frac{1}{\Gamma(\eta)}\left|\int_0^t \frac{(t-s)^{\eta-1}}{1+t^{\eta-1}} s^{\eta-2} g(s, \mathcal{D}^{\eta-1}u(s))\frac{1}{s^{\eta-2}}ds\right| \\
&\leq \frac{u_{\eta-1}}{\Gamma(\eta)} + \frac{1}{\Gamma(\eta)}\int_0^\infty \left|s^{\eta-2}g(s, \mathcal{D}^{\eta-1}u(s))\frac{1}{s^{\eta-2}}\right|ds \\
&\leq \frac{u_{\eta-1}}{\Gamma(\eta)} + \frac{1}{\Gamma(\eta)}\int_0^\infty \frac{a_1(s)|u(s)| + a_2(s)|D^{\eta-2}u(s)| + a_3(s)|D^{\eta-1}u(s)|}{s^{\eta-2}}ds + \frac{1}{\Gamma(\eta)}\int_0^\infty \frac{a_4(s)}{s^{\eta-2}}ds \\
&\leq \frac{u_{\eta-1}}{\Gamma(\eta)} + \frac{1}{\Gamma(\eta)}\int_0^\infty H(s, \mathcal{D}^{\eta-1}u(s))ds + \frac{1}{\Gamma(\eta)}\int_0^\infty \frac{a_4(s)}{s^{\eta-2}}ds \\
&\leq \frac{u_{\eta-1}}{\Gamma(\eta)} + \frac{1}{\Gamma(\eta)}\left[\|u\|_Y \int_0^\infty \frac{[(1+s^{\eta-1})a_1(s) + (1+s)a_2(s) + a_3(s)]}{s^{\eta-2}}ds + \int_0^\infty \frac{a_4(s)}{s^{\eta-2}}ds\right] \\
&\leq \frac{R}{2} < R
\end{aligned} \tag{3.10}$$

for any $u(t) \in U$.

By applying the inequalities $\frac{t-s}{t+1} \leq \frac{t}{t+1} \leq 1$ for all $s \in [0, t]$ and $\Gamma(\eta) \leq 2$ in Eq (3.4), it follows that

$$\begin{aligned}
\frac{|D^{\eta-2}\mathcal{L}u(t)|}{1+t} &\leq \left|\frac{u_{\eta-1}t}{1+t}\right| + \left|\int_0^t \frac{(t-s)}{1+t} s^{\eta-2} g(s, \mathcal{D}^{\eta-1}u(s))\frac{1}{s^{\eta-2}}ds\right| \\
&\leq |u_{\eta-1}| + \int_0^\infty \frac{a_1(s)|u(s)| + a_2(s)|D^{\eta-2}u(s)| + a_3(s)|D^{\eta-1}u(s)| + a_4(s)}{s^{\eta-2}}ds \\
&\leq \Gamma(\eta)\frac{R}{2} \leq R
\end{aligned} \tag{3.11}$$

Finally, from Eq (3.3), we obtain

$$|D^{\eta-1}\mathcal{L}u(t)| \leq |u_{\eta-1}| + \int_0^\infty |s^{\eta-2}g(s,\mathcal{D}^{\eta-1}u(s))\frac{1}{s^{\eta-2}}|ds \leq \Gamma(\eta)\frac{R}{2} \leq R. \tag{3.12}$$

From the inequalities given by Inequalities (3.10)–(3.12) and the definition of the norm $\|.\|_Y$, it follows that

$$\|\mathcal{L}u\|_Y = \max\left\{\sup_{t\in[0,\infty)}\frac{|\mathcal{L}u(t)|}{1+t^{\eta-1}}, \sup_{t\in[0,\infty)}\frac{\left|D^{\eta-2}\mathcal{L}u(t)\right|}{1+t}, \sup_{t\in[0,\infty)}\left|D^{\eta-1}\mathcal{L}u(t)\right|\right\} \leq R.$$

Therefore, this implies that $\mathcal{L}: U \to U$.

Second Step. Let V be a subset of U. We see that $\mathcal{L}V$ is relatively compact by showing that the conditions of Lemma 2.1 are satisfied. Let K be a compact subset of $[0,\infty)$ and $t_1, t_2 \in K$ with $t_1 < t_2$.

- **Equicontinuity of $\mathcal{L}u$:** For any $u(t) \in V$ one can write

$$\begin{aligned}\left|\frac{\mathcal{L}u(t_2)}{1+t_2^{\eta-1}} - \frac{\mathcal{L}u(t_1)}{1+t_1^{\eta-1}}\right| &\leq \frac{u_{\eta-1}}{\Gamma(\eta)}\left|\frac{t_2^{\eta-1}}{1+t_2^{\eta-1}} - \frac{t_1^{\eta-1}}{1+t_1^{\eta-1}}\right| \\ &+ \frac{1}{\Gamma(\eta)}\left|\int_0^{t_2}\frac{(t_2-s)^{\eta-1}}{1+t_2^{\eta-1}}G(s)ds - \int_0^{t_1}\frac{(t_1-s)^{\eta-1}}{1+t_1^{\eta-1}}G(s)ds\right| \\ &\leq \frac{u_{\eta-1}}{\Gamma(\eta)}\left|\frac{t_2^{\eta-1}}{1+t_2^{\eta-1}} - \frac{t_1^{\eta-1}}{1+t_1^{\eta-1}}\right| + \frac{1}{\Gamma(\eta)}\int_{t_1}^{t_2}\frac{(t_2-s)^{\eta-1}}{1+t_2^{\eta-1}}|G(s)|ds \\ &+ \int_0^{t_1}\left|\frac{(t_2-s)^{\eta-1}}{1+t_2^{\eta-1}} - \frac{(t_1-s)^{\eta-1}}{1+t_1^{\eta-1}}\right||G(s)|ds\end{aligned} \tag{3.13}$$

  where, and throughout the text, we used/will use the following abbreviation:

$$G(t) := g(t,\mathcal{D}^{\eta-1}u(t)). \tag{3.14}$$

  For the first term on the right-hand side of the inequality above, it is obvious that $\frac{u_{\eta-1}}{\Gamma(\eta)}\left|\frac{t_2^{\eta-1}}{1+t_2^{\eta-1}} - \frac{t_1^{\eta-1}}{1+t_1^{\eta-1}}\right| \to 0$ when $t_1 \to t_2$. For the second term there, if we define $h(s) = \frac{(t_2-s)^{\eta-1}}{1+t_2^{\eta-1}}|G(s)|s^{\eta-2}$ and $k(s) = \frac{1}{s^{\eta-2}}$, and if we apply the generalized mean value theorem, then one can say that there exists a $c \in (t_1,t_2)$ such that

$$0 \leq \int_{t_1}^{t_2}\frac{(t_2-s)^{\eta-1}}{1+t_2^{\eta-1}}G(s)s^{\eta-2}\frac{1}{s^{\eta-2}}ds = h(c)\int_{t_1}^{t_2}\frac{1}{s^{\eta-2}}ds = h(c)\left(\frac{t_2^{3-\eta}}{3-\eta} - \frac{t_1^{3-\eta}}{3-\eta}\right).$$

  The right-hand side of the equation vanishes as $t_1 \to t_2$ since $3-\eta > 0$. This implies that the above integral converges to zero when $t_1 \to t_2$.

Similarly, if we define $k(s) := \frac{1}{s^{\eta-2}}$ and $h(s) := \left|\frac{(t_2-s)^{\eta-1}}{1+t_2^{\eta-1}} - \frac{(t_1-s)^{\eta-1}}{1+t_1^{\eta-1}}\right| s^{\eta-2}|G(s)|$ in the third integral in Inequality (3.13), then by the generalized mean value theorem, there exists a point $c \in (0, t_1)$ so that the following equation is satisfied:

$$\int_0^{t_1} \left|\frac{(t_2-s)^{\eta-1}}{1+t_2^{\eta-1}} - \frac{(t_1-s)^{\eta-1}}{1+t_1^{\eta-1}}\right| s^{\eta-2}|G(s)|\frac{1}{s^{\eta-2}}ds = h(c)\int_0^{t_1} k(s)ds.$$

From here, we have that $h(c) \to 0$ as $t_1 \to t_2$. This implies that the integral on the right-hand side of the above equation goes to zero.

Following the results obtained above, one can see that $\left|\frac{\mathcal{L}u(t_2)}{1+t_2^{\eta-1}} - \frac{\mathcal{L}u(t_1)}{1+t_1^{\eta-1}}\right| \to 0$ as $t_1 \to t_2$.

- **Equicontinuity of $D^{\eta-2}\mathcal{L}u$:** The integral representation of $D^{\eta-2}\mathcal{L}u$ can be easily obtained from the integral equation in Eq (3.4). From here, one can write

$$\begin{aligned}\left|\frac{D^{\eta-2}\mathcal{L}u(t_2)}{1+t_2} - \frac{D^{\eta-2}\mathcal{L}u(t_1)}{1+t_1}\right| &\le u^{\eta-1}\left|\frac{t_2}{1+t_2} - \frac{t_1}{1+t_1}\right| + \left|\int_0^{t_2}\frac{t_2-s}{1+t_2}G(s)ds - \int_0^{t_1}\frac{t_1-s}{1+t_1}G(s)ds\right| \\ &\le u^{\eta-1}\left|\frac{t_2}{1+t_2} - \frac{t_1}{1+t_1}\right| + \int_{t_1}^{t_2}\left|\frac{t_2-s}{1+t_2}\right||G(s)|ds \\ &+ \int_0^{t_1}\left|\frac{t_2}{1+t_2} - \frac{t_1}{1+t_1}\right||G(s)|ds.\end{aligned}$$

For the first addend just above, one can easily deduce that $u^{\eta-1}\left|\frac{t_2}{1+t_2} - \frac{t_1}{1+t_1}\right| \to 0$ when $t_1 \to t_2$. In the second addend case, if one assumes that $h(s) = \frac{t_2-s}{1+t_2}s^{\eta-2}|G(s)|$ and $k(s) = \frac{1}{s^{\eta-2}}$, then as a consequence of the generalized mean value theorem, one can say that there exists a $c \in (t_1, t_2)$ such that

$$\int_{t_1}^{t_2}\frac{t_2-s}{1+t_2}s^{\eta-2}|G(s)|\frac{1}{s^{\eta-2}}ds = h(c)\int_{t_1}^{t_2}k(s)ds = h(c)\left(\frac{t_2^{3-\eta}}{3-\eta} - \frac{t_1^{3-\eta}}{3-\eta}\right).$$

From here, one can see that $\frac{t_2^{3-\eta}}{3-\eta} - \frac{t_1^{3-\eta}}{3-\eta} \to 0$ as $t_1 \to t_2$.

For the last term, by applying the same argument used above one gets

$$\int_0^{t_1}\left(\frac{t_2}{1+t_2} - \frac{t_1}{1+t_1}\right)s^{\eta-2}|G(s)|\frac{1}{s^{\eta-2}}ds = \left(\frac{t_2}{1+t_2} - \frac{t_1}{1+t_1}\right)c^{\eta-2}|G(c)|\int_0^{t_1}\frac{1}{s^{\eta-2}}ds \to 0$$

when $t_1 \to t_2$.

Consequently, in light of the above calculations, one can deduce that $\left|\frac{D^{\eta-2}\mathcal{L}u(t_2)}{1+t_2^{\eta-1}} - \frac{D^{\eta-2}\mathcal{L}u(t_1)}{1+t_1^{\eta-1}}\right| \to 0$ as $t_1 \to t_2$.

- **Equicontinuity of $D^{\eta-1}\mathcal{L}u$:** After obtaining the integral equation corresponding to $D^{\eta-1}\mathcal{L}u$, one can easily get

$$\left|D^{\eta-1}\mathcal{L}u(t_2) - D^{\eta-1}\mathcal{L}u(t_1)\right| = \left|\int_{t_1}^{t_2}G(s)ds\right|.$$

By the generalized mean value theorem, there is a $c \in (t_1, t_2)$ such that the following equation is satisfied:

$$\int_{t_1}^{t_2} G(s)s^{\eta-2}\frac{1}{s^{\eta-2}}ds = h(c)\int_{t_1}^{t_2}\frac{1}{s^{\eta-2}} = h(c)\left(\frac{t_2^{3-\eta}}{3-\eta} - \frac{t_1^{3-\eta}}{3-\eta}\right).$$

Since the right-hand side of the above equation goes to zero when $t_1 \to t_2$, it means that $\left|D^{\eta-1}\mathcal{L}u(t_2) - D^{\eta-1}\mathcal{L}u(t_1)\right| \to 0$ as $t_1 \to t_2$.

It has been demonstrated that the first hypothesis of Lemma 2.1 is valid. The following estimations are provided to demonstrate that the second assumption of Lemma 2.1 is also satisfied.

Estimations:

- Let $u(t) \in V$. From the condition (K2) we obtain,

$$\int_0^\infty \left|g(s, \mathcal{D}^{\eta-1}u(s))\right| ds \leq \|u\|_Y \int_0^\infty \frac{[(1+s^{\eta-1})a_1(s) + (1+s)a_2(s) + a_3(s)]}{s^{\eta-2}}ds \leq \frac{\Gamma(\eta)}{2}R.$$

Since this integral is over $[0, \infty)$ and finite, one can say that, for all $\varepsilon > 0$, there exists an $L > 0$ such that the following inequality is satisfied:

$$\int_L^\infty \left|g(s, \mathcal{D}^{\eta-1}u(s))\right| ds < \varepsilon. \tag{3.15}$$

- $\lim_{t\to\infty} \frac{t^{\eta-1}}{1+t^{\eta-1}} = 1 \Rightarrow \forall\varepsilon > 0\ \exists T_1 > 0:$

$$\left|\frac{t_1^{\eta-1}}{1+t_1^{\eta-1}} - \frac{t_2^{\eta-1}}{1+t_2^{\eta-1}}\right| \leq \left|1 - \frac{t_1^{\eta-1}}{1+t_1^{\eta-1}}\right| + \left|1 - \frac{t_2^{\eta-1}}{1+t_2^{\eta-1}}\right| < \varepsilon \textit{ whenever } t_1, t_2 \geq T_1. \tag{3.16}$$

- $\lim_{t\to\infty} \frac{(t-L)^{\eta-1}}{1+t^{\eta-1}} = 1 \Rightarrow \forall\varepsilon > 0\ \exists T_2 > 0 : \forall s \in [0, L],$

$$\left|\frac{(t_1-s)^{\eta-1}}{1+t_1^{\eta-1}} - \frac{(t_2-s)^{\eta-1}}{1+t_2^{\eta-1}}\right| \leq \left|1 - \frac{(t_1-s)^{\eta-1}}{1+t_1^{\eta-1}}\right| + \left|1 - \frac{(t_2-s)^{\eta-1}}{1+t_2^{\eta-1}}\right| < \varepsilon \textit{ whenever } t_1, t_2 \geq T_2. \tag{3.17}$$

- $\lim_{t\to\infty} \frac{t}{1+t} = 1 \Rightarrow \forall\varepsilon > 0\ \exists T_3 > 0:$

$$\left|\frac{t_1}{1+t_1} - \frac{t_2}{1+t_2}\right| \leq \left|1 - \frac{t_1}{1+t_1}\right| + \left|1 - \frac{t_2}{1+t_2}\right| < \varepsilon \textit{ whenever } t_1, t_2 \geq T_3. \tag{3.18}$$

- $\lim_{t\to\infty} \frac{(t-L)}{1+t} = 1 \Rightarrow \forall\varepsilon > 0\ \exists T_4 > 0 : \forall s \in [0, L],$

$$\left|\frac{t_1-s}{1+t_1} - \frac{t_2-s}{1+t_2}\right| \leq \left|1 - \frac{t_1-s}{1+t_1}\right| + \left|1 - \frac{t_2-s}{1+t_2}\right| < \varepsilon \textit{ whenever } t_1, t_2 \geq T_4. \tag{3.19}$$

Using the estimations provided above, we can now justify the second assumption of Lemma 2.1. Let $T > \max\{T_1, T_2, T_3, T_4\}$.

First, from the definition of the operator in Eq (3.8), one can get the following for any $t_1, t_2 \geq T$

$$\begin{aligned}\left|\frac{\mathcal{L}u(t_2)}{1+t_2^{\eta-1}} - \frac{\mathcal{L}u(t_1)}{1+t_1^{\eta-1}}\right| \leq{}& \frac{u_{\eta-1}}{\Gamma(\eta)}\left|\frac{t_2^{\eta-1}}{1+t_2^{\eta-1}} - \frac{t_1^{\eta-1}}{1+t_1^{\eta-1}}\right| \\ &+ \frac{1}{\Gamma(\eta)}\left|\int_0^{t_2}\frac{(t_2-s)^{\eta-1}}{1+t_2^{\eta-1}}G(s)ds - \int_0^{t_1}\frac{(t_1-s)^{\eta-1}}{1+t_1^{\eta-1}}G(s)ds\right| \\ \leq{}& \frac{u_{\eta-1}}{\Gamma(\eta)}\left|\frac{t_2^{\eta-1}}{1+t_2^{\eta-1}} - \frac{t_1^{\eta-1}}{1+t_1^{\eta-1}}\right| + \frac{1}{\Gamma(\eta)}\int_0^L\left|\frac{(t_2-s)^{\eta-1}}{1+t_2^{\eta-1}} - \frac{(t_1-s)^{\eta-1}}{1+t_1^{\eta-1}}\right||G(s)|ds \\ &+ \frac{1}{\Gamma(\eta)}\int_L^{t_2}\left|\frac{(t_2-s)^{\eta-1}}{1+t_2^{\eta-1}}\right||G(s)|ds + \frac{1}{\Gamma(\eta)}\int_L^{t_1}\left|\frac{(t_1-s)^{\eta-1}}{1+t_1^{\eta-1}}\right||G(s)|ds. \end{aligned} \tag{3.20}$$

Let us now obtain some estimates for four addends on the right-hand side of Inequality (3.20). For the first one, by Inequality (3.16) it follows that

$$\frac{u_{\eta-1}}{\Gamma(\eta)}\left|\frac{t_2^{\eta-1}}{1+t_2^{\eta-1}} - \frac{t_1^{\eta-1}}{1+t_1^{\eta-1}}\right| < \frac{u_{\eta-1}}{\Gamma(\eta)}\varepsilon. \tag{3.21}$$

Applying Inequality (3.17) to the second one, one can obtain

$$\frac{1}{\Gamma(\eta)}\int_0^L\left|\frac{(t_2-s)^{\eta-1}}{1+t_2^{\eta-1}} - \frac{(t_1-s)^{\eta-1}}{1+t_1^{\eta-1}}\right|\frac{|G(s)|s^{\eta-2}}{s^{\eta-2}}ds \leq \frac{\varepsilon \max_{t\in[0,\infty)}|s^{\eta-2}G(s)|}{\Gamma(\eta)}\int_0^L\frac{1}{s^{\eta-2}}ds \leq \frac{M\varepsilon}{\Gamma(\eta)}\frac{L^{3-\eta}}{3-\eta}. \tag{3.22}$$

For the third and fourth addends, one can have

$$\frac{1}{\Gamma(\eta)}\int_L^{t_i}\left|\frac{(t_i-s)^{\eta-1}}{1+t_i^{\eta-1}}G(s)\right|ds \leq \frac{1}{\Gamma(\eta)}\int_L^{\infty}|G(s)|ds < \frac{\varepsilon}{\Gamma(\eta)}, \quad (i=1,2) \tag{3.23}$$

since $\frac{(t_i-s)^{\eta-1}}{1+t_i^{\eta-1}} \leq \frac{t_i^{\eta-1}}{1+t_i^{\eta-1}} \leq 1$ for $i=1,2$. From the estimations in Inequalities (3.21)–(3.23), one can obtain

$$\left|\frac{\mathcal{L}u(t_2)}{1+t_2^{\eta-1}} - \frac{\mathcal{L}u(t_1)}{1+t_1^{\eta-1}}\right| \leq \frac{1}{\Gamma(\eta)}\varepsilon\left(u_{\eta-1} + M\frac{L^{3-\eta}}{3-\eta} + 2\right). \tag{3.24}$$

Second, from the integral representation of $D^{\eta-2}u$ one can have the following for all $t_1, t_2 \geq T$

$$\begin{aligned}\left|\frac{D^{\eta-2}\mathcal{L}u(t_2)}{1+t_2} - \frac{D^{\eta-2}\mathcal{L}u(t_1)}{1+t_1}\right| \leq{}& u_{\eta-1}\left|\frac{t_2}{1-t_2} - \frac{t_1}{1-t_1}\right| + \left|\int_0^{t_2}\frac{t_2-s}{1+t_2}G(s)ds - \int_0^{t_1}\frac{t_1-s}{1+t_1}G(s)ds\right| \\ \leq{}& u_{\eta-1}\left|\frac{t_2}{1-t_2} - \frac{t_1}{1-t_1}\right| + \int_0^L\left|\frac{t_2-s}{1+t_2} - \frac{t_1-s}{1+t_1}\right||G(s)|ds \\ &+ \int_L^{t_2}\frac{t_2-s}{1+t_2}|G(s)|ds + \int_L^{t_1}\frac{t_1-s}{1+t_1}|G(s)|ds. \end{aligned} \tag{3.25}$$

The first term of Inequality (3.25) can be evaluated with the help of Inequality (3.18) as follows

$$u_{\eta-1}\left|\frac{t_2}{1-t_2}-\frac{t_1}{1-t_1}\right| < u_{\eta-1}\varepsilon. \tag{3.26}$$

For the second term, by Inequality (3.19), we obtain

$$\int_0^L \left|\left[\frac{t_2-s}{1+t_2}-\frac{t_1-s}{1+t_1}\right]\frac{G(s)s^{\eta-2}}{s^{\eta-2}}\right|ds \le \max_{s\in[0,L]}|s^{\eta-2}G(s)|\int_0^L \left|\frac{t_2-s}{1+t_2}-\frac{t_1-s}{1+t_1}\right|\frac{1}{s^{\eta-2}}ds \le M\varepsilon\frac{L^{3-\eta}}{3-\eta}. \tag{3.27}$$

Since $\frac{(t_i-s)}{1+t_i} \le \frac{t_i}{1+t_i} \le 1$ holds for $s \in [0,t_i]$ $(i = 1,2)$ in the third and fourth addends in Inequality (3.25), it follows that

$$\int_L^{t_i}\frac{t_i-s}{1+t_i}|G(s)|ds \le \int_L^{\infty}G(s)ds < \varepsilon, \quad (i=1,2) \tag{3.28}$$

Consequently, using the estimations in Inequalities (3.26)–(3.28) in Inequality (3.25) gives

$$\left|\frac{D^{\eta-2}\mathcal{L}u(t_2)}{1+t_2^{\eta-1}}-\frac{D^{\eta-2}\mathcal{L}u(t_1)}{1+t_1^{\eta-1}}\right| \le u_{\eta-1}\varepsilon + M\varepsilon\frac{L^{3-\eta}}{3-\eta}+2\varepsilon.$$

Third, by using Inequality (3.15) in the integral formula of $D^{\eta-1}u$ in Eq (3.3), it follows that

$$|D^{\eta-1}\mathcal{L}u(t_2)-D^{\eta-1}\mathcal{L}u(t_1)| = \int_{t_1}^{t_2}|G(s)|ds = \int_L^{\infty}|G(s)|ds < \varepsilon.$$

It is shown that the second condition of the lemma is also satisfied. Thus, by this lemma $\mathcal{L}V$ is relatively compact in V.

Step 3. Here, it is shown that the operator $\mathcal{L}: U \to U$ is continuous by its sequentially continuity. Let $\{u_n\}_{n=1}^{\infty} \subset U$ be a sequence such that $\|u_n-u\|_Y \to 0$ for $u \in U$ as $n \to \infty$. We need to show that $\|\mathcal{L}u_n-\mathcal{L}u\|_Y \to 0$. Since $\|u_n-u\|_Y \to 0$, by the definition of Y, we have that $u_n(t) \to u(t)$, $D^{\eta-2}u_n(t) \to D^{\eta-2}u(t)$ ve $D^{\eta-1}u_n(t) \to D^{\eta-1}u(t)$ for all $t \in [0,\infty)$. Moreover, there exists an $r > 0$ such that for all n, $\|u_n\|_Y \le r$ and $\|u\|_Y \le r$. On the other hand, for an $\varepsilon > 0$ there exist an $L_1 > 0$ such that the following inequality is satisfied:

$$\int_{L_1}^{+\infty}\frac{[(1+s^{\eta-1})a_1(s)+(1+s)a_2(s)+a_3(s)]}{s^{\eta-2}}ds < \frac{\Gamma(\eta)}{6r}\varepsilon, \int_{L_1}^{+\infty}\frac{a_4(s)}{s^{\eta-2}}ds < \frac{\Gamma(\eta)}{6}\varepsilon, \tag{3.29}$$

because we supposed that the condition (K2) holds.

According to these arguments, we split the integral, which gives

$$\begin{aligned}\left|\frac{\mathcal{L}u_n(t)}{1+t^{\eta-1}}-\frac{\mathcal{L}u(t)}{1+t^{\eta-1}}\right| &\le \frac{1}{\Gamma(\eta)}\int_0^{\infty}\frac{(t-s)^{\eta-1}}{1+t^{\eta-1}}\frac{1}{s^{\eta-2}}s^{\eta-2}|g(s,\mathcal{D}^{\eta-1}u_n(s))-g(s,\mathcal{D}^{\eta-1}u(s))|ds\\ &\le \frac{1}{\Gamma(\eta)}\int_0^{L_1}\frac{s^{\eta-2}|g(s,\mathcal{D}^{\eta-1}u_n(s))-g(s,\mathcal{D}^{\eta-1}u(s))|}{s^{\eta-2}}ds\\ &+\frac{1}{\Gamma(\eta)}\int_{L_1}^{\infty}\frac{s^{\eta-2}|g(s,\mathcal{D}^{\eta-1}u_n(s))-g(s,\mathcal{D}^{\eta-1}u(s))|}{s^{\eta-2}}ds.\end{aligned} \tag{3.30}$$

An estimation for the first integral on the right-hand side of the above inequality is obtained as follows: By the condition (K1), the function $t^{\eta-2}g(t,u,v,\omega)$ is continuous on $[0,L_1]\times\left[0,\left(1+L_1^{\eta-1}\right)r\right]\times[0,(1+L_1)r]\times[0,r]$, and, therefore, it is uniformly continuous on that compact domain. Thus, for the same $\varepsilon>0$, there exists an n_0 such that, for all $n\geq n_0$ and $t\in[0,L_1]$, the following inequality holds:

$$|g(t,\mathcal{D}^{\eta-1}u_n(t))-g(t,\mathcal{D}^{\eta-1}u(t))|<\varepsilon\frac{\Gamma(\eta)(3-\eta)}{L_1^{3-\eta}}.$$

By this result, we obtain

$$\frac{1}{\Gamma(\eta)}\int_0^{L_1}\frac{s^{\eta-2}|g(s,\mathcal{D}^{\eta-1}u_n(s))-g(s,\mathcal{D}^{\eta-1}u(s))|}{s^{\eta-2}}ds<\varepsilon\frac{\Gamma(\eta)(3-\eta)}{L_1^{3-\eta}}\frac{1}{\Gamma(\eta)}\int_0^{L_1}\frac{1}{s^{\eta-2}}ds=\frac{\varepsilon}{3}. \tag{3.31}$$

The second integral in Eq (3.30) is first written as follows

$$\frac{1}{\Gamma(\eta)}\int_{L_1}^{\infty}\frac{s^{\eta-2}|G(s,u_n)-G(s,u)|}{s^{\eta-2}}ds\leq\frac{1}{\Gamma(\eta)}\int_{L_1}^{\infty}\frac{s^{\eta-2}|G(s,u_n)|}{s^{\eta-2}}ds+\frac{1}{\Gamma(\eta)}\int_{L_1}^{\infty}\frac{s^{\eta-2}|G(s,u)|}{s^{\eta-2}}ds, \tag{3.32}$$

where

$$G(s,u(s))=g(s,\mathcal{D}^{\eta-1}u(s)) \tag{3.33}$$

is taken for simplicity.

By following the same mathematical operations for obtaining Inequality (3.10) for the first integral above and by using Inequalities (3.7) and (3.29), we obtain

$$\begin{aligned}\int_{L_1}^{\infty}\frac{s^{\eta-2}|G(s,u_n)|}{s^{\eta-2}}ds&\leq\frac{\|u_n\|_Y}{\Gamma(\eta)}\int_{L_1}^{+\infty}H(s)ds+\int_{L_1}^{+\infty}\frac{a_4(s)}{\Gamma(\eta)s^{\eta-2}}ds\\&\leq\frac{r}{\Gamma(\eta)}\int_{L_1}^{+\infty}H(s)ds+\int_{L_1}^{+\infty}\frac{a_4(s)}{\Gamma(\eta)s^{\eta-2}}ds<\frac{\varepsilon}{3},\end{aligned}$$

where

$$H(s)=\frac{[(1+s^{\eta-1})a_1(s)+(1+s)a_2(s)+a_3(s)]}{s^{\eta-2}}.$$

By using similar arguments, one can get

$$\frac{1}{\Gamma(\eta)}\int_{L_1}^{\infty}\frac{s^{\eta-2}|G(s,u)|}{s^{\eta-2}}ds\leq\frac{\|u\|_Y}{\Gamma(\eta)}\int_{L_1}^{+\infty}H(s)ds+\frac{1}{\Gamma(\eta)}\int_{L_1}^{+\infty}\frac{a_4(s)}{s^{\eta-2}}ds<\frac{\varepsilon}{3}. \tag{3.34}$$

If the estimations obtained for Inequalities (3.31)–(3.34) are considered for Inequality (3.30), then one can easily see that

$$\left|\frac{\mathcal{L}u_n(t)}{1+t^{\eta-1}}-\frac{\mathcal{L}u(t)}{1+t^{\eta-1}}\right|<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon. \tag{3.35}$$

Furthermore, if the same procedure for revealing Inequality (3.35) is repeated for the following integrals:

$$\left|\frac{D^{\eta-2}\mathcal{L}u_n(t)}{1+t}-\frac{D^{\eta-2}\mathcal{L}u(t)}{1+t}\right|\leq\int_0^{\infty}\frac{(t-s)}{1+t}\frac{s^{\eta-2}|G(s,u_n)-G(s,u)|}{s^{\eta-2}}ds\leq\int_0^{\infty}\frac{s^{\eta-2}|G(s,u_n)-G(s,u)|}{s^{\eta-2}}ds$$

and

$$|D^{\eta-1}\mathcal{L}u_n(t) - D^{\eta-1}\mathcal{L}u(t)| = \int_0^t \frac{1}{s^{\eta-2}} s^{\eta-2}|G(s,u_n) - G(s,u)|ds,$$

then one can obtain the following results:

$$\left|\frac{D^{\eta-2}\mathcal{L}u_n(t)}{1+t} - \frac{D^{\eta-2}\mathcal{L}u(t)}{1+t}\right| < \varepsilon, \quad |D^{\eta-1}\mathcal{L}u_n(t) - D^{\eta-1}\mathcal{L}u(t)| < \varepsilon.$$

Consequently, for a given $\varepsilon > 0$ there exists an n_0 such that, for all $n > n_0$, the following inequality is satisfied:

$$\|\mathcal{L}u_n - \mathcal{L}u\|_Y < \varepsilon,$$

i.e., $\|\mathcal{L}u_n - \mathcal{L}u\|_Y \to 0$ as $n \to \infty$. It implies that $\mathcal{L}$ is sequentially continuous.

The results of the above three steps show that the conditions of Schauder's theorem are fulfilled. As a consequence of this theorem, the initial value problem in Problem (1.3) has at least one solution.

□

Example 3.1. Let us consider the following initial value problem:

$$\begin{cases} D^{5/2}u(t) = \frac{t^{-1/2}}{Ae^{\sqrt{t}}}\left[arctan\left(2\left|u(t)D^{1/2}u(t)\right| + 2\left|u(t)D^{3/2}u(t)\right| + 2\left|D^{1/2}u(t)D^{3/2}u(t)\right|\right)\right], \quad t > 0 \\ u(0) = 0, \quad D^{1/2}u(t)|_{t=0} = 0, \quad D^{3/2}u(t)|_{t=0} = u_{\eta-1} \in \mathbb{R}, \end{cases} \tag{3.36}$$

where $A \geq 11$ is a fixed real number. We examine whether the conditions of Theorem 3.1 are satisfied. From Problem (3.36), we see that

$$g(t,u,v,w) = \frac{t^{-1/2}}{Ae^t}\left[arctan\left(2\left(|uv| + |uw| + |vw|\right)\right)\right]$$

and, it is obvious that the condition (K1) is satisfied. Moreover, the following is obtained:

$$\begin{aligned} \left|t^{1/2}g(t,u,v,w)\right| &= \frac{1}{Ae^t}\left[arctan\left(2\left(|uv| + |uw| + |vw|\right)\right)\right] \leq \frac{1}{Ae^t}\sqrt{2\left(|uv| + |uw| + |vw|\right)} \\ &\leq \frac{1}{Ae^t}\left(|u| + |v| + |w|\right). \end{aligned}$$

This implies that $a_i(t) = \frac{1}{Ae^t}$ for $i = 1, 2, 3$ and $a_4(t) = 0$. Considering condition (K2), we have

$$\int_0^\infty \frac{(1+s^{3/2})a_1(s)}{s^{1/2}}ds = \frac{1+\sqrt{\pi}}{A}, \quad \int_0^\infty \frac{(1+s)a_2(s)}{s^{1/2}}ds = \frac{3\sqrt{\pi}}{A}, \quad \int_0^\infty \frac{a_3(s)}{s^{1/2}}ds = \frac{\sqrt{\pi}}{A}.$$

It follows that

$$\int_0^\infty \frac{[(1+s^{3/2})a_1(s) + (1+s)a_2(s) + a_3(s)]}{s^{1/2}}ds = \frac{7\sqrt{\pi}+2}{2A} \leq \frac{\Gamma(5/2)}{2} \approx 0.66.$$

So, the condition (K2) was also fulfilled. By Theorem 3.1, this problem has at most one solution.

3.3. A generalized fractional mean value theorem

In what follows, we generalize the fractional mean value theorem given in [9] and [16] to prove the uniqueness of a solution to Problem (1.3).

Lemma 3.2. Let $n \in \mathbb{N}$ and $\eta \in (n-1, n)$. Furthermore, suppose that $u \in C^{\eta-1}[0,T]$ such that $D^{\eta}u$ and $t^{\eta-n+1}D^{\eta}u$ are continuous on $(0,T)$ and $[0,T]$, respectively. Then, there exists a function $\Lambda : [0,T] \to [0,T]$ with $0 < \Lambda(t) < x$ such that the following equation is satisfied:

$$u(t) = \sum_{k=1}^{n-1} D^{\eta-k}u(0)\frac{t^{\eta-k}}{\Gamma(\eta-k+1)} + \frac{\Gamma(n-\eta)}{\Gamma(n)}t^{n-1}(\Lambda(t))^{\eta-n+1}D^{\eta}u(\Lambda(t)).$$

Proof. Here we follow the proof for Eq (3.1). In the process of obtained the integral equation in Eq (3.1) in that proof, let us consider for a moment that we do not assign the initial values ($u(0) = 0$ and $D^{\eta-2}u(0) = 0$); then, we have

$$u(t) = I^{3-\eta}u(0)\frac{t^{\eta-3}}{\Gamma(\eta-2)} + D^{\eta-2}u(0)\frac{t^{\eta-2}}{\Gamma(\eta-1)} + D^{\eta-1}u(0)\frac{t^{\eta-1}}{\Gamma(\eta)} + \frac{1}{\Gamma(\eta)}\int_0^t \frac{D^{\eta}u(s)}{(t-s)^{1-\eta}}ds$$

for $\eta \in (2,3)$, where we used $D^{\eta}u(s) = g(s, u(s), D^{\eta-2}u(s), D^{\eta-1}u(s))$. Since u is continuous on $[0,T]$, $I^{3-\eta}u(0) = 0$. Then the above equality turns into the following

$$u(t) = D^{\eta-2}u(0)\frac{t^{\eta-2}}{\Gamma(\eta-1)} + D^{\eta-1}u(0)\frac{t^{\eta-1}}{\Gamma(\eta)} + \frac{1}{\Gamma(\eta)}\int_0^t \frac{D^{\eta}u(s)}{(t-s)^{1-\eta}}ds$$

If the same process applied for $\eta \in (2,3)$ in the proof of the Lemma is carried out for $\eta \in (n-1, n)$, then the following equation is obtained:

$$u(t) = \sum_{k=1}^{n-1} D^{\eta-k}u(0)\frac{t^{\eta-k}}{\Gamma(\eta-k+1)} + \frac{1}{\Gamma(\eta)}\int_0^t \frac{D^{\eta}u(s)}{(t-s)^{1-\eta}}ds. \tag{3.37}$$

In the integrand on the right-hand side of the equation just above, the function $s^{\eta-n+1}D^{\eta}u(s)$ is continuous on $[0,T]$ and the function $s^{n-\eta-1}(t-s)^{\eta-1}$ is in the space of an integrable function, i.e., $L^1[0,t]$. Then, the generalized mean value theorem can be applied to the integral in Eq (3.37), and as a result of this, there exists $\Lambda : [0,T] \to [0,T]$ with $0 < \Lambda(t) < t$ such that the following is satisfied:

$$\begin{aligned}
u(t) &= \sum_{k=1}^{n-1} D^{\eta-k}u(0)\frac{t^{\eta-k}}{\Gamma(\eta-k+1)} + \frac{1}{\Gamma(\eta)}\int_0^t \frac{D^{\eta}u(s)}{(t-s)^{1-\eta}}ds \\
&= \sum_{k=1}^{n-1} D^{\eta-k}u(0)\frac{t^{\eta-k}}{\Gamma(\eta-k+1)} + \frac{1}{\Gamma(\eta)}\int_0^t \frac{s^{\eta-n+1}D^{\eta}u(s)}{s^{\eta-n+1}(t-s)^{1-\eta}}ds \\
&= \sum_{k=1}^{n-1} D^{\eta-k}u(0)\frac{t^{\eta-k}}{\Gamma(\eta-k+1)} + \frac{1}{\Gamma(\eta)}(\Lambda(t))^{\eta-n+1}D^{\eta}u(\Lambda(t))\int_0^t \frac{ds}{s^{\eta-n+1}(t-s)^{1-\eta}} \\
&= \sum_{k=1}^{n-1} D^{\eta-k}u(0)\frac{t^{\eta-k}}{\Gamma(\eta-k+1)} + \frac{\Gamma(n-\eta)}{\Gamma(n)}t^{n-1}(\Lambda(t))^{\eta-2}D^{\eta}u(\Lambda(t))
\end{aligned}$$

That is what we want to show.

□

If we take $n = 3$ in the previous lemma, then we have the following result.

Corollary 3.1. Let $\eta \in (2,3)$. Furthermore, we suppose that $u \in C^{\eta-1}[0,T]$ such that $D^{\eta}u$ and $t^{\eta-n+1}D^{\eta}u$ are continuous on $(0,T]$ and $[0,T]$, respectively. Then, there is a function $\Lambda : [0,T] \to [0,T]$ with $0 < \Lambda(t) < t$ such that the following equation holds:

$$u(t) = D^{\eta-2}u(0)\frac{t^{\eta-2}}{\Gamma(\eta-1)} + D^{\eta-1}u(0)\frac{t^{\eta-1}}{\Gamma(\eta)} + \frac{\Gamma(3-\eta)}{2}t^{2}(\Lambda(t))^{\eta-1}D^{\eta}u(\Lambda(t))$$

for $t \in [0,T]$.

3.4. Uniqueness of the solution to Problem (1.3)

In what follows, we first give a uniqueness theorem for Problem (1.3) by using the mean value theorem given above. This theorem cannot give the uniqueness of the solution on the whole half-plane. This situation is stated in Remark 3.1. Lastly, we illustrate the theorem with an example.

Theorem 3.2 (Nagumo-type uniqueness theorem)**.** Let $2 < \eta < 3$ and $0 < T < \infty$ and let conditions (K1) and (K2) be satisfied. Moreover, we assume that there exists a $\kappa \in \mathbb{R}$ with

$$0 < \kappa \leq \frac{1}{\max(T^{\eta}, T^{\eta-1})\left(\frac{\Gamma(3-\eta)}{2} + \frac{2}{\eta-1}\right)} \tag{3.38}$$

such that, for all $u_i, v_i, w_i \in \mathbb{R}$ $(i = 1, 2)$ the following inequality is satisified:

$$t^{\eta-2}|g(t,u_1,v_1,w_1) - g(t,u_2,v_2,w_2)| \leq \kappa(|u_1-u_2| + |v_1-v_2| + |w_1-w_2|). \tag{3.39}$$

Then, Problem (1.3) has a unique solution $u \in Y_{[0,T]}$.

Before we give the proof of the above problem, it is useful to take a look at the following remark.

Remark 3.1. In Theorem 3.2, one can realize from Inequality (3.38) that $\kappa \to 0$ when T goes to infinity. $\kappa \to 0$ means that Inequality (3.39) can be satisfied only for some function g. Therefore, the uniqueness theorem is applicable only for some function g. Therefore, when we talk about the existence of solutions to Problem (1.3), we are talking about all functions in Y, whereas uniqueness will apply to functions that lie within the restriction of Y to $[0,T]$, i.e., $Y_{[0,T]}$.

Proof. In Theorem 3.1, the existence of the global solution to the problem was proved under conditions (K1) and (K2). Now, the uniqueness of the solution to the problem will be proved. Assume otherwise that the problem has two solutions given by $u_1, u_2 \in Y$ with $u_1 \neq u_2$. Let us show, by the method of contradiction, that this cannot be true. First, we define a function $\Psi : [0,T] \to [0,\infty)$ by

$$\Psi(t) = \begin{cases} |u_1(t) - u_2(t)| + \sum_{k=1}^{2} |D^{\eta-k}u_1(t) - D^{\eta-k}u_2(t)| & , \text{ if } \quad t > 0 \\ 0 & , \text{ if } \quad t = 0 \end{cases}$$

Since $u_1, u_2 \in C^{\eta-1}[0,T]$, it is obvious that Ψ is continuous for $t > 0$. Using integral representations of $u, D^{\eta-2}u$ and $D^{\eta-1}u$ given by Eqs (3.1), (3.3), and (3.4), respectively, its continuity at $t = 0$ can be obtained by performing the following calculation:

$$
\begin{aligned}
0 \leq \lim_{t\to 0^+} \Psi(t) = &\lim_{t\to 0^+} \left| \frac{1}{\Gamma(\eta)} \int_0^t \frac{s^{\eta-2}[G(s,u_1) - G(s,u_2)]}{s^{\eta-2}(t-s)^{1-\eta}} ds \right| \\
&+ \lim_{t\to 0^+} \left| \int_0^t \frac{s^{\eta-2}(t-s)[G(s,u_1) - G(s,u_2)]}{s^{\eta-2}} ds \right| \\
&+ \lim_{t\to 0^+} \left| \int_0^t \frac{s^{\eta-2}[G(s,u_1) - G(s,u_2)]}{s^{\eta-2}} ds \right| \\
&= \lim_{t\to 0^+} \left| \frac{1}{\Gamma(\eta)} t^2 \int_0^1 \frac{(mt)^{\eta-2}[G(mt,u_1) - G(mt,u_2)]}{m^{\eta-2}(1-m)^{1-\eta}} dm \right| \\
&+ \lim_{t\to 0^+} \left| t^{4-\eta} \int_0^1 \frac{(1-m)(mt)^{\eta-2}[G(mt,u_1) - G(mt,u_2)]}{m^{\eta-2}} \right| \\
&+ \lim_{t\to 0^+} \left| t^{3-\eta} \int_0^1 \frac{(mt)^{\eta-2}[G(mt,u_1) - G(mt,u_2)]}{m^{\eta-2}} \right| = 0,
\end{aligned}
$$

where $G(s,u)$ is as in Eq (3.33). Using the sandwich theorem, we can show first that $\lim_{t\to 0^+} \Psi(0) = 0$, and then the continuity of $\Psi(t)$ for $t \geq 0$. Its continuity and non-negativity of $\Psi(t)$, and given that $\Psi(0) = 0$ guarantee that there exists a $t_0 \in (0,T]$ such that the following equation holds:

$$
0 < \Psi(t_0) = |u_1(t_0) - u_2(t_0)| + \sum_{k=1}^{2} |D^{\eta-k}u_1(t_0) - D^{\eta-k}u_2(t_0)|.
$$

Let us now obtain an upper estimation for the right-hand side of the inequality just above. From the mean value theorem given by Corollary 3.1, one can say that there exists a $t_1 \in (0,t_0)$ such that the following equality is satisfied:

$$
\begin{aligned}
|u_1(t_0) - u_2(t_0)| &= \frac{\Gamma(3-\eta)}{2} t_0^2 |t_1^{\eta-2} D^{\eta}(u_1 - u_2)(t_1)| \\
&= \frac{\Gamma(3-\eta)}{2} t_0^2 t_1^{\eta-2} |G(t_1,u(t_1)) - G(t_2,u(t_0))|. \qquad (3.40)
\end{aligned}
$$

For the second summation, from Eq (3.4) we have

$$
|D^{\eta-2}u_1(t_0) - D^{\eta-2}u_2(t_0)| = \left| \int_0^{t_0} \frac{(t_0 - s)}{s^{\eta-2}} s^{\eta-2}[G(s,u_1(s)) - G(s,u_2(s))] ds \right|. \qquad (3.41)
$$

Since $\frac{(t-s)}{s^{\eta-2}}$ is integrable and $t^{\eta-2}G(t,u(t))$ is continuous, from the generalized mean value theorem and Eq (3.41), there exists a $t_2 \in (0,t_0)$ such that the following holds:

$$
\begin{aligned}
|D^{\eta-2}u_1(t_0) - D^{\eta-2}u_2(t_0)| &= t_2^{\eta-2} |G(t_2,u_1(t_2)) - G(t_2,u_2(t_2))| \int_0^{t_0} \frac{(t_0 - s)}{s^{\eta-2}} ds \\
&\leq \frac{t_0^{\eta}}{\eta - 1} t_2^{\eta-2} |G(t_2,u_1(t_2)) - G(t_2,u_2(t_2))|. \qquad (3.42)
\end{aligned}
$$

Finally, using Eq (3.3) and the generalized mean value theorem for the third summation, we have the following estimation:

$$
\begin{aligned}
|D^{\eta-1}u_1(t_0) - D^{\eta-1}u_2(t_0)| &= t_3^{\eta-2}|G(t_3, u_1(t_3) - G(t_3, u_2(t_3))| \int_0^{t_0} \frac{1}{s^{\eta-2}} ds \\
&\leq t_3^{\eta-2}|G(t_3, u_1(t_3) - G(t_3, u_2(t_3))| \frac{t_0^{\eta-1}}{\eta - 1}
\end{aligned} \tag{3.43}
$$

where $t_3 \in (0, t_0)$.

As a result of Inequalities (3.40)–(3.43), we assign the point t_4 as follows:

$$
t_4^{\eta-2}|G(t_4, u_1(t_4) - G(t_4, u_2(t_4))| = \max_{i=1,2,3}(t_i^{\eta-2}|G(t_i, u_1(t_i) - G(t_i, u_2(t_i)|)
$$

and, in this way, one can obtain

$$
\begin{aligned}
0 < \Psi(t_0) &= |u_1(t_0) - u_2(t_0)| + \sum_{k=1}^{2} |D^{\eta-k}u_1(t_0) - D^{\eta-k}u_2(t_0)| \\
&\leq \frac{\Gamma(3-\eta)}{2} t_0^2 t_1^{\eta-2}|G(t_1, u(t_1)) - G(t_2, u(t_0))| + \frac{t_0^{\eta}}{\eta - 1} t_2^{\eta-2}|G(t_2, u_1(t_2)) - G(t_2, u_2(t_2))| \\
&+ \frac{t_0^{\eta-1}}{\eta - 1} t_3^{\eta-2}|G(t_3, u_1(t_3) - G(t_3, u_2(t_3))| \\
&\leq \left(\frac{\Gamma(3-\eta)t_0^2}{2} + \frac{t_0^{\eta}}{\eta - 1} + \frac{t_0^{\eta-1}}{\eta - 1} \right) t_4^{\eta-2}|G(t_4, u_1(t_4) - G(t_4, u_2(t_4))|.
\end{aligned}
$$

In this last inequality, by considering the definition of G and the Nagumo constant with Inequality (3.39), it follows that

$$
\begin{aligned}
0 < \Psi(t_0) &\leq \left(\frac{\Gamma(3-\eta)t_0^2}{2} + \frac{t_0^{\eta}}{\eta - 1} + \frac{t_0^{\eta-1}}{\eta - 1} \right) t_4^{\eta-2}|G(t_4, u_1(t_4) - G(t_4, u_2(t_4))| \\
&\leq \max(T^{\eta}, T^{\eta-1}) \left(\frac{\Gamma(3-\eta)}{2} + \frac{2}{\eta - 1} \right) t_4^{\eta-2}|G(t_4, u_1(t_4) - G(t_4, u_2(t_4))| \\
&\leq |u_1(t_4) - u_2(t_4)| + \sum_{k=1}^{2} |D^{\eta-k}u_1(t_4) - D^{\eta-k}u_2(t_4)| = \Psi(t_4).
\end{aligned}
$$

In conclusion, at the end of all of these operations, one can say that there is a $t_4 \in (0, t_0)$ such that the following inequality is satisfied:

$$
0 < \Psi(t_0) \leq \Psi(t_4)
$$

.

If all of these operations applied to the point t_0 are repeated for the point t_4, then there exists a point $t_8 \in (0, t_4)$ such that $0 < \Psi(t_0) \leq \Psi(t_4) \leq \Psi(t_8)$ is satisfied. Continuing similarly, one can construct a sequence $\{t_{4n}\}_{n=1}^{\infty} \subset [0, t_0]$ satisfying $t_{4n} \to 0$ such that $\Psi(t_{4n})$ moves away from 0 as n grows.

However, the continuity of the function Ψ implies that it is sequentially continuous and therefore, that $\Psi(t_{4n}) \to \Psi(0) = 0$ when $t_{4n} \to 0$. This is a contradiction, so our first assumption is wrong, and $u_1 = u_2$. So there is only one solution to the problem.

□

In what follows, a supportive example is given to demonstrate the applicability of the uniqueness theorem.

Example 3.2. Let $T \geq 1$ be a fixed real number. Consider the following initial value problem

$$\begin{cases} D^{5/2}u(t) = \frac{t^{-1/2}}{Ae^t}\left(u(t) + D^{1/2}u(t) + D^{3/2}u(t) - \psi(t)\right), & t > 0 \\ u(0) = 0, \quad D^{1/2}u(t)\,|_{t=0} = 0, \quad D^{3/2}u(t)\,|_{t=0} = \Gamma(5/2) \in \mathbb{R}. \end{cases} \tag{3.44}$$

where $\psi(t) = t^{3/2} + \Gamma(5/2)t + \Gamma(5/2)$ and A is a fixed real number with

$$A \geq max\left\{11, T^{5/2}\left(\frac{\sqrt{\pi}}{2} + \frac{4}{3}\right)\right\}.$$

We shall try to establish that the conditions of Theorem 3.1 and Theorem 3.2 are satisfied. From Problem (3.44), we observe that

$$g(t,u,v,w) = \frac{t^{-1/2}}{Ae^t}\,(u + v + w - \psi(t))$$

and, it is clear that the condition (K1) is fulfilled. In addition to this, the following inequality is obtained:

$$\left|t^{1/2}g(t,u,v,w)\right| \leq \frac{1}{Ae^t}\,(|u| + |v| + |w| + |\psi(t)|)\,.$$

This gives $a_i(t) = \frac{1}{Ae^t}$ for $i = 1, 2, 3$ and $a_4(t) = \frac{\psi(t)}{Ae^t}$. Considering condition (K2), we have

$$\int_0^\infty \frac{(1+s^{3/2})a_1(s)}{s^{1/2}}ds = \frac{1+\sqrt{\pi}}{A},\quad \int_0^\infty \frac{(1+s)a_2(s)}{s^{1/2}}ds = \frac{3\sqrt{\pi}}{A},\quad \int_0^\infty \frac{a_3(s)}{s^{1/2}}ds = \frac{\sqrt{\pi}}{A}.$$

and $\int_0^\infty \frac{a_4(s)}{s^{1/2}}ds = \int_0^\infty \frac{\psi(s)}{Ae^s s^{1/2}}ds = \int_0^\infty \frac{a_4(s)}{s^{1/2}}ds = \int_0^\infty \frac{s^{3/2}+\Gamma(5/2)s+\Gamma(5/2)}{Ae^s s^{1/2}}ds < \infty$. It follows that

$$\int_0^\infty \frac{[(1+s^{3/2})a_1(s) + (1+s)a_2(s) + a_3(s)]}{s^{1/2}}ds = \frac{7\sqrt{\pi}+2}{2A} \leq \frac{\Gamma(5/2)}{2} \approx 0.66.$$

So, the condition (K2) is also satisfied. Finally,

$$\begin{aligned} t^{1/2}|g(t,u_1,v_1,w_1) - g(t,u_2,v_2,w_2)| &\leq \frac{1}{Ae^t}(|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|) \\ &\leq \kappa(|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|) \end{aligned} \tag{3.45}$$

where $0 < \kappa \leq \frac{1}{T^{5/2}\left(\frac{\sqrt{\pi}}{2}+\frac{4}{3}\right)}$ can be calculated from Inequality (3.39). So, the hypotheses of Theorem 3.2 are justified. By this theorem, the problem has a unique solution on $[0, T]$. Moreover, the unique solution to this problem is $u(t) = t^{3/2}$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflicts of interest.

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