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Research article

Sufficient and necessary conditions of near-optimal controls for a stochastic listeriosis model with spatial diffusion

Zhaoyan Meng[1], **Shuting Lyu**[1,*], **Mengqing Zhang**[1], **Xining Li**[2] **and Qimin Zhang**[1,2,*]

[1] School of Mathematics and Information Science, North Minzu University, Yinchuan 750021, China
[2] School of Mathematics and Statistics, Ningxia University, Yinchuan 750021, China

* **Correspondence:** Email: xjlsht@163.com, zhangqimin64@sina.com.

Abstract: Random environment and human activities have important effects on the survival of listeria. In this paper, treating infected people and removing bacteria from the environment as control strategies, we developed a listeriosis model that considers random noise and spatial diffusion. By constructing a Lyapunov function, we demonstrated the existence and uniqueness of the global positive solution of the model. However, it was a challenging task to realize the optimal control of the model by solving the Pontryagin random maximum principle with the lowest control cost. Therefore, our study on near-optimal controls is of great significance for controlling the spread of listeriosis. Initially, we gave some adjoint equations and a priori estimates. Subsequently, the Pontryagin random maximum principle was utilized to establish the sufficient and necessary conditions for achieving near-optimal controls. Ultimately, the theoretical findings are corroborated through numerical analysis.

Keywords: listeriosis model; near-optimal controls; sufficient and necessary conditions; spatial diffusion

1. Introduction

Listeriosis is a serious foodborne bacterial infection in humans and animals, caused by Listeria monocytogenes [1, 2]. Listeria is mostly found in soil, unclean water (lakes, rivers), plants, refrigerators, and food [3]. Listeriosis in humans is caused by eating food contaminated with inappropriate listeria or by direct contact with bacteria in the environment. Animals that eat plants and water contaminated with listeria are susceptible to listeriosis. Listeriosis can present as bacteremia, meningitis or meningoencephalitis [4]. The World Health Organization reports that the 2017–2018 listeriosis outbreak in South Africa is the largest global outbreak recorded to date, with 978 confirmed cases reported by the National Institute of Communicable Diseases between 1 January 2017 and 14 March 2018 [5]. Listeriosis is a rare disease, with a reported occurrence of 0.1 to 10 cases per 1 million people annually, depending on the geographic location and the prevalence of infection [6]. Nevertheless, the substantial fatality rate and the expenses related to its treatment underscore its importance as a significant public health concern. Therefore, the study of listeriosis has attracted the attention of many scholars.

Mathematical models are essential in the examination of the dynamic characteristics of listeriosis. So far, several ordinary differential listeriosis models have been established and their dynamic behavior has been discussed in detail [7–10]. For example, Asamoah et al. [7] established the presence and singular nature of the solutions, and conducted an examination of their stability using the Hyers-Ulam method. Chukwu et al. [8] considered the factors of listeria cross-contamination in food processing plants, determined the pollution threshold and performed a time-dependent sensitivity analysis. Osman et al. [9] conducted a study to investigate the presence of a disease-free equilibrium point and the basic reproduction number. They employed sensitivity analysis to assess the impact of individual parameters on the basic reproduction number. Chukwu et al. [10] identified food contamination thresholds, discussed local stability, and performed numerical simulations to assess the impact of media campaigns on listeriosis transmission. The references [7–10] discussed above does not consider the effects of random noise and spatial diffusion. Indeed, it is understood that the dissemination of listeria is influenced by stochastic variables such as temperature, humidity, and meteorological conditions. When the environment is suitable for the growth of Listeria, Listeria will multiply and spread rapidly. A great deal of research has been done on the effects of random noise on infectious diseases [11–13] . On the other hand, we know that listeria mainly attaches to food and water. With the rapid development of modern logistics and transportation, food flows between regions are becoming more and more frequent, which promotes the spread of listeria in different regions. Therefore, it is necessary to consider random noise and spatial diffusion during modeling.

From the perspective of control, when a disease breaks out, in order to minimize the cost of health care, we tend to artificially add some control measures, such as isolation treatment, vaccination, etc. Studies on listeriosis mainly focus on dynamic behavior, and there are few studies on this disease control, such as [14, 15]. They use Pontryagin maximum principle to give the expression of the optimal control theoretically. Compared with near-optimal control, first of all, the optimal control is sensitive to the disturbance of the external random environment, while the near-optimal control has a larger range and is more inclusive to the disturbance of the random environment. Second, the optimal control can generally be obtained by solving Hamilton-Jacobi-Bellman equation, but its exact solution is difficult to obtain, so the most feasible method is to determine the near-optimal control by numerical calculation. Furthermore, it is possible that an optimal control solution may not be attainable. As an illustration, scholars have discussed in [16] that in the optimal production plan of a manufacturing system, in the random scenario, it is not possible to determine the most effective approach for the zero-inventory strategy. Finally, from a practical point of view, the optimal control [17] is to prevent and control all susceptible people, but it will cause more waste of medical resources in real life, and the solution that is close to the optimal can meet the prevention and control purpose of decision makers.

In light of the preceding discourse, we present a listeriosis model that incorporates the impacts of stochastic noise and spatial diffusion. The study aims to obtain the sufficient and necessary conditions for the near-optimal controls of listeriosis. Initially, the existence and uniqueness of the global positive solution are established through the utilization of a Lyapunov function. Subsequently, the adjoint equation and prior estimates are presented. Ultimately, the Pontryagin maximum principle is employed to derive the sufficient and necessary conditions for near-optimal controls. Below, we provide a concise overview of the major contributions presented in this article:

- Taking into account the impact of environmental variables, including temperature, humidity, weather, and human factors such as logistics, a model for listeriosis is developed. This model incorporates stochastic noise and spatial diffusion, building upon the framework discussed in the reference [4].
- The Pontryagin maximum principle is utilized to derive both the sufficient and necessary conditions, and a computational algorithm is provided. The resulting conclusion pertains to the expansion of the findings in the referenced work by [14].

The subsequent sections of this article are structured in the following manner. In Section 2, the model is established and the objective function is constructed. In Section 3, the existence and uniqueness of global positive solutions are demonstrated, and the adjoint equations of stochastic systems are given, along with the provision of some preliminary estimates. In Section 4, the research investigates the sufficient and necessary conditions for near-optimal controls in models of listeriosis with spatial diffusion and stochastic noise. In Section 5, numerical simulations are presented to prove the theoretical results. Ultimately, the paper concludes with the findings presented in Section 6.

2. Model formulation

In this section, we will describe the process of building the model and give some assumptions. In recent years, models of various forms of listeriosis have been established, for example [4, 7–10]. In [4], Chukwu et al. mentioned the listeriosis model.

$$\begin{cases} \frac{\mathrm{dS(t)}}{\mathrm{dt}} = \Lambda + \theta_L R_L(t) - (\mu + \beta_L L(t))S(t), \\ \frac{\mathrm{dI_L(t)}}{\mathrm{dt}} = \beta_L L(t) S(t) - (\sigma_L + \delta_L + \mu) I_L(t), \\ \frac{\mathrm{dR_L(t)}}{\mathrm{dt}} = \sigma_L I_L(t) - (\theta_L + \mu) R_L(t), \\ \frac{\mathrm{dL(t)}}{\mathrm{dt}} = r_L L(t)(1 - \frac{L(t)}{K_L}) - (\varepsilon + r) L(t), \end{cases} \tag{2.1}$$

where $S(t)$, $I_L(t)$, $R_L(t)$, $L(t)$ represent the population density of susceptible, infected, recovered persons and the listeria contaminated environment at time t, respectively. The other parameters of the model as Table 1.

Observations show that the system (2.1) does not take into account the effects of spatial diffusion. However, it is understood that the scope of human activities is not static. On the other hand, in modern society, logistics and transportation are convenient, and Listeria is generally attached to meat, vegetables, and other foods, which accelerates the spread of Listeria between regions. Listeria also exists in large quantities in water and soil, and the flow of water can also cause the spread of listeria. Because bacteria can flow through humans and the environment, we introduce spatial diffusion to the system (2.1) using the same method as in article [17].

Table 1. Parameter values of numerical experiments for system (2.1).

Parameter	Meaning
Λ	the migration rate
σ_L	the transfer rate from I_L to R_L
μ	the natural death rate of a person
β_L	the infection rate of listeriosis
r_L	the growth rate of listeria
K_L	the carrying capacity of listeria
δ_L	the mortality rate of listeriosis
θ_L	the listeriosis immunity loss rate
ε	the mortality rate of bacteria in the environment
r	the hygiene clearance rate

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_1 \Delta S(x,t) + \Lambda + \theta_L R_L(x,t) - (\mu + \beta_L L(x,t)) S(x,t), \\ \dfrac{\partial I_L(x,t)}{\partial t} = d_2 \Delta I_L(x,t) + \beta_L L(x,t) S(x,t) - (\sigma_L + \delta_L + \mu) I_L(x,t), \\ \dfrac{\partial R_L(x,t)}{\partial t} = d_3 \Delta R_L(x,t) + \sigma_L I_L(x,t) - (\theta_L + \mu) R_L(x,t), \\ \dfrac{\partial L(x,t)}{\partial t} = d_4 \Delta L(x,t) + r_L L(x,t)(1 - \dfrac{L(x,t)}{K_L}) - (\varepsilon + r) L(x,t), \end{cases} \tag{2.2}$$

with boundary condition

$$\frac{\partial S(x,t)}{\partial \nu} = \frac{\partial I_L(x,t)}{\partial \nu} = \frac{\partial R_L(x,t)}{\partial \nu} = \frac{\partial L(x,t)}{\partial \nu} = 0, x \in \partial\Omega, t > 0, \tag{2.3}$$

and initial condition

$$S(x,0) = S_0(x) \geq 0, I_L(x,0) = I_{L,0}(x) \geq 0, R_L(x,0) = R_{L,0}(x) \geq 0, L(x,0) = L_0(x) \geq 0, \tag{2.4}$$

where Ω represents a bounded field with smooth boundary; Δ is Laplacian; the variables $S(x,t)$, $I_L(x,t)$, and $R_L(x,t)$ denote the population density of individuals who are susceptible, infected, and recovered at location x and time t; $L(x,t)$ stands for the density of the contaminated environment; d_1, d_2, and d_3 are human diffusion coefficients; and d_4 represents the diffusion coefficient of bacteria in the environment.

For listeria, both climatic and human factors can cause changes in the number of bacteria in the environment. For example, high-temperature disinfection can effectively reduce the number of bacteria in the environment; for humans, areas with high concentrations of listeria have higher rates of disease and increased mortality. Hence, the mortality rate of bacteria in the environment, denoted as ε, and the inherent mortality rate of humans, denoted as μ, will be disrupted. We express the disturbance in the following form:

$$\mu dt = \mu dt + \xi_1 dB_1(t), \quad \varepsilon dt = \varepsilon dt + \xi_2 dB_2(t). \tag{2.5}$$

The variable ξ_i for $i = (1,2)$ represents the noise intensity, while $B_i(t)$ for $i = (1,2)$ denotes independent standard Brownian motion. We can express system (2.2) in the following form:

$$\begin{cases} dS(x,t) = [d_1\Delta S(x,t) + \Lambda + \theta_L R_L(x,t) - (\mu + \beta_L L(x,t))S(x,t)]dt - \xi_1 S(x,t)dB_1(t), \\ dI_L(x,t) = [d_2\Delta I_L(x,t) + \beta_L L(x,t)S(x,t) - (\sigma_L + \delta_L + \mu)I_L(x,t)]dt - \xi_1 I_L(x,t)dB_1(t), \\ dR_L(x,t) = [d_3\Delta R_L(x,t) + \sigma_L I_L(x,t) - (\theta_L + \mu)R_L(x,t)]dt - \xi_1 R_L(x,t)dB_1(t), \\ dL(x,t) = [d_4\Delta L(x,t) + r_L L(x,t)(1 - \frac{L(x,t)}{K_L}) - (\varepsilon + r)L(x,t)]dt - \xi_2 L(x,t)dB_2(t). \end{cases} \tag{2.6}$$

Next, incorporate control variables into the system. If U denotes a closed set that is both bounded and nonempty, and for a specified terminal time T, the control $u(\cdot) : [0,T]\times\Omega \to U$ is $\mathcal{F}_t$-predictable. Then, we denote $U \in R$ as admissible. Assuming $u(\cdot) = u_i(x,t) = (u_1(x,t), u_2(x,t))^T$, where $u_i(x,t) \in U_{ad}(\Omega\times[0,T]), i = 1,2$, where $U_{ad}(\Omega\times[0,T])$ is an allowable control set. The function $u_1(x,t)$ denotes the proportion of individuals who are infected. To study the effect of sanitary removal on bacteria in the environment, we may wish to consider replacing sanitary removal r with $u_2(x,t)$ to better highlight the impact of control measures on disease transmission, where $u_2(x,t)$ indicates the extent to which sanitary measures remove bacteria, $0 \le u_i(x,t) \le 1, i = 1,2$. According to reference [18], the number of people who have recovered due to medication treatment is $\frac{cu_1(x,t)I_L(x,t)}{1+\alpha I_L(x,t)}$, where $c > 0, \alpha \ge 0$. Therefore, we have the following random system:

$$\begin{cases} dS(x,t) = [d_1\Delta S(x,t) + \Lambda + \theta_L R_L(x,t) - (\mu + \beta_L L(x,t))S(x,t)]dt - \xi_1 S(x,t)dB_1(t), \\ dI_L(x,t) = [d_2\Delta I_L(x,t) + \beta_L L(x,t)S(x,t) - (\sigma_L + \delta_L + \mu)I_L(x,t) - \frac{cu_1(x,t)I_L(x,t)}{1+\alpha I_L(x,t)}]dt - \xi_1 I_L(x,t)dB_1(t), \\ dR_L(x,t) = [d_3\Delta R_L(x,t) + \sigma_L I_L(x,t) - (\theta_L + \mu)R_L(x,t) + \frac{cu_1(x,t)I_L(x,t)}{1+\alpha I_L(x,t)}]dt - \xi_1 R_L(x,t)dB_1(t), \\ dL(x,t) = [d_4\Delta L(x,t) + r_L L(x,t)(1 - \frac{L(x,t)}{K_L}) - (\varepsilon + u_2(x,t))L(x,t)]dt - \xi_2 L(x,t)dB_2(t), \\ Z(0) = Z_0. \end{cases} \tag{2.7}$$

Remark 1. *At the end of the modeling process, the final system* (2.7) *is based on the system* (2.1)*, which adds the influence of random noise and spatial diffusion, and introduces the control variables $u_1(x,t)$ and $u_2(x,t)$. We know that in the real environment, temperature, humidity, climate, and other factors are important factors affecting the growth of Listeria, and logistics transportation will also speed up the spread of Listeria between regions. On the other hand, it is very meaningful to study how to control the spread of infectious diseases, so we introduce control variables $u_1(x,t)$ and $u_2(x,t)$ in the model.*

The set $U_{ad}[0,T]$ denotes the collection of permissible controls, where the control $u(\cdot) \in U_{ad}[0,T]$, and the system described by Eq (2.7) possesses a solution that is adapted to the filtration $\mathcal{F}_t$. $\|\cdot\|$ represents the Euclidean space norm, where $\|x\|_2 = \sqrt{\langle x, x\rangle}$. For convenience, define $Z(x,t) = (S(x,t), I_L(x,t), R_L(x,t), L(x,t))^T$.

The objective function is represented as follows:

$$J(0,Z_0;u_1(x,t),u_2(x,t)) = E\{\int_0^T\int_\Omega Y(Z(x,t),u_1(x,t),u_2(x,t))dxdt + \int_\Omega h(Z(x,T))dx\}, \tag{2.8}$$

where $Y(Z(x,t),u_1(x,t),u_2(x,t))= B_1I_L(x,t)+B_2L(x,t)+\frac{1}{2}B_3u_1^2(x,t)+\frac{1}{2}B_4u_2^2(x,t)$. $\int_0^T\int_\Omega(\frac{1}{2}B_3u_1^2(x,t)+\frac{1}{2}B_4u_2^2(x,t))dxdt$ represents the total cost of using the control policy. $\int_0^T\int_\Omega(B_1I_L(x,t)+B_2L(x,t))dxdt$ indicates the number of infected people and contaminated environments. $\int_\Omega h(Z(x,T))dx$ indicates the number of infected individuals at terminal time T and the number of contaminated environments at that time. The variables $B_i(i=1,2,3,4)$ represent the weights assigned to the functions $I_L(x,t), L(x,t), u_1(x,t)$, and $u_2(x,t)$. Ultimately, the primary challenge in the control problem lies in identifying an admissible control that minimizes the objective function $J(0,Z_0;u_1(x,t),u_2(x,t))$ for $u_1(x,t),u_2(x,t)\in U_{ad}$. The value function is formally defined as

$$V(0,Z_0)=\inf_{u_1,u_2\in U_{ad}} J(0,Z_0;u_1(x,t),u_2(x,t)). \tag{2.9}$$

Figure 1 depicts the schematic representation of the system described by Eq (2.7), designed to minimize the prevalence of illness and environmental pollution.

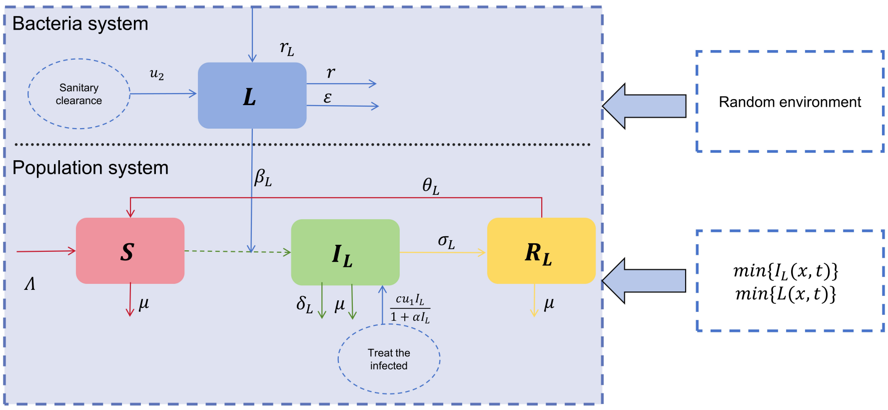

Figure 1. Schematic diagram of system (2.7).

In this article, the the stated assumptions are considered to be valid.
(H1): For all $0\le t\le T$, $Z(x,t),\bar{Z}(x,t)\in R_+^4$. The function $Z(x,t)$ and $\bar{Z}(x,t)$ are continuous, and there exists a constant C satisfying the given condition.

$$|\ h(Z(x,t))-h(\bar{Z}(x,t))\ |\le C\ |\ Z(x,t)-\bar{Z}(x,t)\ |,$$

and

$$|\ h_{Z(x,t)}(Z(x,t))-h_{\bar{Z}(x,t)}(Z(x,t))\ |\le C\ |\ Z(x,t)-\bar{Z}(x,t)\ |,$$

(H2): The set of controls, denoted as $\mathcal{U}_{ad}$, exhibits convexity.

In the above, we mainly established a listeriosis model (2.7) and established the objective function (2.8), so as to study the near-optimal controls theory. Since (2.7) is a newly established model, it is imperative to demonstrate the existence and uniqueness of the positive solution.

3. Existence and uniqueness of the global positive solution and some prior estimates

Before determining the sufficient and necessary conditions for near-optimal controls, it is crucial to verify the existence and uniqueness of the global positive solution. Therefore, we present the following lemma. Then, we give the adjoint equations and some lemmas, which lay the foundation for the subsequent proof.

3.1. Existence and uniqueness of the global positive solution

Lemma 3.1. *For any initial data* $(S_0, I_{L,0}, R_{L,0}, L_0)$*, the solution* $Z(x,t) = (S(x,t), I_L(x,t), R_L(x,t), L(x,t))$ *of system* (2.7) *satisfies that*

$$\limsup_{t\to\infty}((S(x,t), I_L(x,t), R_L(x,t), L(x,t)) < \infty. \tag{3.1}$$

The evidence is presented in Appendix A1.

Subsequently, we establish the existence and uniqueness of the global positive solution for the system represented by Eq (2.7).

Theorem 3.1. *For any given initial data set* $(S_0, I_{L,0}, R_{L,0}, L_0) > 0$*, there exists a singular positive solution* $(S(x,t), I_L(x,t), R_L(x,t), L(x,t))$ *to system* (2.7) *for* $t > 0$ *on* Ω*.*

Proof. The coefficients of system (2.7) satisfy the local Lipschitz condition, guaranteeing the existence of a unique local solution $t \in [0, \tau_e)$, where τ_e denotes the explosion time. Let $l_0 > 1$, such that it is sufficiently large to ensure that every initial value falls within the range of $\frac{1}{l_0}$ to l_0. For every integer $l > l_0$, establish the stopping time.

$$\begin{aligned}\tau_l = \inf\{t \in [0,\tau_e] : &\min(S(x,t), I_L(x,t), R_L(x,t), L(x,t)) \le \frac{1}{l} \\ &\text{or } \max(S(x,t), I_L(x,t), R_L(x,t), L(x,t)) \ge l\}.\end{aligned} \tag{3.2}$$

The empty set is assigned a value of infinity, which is represented as $inf\ \emptyset = \infty$, with $\emptyset$ indicating the empty set. As the parameter l approaches infinity, the function τ_l is increasing. $\tau_\infty = \lim_{l\to\infty} \tau_l$, then $\tau_\infty < \tau_e$, a.s. In the subsequent analysis, it is necessary to demonstrate that $\tau_\infty = \infty$ almost surely. Therefore, using the Itô formula, the following formula can be obtained

$$\begin{aligned}
&d(\|S(x,t)\|^2 + \|I_L(x,t)\|^2 + \|R_L(x,t)\|^2 + \|L(x,t)\|^2) \\
&= \{2\langle S(x,t), d_1\Delta S(x,t) + \Lambda + \theta_L R_L(x,t) - (\mu + \beta_L L(x,t))S(x,t)\rangle + 2\langle I_L(x,t), d_2\Delta I_L(x,t) \\
&+ \beta_L L(x,t)S(x,t) - (\sigma_L + \delta_L + \mu)I_L(x,t) - \frac{cu_1(x,t)I_L(x,t)}{1+\alpha I_L(x,t)}\rangle + 2\langle R_L(x,t), d_3\Delta R_L(x,t) \\
&+ \sigma_L I_L(x,t) - (\theta_L + \mu)R_L(x,t) + \frac{cu_1(x,t)I_L(x,t)}{1+\alpha I_L(x,t)}\rangle + 2\langle L(x,t), d_4\Delta L + r_L L(x,t)(1 - \frac{L(x,t)}{K_L}) \\
&- \varepsilon L(x,t)\rangle + \xi_1^2(t)\|S(x,t)\|^2 + \xi_1^2(t)\|I_L(x,t)\|^2 + \xi_1^2(t)\|R_L(x,t)\|^2 + \xi_2^2(t)\|L(x,t)\|^2\}dt \\
&+ 2\langle S(x,t), -\xi_1(t)S(x,t)dB_1(t)\rangle + 2\langle I_L(x,t), -\xi_1(t)I_L(x,t)dB_1(t)\rangle \\
&+ 2\langle R_L(x,t), -\xi_1(t)R_L(x,t)dB_1(t)\rangle + 2\langle L(x,t), -\xi_2(t)L(x,t)dB_2(t)\rangle.
\end{aligned} \tag{3.3}$$

May as well assume $l > l_0$ and $T > 0$; Next, we will perform the integration of both sides of Eq (3.3) from 0 to $\tau_l \wedge T$ and subsequently calculate the expectations.

$$
\begin{aligned}
& E[\|S(x,\tau_l \wedge T)\|^2 + \|I_L(x,\tau_l \wedge T)\|^2 + \|R_L(x,\tau_l \wedge T)\|^2 + \|L(x,\tau_l \wedge T)\|^2] - (\|S_0\|^2 + \|I_{L,0}\|^2 + \|R_{L,0}\|^2 + \|L_0\|^2) \\
& \leq E\int_0^{\tau_l \wedge T} \{-2\langle \nabla S(x,s), d_1 \nabla S(x,s)\rangle + 2\langle \Lambda, S(x,s)\rangle + 2\langle \theta_L I_L(x,s), S(x,s)\rangle - 2\langle \nabla I_L(x,s), d_2 \nabla I_L(x,s)\rangle \\
& + 2\langle I_L(x,s), \beta_L L(x,s) S(x,s)\rangle - 2\langle \nabla R_L(x,s), d_3 \nabla R_L(x,s)\rangle + 2\langle R_L(x,s), \sigma_L I_L(x,s)\rangle \\
& + 2\langle R_L(x,s), \frac{cu_1(x,s) I_L(x,s)}{1+\alpha I_L(x,s)}\rangle - 2\langle \nabla L(x,s), d_4 \nabla L(x,s)\rangle + 2\langle L(x,s), r_L L(x,s)(1 - \frac{L(x,s)}{K_L})\rangle \\
& + \xi_1^2 \|S(x,s)\|^2 + \xi_1^2 \|I_L(x,s)\|^2 + \xi_1^2 \|R_L(x,s)\|^2 + \xi_2^2 \|L(x,s)\|^2\} ds \\
& \leq E\int_0^{\tau_l \wedge T} \{-2 d_1 \lambda_0 \|S(x,s)\|^2 + 2\langle \Lambda, S(x,s)\rangle + 2\langle \theta_L R_L(x,s), S(x,s)\rangle - 2 d_2 \lambda_0 \|I_L(x,s)\|^2 \\
& + 2\langle I_L(x,s), \beta_L L(x,s) S(x,s)\rangle - 2 d_3 \lambda_0 \|R_L(x,s)\|^2 + 2\langle L(x,s), \sigma_L I_L(x,s)\rangle + 2\langle R_L(x,s), \frac{cu_1(x,s) I_L(x,s)}{1+\alpha I_L(x,s)}\rangle \\
& - 2 d_4 \lambda_0 \|L(x,s)\|^2 + 2\langle L(x,s), r_L L(x,s)(1 - \frac{L(x,s)}{K_L})\rangle + \xi_1^2 \|S(x,s)\|^2 \\
& + \xi_1^2 \|I_L(x,s)\|^2 + \xi_1^2 \|R_L(x,s)\|^2 + \xi_2^2 \|L(x,s)\|^2\} ds,
\end{aligned}
$$

where $\lambda_0 = \inf_{u \in \mathcal{H}} \|\nabla u(x,s)\|^2 / \|u(x,s)\|^2$.

Then according to Lemma 3.1 and fundamental inequality, we have

$$
\begin{aligned}
& E[\|S(x,\tau_l \wedge T)\|^2 + \|I_L(x,\tau_l \wedge T)\|^2 + \|R_L(x,\tau_l \wedge T)\|^2 + \|L(x,\tau_l \wedge T)\|^2] \\
& \leq (\|S_0\|^2 + \|I_{L,0}\|^2 + \|R_{L,0}\|^2 + \|L_0\|^2) + E\int_0^{\tau_l \wedge T} \{-2 d_1 \lambda_0 \|S(x,s)\|^2 + \Lambda^2 + 2\|S(x,s)\|^2 + \theta_L^2 \|R_L(x,s)\|^2 \\
& - 2 d_2 \lambda_0 \|I_L(x,s)\|^2 + \beta_L^2 S^2 \|L(x,s)\|^2 + \|I_L(x,s)\|^2 - 2 d_3 \lambda_0 \|R_L(x,s)\|^2 + 2\|R_L(x,s)\|^2 + \sigma_L^2 \|I_L(x,s)\|^2 \\
& + c^2 u_1(x,s) \|I_L(x,s)\|^2 - 2 d_4 \lambda_0 \|L(x,s)\|^2 + \|L(x,s)\|^2 + r_L \|L(x,s)\|^2 + \xi_1^2 \|S(x,s)\|^2 + \xi_1^2 \|I_L(x,s)\|^2 \\
& + \xi_1^2 \|R_L(x,s)\|^2 + \xi_2^2 \|L(x,s)\|^2\} ds \\
& \leq M_2 + M_3 E\int_0^{\tau_l \wedge T} \|S(x,s)\|^2 + \|I_L(x,s)\|^2 + \|R_L(x,s)\|^2 + \|L(x,s)\|^2 ds,
\end{aligned}
$$

where

$$M_2 = \|S_0\|^2 + \|I_{L,0}\|^2 + \|R_{L,0}\|^2 + \|L_0\|^2 + \Lambda^2 \tau_l,$$

$$M_3 = max\{(2+\xi_1^2 - 2d_1\lambda_0), (1+\sigma_L^2+\xi_1^2+c^2u_1^2-2d_2\lambda_0), (2+\xi_1^2+\theta_L^2-2d_3\lambda_0), (1+\beta_L^2 M_1^2 + r_L + \xi_2^2 - 2d_4\lambda_0)\}.$$

By the Gronwall inequality

$$E[\|S(x,\tau_l \wedge T)\|^2 + \|I_L(x,\tau_l \wedge T)\|^2 + \|R_L(x,\tau_l \wedge T)\|^2 + \|L(x,\tau_l \wedge T)\|^2] \leq M_2 e^{M_3 T}. \tag{3.4}$$

Define

$$\lambda_l = \inf_{\|Z(x,t)\| > l, 0 < t < \infty} (\|S(x,t)\|^2 + \|I_L(x,t)\|^2 + \|R_L(x,t)\|^2 + \|L(x,t)\|^2), for\ any\ l > l_0. \tag{3.5}$$

Combine Eqs (3.4) and (3.5) to get

$$\lambda_l P(\tau_l \leq T) \leq M_2 e^{M_3 T}. \tag{3.6}$$

Because $\lim_{l\to\infty} \lambda_l = \infty$, for inequality (3.6), let $l \to \infty$,we can get $P(\tau_\infty \le T) = 0$, namely,

$$P(\tau_l \ge T) = 1. \tag{3.7}$$

The evidence has been effectively demonstrated, thereby confirming the presence of a unique worldwide positive solution for the system delineated by Eq (2.7).

3.2. Some prior estimates

In this article, the focus has been on the examination of near-optimal controls as opposed to optimal control, and a number of fundamental definitions have been outlined to differentiate between the two.

Definition 3.2. [19] An admissible pair $\{Z^*(\cdot), u^*(\cdot)\}$ or an admissible control $u^*(\cdot)$ is referred to as an optimal control if it minimizes the value of $J(0, Z_0; u^*(\cdot))$.

Definition 3.3. [19] For any value of $\varepsilon > 0$, an admissible pair $\{Z^\varepsilon(\cdot), u^\varepsilon(\cdot)\}$ or an admissible control $u^\varepsilon(\cdot)$ is called ε-optimal, if

$$|J(0, Z_0; u^\varepsilon(\cdot)) - V(0, Z_0)| \le \epsilon.$$

Definition 3.4. [19] A set of admissible pairs $\{Z^\varepsilon(\cdot), u^\varepsilon(\cdot)\}$ is indexed by the parameter $\varepsilon > 0$. Any member $u^\varepsilon(\cdot)$ within this set is referred to as near-optimal if

$$|J(0, Z_0; u^\varepsilon(\cdot)) - V(0, Z_0)| \le \delta(\varepsilon).$$

This statement remains valid for a sufficiently small value of ε, where δ is a function of ε satisfying $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. The value $\delta(\varepsilon)$ is referred to as a bound for the error. If $\delta(\varepsilon) = a\varepsilon^m$, where a is a constant and $m > 0$, then $u^\varepsilon(\cdot)$ is referred to as near-optimal with an order of ε^m.

Next, we give Clark's generalized gradient definition and Ekland's variational principle.

Definition 3.5. [20] Let D be a convex set in R^d and let $V(\cdot) : D \to R^1$ be a locally Lipschitz function. The generalized gradient of the function V at the point $\widehat{x} \in Y$, denoted as $\partial_x V(\hat{x})$, is defined as a specific set according to the following formulation:

$$\partial_x V(\hat{x}) = \{p \in R^d | p \cdot \xi \le V^0(\hat{x}; \xi),\ for\ any\ \xi \in R^d\},$$

where $V^0(\hat{x}; \xi) = \limsup_{x\in D, x+h\xi\in D, x\to\hat{x}, h\to 0} \frac{V(x+h\xi)-V(x)}{h}$.

Lemma 3.2. *[21] Let (P, d) represent a complete metric space, and let $k(\cdot) : P \to \mathbb{R}$ be a function that is both bounded from below and has lower-semicontinuity. For any value $\varepsilon \ge 0$, suppose there is a function $u^\varepsilon(\cdot) \in p$ that satisfies the following condition:*

$$k(u^\epsilon(\cdot)) \le \inf_{u(\cdot)\in p} k(u(\cdot)) + \epsilon.$$

For any value of $\phi > 0$, there is a corresponding element $u^\phi \in P$ such that

$$k(u^\phi(\cdot)) \le k(u^\varepsilon(\cdot)), d(u^\phi(\cdot), u^\varepsilon(\cdot)) \le \phi\ and\ k(u^\phi(\cdot)) \le k(u(\cdot)) + \frac{\varepsilon}{\phi} d(u(\cdot), u^\phi)(\cdot).$$

Based on the preceding discussion, it can be inferred that $u(\cdot) \in U_{ad}[0,T]$. This implies the existence of a solution pair $(p(\cdot,t), q(\cdot,t)) \in \mathbb{L}^2_{\mathcal{F}}(\Omega \times [0,T]; R^4) \times \mathbb{L}^2_{\mathcal{F}}(\Omega \times [0,T]; R^4)$, where $\mathbb{L}^2_{\mathcal{F}}(\Omega \times [0,T]; R^4)$ represents the Hilbert space of $\mathcal{F}_t$-adapted processes. This solution pair satisfies the adjoint equation as follows:

$$\begin{cases} dp_1(x,t) = -g_1(Z(x,t), u_1(x,t), u_2(x,t), p(x,t), q(x,t))dt + q_1(x,t)dB_1(t), \\ dp_2(x,t) = -g_2(Z(x,t), u_1(x,t), u_2(x,t), p(x,t), q(x,t))dt + q_2(x,t)dB_1(t), \\ dp_3(x,t) = -g_3(Z(x,t), u_1(x,t), u_2(x,t), p(x,t), q(x,t))dt + q_3(x,t)dB_1(t), \\ dp_4(x,t) = -g_4(Z(x,t), u_1(x,t), u_2(x,t), p(x,t), q(x,t))dt + q_4(x,t)dB_2(t), \\ p_1(x,T) = h_S(S(x,T)), p_2(x,T) = h_{I_L}(I_L(x,T)), p_3(x,T) = h_{R_L}(R_L(x,T)), \\ p_4(x,T) = h_L(L(x,T)), \end{cases} \tag{3.8}$$

where

$$\begin{aligned} g_1(Z(x,t), u_1(x,t), u_2(x,t), p(x,t), q(x,t)) =& [d_1\Delta - \mu - \beta_L L(x,t)]p_1(x,t) + \beta_L L(x,t)p_2(x,t) - \xi_1 q_1(x,t), \\ g_2(Z(x,t), u_1(x,t), u_2(x,t), p(x,t), q(x,t)) =& [d_2\Delta - (\sigma_L + \delta_L + \mu) - \frac{cu_1(x,t)}{(1+\alpha I_L(x,t))^2}]p_2(x,t) + [\sigma_L \\ &+ \frac{cu_1(x,t)}{(1+\alpha I_L(x,t))^2}]p_3(x,t) - \xi_1 q_2(x,t), \\ g_3(Z(x,t), u_1(x,t), u_2(x,t), p(x,t), q(x,t)) =& \theta_L p_1(x,t) + [d_3\Delta - (\theta_L + \mu)]p_3(x,t) - \xi_1 q_3(x,t), \\ g_4(Z(x,t), u_1(x,t), u_2(x,t), p(x,t), q(x,t)) =& -\beta_L S(x,t)p_1(x,t) + \beta_L S(x,t)p_2(x,t) + [d_4\Delta + r_L \\ &- 2\frac{r_L L(x,t)}{K_L} - (\varepsilon + u_2(x,t))]p_4(x,t) - \xi_2 q_4(x,t). \end{aligned} \tag{3.9}$$

Subsequently, in order to establish the sufficient and necessary conditions for the listeriosis model to near-optimal controls, it is important to make appropriate conclusions about the state and adjoint equations in connection with the control variables. Next, we demonstrate that the system described by Eq (2.7) is limited in its behavior.

Lemma 3.3. *For any $p \geq 0$, if $u_1(x,t), u_2(x,t) \in U_{ad}(\Omega \times [0,T])$, there is a constant C such that the following inequality holds:*

$$E \sup_{0 \leq t \leq T} \int_\Omega \{|S(x,t)|^p + |I_L(x,t)|^p + |R_L(x,t)|^p + |L(x,t)|^p\} dx \leq C. \tag{3.10}$$

Proof. Similar to Theorem 3.1, we can conclude that the inequality (3.10) is true.

Lemma 3.4. *For any $u_1(x,t), u_2(x,t) \in U_{ad}(\Omega \times [0,T])$, provided that the solution of the adjoint equation displays the following feature.*

$$\sum_{i=1}^4 E\{\sup_{0 \leq t \leq T} \int_\Omega |p_i(x,t)|^2 dx\} + \sum_{i=1}^4 E \int_0^T \int_\Omega |q_i(x,t)|^2 dxdt \leq C, \tag{3.11}$$

where C is a constant.

The evidence is presented in Appendix A2.

To obtain the necessary conditions for the near-optimal controls of system (2.7), we introduce the following two lemmas to evaluate the difference between the equation of state and the adjoint equation. Initially, we must establish a definition for distance within the space of admissible control in order for (U_{ad}, d) to form a comprehensive metric space. For any $u(x,t)$,$\bar{u}(x,t) \in U_{ad}(\Omega \times [0,T])$, we set

$$d(u(t), \bar{u}(t)) = \mathbb{E}[mes(Z,t) \in \Omega \times [0,T] : u(Z,t) \neq \bar{u}(Z,t)].$$

The symbol "mes" represents the Lebesgue measure. Additionally, it is also confirmed that the pair (U_{ad}, d) constitutes a fully complete metric space [22]. Following this, we introduce the lemma regarding the continuity of the state process in relation to the metric distance d.

Lemma 3.5. *For any $\eta \geq 0$ and a value of $k \in (0,1)$ such that $k\eta < 1$, there exists a constant $C = C(\eta, k)$ such that for any functions $u(x,t)$ and $\bar{u}(x,t)$ in the set $U_{ad}(\Omega \times [0,T])$, and their corresponding trajectories $Z(x,t)$ and $\bar{Z}(x,t)$, the following equation is satisfied:*

$$\begin{aligned} &E\{\sup_{0\leq t\leq T} \int_\Omega (|S(x,t) - \bar{S}(x,t)|^{2\eta} + |I_L(x,t) - \bar{I}_L(x,t)|^{2\eta} + |R_L(x,t) - \bar{R}_L(x,t)|^{2\eta} + |L(x,t) - \bar{L}(x,t)|^{2\eta})dx\} \\ &\leq C[d(u_1(x,t), \bar{u}_1(x,t))^{k\eta} + d(u_2(x,t), \bar{u}_2(x,t))^{k\eta}]. \end{aligned} \tag{3.12}$$

The evidence is presented in Appendix A3.

Lemma 3.6. *For any value $1 < \eta < 2$, and $0 < \kappa < 1$ that satisfies the condition $(1+\kappa)\eta < 2$, there exists a constant $C = C(\eta, \kappa)$ such that for any functions $u(x,t)$ and $\bar{u}(x,t)$ belonging to the set $U_{ad}(\Omega \times [0,T])$, along with their corresponding trajectories $(Z(x,t), \bar{Z}(x,t))$ and the solution of the corresponding adjoint equation, the following inequality holds:*

$$\sum_{i=1}^{4} \int_0^T \int_\Omega (|p_i(x,t) - \tilde{p}_i(x,t)|^\eta + |q_i(x,t) - \tilde{q}_i(x,t)|^\eta)dxdt \leq Cd(u(x,t), \bar{u}(x,t))^{\frac{\kappa\eta}{2}}. \tag{3.13}$$

The evidence is presented in Appendix A4.

The theorems and lemmas outlined previously provide a theoretical foundation for establishing the sufficient and necessary conditions for near-optimality. Subsequently, we will initially establish the sufficient conditions for near-optimality, followed by the demonstration of the necessary conditions.

4. Sufficient and necessary conditions of near-optimal

4.1. Sufficient conditions of near-optimal

In this particular section, our objective is to establish sufficient conditions for near-optimality. Subsequently, the functions outlined can be derived based on the definition of the Hamiltonian function as described in the work by [23].

$$
\begin{aligned}
&H(Z(x,t),u_1(x,t),u_2(x,t),p(x,t),q(x,t))\\
&=[d_1\Delta S(x,t)+\Lambda+\theta_L R_L(x,t)-(\mu+\beta_L L(x,t))S(x,t)]p_1(x,t)+[d_2\Delta I_L(x,t)+\beta_L L(x,t)S(x,t)\\
&-(\sigma_L+\delta_L+\mu)I_L(x,t)-\frac{cu_1(x,t)I_L(x,t)}{1+\alpha I_L(x,t)}]p_2(x,t)+[d_3\Delta R_L(x,t)+\sigma_L I_L(x,t)-(\theta_L+\mu)R_L(x,t)\\
&+\frac{cu_1(x,t)I_L(x,t)}{1+\alpha I_L(x,t)}]p_3(x,t)+[d_4\Delta L(x,t)+r_L L(x,t)(1-\frac{L(x,t)}{K_L})-(\varepsilon+u_2(x,t))L(x,t)]p_4(x,t)\\
&-\xi_1 S(x,t)q_1(x,t)-\xi_1 I_L(x,t)q_2(x,t)-\xi_1 R_L(x,t)q_3(x,t)-\xi_2 L(x,t)q_4(x,t)+B_1 I_L(x,t)+B_2 L(x,t)\\
&+\frac{1}{2}B_3u_1^2(x,t)+\frac{1}{2}B_4u_2^2(x,t).
\end{aligned}
\tag{4.1}
$$

Subsequently, by employing the Eq (4.1) presented below, we derive sufficient conditions for the near-optimality of the system (2.7), as follows Theorem 4.1.

Theorem 4.1. *Suppose (H1) and (H2) hold and $H(Z(x,t),u_1(x,t),u_2(x,t),p(x,t),q(x,t))$ and h is convex, for any $\varepsilon>0$, let u^ε be represent an admissible control, and $(p^\varepsilon(x,t),q^\varepsilon(x,t))$ denote the adjoint Eq* (3.8) *associated with u^ε,*

$$
\begin{aligned}
&\inf_{u_1,u_2\in U_{ad}} E\{\int_0^T\int_\Omega(\frac{1}{2}B_3(u_1(x,t))^2+\frac{1}{2}B_4(u_2(x,t))^2+\frac{cu_1(x,t)I_L^\varepsilon(x,t)}{1+\alpha I_L^\varepsilon(x,t)}(p_3^\varepsilon(x,t)-p_2^\varepsilon(x,t))\\
&-u_2(x,t)L^\varepsilon(x,t)p_4^\varepsilon(x,t))dxdt\}\\
&\geq E\int_0^T\int_\Omega(\frac{1}{2}B_3(u_1^\varepsilon(x,t))^2+\frac{1}{2}B_4(u_2^\varepsilon(x,t))^2+\frac{cu_1^\varepsilon(x,t)I_L^\varepsilon(x,t)}{1+\alpha I_L^\varepsilon(x,t)}(p_3^\varepsilon(x,t)-p_2^\varepsilon(x,t))\\
&-u_2^\varepsilon(x,t)L^\varepsilon(x,t)p_4^\varepsilon(x,t))dxdt-\varepsilon,
\end{aligned}
\tag{4.2}
$$

then we have

$$
J(0,Z_0;u^\varepsilon(x,t))\leq\inf_{u_1,u_2\in U_{ad}}J(0,Z_0;u(x,t))+C\varepsilon^{\frac{1}{2}},
\tag{4.3}
$$

where B_3 and B_4 represent the weights that control $u_1(x,t)$ and $u_2(x,t)$.

Proof. First, redefine a new metric $\bar{d}$ *on* U_{ad} so that $H_u(Z^\varepsilon(x,t),u_1^\varepsilon(x,t),u_2^\varepsilon(x,t),p^\varepsilon(x,t),q^\varepsilon(x,t))$ can be estimated in terms of ε. The new metric $\bar{d}$ is as follows:

$$
\bar{d}(u,\tilde{u})=E\int_0^T\int_\Omega Z^\varepsilon(x,t)|u(x,t)-\tilde{u}(x,t)|dxdt,
$$

where

$$
Z^\varepsilon(x,t)=1+\sum_{i=1}^4|p_i^\varepsilon(x,t)|+\sum_{i=1}^4|q_i^\varepsilon(x,t)|.
$$

Clearly, the metric $\bar{d}$ fulfills the condition $Z^\varepsilon(x,t)>1$, and it is a complete metric when considered as a weighted L^1 norm.

Based on the objective function 2.8 and the Hamiltonian function 4.1 previously established, the estimation of $J(0,Z_0;u^\varepsilon(x,t))-J(0,Z_0;u(x,t))$ can be conducted using the approach outlined in the work of [24].

$$
\begin{aligned}
&J(0,Z_0;u^{\varepsilon}(x,t)) - J(0,Z_0;u(x,t)) \\
&\leq E\int_0^T\int_\Omega(-\frac{cI_L^{\varepsilon}(x,t)}{1+\alpha I_L^{\varepsilon}(x,t)}p_2^{\varepsilon}(x,t) + \frac{cI_L^{\varepsilon}(x,t)}{1+\alpha I_L^{\varepsilon}(x,t)}p_4^{\varepsilon}(x,t) + B_3u_1^{\varepsilon}(x,t))(u_1^{\varepsilon}(x,t) - u_1(x,t)) \\
&+ (-L^{\varepsilon}(x,t)p_4^{\varepsilon}(x,t) + B_4u_2^{\varepsilon}(x,t))(u_2^{\varepsilon}(x,t) - u_2(x,t))dxdt.
\end{aligned}
\tag{4.4}
$$

Afterward, in order to determine the numerical value of the expression presented on the right side of Eq (4.4), a function $M(\cdot) : U_{ad}[0,T] \to R$ is introduced.

$$
M(u(x,t)) = E\int_0^T\int_\Omega H(Z^{\varepsilon}(x,t), u(x,t), p^{\varepsilon}(x,t), q^{\varepsilon}(x,t))dxdt.
$$

According to hypothesis (H1), the function $M(\cdot)$ exhibits continuity on the matrix $\bar{d}$. As a result, based on the criteria specified in Theorem 4.1 and the Ekeland principle, there is a $\bar{u}^{\varepsilon} \in U_{ad}$ that fulfills the specified conditions.

$$
\bar{d}(u^{\varepsilon}(x,t), \bar{u}^{\varepsilon}(x,t)) \leq \varepsilon^{\frac{1}{2}} \text{ and } F(\bar{u}^{\varepsilon}(x,t)) - F(u(x,t) \leq \varepsilon^{\frac{1}{2}}\bar{d}(u(x,t), \bar{u}^{\varepsilon}(x,t)).
\tag{4.5}
$$

Thus

$$
\begin{aligned}
&E\int_0^T\int_\Omega\{-\frac{c\bar{u}_1^{\varepsilon}(x,t)I_L^{\varepsilon}(x,t)}{1+\alpha I_L^{\varepsilon}(x,t)}p_2^{\varepsilon}(x,t) + \frac{c\bar{u}_1^{\varepsilon}(x,t)I_L^{\varepsilon}(x,t)}{1+\alpha I_L^{\varepsilon}(x,t)}p_3^{\varepsilon}(x,t) - \bar{u}_2^{\varepsilon}(x,t)L^{\varepsilon}(x,t)p_4^{\varepsilon}(x,t) \\
&+ \frac{1}{2}B_3(\bar{u}_1^{\varepsilon}(x,t))^2 + \frac{1}{2}B_4(\bar{u}_2^{\varepsilon}(x,t))^2\}dxdt \\
&= \min_{u_1,u_2\in U_{ad}[0,T]} E\int_0^T\int_\Omega\{-\frac{cu_1(x,t)I_L^{\varepsilon}(x,t)}{1+\alpha I_L^{\varepsilon}(x,t)}p_2^{\varepsilon}(x,t) + \frac{cu_1(x,t)I_L^{\varepsilon}(x,t)}{1+\alpha I_L^{\varepsilon}(x,t)}p_3^{\varepsilon}(x,t) - u_2(x,t)L^{\varepsilon}(x,t)p_4^{\varepsilon}(x,t) \\
&+ \frac{1}{2}B_3u_1^2(x,t) + \frac{1}{2}B_4u_2^2(x,t) + \varepsilon^{\frac{1}{2}}Z^{\varepsilon}(x,t)|u_1(x,t) - \bar{u}_1^{\varepsilon}(x,t) + \varepsilon^{\frac{1}{2}}Z^{\varepsilon}(x,t)|u_2(x,t) - \bar{u}_2^{\varepsilon}(x,t)|\}dxdt.
\end{aligned}
$$

Applying in [25], can be obtained

$$
\begin{aligned}
&0 \in -\frac{cI_L^{\varepsilon}(x,t)}{1+\alpha I_L^{\varepsilon}(x,t)}p_2^{\varepsilon}(x,t) + \frac{cI_L^{\varepsilon}(x,t)}{1+\alpha I_L^{\varepsilon}(x,t)}p_3^{\varepsilon}(x,t) + B_3\bar{u_1}^{\varepsilon}(x,t) - L^{\varepsilon}(x,t)p_4^{\varepsilon}(x,t) + B_4\bar{u_2}^{\varepsilon}(x,t) \\
&\subset -\frac{cI_L^{\varepsilon}(x,t)}{1+\alpha I_L^{\varepsilon}(x,t)}p_2^{\varepsilon}(x,t) + \frac{cI_L^{\varepsilon}(x,t)}{1+\alpha I_L^{\varepsilon}(x,t)}p_3^{\varepsilon}(x,t) + B_3\bar{u_1}^{\varepsilon}(x,t) - L^{\varepsilon}(x,t)p_4^{\varepsilon}(x,t) + B_4\bar{u_2}^{\varepsilon}(x,t) \\
&+ [-\varepsilon^{\frac{1}{2}}Z^{\varepsilon}(x,t), \varepsilon^{\frac{1}{2}}Z^{\varepsilon}(x,t)].
\end{aligned}
\tag{4.6}
$$

Since the Hamiltonian function equation $H(Z(x,t), u_1(x,t), u_2(x,t), p(x,t), q(x,t))$ is differentiable in $u_1(x,t), u_2(x,t)$, therefore Eq (4.6) implies that there exists $\tau^{\varepsilon}(x,t) \in [-\varepsilon^{\frac{1}{2}}Z^{\varepsilon}(x,t), \varepsilon^{\frac{1}{2}}Z^{\varepsilon}(x,t)]$ such that

$$
\begin{aligned}
&-\frac{cI_L^{\varepsilon}(x,t)}{1+\alpha I_L^{\varepsilon}(x,t)}p_2^{\varepsilon}(x,t) + \frac{cI_L^{\varepsilon}(x,t)}{1+\alpha I_L^{\varepsilon}(x,t)}p_3^{\varepsilon}(x,t) + B_3\bar{u_1}^{\varepsilon}(x,t) - L^{\varepsilon}(x,t)p_4^{\varepsilon}(x,t) + B_4\bar{u_2}^{\varepsilon}(x,t) \\
&+ \tau^{\varepsilon}(x,t) = 0.
\end{aligned}
$$

This can be proved using hypothesis (H1)

$$
\begin{aligned}
&|-\frac{cI_L^\varepsilon(x,t)}{1+\alpha I_L^\varepsilon(x,t)}p_2^\varepsilon(x,t)+\frac{cI_L^\varepsilon(x,t)}{1+\alpha I_L^\varepsilon(x,t)}p_3^\varepsilon(x,t)+B_3u_1^\varepsilon(x,t)-L^\varepsilon(x,t)p_4^\varepsilon(x,t)+B_4u_2^\varepsilon(x,t)| \\
&\leq |B_3(u_1^\varepsilon(x,t)-\bar{u}_1^\varepsilon(x,t))+B_4(u_2^\varepsilon(x,t)-\bar{u}_2^\varepsilon(x,t))|+|-\frac{cI_L^\varepsilon(x,t)}{1+\alpha I_L^\varepsilon(x,t)}p_2^\varepsilon(x,t)+\frac{cI_L^\varepsilon(x,t)}{1+\alpha I_L^\varepsilon(x,t)}p_3^\varepsilon(x,t) \\
&+B_3\bar{u}_1^\varepsilon(x,t)-L^\varepsilon(x,t)p_4^\varepsilon(x,t)+B_4\bar{u}_2^\varepsilon(x,t)| \\
&\leq CZ^\varepsilon(x,t)|u_1(x,t)-\bar{u}_1(x,t)|+Z^\varepsilon(x,t)|u_2(x,t)-\bar{u}_2(x,t)|)+\tau^\varepsilon(x,t) \\
&\leq C(Z^\varepsilon(x,t)|u_1(x,t)-\bar{u}_1(x,t)|+Z^\varepsilon(x,t)|u_2(x,t)-\bar{u}_2(x,t)|)+2\varepsilon^{\frac{1}{2}}Z^\varepsilon(x,t).
\end{aligned}
\tag{4.7}
$$

By utilizing Lemma 3.4 in conjunction with the definition of $\bar{d}$, the intended outcome can be derived from Eqs (4.4) and (4.7).

Remark 2. *As can be seen from Eq* (4.2)*, the sufficient conditions near-optimal is affected by the parameters of the system. When* $\varepsilon = 0$*, we can obtain the exact optimality of system* (2.7)*, in simpler terms, Theorem 4.1 represents the necessary conditions for achieving optimal control.*

4.2. Necessary conditions of near-optimal

In this segment, we utilize Lemmas 3.5 and 3.6 to establish the necessary conditions for the system described in Eq (2.7) to near-optimal controls.

Theorem 4.2. *Given that conditions (H1) and (H2) are satisfied, it is established that the pair* $(p^\varepsilon(x,t), q^\varepsilon(x,t))$ *serves as the solution to the adjoint Eq* (3.8) *corresponding to* $u^\varepsilon(x,t)$*. For any* $\eta \in [0,1)$*, there is a positive constant C, for any* $\varepsilon > 0$ *and any* ε*-optimal pair* $(Z^\varepsilon(x,t), u^\varepsilon(x,t))$*, the given condition holds.*

$$
\begin{aligned}
&\inf_{u_1,u_2\in U_{ad}} E\{\int_0^T\int_\Omega(\frac{1}{2}B_3(u_1(x,t))^2+\frac{1}{2}B_4(u_2(x,t))^2+\frac{cu_1(x,t)I_L^\varepsilon(x,t)}{1+\alpha I_L^\varepsilon(x,t)}(p_3^\varepsilon(x,t)-p_2^\varepsilon(x,t)) \\
&-u_2(x,t)L^\varepsilon(x,t)p_4^\varepsilon(x,t))dxdt\} \\
&\geq E\int_0^T\int_\Omega(\frac{1}{2}B_3(u_1^\varepsilon(x,t))^2+\frac{1}{2}B_4(u_2^\varepsilon(x,t))^2+\frac{cu_1^\varepsilon(x,t)I_L^\varepsilon(x,t)}{1+\alpha I_L^\varepsilon(x,t)}(p_3^\varepsilon(x,t)-p_2^\varepsilon(x,t)) \\
&-u_2^\varepsilon(x,t)L^\varepsilon(x,t)p_4^\varepsilon(x,t))dxdt-C\varepsilon^{\frac{\eta}{3}},
\end{aligned}
\tag{4.8}
$$

where B_3 *and* B_4 *represent the weights that control* $u_1(x,t)$ *and* $u_2(x,t)$*.*

Proof. According to hypothesis (H1), it is evident that the function $J(0,Z_0;u(x,t)) : U_{ad} \to R$ remains continuous with respect to the metric d. In accordance with Ekeland's variational principle, it is established that there is a feasible pair $(Z^\varepsilon(x,t), u^\varepsilon(x,t))$ that satisfies the given conditions,

$$
d(u^\varepsilon(x,t),\bar{u}^\varepsilon(x,t)) \leq \varepsilon^{\frac{2}{3}}, and\ \bar{J}(0,Z_0;\bar{u}^\varepsilon(x,t)) \leq \bar{J}(0,Z_0;u(x,t)),\ for\ u(x,t)\in U_{ad}, \tag{4.9}
$$

where

$$
\bar{J}(0,Z_0;u(x,t)) = J(0,Z_0;u(x,t))+\varepsilon^{\frac{1}{3}}d(u(x,t),\bar{u}^\varepsilon(x,t)).
$$

This indicates that the pair $(\bar{Z}^{\varepsilon}(x,t), \bar{u}^{\varepsilon}(x,t))$ represents the optimal solution for the system described by Eq (2.7). Next, we will push out a necessary condition of $(\bar{Z}^{\varepsilon}(x,t), \bar{u}^{\varepsilon}(x,t))$ by defining a spike variation. Select $\rho > 0$ and $\bar{t} \in [0,T]$, we define $\bar{u}^{\varepsilon}(x,t) \in U_{ad}[0,T]$ as follows:

$$u^{\rho}(x,t) = \begin{cases} u(x,t), \text{ if } (x,t) \in \Omega \times [\bar{t}, \bar{t}+\rho], \\ \bar{u}^{\varepsilon}(x,t), \text{ if } (x,t) \in [0,T]\backslash[\bar{t}, \bar{t}+\rho] \times \Omega. \end{cases} \tag{4.10}$$

The pair $(Z^{\rho}(x,t), u^{\rho}(x,t))$ denotes the solution to the system described by Eq (2.7). In accordance with Eq (4.9), it follows that

$$d(u^{\rho}(x,t), \bar{u}^{\varepsilon}(x,t)) \le \rho, and\ \bar{J}(0, Z_0; \bar{u}^{\varepsilon}(x,t)) \le \bar{J}(0, Z_0; u^{\rho}(x,t)). \tag{4.11}$$

Hence

$$J(0, Z_0; \bar{u}^{\varepsilon}(x,t)) = \bar{J}(0, Z_0; \bar{u}^{\varepsilon}(x,t)) \le \bar{J}(0, Z_0; u^{\rho}(x,t)) = J(0, Z_0; u^{\rho}(x,t)) + \rho\varepsilon^{\frac{1}{3}}. \tag{4.12}$$

It follows from Eq (4.12), Lemma 3.5 and Taylor's expansion that

$$\begin{aligned}
&-\rho\varepsilon^{\frac{1}{3}} \le J(0, Z_0; u^{\rho}(x,t)) - J(0, Z_0; \bar{u}^{\varepsilon}(x,t)) \\
&= E\int_0^T \int_{\Omega} [B_1 I_L^{\rho}(x,t) + B_2 L^{\rho}(x,t) + \frac{1}{2}B_3(u_1^{\rho}(x,t))^2 + \frac{1}{2}B_4(u_2^{\rho}(x,t))^2 - B_1\bar{I}_L^{\varepsilon}(x,t) \\
&- B_2\bar{L}^{\varepsilon}(x,t) - \frac{1}{2}B_3(\bar{u}_1^{\varepsilon}(x,t))^2 - \frac{1}{2}B_4(\bar{u}_2^{\varepsilon}(x,t))^2]dxdt + E\int_{\Omega}[h(Z^{\rho}(x,T)) - h(\bar{Z}^{\varepsilon}(x,T))]dx \\
&\le E\int_0^T \int_{\Omega}\{B_1(I_L^{\rho}(x,t) - \bar{I}_L^{\varepsilon}(x,t)) + B_2(L^{\rho}(x,t) - \bar{L}^{\varepsilon}(x,t))\}dxdt \\
&+ E\int_{\bar{t}}^{\bar{t}+\rho} \int_{\Omega}\{\frac{1}{2}B_3(u_1^2(x,t)) - \frac{1}{2}B_3(\bar{u}_1^{\varepsilon}(x,t))^2 + \frac{1}{2}B_4(u_2^2(x,t)) - \frac{1}{2}B_4(\bar{u}_2^{\varepsilon}(x,t))^2\}dxdt \\
&+ E\int_{\Omega}\{[h_S(S^{\rho}(x,T) - \bar{S}^{\varepsilon}(x,T))dx] + [h_{I_L}(I_L^{\rho}(x,T) - \bar{I}_L^{\varepsilon}(x,T))dx] \\
&+ [h_{R_L}(R_L^{\rho}(x,T) - \bar{R}_L^{\varepsilon}(x,T))dx] + [h_L(L^{\rho}(x,T) - \bar{L}^{\varepsilon}(x,T))dx]\} + \circ(\rho).
\end{aligned} \tag{4.13}$$

The following formula is obtained using It$\hat{o}$'s formula for $\sum_{i=1}^{4}\bar{p}_i^{\varepsilon}(x,t)[S^{\rho}(x,t) - \bar{S}^{\varepsilon}(x,t) + I_L^{\rho}(x,t) - \bar{I}_L^{\varepsilon}(x,t) + R_L^{\rho}(x,t) - \bar{R}_L^{\varepsilon}(x,t) + S^{\rho}(x,t) - \bar{S}^{\varepsilon}(x,t)]$,

$$\begin{aligned}
&E\int_{\Omega}\{h_S[S^{\rho}(x,T) - \bar{S}^{\varepsilon}(x,T)] + h_{I_L}[I_L^{\rho}(x,T) - \bar{I}_L^{\varepsilon}(x,T)] + h_{R_L}[R_L^{\rho}(x,T) - \bar{R}_L^{\varepsilon}(x,T)] \\
&+ h_L[L^{\rho}(x,T) - \bar{L}^{\varepsilon}(x,T)]\}dx \\
&\le E\int_{\bar{t}}^{\bar{t}+\rho}\int_{\Omega}[(\frac{c\bar{u}_1^{\varepsilon}(x,t)}{1+\alpha\bar{I}_L^{\varepsilon}(x,t)}\bar{I}_L^{\varepsilon}(x,t) - \frac{cu_1^{\rho}(x,t)}{1+\alpha I_L^{\rho}(x,t)}I_L^{\rho}(x,t))\bar{p}_2^{\varepsilon}(x,t) + (\frac{cu_1^{\rho}(x,t)}{1+\alpha I_L^{\rho}(x,t)}I_L^{\rho}(x,t) \\
&- \frac{c\bar{u}_1^{\varepsilon}(x,t)}{1+\alpha\bar{I}_L^{\varepsilon}(x,t)}\bar{I}_L^{\varepsilon}(x,t))\bar{p}_3^{\varepsilon}(x,t) + (\bar{u}_2^{\varepsilon}(x,t)\bar{L}^{\varepsilon}(x,t) - u_2^{\rho}(x,t)L^{\rho}(x,t))\bar{p}_4^{\varepsilon}(x,t)]dxdt \\
&- E\int_0^T \int_{\Omega}\{B_1(I_L^{\rho}(x,t) - \bar{I}_L^{\varepsilon}(x,t)) + B_2(L^{\rho}(x,t) - \bar{L}^{\varepsilon}(x,t))\}dxdt.
\end{aligned} \tag{4.14}$$

Together with Eqs (4.13) and (4.14), we have

$$
\begin{aligned}
&-\rho\varepsilon^{\frac{1}{3}} \leq J(0,Z_0;u^{\rho}(x,t)) - J(0,Z_0;\bar{u}^{\varepsilon}(x,t))\\
&\leq E\int_{\bar{t}}^{\bar{t}+\rho}\int_{\Omega}\{\frac{1}{2}B_3(u_1(x,t))^2 - \frac{1}{2}B_3(\bar{u_1}^{\varepsilon}(x,t))^2 + \frac{1}{2}B_4(u_2(x,t))^2 - \frac{1}{2}B_4(\bar{u_2}^{\varepsilon}(x,t))^2\}dxdt\\
&+E\int_{\bar{t}}^{\bar{t}+\rho}\int_{\Omega}[(\frac{c\bar{u}_1^{\varepsilon}(x,t)}{1+\alpha\bar{I}_L^{\varepsilon}(x,t)}\bar{I}_L^{\varepsilon}(x,t) - \frac{cu_1^{\rho}(x,t)}{1+\alpha I_L^{\rho}(x,t)}I_L^{\rho}(x,t))\bar{p}_2^{\varepsilon}(x,t)\\
&+(\frac{cu_1^{\rho}(x,t)}{1+\alpha I_L^{\rho}(x,t)}I_L^{\rho}(x,t) - \frac{c\bar{u}_1^{\varepsilon}(x,t)}{1+\alpha\bar{I}_L^{\varepsilon}(x,t)}\bar{I}_L^{\varepsilon}(x,t))\bar{p}_3^{\varepsilon}(x,t)\\
&+(\bar{u}_2^{\varepsilon}(x,t)\bar{L}^{\varepsilon}(x,t) - u_2^{\rho}(x,t)L^{\rho}(x,t))\bar{p}_4^{\varepsilon}(x,t)]dxdt + \circ(\rho).
\end{aligned}
\tag{4.15}
$$

By dividing both sides of Eq (4.15) by the variable ρ and then taking $\rho \to 0$.

$$
\begin{aligned}
&-\varepsilon^{\frac{1}{3}} \leq E[\frac{1}{2}B_3(u_1(x,t))^2 - \frac{1}{2}B_3(\bar{u}_1^{\varepsilon}(x,t))^2 + \frac{1}{2}B_4(u_2(x,t))^2 - \frac{1}{2}B_4(\bar{u}_2^{\varepsilon}(x,t))^2]\\
&+E[(\frac{c\bar{u}_1^{\varepsilon}(x,t)}{1+\alpha\bar{I}_L^{\varepsilon}(x,t)}\bar{I}_L^{\varepsilon}(x,t) - \frac{cu_1^{\rho}(x,t)}{1+\alpha I_L^{\rho}(x,t)}I_L^{\rho}(x,t))\bar{p}_2^{\varepsilon}(x,t)\\
&+(\frac{cu_1^{\rho}(x,t)}{1+\alpha I_L^{\rho}(x,t)}I_L^{\rho}(x,t) - \frac{c\bar{u}_1^{\varepsilon}(x,t)}{1+\alpha\bar{I}_L^{\varepsilon}(x,t)}\bar{I}_L^{\varepsilon}(x,t))\bar{p}_3^{\varepsilon}(x,t) + (\bar{u}_2^{\varepsilon}(x,t)\bar{L}^{\varepsilon}(x,t) - u_2^{\rho}(x,t)L^{\rho}(x,t))\bar{p}_4^{\varepsilon}(x,t)].
\end{aligned}
\tag{4.16}
$$

Furthermore, we compute a component from the right-hand side of Eq (4.16) and replace $(\bar{Z}^{\varepsilon}(x,t),\bar{u}^{\varepsilon}(x,t))$ with $(Z^{\varepsilon}(x,t),u^{\varepsilon}(x,t))$. Subsequently, we assess the estimation.

$$
\begin{aligned}
&E\int_0^T\int_{\Omega}[(\frac{c\bar{u}_1^{\varepsilon}(x,t)}{1+\alpha\bar{I}_L^{\varepsilon}(x,t)}\bar{I}_L^{\varepsilon}(x,t) - \frac{cu_1^{\rho}(x,t)}{1+\alpha I_L^{\rho}(x,t)}I_L^{\rho}(x,t))\bar{p}_2^{\varepsilon}(x,t) - (\frac{cu_1^{\varepsilon}(x,t)}{1+\alpha I_L^{\varepsilon}(x,t)}I_L^{\varepsilon}(x,t)\\
&-\frac{cu_1^{\rho}(x,t)}{1+\alpha I_L^{\rho}(x,t)}I_L^{\rho}(x,t))p_2^{\varepsilon}(x,t)]dxdt\\
&=E\int_0^T\int_{\Omega}(\frac{c\bar{u}_1^{\varepsilon}(x,t)}{1+\alpha\bar{I}_L^{\varepsilon}(x,t)}\bar{I}_L^{\varepsilon}(x,t) - \frac{cu_1^{\varepsilon}(x,t)}{1+\alpha I_L^{\varepsilon}(x,t)}I_L^{\varepsilon}(x,t))p_2^{\varepsilon}(x,t)dxdt\\
&+E\int_0^T\int_{\Omega}(\frac{c\bar{u}_1^{\varepsilon}(x,t)}{1+\alpha\bar{I}_L^{\varepsilon}(x,t)}\bar{I}_L^{\varepsilon}(x,t) - \frac{cu_1^{\rho}(x,t)}{1+\alpha I_L^{\rho}(x,t)}I_L^{\rho}(x,t))(\bar{p}_2^{\varepsilon}(x,t) - p_2^{\varepsilon}(x,t))dxdt\\
&=I_1+I_2.
\end{aligned}
\tag{4.17}
$$

According to Lemma 3.6 and Eq (4.9), and applying the Holder inequality, it is possible to derive that for any values of k and η within the ranges of $0 < k < 1$ and $1 < \eta < 2$, respectively, and satisfying the condition $(1+k)\eta < 2$, there exists an indeterminate parameter C,

$$
\begin{aligned}
I_1 &\leq C\{E\int_0^T\int_{\Omega}|p_2^{\varepsilon}(x,t)|^2dxdt\}^{\frac{1}{2}}\{E\int_0^T\int_{\Omega}|\bar{u}_1^{\varepsilon}(x,t) - u_1^{\varepsilon}(x,t)|^2\chi_{u^{\varepsilon}\neq\bar{u}^{\varepsilon}}(x,t)dxdt\}^{\frac{1}{2}}\\
&\leq C\{E\int_0^T\int_{\Omega}(|u_1^{\varepsilon}(x,t)|^4 + |\bar{u}_1^{\varepsilon}(x,t)|^4)dxdt\}^{\frac{1}{4}}E\int_0^T\int_{\Omega}(\chi_{u^{\varepsilon}\neq\bar{u}^{\varepsilon}}(x,t)dxdt)^{\frac{1}{4}}\\
&\leq C[d(u(x,t),\bar{u}(x,t))]^{\frac{1}{4}}\\
&\leq C\varepsilon^{\frac{\eta}{3}},
\end{aligned}
\tag{4.18}
$$

and

$$
\begin{aligned}
I_2 &\leq \{E\int_0^T\int_\Omega|\frac{cu_1^\varepsilon(x,t)}{1+\alpha I_L^\varepsilon(x,t)}I_L^\varepsilon(x,t)-\frac{cu_1^\rho(x,t)}{1+\alpha I_L^\rho(x,t)}I_L^\rho(x,t)|^{\frac{\eta}{\eta-1}}dxdt\}^{\frac{\eta-1}{\eta}}\{E\int_0^T\int_\Omega|\bar{p}_2^\varepsilon(x,t)-p_2^\varepsilon(x,t)|^\eta dxdt\}^{\frac{1}{\eta}} \\
&\leq C[d(u(x,t),\bar{u}(x,t))^{\frac{\kappa\eta}{2}}]^{\frac{1}{\eta}}\{E\int_0^T\int_\Omega|\frac{cu_1^\varepsilon(x,t)}{1+\alpha I_L^\varepsilon(x,t)}I_L^\varepsilon(x,t)|^{\frac{\eta}{\eta-1}}+|\frac{cu_1^\rho(x,t)}{1+\alpha I_L^\rho(x,t)}I_L^\rho(x,t)|^{\frac{\eta}{\eta-1}}dxdt\}^{\frac{\eta-1}{\eta}} \\
&\leq C\varepsilon^{\frac{\eta}{3}}.
\end{aligned}
\tag{4.19}
$$

Therefore

$$
\begin{aligned}
&E\int_0^T\int_\Omega[(\frac{c\bar{u}_1^\varepsilon(x,t)}{1+\alpha\bar{I}_L^\varepsilon(x,t)}\bar{I}_L^\varepsilon(x,t)-\frac{cu_1^\rho(x,t)}{1+\alpha I_L^\rho(x,t)}I_L^\rho(x,t))\bar{p}_2^\varepsilon(x,t)-(\frac{cu_1^\varepsilon(x,t)}{1+\alpha I_L^\varepsilon(x,t)}I_L^\varepsilon(x,t) \\
&-\frac{cu_1^\rho(x,t)}{1+\alpha I_L^\rho(x,t)}I_L^\rho(x,t))p_2^\varepsilon(x,t)]dxdt \\
&\leq C\varepsilon^{\frac{\eta}{3}}.
\end{aligned}
\tag{4.20}
$$

In the same way, we can estimate

$$
\begin{aligned}
&E\int_0^T\int_\Omega[(\bar{u}_2^\varepsilon(x,t)\bar{L}^\varepsilon(x,t)-u_2^\rho(x,t)L^\rho(x,t))\bar{p}_4^\varepsilon(x,t)-(u_2^\varepsilon(x,t)L^\varepsilon(x,t)-u_2^\rho(x,t)L^\rho(x,t))p_4^\varepsilon(x,t)]dxdt \\
&\leq C\varepsilon^{\frac{\eta}{3}}.
\end{aligned}
\tag{4.21}
$$

Similarly

$$
E\int_0^T\int_\Omega\frac{1}{2}B_3(u_1(x,t))^2+\frac{1}{2}B_4(u_2(x,t))^2-\frac{1}{2}B_3(\bar{u}_1^\varepsilon(x,t))^2-\frac{1}{2}B_4(\bar{u}_2^\varepsilon(x,t))^2dxdt\leq C\varepsilon^{\frac{\eta}{3}}. \tag{4.22}
$$

Utilizing the definition of the Hamiltonian function and combining Eqs (4.16) and (4.20)–(4.22), the inequality (4.8) can be obtained.

Remark 3. *In practical terms, when an outbreak of listeriosis occurs, it is imperative to promptly implement measures to contain the spread of the disease. Theoretically, if we take the ε between optimal control and near-optimal controls to be larger, then in reality, our control scope will be larger, and it will be easier to find near-optimal controls, thus curbing the spread of the disease. Thus, depending on the extent of the outbreak, we take a different value of ε.*

5. Numerical simulations

In order to confirm the aforementioned findings, the numerical simulation of the solution to the system (2.7) is conducted. Employing Milstein's method as described in the work by [26], the discretization of system (2.7) yields the subsequent equation.

$$
\begin{aligned}
S_{(i,j+1)} &= S_{(i,j)} + [d_1 \frac{S_{(i,j+1)} - 2S_{(i,j)} + S_{(i,j-1)}}{(\Delta x)^2} + \Lambda + \theta_L R_{L(i,j)} \\
&\quad - \mu S_{(i,j)} - \beta_L L_{(i,j)} S_{(i,j)}]\Delta t - \xi_1 S_{(i,j)} \varsigma_i \sqrt{\Delta t} - \frac{1}{2}\xi_1^2 S_{(i,j)}(\varsigma_i^2 - 1)\Delta t, \\
I_{L(i,j+1)} &= I_{L(i,j)} + [d_2 \frac{I_{L(i,j+1)} - 2I_{L(i,j)} + I_{L(i,j-1)}}{(\Delta x)^2} + \beta_L L_{(i,j)} S_{(i,j)} \\
&\quad - (\sigma_L + \delta_L + \mu) I_{L(i,j)} - \frac{cu_1}{1 + \alpha I_{L(i,j)}} I_{L(i,j)}]\Delta t - \xi_1 I_{L(i,j)} \varsigma_i \sqrt{\Delta t} - \frac{1}{2}\xi_1^2 I_{L(i,j)}(\varsigma_i^2 - 1)\Delta t, \\
R_{L(i,j+1)} &= R_{L(i,j)} + [d_3 \frac{R_{L(i,j+1)} - 2R_{L(i,j)} + R_{L(i,j-1)}}{(\Delta x)^2} + \sigma_L I_{L(i,j)} \\
&\quad - (\theta_L + \mu) R_{L(i,j)} + \frac{cu_1}{1 + \alpha I_{L(i,j)}} I_{L(i,j)}]\Delta t - \xi_1 R_{L(i,j)} \varsigma_i \sqrt{\Delta t} - \frac{1}{2}\xi_1^2 R_L(i,j)(\varsigma_i^2 - 1)\Delta t, \\
L_{(i,j+1)} &= L_{(i,j)} + [d_4 \frac{L_{(i,j+1)} - 2L_{(i,j)} + L_{(i,j-1)}}{(\Delta x)^2} + r_L L_{(i,j)}(1 - \frac{L_{(i,j)}}{K_L}) \\
&\quad - (\varepsilon + u_2) L_{(i,j)}]\Delta t - \xi_2 L_{(i,j)} \varsigma_i \sqrt{\Delta t} - \frac{1}{2}\xi_2^2 L_{(i,j)}(\varsigma_i^2 - 1)\Delta t,
\end{aligned}
\tag{5.1}
$$

where ς_i $i = 1, 2, \cdots, n$, are independent Gaussian random variables $N(0, 1)$.

The calculation algorithm is shown as follows:

Algorithm 1

Step 1:

Choose an initial $u_0 = (u_{1,0}, u_{2,0})$, an initial step size S_0,and stopping tolerances T_1 and T_2
initial states $(S_0, I_{L,0}, R_{L,0}, L_0)$
initial adjoints $p_0 = (p_1(0), p_2(0), p_3(0), p_4(0)), q_0 = (q_1(0), q_2(0), q_3(0), q_4(0))$
gradient of J, i.e., $g_0^* = B_3 u_{1,0} + \frac{cu_{1,0} I_L}{1+\alpha I_L}(p_3(x, 0) - p_2(x, 0)), g_0 = B_4 u_{2,0} - Lp_4(x, 0)$
anti-gradient J, i.e., $d_0^* = -g_0^*, d_0 = -g_0$

Step 2:

control, i.e., $u_{k+1} = u_k + s_k d_k$
states $(S_k + 1, I_{L,k+1}, R_{L,k+1}, L_{k+1}) = (S_{u_{k+1}}, I_{L,u_{k+1}}, R_{L,u_{k+1}}, L_{u_{k+1}})$
by solving the discrete form of system (2.7),
adjoints $(p_{k+1}, q_{k+1}) = (p_{S_{k+1}, I_{L,k+1}, R_{L,k+1}, L_{k+1}}, q_{S_{k+1}, I_{L,k+1}, R_{L,k+1}, L_{k+1}})$,
by solving the discrete form of system (4.1),
gradient of J, i.e., $g_{k+1}^* = B_3 u_{1,k+1} + \frac{cu_{1,k+1} I_{L_{k+1}}}{1+\alpha I_{L_{k+1}}}(p_{3_{k+1}} - p_{2_{k+1}}), g_{k+1} = B_4 u_{2,k+1} - L_{k+1} p_{4_{k+1}}$.

Step 3:

Stop if $\|g_{k+1}^*\| < T_1, \|g_{k+1}\| < T_1 \; or \; \|J_{K+1} - J_k\| \leq T_2$.
Compute the conjugate direction ϖ_{k+1} according to one of the updated formulas [27, 28],
$d_{k+1} = -g_{k+1} + \varpi_{k+1} d_k$. Select step size S_{k+1} in terms of some standard options.
Set $k =: k + 1$ and go to Step 1.

The selection of parameter values is as follows:

Table 2. Parameter values of numerical experiments for system (2.7).

Parameter	Value	units	Source of date	Parameter	Value	units	Source of date
Λ	1	people	Assumed	σ_L	0.034	$day^{-1}people^{-1}$	[29]
μ	0.25	day^{-1}	Assumed	β_L	0.085	day^{-1}	Assumed
r_L	0.32	day^{-1}	[29]	K_L	0.08	CFU/cm	[29]
δ_L	0.02	day^{-1}	[4]	θ_L	0.2	day^{-1}	[29]
α	0.04	n.a.	Assumed	c	0.05	n.a.	Assumed
ε	0.2406	day^{-1}	[4]	r	0.0094	mg/L	[30]
ξ_1	0.15	n.a.	Assumed	ξ_2	0.1	n.a.	Assumed

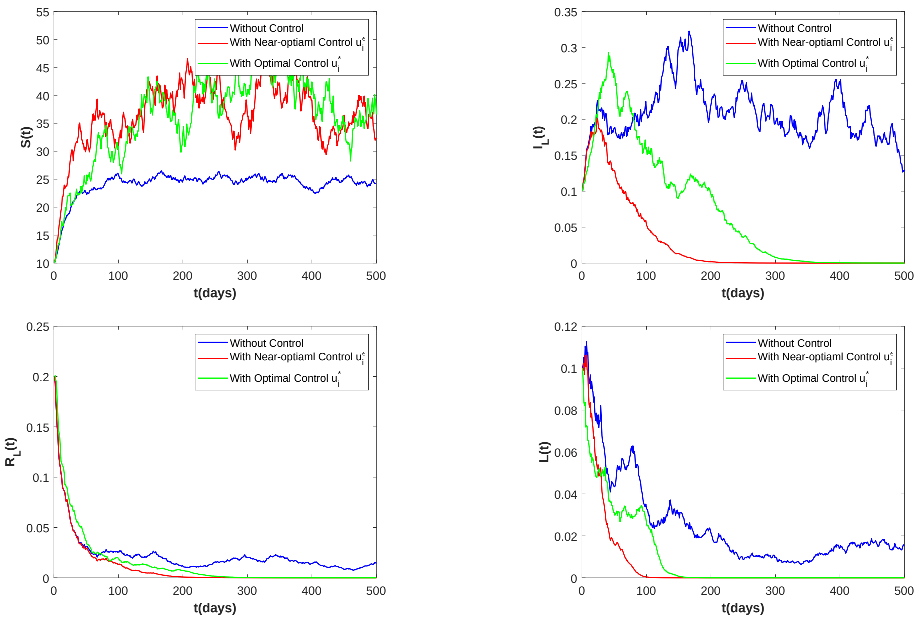

Figure 2. State variable comparison diagram of without control, near-optimal controls u_i^{ε} and optimal control $u_i^*(i = 1, 2)$.

In Figure 2, we show the contrast plots of variables not under control, versus near-optimal controls and optimal control, with curves of different colors representing changes in different Spaces. The values of the diffusion coefficient are $d_1 = 0.02, d_2 = 0.024, d_3 = 0.025, d_4 = 0.036$. Before treatment of infected persons and elimination of environmental bacteria, the number of susceptible persons increased first and then leveled off. The number of infected people remained stable; because the number of infected people has remained roughly the same, the number of recovered people is getting smaller and smaller. Due to the limited survival time of bacteria and the competition between bacteria, the amount of contaminated environment first decreases and then remains stable. After the infected person is treated and the bacteria in the environment are removed, it can be seen from the observation that, the number of susceptible people is higher than the number of people without control means, because after the infected person is treated and the bacteria in the environment are removed, the number of sick people is reduced and the number of susceptible people is increased. Second, for infected people, both kinds of control make the number of susceptible people increase first and then decrease, and the number of sick people decreases significantly around 60 days, and the number of sick people gradually approaches 0 after 180 days of near optimal control, and gradually approaches 0 after 300 days of optimal control. Then, for the recovery, because the number of infected people is decreasing, the number of recovery people is also decreasing, and gradually approaches 0 after 200 days; finally, removing bacteria from the environment can reduce the spread of bacteria and thus reduce the polluted environment. As can be clearly seen from the picture above, treating infected people and removing germs in the environment can effectively inhibit the spread and spread of the disease; Moreover, we find that the effect of near-optimal controls at the same time is better than that of optimal control.

In Figure 3, we show the paths of $S(x,t), I_L(x,t), R_L(x,t), L(x,t)$ for the system (2.7). Add controls to the system (2.7), i.e., treatment of the patient $u_1(x,t)$ and removal of germs from the environment $u_2(x,t)$. By observing Figure 3, it can be found that after increasing control, the number of susceptible people first increases and then becomes stable; control $u_1(x,t)$ is to treat infected people, and the number of infected people decreases after treatment; as the number of infected persons decreases, so does the number of recovered persons; and controlling $u_2(x,t)$ is removing bacteria from the environment, so the bacteria in the environment are constantly decreasing, so the amount of polluted environment is also decreasing.

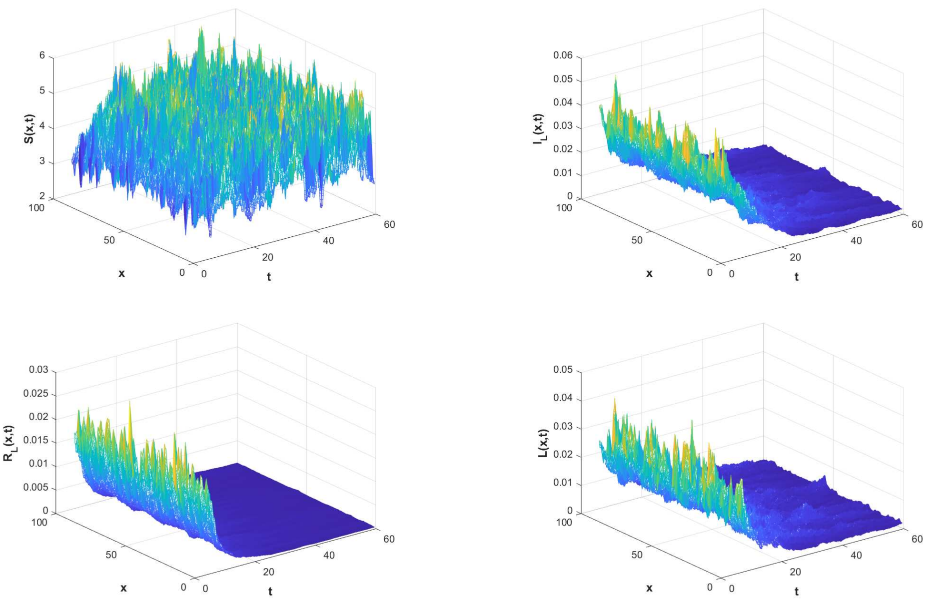

Figure 3. The path of S, I_L, R_L, L for system (2.7).

The discrepancies in optimal control and near-optimal controls are illustrated in Figure 4. Optimal control of $u_1(x,t)$ and $u_2(x,t)$ at $\varepsilon = 0.5$ indicates that the infected person and the virus in the environment have different optimal processing rates at different times. For the control variable $u_1(x,t)$, both the optimal control and near-optimal control intensity are large when a disease breaks out, and the control intensity decreases when the number of sick people begins to decrease. For the control variable $u_2(x,t)$, the optimal control of bacteria in the environment first increases and then gradually levels off. Due to the wide distribution range of bacteria in the environment, it is impossible to measure accurately, and the near-optimal control is always smooth control strategy. It is not difficult to find that the approximate optimal is easier to realize than the optimal, and also more in line with the real conditions. The discrepancy in the optimal and near-optimal controls of $u_1(x,t)$ is below 1, while the discrepancy in $u_2(x,t)$ is below 0.018.

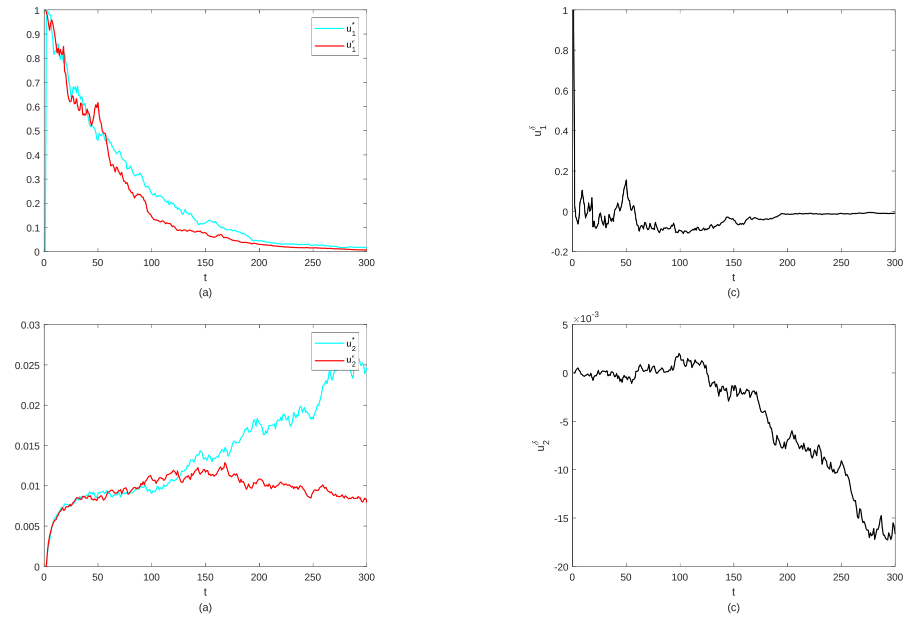

Figure 4. If $\varepsilon = 0.5$, u_i^* and u_i^ε are the optimal control and near-optimal controls, respectively; u_i^δ indicates the error (where $u_i^\delta = u_i^\varepsilon - u_i^*)(i = 1, 2)$.

6. Conclusions

In this paper, we develop a listeriosis model (2.7) that takes into account random noise and spatial diffusion by treating infected people and removing bacteria from the environment as control strategies. Under the goal of controlling the lowest medical cost, we establish the objective function (2.8). Next, Theorem 3.1 proves the existence and uniqueness of the global positive solution of system (2.7). In order to investigate the sufficient and necessary conditions for achieving near-optimal controls, we also provide the adjoint equations and prior eatimates in Section 3. The Lemmas 3.3 and 3.4 lay the foundation for proving sufficient conditions for near-optimality. Lemmas 3.5 and 3.6 make theoretical preparations for proving the conditions necessary for near-optimality. In Section 4, we establish both the sufficient and necessary conditions for near-optimal, as detailed in Theorems 4.1 and 4.2. When $\varepsilon = 0$, the conditions outlined in Theorems 4.1 and 4.2 are both sufficient and necessary for achieving optimal control. Ultimately, through numerical simulation, the transmission of listeriosis can be effectively inhibited by treating infected persons and removing bacteria in the environment at the lowest medical cost.

In the future work, the following two questions are worth addressing:

- The best survival temperature of listeria is 30–37 degrees Celsius, as too high and too low temperature will make Listeria perish [8]. In the previous models, the influence of temperature on bacteria was not considered. In the system (2.7), we only considered white noise, and we could also consider introducing color noise.
- Whether event triggers can be used to control the optimal control of nonlinear stochastic systems [31] to determine how to control the spread of disease.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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Appendix

A1. Proof of Lemma 3.1

Proof. Let

$$Z(t)=\int_{\Omega}[(S(x,t),I_L(x,t),R_L(x,t),L(x,t)]dx.$$

It can be obtained by system (2.7)

$$\begin{aligned}
\frac{\partial Z(x,t)}{\partial t} &= \int_{\Omega}[\frac{\partial S(x,t)}{\partial t}+\frac{\partial I_L(x,t)}{\partial t}+\frac{\partial R_L(x,t)}{\partial t}+\frac{\partial L(x,t)}{\partial t}]dx\\
&= \int_{\Omega}[d_1\Delta S(x,t)+\Lambda+\theta_L R_L(x,t)-(\mu+\beta_L L(x,t))S(x,t)-\xi_1(t)S(x,t)\dot{B}_1(t)+d_2\Delta I_L(x,t)\\
&+\beta_L L(x,t)S(x,t)-(\sigma_L+\delta_L+\mu)I_L(x,t)-\frac{cu_1(x,t)I_L(x,t)}{1+\alpha I_L(x,t)}-\xi_1(t)I_L(x,t)\dot{B}_1(t)\\
&+d_3\Delta R_L(x,t)+\sigma_L I_L(x,t)-(\theta_L+\mu)R_L(x,t)+\frac{cu_1(x,t)I_L(x,t)}{1+\alpha I_L(x,t)}-\xi_1(t)R_L(x,t)\dot{B}_1(t)\\
&+d_4\Delta L(x,t)+r_L L(x,t)(1-\frac{L(x,t)}{K_L})-(\varepsilon+u_2(x,t))L(x,t)-\xi_2(t)L(x,t)\dot{B}_2(t)]dx.
\end{aligned}$$

Next, we continue our process,

$$
\begin{aligned}
\frac{\partial Z(x,t)}{\partial t} &\le d_1\int_{\partial\Omega}(\frac{\partial S(x,t)}{\partial \nu})dx + d_2\int_{\partial\Omega}(\frac{\partial I_L(x,t)}{\partial \nu})dx + d_3\int_{\partial\Omega}(\frac{\partial R_L(x,t)}{\partial \nu})dx\\
&+ d_4\int_{\partial\Omega}(\frac{\partial L(x,t)}{\partial \nu})dx + \int_{\Omega}[\Lambda - \mu S(x,t) - (\delta_L+\mu)I_L(x,t)\\
&- \mu R_L(x,t) + r_L L(x,t)(1-\frac{L(x,t)}{K_L}) - (\varepsilon + u_2(x,t))L(x,t) - \xi_1(t)S(x,t)\dot{B}_1(t)\\
&- \xi_1(t)I_L(x,t)\dot{B}_1(t) - \xi_1(t)R_L(x,t)\dot{B}_1(t) - \xi_2(t)L(x,t)\dot{B}_2(t)]dx\\
&= d_1\int_{\partial\Omega}(\frac{\partial S(x,t)}{\partial \nu})dx + d_2\int_{\partial\Omega}(\frac{\partial I_L(x,t)}{\partial \nu})dx + d_3\int_{\partial\Omega}(\frac{\partial R_L(x,t)}{\partial \nu})dx\\
&+ d_4\int_{\partial\Omega}(\frac{\partial L(x,t)}{\partial \nu})dx + \int_{\Omega}[\Lambda + r_L L(x,t)(1-\frac{L(x,t)}{K_L}) - \mu S(x,t) - (\delta_L+\mu)I_L(x,t)\\
&- \mu R_L(x,t) - (\varepsilon + u_2(x,t))L(x,t)]dx - \int_{\Omega}[\xi_1(t)S(x,t)\dot{B}_1(t)\\
&+ \xi_1(t)I_L(x,t)\dot{B}_1(t) + \xi_1(t)R_L(x,t)\dot{B}_1(t) + \xi_2(t)S(x,t)\dot{B}_2(t)]dx\\
&\le [\Lambda + r_L L(x,t)(1-\frac{L(x,t)}{K_L})]|\Omega| - BZ(t) - \int_{\Omega}[\xi_1(t)S(x,t)\dot{B}_1(t)\\
&+ \xi_1(t)I_L(x,t)\dot{B}_1(t) + \xi_1(t)R_L(x,t)\dot{B}_1(t) + \xi_2(t)L(x,t)\dot{B}_2(t)]dx,
\end{aligned}
$$

where $|\Omega|$ denotes the volume of Ω , $B = \min\{\mu, \varepsilon\}$. The solution to the given stochastic differential equation is represented by the variable $X(t)$.

$$
\begin{cases}
dX(t) = [\Lambda + r_L L(x,s)(1-\frac{L(x,s)}{K_L} - BX(t))]dt - \int_{\Omega}\xi_1(s)S(x,s)dxdB_1(s) - \int_{\Omega}\xi_1(s)I_L(x,s)dxdB_1(s)\\
\qquad - \int_{\Omega}\xi_1(s)R_L(x,s)dxdB_1(s) - \int_{\Omega}\xi_2(s)L(x,s)dxdB_2(s),\\
X(0) = Z(0).
\end{cases}
\tag{A1}
$$

We can get the solution of Eq (A1) as follows

$$
X(T) = \frac{\Lambda + r_L m_4(1-\frac{L(x,t)}{K_L})}{B} + (X(0) - \frac{\Lambda + r_L L(x,t)(1-\frac{L(x,t)}{K_L})}{B})e^{-Bt} + F(t), \tag{A2}
$$

where

$$
\begin{aligned}
F(t) = &-\int_0 te^{-B(t-s)}\int_{\Omega}\xi_1(s)S(x,s)dxdB_1(s) - \int_0 te^{-B(t-s)}\int_{\Omega}\xi_1(s)I_L(x,s)dxdB_1(s)\\
&- \int_0 te^{-B(t-s)}\int_{\Omega}\xi_1(s)R_L(x,s)dxdB_1(s) - \int_0 te^{-B(t-s)}\int_{\Omega}\xi_2(s)L(x,s)dxdB_2(s).
\end{aligned}
\tag{A3}
$$

The function $F(t)$ is a continuous local martingale with the initial value at $L(x,0) = 0$ almost surely. As per the stochastic comparison theorem, it can be inferred that the process $Z(t) \le X(t)$. Let us proceed to define

$$
X(t) = X(0) + G(t) - U(t) + F(t). \tag{A4}
$$

Among them $G(t) = \frac{\Lambda + r_L L(x,t)(1-\frac{L(x,t)}{K_L})}{B}(1 - e^{-Bt})$ and $U(t) = X(0)(1 - e^{-Bt})$. Clearly, for all values of $t \geq 0$, the functions $G(t)$ and $U(t)$ exhibit continuity, adaptability, and a monotonically increasing behavior. Additionally, at the initial time $t = 0$, it holds that $G(0) = U(0) = 0$. Apply the non-negative semimartingale convergence theorem [32], $\lim_{t\to\infty} X(t) < \infty$ can be obtained, a.s. Therefore, $\limsup_{t\to\infty} Z(t) < \infty$, a.s. The evidence has been finalized.

A2. Proof of Lemma 3.4

Proof. Initially, we proceed by combining the left and right sides of the initial equation in the adjoint Eq (3.8) over the interval from t to T, resulting in the following expression.

$$p_1(x,T) - p_1(x,t) = -\int_t^T g_1(Z(x,s), u_1(x,s), u_2(x,s), p(x,s), q(x,s))ds + \int_t^T q_1(x,s)dB_1(s), \quad \text{(A5)}$$

the above formula is equivalent to

$$p_1(x,t) + \int_t^T q_1(x,s)dB_1(s) = p_1(x,T) + \int_t^T g_1(Z(x,s), u_1(x,s), u_2(x,s), p(x,s), q(x,s))ds. \quad \text{(A6)}$$

Then square it, take the expectation, and integrate x on both sides on Ω,we can get

$$\begin{aligned}
&E|p_1(x,t)|^2 + E\int_t^T |q_1(x,s)|^2 ds \\
&\leq CE|p_1(x,T)|^2 + C(T-t)E\int_t^T |g_1(Z(x,s), u_1(x,s), u_2(x,s), p(x,s), q(x,s)|^2 ds \\
&\leq CE|p_1(x,T)|^2 + C(T-t)\sum_{i=1}^{2} E\int_t^T |p_i(x,s)|^2 ds + C(T-t)E\int_t^T |q_1(x,s)|^2 ds.
\end{aligned} \quad \text{(A7)}$$

In the same way

$$\begin{aligned}
&E|p_2(x,t)|^2 + E\int_t^T |q_2(x,s)|^2 ds \\
&\leq CE|p_2(x,T)|^2 + C(T-t)E\int_t^T |g_2(Z(x,s), u_1(x,s), u_2(x,s), p(x,s), q(x,s)|^2 ds \\
&\leq CE|p_2(x,T)|^2 + C(T-t)\sum_{i=2}^{3} E\int_t^T |p_i(x,s)|^2 ds + C(T-t)E\int_t^T |q_2(x,s)|^2 ds,
\end{aligned} \quad \text{(A8)}$$

$$\begin{aligned}
&E|p_3(x,t)|^2 + E\int_t^T |q_3(x,s)|^2 ds \\
&\leq CE|p_3(x,T)|^2 + C(T-t)E\int_t^T |g_3(Z(x,s), u_1(x,s), u_2(x,s), p(x,s), q(x,s)|^2 ds \\
&\leq CE|p_3(x,T)|^2 + C(T-t)E\int_t^T |p_1(x,s)|^2 ds \\
&+ C(T-t)E\int_t^T |p_3(x,s)|^2 ds + C(T-t)E\int_t^T |q_3(x,s)|^2 ds,
\end{aligned} \quad \text{(A9)}$$

and

$$\begin{aligned}
&E|p_4(x,t)|^2 + E\int_t^T |q_4(x,s)|^2 ds \\
&\leq CE|p_4(x,T)|^2 + C(T-t)E\int_t^T |g_4(Z(x,s),u_1(x,s),u_2(x,s),p(x,s),q(x,s)|^2 ds \\
&\leq CE|p_4(x,T)|^2 + C(T-t)E\int_t^T |p_1(x,s)|^2 ds + C(T-t)E\int_t^T |p_2(x,s)|^2 ds \\
&+ C(T-t)E\int_t^T |p_4(x,s)|^2 ds + C(T-t)E\int_t^T |q_4(x,s)|^2 ds.
\end{aligned} \tag{A10}$$

Combine Eqs (A7)–(A10), we can obtain

$$\begin{aligned}
&\sum_{i=1}^4 E|p_i(x,t)|^2 + \sum_{i=1}^4 E\int_t^T |q_i(x,t)|^2 ds \\
&\leq C\sum_{i=1}^4 E|p_i(x,T)|^2 + C(T-t)\sum_{i=1}^4 \int_t^T |p_i(x,s)|^2 ds + C(T-t)\sum_{i=1}^4 E\int_t^T |q_i(x,s)|^2 ds.
\end{aligned} \tag{A11}$$

$$\begin{aligned}
&\sum_{i=1}^4 E|p_i(x,t)|^2 + \frac{1}{2}\sum_{i=1}^4 E\int_t^T |q_i(x,s)|^2 ds, \\
&\leq C\sum_{i=1}^4 E|p_i(x,T)|^2 + C(T-t)\sum_{i=1}^4 E\int_t^T |p_i(x,s)|^2 ds,
\end{aligned} \tag{A12}$$

where $t \in [T-\varepsilon, T]$, and $\varepsilon = \frac{1}{2C}$. Apply the Gronwall's inequality and integrate them Ω, from Eq (A12), the following formula can be obtained

$$\sum_{i=1}^4 E\sup_{0\leq t\leq T}\int_\Omega |p_i(x,t)|^2 dx \leq C \ \ and \ \ \sum_{i=1}^4 E\int_0^T\int_\Omega |q_i(x,s)|^2 dxds \leq C. \tag{A13}$$

For any value of $t \in [T-2\varepsilon, T]$, the procedure outlined in Eqs (A7)–(A9) should be repeated. To achieve the Eq (A12) for all values of t within the interval $[0,T]$, the aforementioned procedures should be reiterated. Furthermore, Eq (A6) can be expressed as follows:

$$\begin{aligned}
p_1(x,t) = p_1(x,T) &+ \int_t^T g_1(Z(x,s),u_1(x,s),u_2(x,s),p(x,s),q(x,s))ds \\
&- \int_0^T q_1(x,s)dB_1(s) + \int_0^t q_1(x,s)dB_1(s).
\end{aligned} \tag{A14}$$

Using Eq (A14), we have

$$\begin{aligned}
&|p_1(x,t)|^2 \\
&\leq C[|p_1(x,T)|^2 + \int_0^T (\sum_{i=1}^2 |p_i(x,s)|^2 + |q_1(x,s)|^2)ds + (\int_0^T q_1(x,s)dB_1(s))^2 + (\int_0^t q_1(x,s)dB_1(s))^2],
\end{aligned} \tag{A15}$$

$$
\begin{aligned}
&|p_2(x,t)|^2 \\
&\le C[|p_2(x,T)|^2 + \int_0^T (\sum_{i=2}^{3} |p_i(x,s)|^2 + |q_2(x,s)|^2) ds + (\int_0^T q_2(x,s) dB_1(s))^2 + (\int_0^t q_2(x,s) dB_1(s))^2],
\end{aligned} \tag{A16}
$$

$$
\begin{aligned}
&|p_3(x,t)|^2 \\
&\le C[|p_3(x,T)|^2 + \int_0^T (|p_1(x,s)|^2 + |p_3(x,s)|^2 + |q_3(x,s)|^2) ds + (\int_0^T q_3(x,s) dB_1(s))^2 \\
&+ (\int_0^t q_3(x,s) dB_1(s))^2],
\end{aligned} \tag{A17}
$$

and

$$
\begin{aligned}
&|p_4(x,t)|^2 \\
&\le C[|p_4(x,T)|^2 + \int_0^T (|p_1(x,s)|^2 + |p_2(x,s)|^2 + |p_4(x,s)|^2 + |q_4(x,s)|^2) ds + (\int_0^T q_4(x,s) dB_2(s))^2 \\
&+ (\int_0^t q_4(x,s) dB_2(s))^2].
\end{aligned} \tag{A18}
$$

Combining Eqs (A15)–(A18) gives the following equation

$$
\begin{aligned}
&\sum_{i=1}^{4} |p_i(x,t)|^2 \\
&\le C[\sum_{i=1}^{4} |p_i(x,T)|^2 + \int_0^T (\sum_{i=1}^{4} |p_i(x,s)|^2 + \sum_{i=1}^{4} |q_i(x,s)|^2) ds + \sum_{i=1}^{4} (\int_0^T q_i(x,s) dB(s))^2 \\
&+ \sum_{i=1}^{4} (\int_0^t q_i(x,s) dB(s))^2].
\end{aligned} \tag{A19}
$$

Initially, compute the expectations of both sides of Eq (A19), followed by the application of the Burkhold-Davis-Gondy inequality, resulting in the subsequent equation.

$$
\begin{aligned}
&\sum_{i=1}^{4} E \sup_{0\le t\le T} |p_i(x,t)|^2 \\
&\le C[\sum_{i=1}^{4} E|p_i(x,T)|^2 + \sum_{i=1}^{4} E\{\int_0^T \sup_{0\le s\le T} |p_i(x,s)|^2\} ds \\
&+ \sum_{i=1}^{4} E\{\int_0^T \sup_{0\le s\le T} |q_i(x,s)|^2\} ds].
\end{aligned} \tag{A20}
$$

By utilizing Gronwall's inequality and performing integration across the domain Ω, we obtain the outcome as expressed in Eq (3.11). This establishes the completion of the proof.

A3. Proof of Lemma 3.5

Proof. If $\eta \geq 1$. For any $r > 0$, an estimate of $|S(x,t) - \bar{S}(x,t)|^{2\eta}$ can be obtained as follows

$$
\begin{aligned}
&E \sup_{0\leq t\leq r} \int_\Omega |S(x,t) - \bar{S}(x,t)|^{2\eta} dx \\
&\leq CE \int_0^r \int_\Omega [(d_1\Delta - \mu)^{2\eta} |S(x,t) - \bar{S}(x,t)|^{2\eta} + \beta_L^{2\eta} |S(x,t)L(x,t) - \bar{S}(x,t)\bar{L}(x,t)|^{2\eta} \\
&+ \xi_1^{2\eta} |S(x,t) - \bar{S}(x,t)|^{2\eta} + \theta_L^{2\eta} |R_L(x,t) - \bar{R}_L(x,t)|] dxdt \\
&\leq CE[\int_0^r \int_\Omega (|S(x,t) - \bar{S}(x,t)|^{2\eta} + |R_L(x,t) - \bar{R}_L(x,t)|^{2\eta} + |L(x,t) - \bar{L}(x,t)|^{2\eta}) dxdt].
\end{aligned}
\tag{A21}
$$

Let $\bar{q} = \frac{1}{k\eta} > 1$,$\bar{p} > 1$ satisfy that $\frac{1}{\bar{p}} + \frac{1}{\bar{q}} = 1$, then we can obtain

$$
\begin{aligned}
&E \sup_{0\leq t\leq r} \int_\Omega |I_L(x,t) - \bar{I}_L(x,t)|^{2\eta} dx \\
&\leq CE \int_0^r \int_\Omega [(d_2\Delta - \sigma_L - \delta_L - \mu)^{2\eta} |I_L(x,t) - \bar{I}_L(x,t)|^{2\eta} + c^{2\eta} |\frac{u_1 I_L(x,t)}{1 + \alpha I_L(x,t)} - \frac{u_1 \bar{I}_L(x,t)}{1 + \alpha \bar{I}_L(x,t)}|^{2\eta} \\
&+ \beta_L^{2\eta} |S(x,t)L(x,t) - \bar{S}(x,t)\bar{L}(x,t)|^{2\eta} + \xi_1^{2\eta} |I_L(x,t) - \bar{I}_L(x,t)|^{2\eta}] dxdt \\
&\leq CE \int_0^r \int_\Omega (|S(x,t) - \bar{S}(x,t)|^{2\eta} + |I_L(x,t) - \bar{I}_L(x,t)|^{2\eta} + |L(x,t) - \bar{L}(x,t)|^{2\eta}) dxdt \\
&+ C[E \int_0^r 1^{\bar{p}}]^{\frac{1}{\bar{p}}} [E \int_0^r m_{u_1(t)\neq \bar{u}_1(t)}(t) dt]^{\frac{1}{\bar{q}}} \\
&\leq C[E \int_0^r \int_\Omega (|S(x,t) - \bar{S}(x,t)|^{2\eta} + |I_L(x,t) - \bar{I}_L(x,t)|^{2\eta} + |L(x,t) - \bar{L}(x,t)|^{2\eta}) dxdt \\
&+ d(u_1(x,t), \bar{u}_1(x,t))^{k\eta}].
\end{aligned}
\tag{A22}
$$

We can get this by doing something similar

$$
\begin{aligned}
&E \sup_{0\leq t\leq r} \int_\Omega |R_L(x,t) - \bar{R}_L(x,t)|^{2\eta} dx \\
&\leq CE \int_0^r \int_\Omega [(d_3\Delta - \theta_L - \mu)^{2\eta} |R_L(x,t) - \bar{R}_L(x,t)|^{2\eta} + \sigma_L^{2\eta} |I_L(x,t) - \bar{I}_L(x,t)|^{2\eta} \\
&+ c^{2\eta} |\frac{u_1 I_L(x,t)}{1 + \alpha I_L(x,t)} - \frac{u_1 \bar{I}_L(x,t)}{1 + \alpha \bar{I}_L(x,t)}|^{2\eta} + \xi_1^{2\eta} |R_L(x,t) - \bar{R}_L(x,t)|^{2\eta}] dxdt \\
&\leq CE \int_0^r \int_\Omega (|I_L(x,t) - \bar{I}_L(x,t)|^{2\eta} + |R_L(x,t) - \bar{R}_L(x,t)|^{2\eta}) dxdt + C[E \int_0^r 1^{\bar{p}} dt]^{\frac{1}{\bar{p}}} [E \int_0^r w_{u_1(t)\neq \bar{u}_1(t)}(t) dt]^{\frac{1}{\bar{q}}} \\
&\leq C[E \int_0^r \int_\Omega (|I_L(x,t) - \bar{I}_L(x,t)|^{2\eta} + |R_L(x,t) - \bar{R}_L(x,t)|^{2\eta}) dxdt + d(u_1(x,t), \bar{u}_1(x,t))^{k\eta}],
\end{aligned}
\tag{A23}
$$

$$
\begin{aligned}
&E\sup_{0\le t\le r}\int_\Omega |L(x,t)-\bar{L}(x,t)|^{2\eta}dx\\
&\le CE\int_0^r\int_\Omega[(d_4\Delta-\varepsilon)^{2\eta}|L(x,t)-\bar{L}(x,t)|^{2\eta}\\
&+r_L^{2\eta}|L(x,t)-\bar{L}(x,t)|^{2\eta}+\xi_L^{2\eta}|L(x,t)-\bar{L}(x,t)|^{2\eta}+|u_2(x,t)-\bar{u}_2(x,t)|^{2\eta}]dxdt\\
&\le CE\int_0^r\int_\Omega(|L(x,t)-\bar{L}(x,t)|^{2\eta})dxdt+C[E\int_0^r 1^{\bar{p}}dt]^{\frac{1}{\bar{p}}}[E\int_0^r w_{u_2(t)\neq\bar{u}_2(t)}(t)dt]^{\frac{1}{\bar{q}}}\\
&\le C[E\int_0^r\int_\Omega(|L(x,t)-\bar{L}(x,t)|^{2\eta})dxdt+d(u_2(x,t),\bar{u}_2(x,t))^{k\eta}].
\end{aligned} \tag{A24}
$$

From Eq (A21) to (A24), the following formula is true

$$
\begin{aligned}
&E\{\sup_{0\le t\le r}\int_\Omega(|S(x,t)-\bar{S}(x,t)|^{2\eta}+|I_L(x,t)-\bar{I}_L(x,t)|^{2\eta}+|R_L(x,t)-\bar{R}_L(x,t)|^{2\eta}+|L(x,t)-\bar{L}(x,t)|^{2\eta})\}dx\\
&\le C[\int_0^r\int_\Omega E(|S(x,t)-\bar{S}(x,t)|^{2\eta}+|I_L(x,t)-\bar{I}_L(x,t)|^{2\eta}+|R_L(x,t)-\bar{R}_L(x,t)|^{2\eta}\\
&+|L(x,t)-\bar{L}(x,t)|^{2\eta})dxdt+d(u_1(x,t),\bar{u}_1(x,t))^{k\eta}+d(u_2(x,t),\bar{u}_2(x,t))^{k\eta}].
\end{aligned} \tag{A25}
$$

Subsequently, we examine the scenario where $0\le\eta\le 1(k\eta<1)$. By applying Cauchy-Schwartz's inequality, we obtain the following result

$$
\begin{aligned}
&E\{\sup_{0\le t\le r}\int_\Omega(|S(x,t)-\bar{S}(x,t)|^{2\eta}+|I_L(x,t)-\bar{I}_L(x,t)|^{2\eta}+|R_L(x,t)-\bar{R}_L(x,t)|^{2\eta}+|L(x,t)-\bar{L}(x,t)|^{2\eta})\}dx\\
&\le C\int_0^r\int_\Omega E(|S(x,t)-\bar{S}(x,t)|^{2\eta}+|I_L(x,t)-\bar{I}_L(x,t)|^{2\eta}+|R_L(x,t)-\bar{R}_L(x,t)|^{2\eta}\\
&+|L(x,t)-\bar{L}(x,t)|^{2\eta})dxdt\le C[d(u_1(x,t),\bar{u}_1(x,t))^{k\eta}+d(u_2(x,t),\bar{u}_2(x,t))^{k\eta}].
\end{aligned} \tag{A26}
$$

The lemma's proof is concluded by utilizing the aforementioned inequalities and the Gronwall inequality.

A4. Proof of Lemma 3.6

Proof. Let $\bar{p}_i(x,t)=p_i(x,t)-\tilde{p}_i(x,t), \bar{q}_i(x,t)=q_i(x,t)-\tilde{q}_i(x,t), (i=1,2,3,4)$, then according to the adjoint Eq (3.8), we have

$$
\begin{cases}
d\bar{p}_1(x,t)=-[(d_1\Delta-\mu-\beta_L L)\bar{p}_1(x,t)+\beta_L L\bar{p}_2(x,t)-\xi_1\bar{q}_1(x,t)+\hat{f}_1]dt+\bar{q}_1(x,t)dB_1,\\
d\bar{p}_2(x,t)=-\{[d_2\Delta-(\sigma_L+\delta_L+\mu)-\dfrac{cu_1(x,t)}{(1+\alpha I_L)^2}]\bar{p}_2(x,t)+[\sigma_L+\dfrac{cu_1(x,t)}{(1+\alpha I_L)^2}]\bar{p}_3(x,t)\\
\qquad -\xi_1\bar{q}_2(x,t)+\hat{f}_2\}dt+\bar{q}_2(x,t)dB_1,\\
d\bar{p}_3(x,t)=-\{\theta_L\bar{p}_1(x,t)+[d_3\Delta-(\theta_L+\mu)]\bar{p}_3(x,t)-\xi_1\bar{q}_3(x,t)+\hat{f}_3\}dt+\bar{q}_3(x,t)dB_1,\\
d\bar{p}_4(x,t)=-\{-\beta_L S\bar{p}_1(x,t)+\beta_L S\bar{p}_2(x,t)+[d_4\Delta+r_L-\dfrac{2r_L L}{k_L}-(\varepsilon+u_2(x,t))]\bar{p}_4(x,t)-\xi_2\bar{q}_4(x,t)\\
\qquad +\hat{f}_4\}dt+\bar{q}_4(x,t)dB_2,
\end{cases} \tag{A27}
$$

where

$$\begin{cases}\hat{f}_1 = \beta_L(\bar{L}(x,t) - L(x,t))(\tilde{p}_1(x,t) - \tilde{p}_2(x,t)),\\ \hat{f}_2 = \sigma_L(\tilde{p}_2(x,t) - \tilde{p}_3(x,t)) + c(\frac{\bar{u}_1(x,t)}{(1+\alpha\bar{I}_L)^2} - \frac{u_1(x,t)}{(1+\alpha I_L)^2})(\tilde{p}_2(x,t) - \tilde{p}_3(x,t)),\\ \hat{f}_3 = -\theta_L(\tilde{p}_1(x,t) - \tilde{p}_2(x,t)),\\ \hat{f}_4 = \beta_L(\bar{S}(x,t) - S(x,t))(\tilde{p}_1(x,t) - \tilde{p}_2(x,t)) + \tilde{p}_4(x,t)(\bar{u}_2(x,t) - u_2(x,t)).\end{cases} \tag{A28}$$

Let us consider the function $\phi(x,t) = (\phi_1(x,t), \phi_2(x,t), \phi_3(x,t), \phi_4(x,t))^T$ as the solution to the given linear stochastic differential equation.

$$\begin{cases} d\phi_1(x,t) = [(d_1\Delta - \mu - \beta_L L)\phi_1(x,t) + \theta_L\phi_3(x,t) - \beta_L S\phi_4(x,t) + |\bar{p}_1(x,t)|^{\eta-1}sgn(\bar{p}_1(x,t))]dt\\ \qquad + [-\xi_1\phi_1(x,t) + |\bar{q}_1(x,t)|^{\eta-1}sgn(\bar{q}_1(x,t))]dB_1,\\ d\phi_2(x,t) = \{\beta_L L\phi_1(x,t) + [d_2\Delta - (\sigma_L + \delta_L + \mu) - \frac{cu_1(x,t)}{(1+\alpha I_L)^2}]\phi_2(x,t) + \beta_L S\phi_4(x,t)\\ \qquad + |\bar{p}_2(x,t)|^{\eta-1}sgn(\bar{p}_2(x,t))\}dt + [-\xi_1\phi_2(x,t) + |\bar{q}_2(x,t)|^{\eta-1}sgn(\bar{q}_2(x,t))]dB_1,\\ d\phi_3(x,t) = [(\sigma_L + \frac{cu_1(x,t)}{(1+\alpha I_L)^2})\phi_2(x,t) + [d_3\Delta - (\theta_L + \mu)]\phi_3(x,t) + |\bar{p}_3(x,t)|^{\eta-1}sgn(\bar{p}_3(x,t))]dt\\ \qquad + [-\xi_1\phi_3(x,t) + |\bar{q}_3(x,t)|^{\eta-1}sgn(\bar{q}_3(x,t))]dB_1,\\ d\phi_4(x,t) = \{[d_4\Delta + r_L - 2\frac{r_L L}{K_L} - (\varepsilon + u_2(x,t))]\phi_4(x,t)\\ \qquad + |\bar{p}_4(x,t)|^{\eta-1}sgn(\bar{p}_4(x,t))\}dt + [-\xi_2\phi_4(x,t) + |\bar{q}_4(x,t)|^{\eta-1}sgn(\bar{q}_4(x,t))]dB_2.\end{cases} \tag{A29}$$

The function sgn(·) is a symbolic function. Drawing from the premise and the lemma denoted as Lemma 3.5, it can be deduced that the Eq (A29) possesses a unique solution.

$$\begin{aligned} &E\int_0^T\int_\Omega(\| |\bar{p}_1(x,t)|^{\eta-1}sgn(\bar{p}_1(x,t)))|^2 + \| |\bar{p}_2(x,t)|^{\eta-1}sgn(\bar{p}_2(x,t))|^2\\ &+ \| |\bar{p}_3(x,t)|^{\eta-1}sgn(\bar{p}_3(x,t))|^2 + \| |\bar{p}_4(x,t)|^{\eta-1}sgn(\bar{p}_4(x,t))|^2 + \| |\bar{q}_1(x,t)|^{\eta-1}sgn(\bar{q}_1(x,t))|^2\\ &+ \| |\bar{q}_2(x,t)|^{\eta-1}sgn(\bar{q}_2(x,t))|^2 + \| |\bar{q}_3(x,t)|^{\eta-1}sgn(\bar{q}_3(x,t))|^2 + \| |\bar{q}_4(x,t)|^{\eta-1}sgn(\bar{q}_4(x,t))|^2 < +\infty.\end{aligned} \tag{A30}$$

Because $1 < \eta < 2$, there exist $\eta_1 > 2$ *and* $\frac{1}{\eta} + \frac{1}{\eta_1} = 1$. Using Cauchy-Schwartz's inequality

$$\begin{aligned}\sum_{i=1}^4 E\sup_{0\le t\le T}|\phi_i(x,t)|^{\eta_1} \le\ & E\{\int_0^T\int_\Omega(|\bar{p}_1(x,t)|^{(\eta-1)\eta_1} + |\bar{p}_2(x,t)|^{(\eta-1)\eta_1} + |\bar{p}_3(x,t)|^{(\eta-1)\eta_1} + |\bar{p}_4(x,t)|^{(\eta-1)\eta_1}\\ &+ |\bar{q}_1(x,t)|^{(\eta-1)\eta_1} + |\bar{q}_2(x,t)|^{(\eta-1)\eta_1} + |\bar{q}_3(x,t)|^{(\eta-1)\eta_1} + |\bar{q}_4(x,t)|^{(\eta-1)\eta_1})dxdt\}\\ =\ & E\{\int_0^T\int_\Omega(|\bar{p}_1(x,t)|^{\eta} + |\bar{p}_2(x,t)|^{\eta} + |\bar{p}_3(x,t)|^{\eta} + |\bar{p}_4(x,t)|^{\eta} + |\bar{q}_1(x,t)|^{\eta}\\ &+ |\bar{q}_2(x,t)|^{\eta} + |\bar{q}_3(x,t)|^{\eta} + |\bar{q}_4(x,t)|^{\eta})dxdt\}\\ =\ & E\int_0^T\int_\Omega(\hat{f}_1(x,t)\phi_1(t) + \hat{f}_2(x,t)\phi_2(t) + \hat{f}_3(x,t)\phi_3(t) + \hat{f}_4(x,t)\phi_4(t))dxdt\\ \le\ & C\int_0^T\int_\Omega(|\hat{f}_1(t)|^\eta + |\hat{f}_2(t)|^\eta + |\hat{f}_3(t)|^\eta + |\hat{f}_4(t)|^\eta)dxdt.\end{aligned} \tag{A31}$$

In order to derive the outcome of the lemma, we establish the function $V(\bar{p},\phi) = \sum_{i=1}^{4} \bar{p}_i(x,t)\phi_i(x,t)$, and utilize Itô's formula and the martingale theorem to obtain the result.

$$
\begin{aligned}
&\sum_{i=1}^{4} E\int_{i=1}^{4}\int_{\Omega}|\bar{p}_i(x,t)|^{\eta}dxdt + \sum_{i=1}^{4} E\int_{i=1}^{4}\int_{\Omega}|\bar{q}_i(x,t)|^{\eta}dxdt \\
&= \sum_{i=1}^{4} E\int_{0}^{T}\int_{\Omega}\hat{f}_i(x,t)\phi_i(x,t)dxdt + E\int_{\Omega}\{\phi_i(x,T)[h_{S(x,t)}(Z(x,T)) - h_{\bar{S}(x,t)}(\bar{Z}(x,T))] \\
&+ \phi_i(x,T)[h_{I_L(x,t)}(Z(x,T)) - h_{\bar{I}_L(x,t)}(\bar{Z}(x,T))] + \phi_i(x,T)[h_{R_L(x,t)}(Z(x,T)) - h_{\bar{R}_L(x,t)}(\bar{Z}(x,T))] \\
&+ \phi_i(x,T)[h_{L(x,t)}(Z(x,T)) - h_{\bar{L}(x,t)}(\bar{Z}(x,T))]\}dx \\
&\le \sum_{i=1}^{4}[E\int_{0}^{T}|\int_{\Omega}\hat{f}_i(x,t)|^{\eta})^{\frac{1}{\eta}}(E\int_{0}^{T}|\int_{\Omega}\phi_i(x,t)|^{\eta_1})^{\frac{1}{\eta_1}}]dxdt + E\int_{\Omega}\{[|h_S(x,t)(Z(x,T)) - h_{\bar{S}(x,t)}(\bar{Z}(x,T))|^{\eta}]^{\frac{1}{\eta}}, \\
&+ [|h_{I_L}(x,t)(Z(x,T)) - h_{\bar{I}_L(x,t)}(\bar{Z}(x,T))|^{\eta}]^{\frac{1}{\eta}} + [|h_{R_L}(x,t)(Z(x,T)) - h_{\bar{R}_L(x,t)}(\bar{Z}(x,T))|^{\eta}]^{\frac{1}{\eta}} \\
&+ [|h_L(x,t)(Z(x,T)) - h_{\bar{L}(x,t)}(\bar{Z}(x,T))|^{\eta}]^{\frac{1}{\eta}}\}dx,
\end{aligned} \tag{A32}
$$

where

$$
\begin{aligned}
&(E|\phi_i(x,T)|^{\eta_1})^{\frac{1}{\eta_1}}dx \\
&\le C(\sum_{i=1}^{4} E\int_{0}^{T}\int_{\Omega}|\bar{p}_i(x,t)|^{\eta}dxdt + \sum_{i=1}^{4}\int_{0}^{T}\int_{\Omega}|\bar{q}_i(x,t)|^{\eta}dxdt)^{\frac{1}{\eta_1}} \times \{\sum_{i=1}^{4}(E\int_{0}^{T}\int_{\Omega}|\hat{f}_i|^{\eta}dxdt)^{\frac{1}{\eta}} \\
&+ E\int_{\Omega}[|h_S(x,t)(Z(x,T)) - h_{\bar{S}}(x,t)(\bar{Z}(x,T))|^{\eta}dx]^{\frac{1}{\eta}} + [|h_{I_L}(x,t)(Z(x,T)) - h_{\bar{I}_L}(x,t)(\bar{Z}(x,T))|^{\eta}dx]^{\frac{1}{\eta}} \\
&+ [|h_{R_L}(x,t)(Z(x,T)) - h_{\bar{R}_L}(x,t)(\bar{Z}(x,T))|^{\eta}dx]^{\frac{1}{\eta}} + [|h_L(x,t)(Z(x,T)) - h_{\bar{L}}(x,t)(\bar{Z}(x,T))|^{\eta}dx]^{\frac{1}{\eta}}\}.
\end{aligned} \tag{A33}
$$

Both sides of Eq (A32) are multiplied by the exponent η, resulting in the following expression.

$$
\begin{aligned}
&\sum_{i=1}^{4} E\int_{0}^{T}\int_{\Omega}|\bar{p}_i(x,t)|^{\eta}dxdt + \sum_{i=1}^{4} E\int_{0}^{T}\int_{\Omega}|\bar{q}_i(x,t)|^{\eta}dxdt \\
&\le \sum_{i=1}^{4}(E\int_{0}^{T}\int_{\Omega}|\hat{f}_i(x,t)|^{\eta}dxdt) + E\int_{\Omega}\{[|h_S(x,t)(Z(x,T)) - h_{\bar{S}}(x,t)(\bar{Z}(x,T))|^{\eta}dx]^{\frac{1}{\eta}} \\
&+ [|h_{I_L}(x,t)(Z(x,T)) - h_{\bar{I}_L}(x,t)(\bar{Z}(x,T))|^{\eta}dx]^{\frac{1}{\eta}} + [|h_{R_L}(x,t)(Z(x,T)) - h_{\bar{R}_L}(x,t)(\bar{Z}(x,T))|^{\eta}dx]^{\frac{1}{\eta}} \\
&+ [|h_L(x,t)(Z(x,T)) - h_{\bar{L}}(x,t)(\bar{Z}(x,T))|^{\eta}dx]^{\frac{1}{\eta}}\}.
\end{aligned} \tag{A34}
$$

Following this, we proceed with the computation of the value of the expression located on the right-hand side of inequality (A34). Utilizing hypothesis (H1) and referencing Lemma 3.5, we derive the subsequent equation.

$$
\begin{aligned}
&E[|h_{S(x,t)}(Z(x,T)) - h_{\bar{S}(x,t)}(\bar{Z}(x,T))|^{\eta} + |h_{I_L(x,t)}(Z(x,T)) - h_{\bar{I}_L(x,t)}(\bar{Z}(x,T))|^{\eta} \\
&+ |h_{R_L(x,t)}(Z(x,T)) - h_{\bar{R}_L(x,t)}(\bar{Z}(x,T))|^{\eta} + |h_{L(x,t)}(Z(x,T)) - h_{\bar{L}(x,t)}(\bar{Z}(x,T))|^{\eta}] \\
&\le C^{\eta} E\{|S(x,T) - \bar{S}(x,T)|^{\eta} + |I_L(x,T) - \bar{I}_L(x,T)|^{\eta} + |R_L(x,T) - \bar{R}_L(x,T)|^{\eta} + |L(x,T) - \bar{L}(x,T)|^{\eta}\} \\
&\le Cd(u,\tilde{u})^{\frac{\kappa\eta}{2}}.
\end{aligned}
\tag{A35}
$$

Considering the Cauchy-Schwartz inequality, it follows that

$$
\begin{aligned}
&E\int_0^T\int_\Omega |\hat{f}_1(x,t)|^{\eta} dxdt \\
&\le CE[\int_0^T\int_\Omega |L(x,t) - \bar{L}(x,t)|^{\eta}(|\bar{p}_1(x,t)^{\eta}| + |\bar{p}_2(x,t)|^{\eta})]dxdt \\
&\le CE[\int_0^T\int_\Omega |L(x,t) - \bar{L}(x,t)|^{\frac{2\eta}{2-\eta}} dxdt]^{1-\frac{\eta}{2}}(\int_0^T\int_\Omega (|\bar{p}_1(x,t)|^2 + |\bar{p}_2(x,t)|^2)dxdt)^{\frac{\eta}{2}}.
\end{aligned}
\tag{A36}
$$

Note that $\frac{2\eta}{1-\eta} < 1$, $1 - \frac{\eta}{2} > \frac{k\eta}{2}$ and $d(u,\bar{u}) < 1$. In the same way

$$
E\int_0^T\int_\Omega |\hat{f}_2(x,t)|^{\eta} dxdt \le Cd(u(x,t),\bar{u}(x,t))^{\frac{\kappa\eta}{2}}, \tag{A37}
$$

$$
E\int_0^T\int_\Omega |\hat{f}_3(x,t)|^{\eta} dxdt \le Cd(u(x,t),\bar{u}(x,t))^{\frac{\kappa\eta}{2}}, \tag{A38}
$$

and

$$
E\int_0^T\int_\Omega |\hat{f}_4(x,t)|^{\eta} dxdt \le Cd(u(x,t),\bar{u}(x,t))^{\frac{\kappa\eta}{2}}, \tag{A39}
$$

then the following can be obtained

$$
\begin{aligned}
&\sum_{i=1}^{4}\int_0^T\int_\Omega (|\bar{p}_i(x,t)|^{\eta})dxdt \\
&\le CE\sum_{i=1}^{4}\int_0^T\int_\Omega (|\hat{f}_i(x,t)|^{\eta} dxdt) \\
&\le C[d(u(x,t),\bar{u}(x,t))]^{\frac{\kappa\eta}{2}}.
\end{aligned}
\tag{A40}
$$