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Research article

Orientable vertex imprimitive complete maps

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Abstract: In the work by Li (*J. Combin. Theory Ser. B*, **99** (2009), 447–454.), the author characterized the classification of vertex transitive embeddings of complete graphs, and proposed how to enumerate such maps. In this paper, we study the counting problem of orientable vertex imprimitive complete maps, which is the automorphism group of this map acts imprimitively on its vertex set. Moreover, we obtain the number of non-isomorphic embeddings when the vertex-stabilizer subgroups of the automorphism groups of maps are isomorphic to Z_{p-1} with odd prime p.

Keywords: vertex imprimitive embeddings; complete graphs; Frobenius groups

1. Introduction

If a graph can be embedded in a surface, then naturally there will be a problem: How many non-isomorphic ways can it be done? One of the main aims of topological graph theory is to enumerate all the symmetrical embeddings of a given class of graphs in closed surfaces, see [1–3]. As one of a series of papers toward solving the problem of counting the number vertex transitive embeddings of complete graphs, we restrict our attention here to the orientable vertex imprimitive embeddings of complete graphs.

Let $\mathcal{M} = (V, E, F)$ be an *orientable map* with vertex set V, edge set E, and face set F, that is, $\mathcal{M}$ is a 2-cell embedding of the finite underlying graph $\Gamma = (V, E)$ in an orientable surface. For convenience, a map $\mathcal{M}$ is called a *complete map* if its underlying graph is a complete graph.

An *automorphism* of a map $\mathcal{M}$ is a permutation of $V \cup E \cup F$, which preserves V, E, F, and their incidence relations, so it is exactly an automorphism of the underlying graph which preserves the supporting surface. All automorphisms of $\mathcal{M}$ form the *automorphism group* $\mathrm{Aut}(\mathcal{M})$ under composition.

A map $\mathcal{M}$ is said to be *G-vertex-imprimitive* (or a *vertex-imprimitive embedding* of its underlying graph) if $G = \mathrm{Aut}(\mathcal{M})$ acts imprimitively (but transitively) on the vertex set V. Furthermore, if G also preserves the orientation of the supporting surface, then $\mathcal{M}$ is called *orientable vertex-imprimitive*. Here, a permutation group G acting transitively on a set Ω is *imprimitive*, which means that G preserves a nontrivial partition of Ω.

Recent development concerning the theory of maps was closely related to the theory of map colorings, with the topic of highly 'symmetrical' maps always at the center of interest, and recent investigation began with Biggs [4,5]. In the past fifty years, plenty of results about 'symmetrical' maps have been obtained, see [2,6,7] and the references therein. In particular, see [2,3,5] for arc transitive complete maps, see [8–10] for vertex transitive complete maps, and see [11,12] for edge transitive complete maps. Very recently, some special families of edge-transitive embeddings of complete bipartite graphs are classified in [13–18]. Additionally, the Cayley map of the quaternion group Q_8 constructed in [19, Figure 7] is a complete 4-partite map with arcs colored i, j, k; the map constructed has the property that removing the arcs colored i creates an imprimitive map whose automorphism group action preserves the partition of the edges into those colored j and those colored k. For more information about the embeddings of complete graphs, see [20–22].

The purpose of this paper is to enumerate the number of orientable vertex imprimitive maps with underlying graphs being complete graphs. Recall that $\phi(n)$ is the Euler phi-function, i.e. the number of positive integers which is less than and co-prime to n, where n is a positive integer. The main result of this paper is now stated as follows.

Theorem 1.1. *Let $\mathcal{M}$ be an orientable, vertex imprimitive, complete map with automorphism group $G = Aut(\mathcal{M})$, and let G_α be the stabilizer of a vertex α of $\mathcal{M}$. Then $\mathcal{M}$ is a Cayley map of $\mathbb{Z}_p^d$ for some integer $d \geq 2$ and odd prime p. In addition, the group $G \cong \mathbb{Z}_p^d{:}G_\alpha$ is a Frobenius group whenever G_α is a cyclic group. Furthermore, if $G_\alpha \cong \mathbb{Z}_{p-1}$ acting on the neighborhood of α has λ orbits with $\lambda(p-1) = p^d - 1$ and $\lambda \geq 4$ is a prime, then the number of non-isomorphic orientable vertex imprimitive complete maps equals*

$$\frac{|\mathcal{A}_\lambda| - |\mathcal{A}_1|}{|SL(d,p)|},$$

where $|\mathcal{A}_\lambda| = (\lambda-1)!(p-1)^{\lambda-1}\phi(p-1)$ and $|\mathcal{A}_1| = \phi(p^d-1)$.

As a by-product of the Theorem 1.1, we can deduce the following conclusions when λ is a composite integer.

Corollary 1.2. *Let p_1 and p_2 be two different primes.*

(i) If $\lambda = p_1p_2$, then the number of non-isomorphic orientable vertex imprimitive complete maps equals $\frac{|\mathcal{A}_{p_1p_2}|-|\mathcal{A}_{p_1}|-|\mathcal{A}_{p_2}|+|\mathcal{A}_1|}{|\mathrm{SL}(d,p)|}$.

(ii) If $\lambda = p_1^2p_2$, then the number of non-isomorphic orientable vertex imprimitive complete maps equals $\frac{|\mathcal{A}_{p_1^2p_2}|-|\mathcal{A}_{p_1p_2}|-|\mathcal{A}_{p_1^2}|+|\mathcal{A}_{p_1}|}{|\mathrm{SL}(d,p)|}$.

This paper is organized as follows. After this introductory section, some preliminary results are given in Section 2, then the enumeration of different and non-isomorphic orientable vertex imprimitive complete maps is given in Sections 3 and 4, respectively. Finally, we give the complete proof of Theorem 1.1 in Section 5.

2. Preliminaries

In this section, we need some notations for convenience and give some results that will be used in the following discussion.

Let p be an odd prime and let $d \geq 2$ be an integer. The elementary abelian p-group of order p^d will be denoted by Z_p^d. We use $o(a)$ and $\langle a \rangle$ to denote the order of a and the group generated by a, respectively. We use Z_{p-1} and D_{2p} to denote the cyclic group of order $p-1$ and the dihedral group of order $2p$, respectively. The general linear group and the special linear group of the field $\mathbb{F}_{p^d}$ are denoted by $\mathrm{GL}(d, p)$ and $\mathrm{SL}(d, p)$, respectively. The centralizer and the normalizer of Z_{p-1} in $\mathrm{GL}(d, p)$ are denoted by $C_{\mathsf{GL}(d,p)}(\mathrm{Z}_{p-1})$ and $N_{\mathsf{GL}(d,p)}(\mathrm{Z}_{p-1})$, respectively. Let H and K be two groups. Then we use $H{:}K$ to denote a semi-direct product of H by K, in which H is a normal subgroup. We use $\mathrm{Z}(H)$, $\mathrm{Aut}(H)$, and $\mathrm{Inn}(H)$ to denote the center, the automorphism group, and the inner automorphism group of H, respectively.

Let $F = \mathbb{F}_{p^d}$ be the field of order p^d. Let $F^+ = \mathbb{F}^+_{p^d}$ and $F^\times = \mathbb{F}^\times_{p^d}$ be the additive group and the multiplicative group of F, respectively. It follows that

$$F^+ \cong \mathbb{Z}_p^d, \quad F^\times \cong \mathbb{Z}_{p^d-1}.$$

Let $\mathbf{0}$ be the identity of the F^+. Let $F^\#$ be the set of all nonidentity elements of F^+, namely, $F^\# = F^+ \setminus \{\mathbf{0}\}$. Then the complete graph K_{p^d} may be represented as a Cayley graph

$$\mathrm{K}_{p^d} = \mathrm{Cay}(F^+, F^\#).$$

A *Cayley map* $\mathcal{M}$ is an embedding of a Cayley graph $\Sigma = \mathrm{Cay}(H, S)$ into a surface, such that $\mathrm{Aut}(\mathcal{M})$ contains a subgroup N acting regularly on the vertices and $\mathcal{M}$ is called a Cayley map of N (or a Cayley embedding of Σ with respect to N). Moreover, Cayley maps form a very interesting family of vertex-transitive maps [6, 9].

For a vertex v, a cyclic permutation of the neighbour set $\mathit{\Gamma}(v)$ of v is called a *rotation* at v and denoted by R_v. A *rotation system* $R(\mathit{\Gamma})$ of a graph $\mathit{\Gamma}$ is the set of rotations at all vertices, that is, $R(\mathit{\Gamma}) = \{R_v\}_{v \in V}$. Hence, each rotation system $R(\mathit{\Gamma})$ defines an orientable embedding of $\mathit{\Gamma}$, refer to [23, pp.104–108].

Noting that the vertex rotations R_v can be regarded as permutations not only of the set $\mathit{\Gamma}(v)$ but also of the generating set S, the Cayley maps have another equivalent definition [24]. A map with an underlying graph being Cayley graph $\Sigma = \mathrm{Cay}(H, S)$ is a *Cayley map* if the induced local cyclic permutations of S are all the same. Moreover, each cyclic permutation ρ of $F^\#$ gives rise to a unique orientable Cayley embedding of K_{p^d} with the underlying graph $\Gamma = \mathrm{Cay}(F^+, F^\#)$. Thus, if two cyclic permutations ρ_1 and ρ_2 of $F^\#$ are different, then the orientable vertex imprimitive complete maps generated by them are also considered to be different.

3. Enumeration of different embeddings

In this section, we determine enumeration of different vertex imprimitive embeddings of a complete graph. Now, we begin by citing the well-known conclusion about vertex transitive maps.

Lemma 3.1. ([8, Lemma 2.2]) *Let $\mathcal{M}$ be a vertex-transitive map and let $G = Aut(\mathcal{M})$. Then the stabilizer $G_\alpha \cong \mathbb{Z}_k$ or D_{2k} for a vertex α, and each orbit of G_α acting on the neighborhood of α has length k.*

Next, from [8, Theorem 1.1] and [23, Lemma 5.4.1], we can directly obtain the following lemma.

Lemma 3.2. *Let $\mathcal{M}$ be an orientable vertex imprimitive complete map. Let $G = Aut(\mathcal{M})$. Then $\mathcal{M}$ is a Cayley map of $\mathbb{Z}_p^d$, $G \cong \mathbb{Z}_p^d{:}\mathbb{Z}_k$ is a Frobenius group, and $G_\alpha \cong \mathbb{Z}_k$ such that $(k, p) = 1$.*

Assume that $G_\alpha := \langle a\rangle$ with $o(a) \geq 2$. Then $G \cong F^+{:}\langle a\rangle$ is a Frobenius group by Lemma 3.2. It follows that $\langle a\rangle$ is half-transitive on F^+, and $|\langle a\rangle| = k$ is a divisor of $|F^+| - 1 = p^d - 1$. Specially, if $|\langle a\rangle| = p - 1$, thus, G_α acting on $\Gamma(\alpha)$ has λ orbits with $\lambda = (p^d - 1)/(p - 1) \geq 4$, then we get the lemma as follows.

Lemma 3.3. *If $G \cong F^+{:}\mathbb{Z}_{p-1}$, then there are exactly $(\lambda-1)!(p-1)^{\lambda-1}\phi(p-1)$ different orientable vertex imprimitive complete maps.*

Proof. Taking $\alpha = \mathbf{0}$ for convenience with $\mathbf{0}$ the identity element of F^+. It follows that $G_\mathbf{0}$ partitions the neighborhood $\Gamma(\mathbf{0})$ of $\mathbf{0}$ into λ orbits with $\lambda \geq 4$, and the length of each orbit is $p - 1$ in view of Lemma 3.1. Since $F^\#$ is the set of all nonidentity elements of F^+, then $|F^\#| = p^d - 1$, and, further, we have

$$F^\# = \Delta_1 \dot{\cup} \Delta_2 \dot{\cup} \cdots \dot{\cup} \Delta_\lambda,$$

where Δ_i is an orbit of $G_\mathbf{0}$ acting on $\Gamma(\mathbf{0})$, and $|\Delta_i| = p - 1$ with $1 \leq i \leq \lambda$.

Note that the vertex $\mathbf{0}$ and the neighbors can be lied on a disc such that $\mathbf{0}$ is in the center and the neighbors of $\mathbf{0}$ are around $\mathbf{0}$. Without loss of generality, we may assume that the $p^d - 1$ neighbors of $\mathbf{0}$ (i.e. all the elements of $F^\#$) are in clockwise order around $\mathbf{0}$ and denoted by $\beta_1, \beta_2, \cdots, \beta_{p^d-1}$, then

$$\rho := (\beta_1, \beta_2, \cdots, \beta_{p^d-1})$$

is a cyclic permutation of $F^\#$. Further, we can obtain that the number of the cyclic permutations of $F^\#$ equals the number of arrangements of β_i, and it follows that to determine the number of the orientable vertex imprimitive complete maps, we only need to determine the different choices of β_i with $1 \leq i \leq p^d - 1$.

Set $\beta_1 = \mathbf{1}$ and $\beta_1 \in \Delta_1$ for convenience, where $\mathbf{1}$ is the identity element of $F^\times$. If $\beta_2 \in \Delta_1$, then $\mathbf{1}^{a^l} = \beta_2$ for some a^l, where $0 < l \leq p - 2$. It follows that $a^l : \beta_2 \mapsto \beta_3 \mapsto \beta_4 \mapsto \cdots \mapsto \mathbf{1}$. Thus, $(l, p - 1) = 1$ and $G_\mathbf{0} = \langle a^l\rangle$ acting on $\Gamma(\mathbf{0})$ has only one orbit, which is a contradiction, so $\beta_2 \notin \Delta_1$ and $\beta_2 \in \Delta_i$ with $2 \leq i \leq \lambda$. Without loss of generality, set $\beta_2 \in \Delta_2$ for convenience, then β_2 has $(\lambda-1)(p-1)$ different choices.

If $\beta_3 \in \Delta_2$, then $\beta_2^{a^j} = \beta_3$ for some a^j, where $0 < j \leq p - 2$. Furthermore, there is $a^j : \beta_2 \mapsto \beta_3 \mapsto \beta_4 \mapsto \cdots \mapsto \mathbf{1}$. It follows that $(j, p - 1) = 1, \beta_1 \in \Delta_2$ and $G_\mathbf{0}$ acting on $\Gamma(\mathbf{0})$ has only one orbit, which is a contradiction. Thus, $\beta_3 \notin \Delta_2$.

If $\beta_3 \in \Delta_1$, then $\mathbf{1}^{a^{j'}} = \beta_3$ for some $a^{j'}$, where $0 < j' \leq p - 2$. It follows that there are $a^{j'} : \mathbf{1} \mapsto \beta_3 \mapsto \beta_5 \mapsto \cdots \mapsto \mathbf{1}$ and $a^{j'} : \beta_2 \mapsto \beta_4 \mapsto \beta_6 \mapsto \cdots \mapsto \beta_2$. Thus, $G_\mathbf{0} = \langle a^{j'}\rangle$ acting on $\Gamma(\mathbf{0})$ has two orbits, which is a contradiction. It follows that $\beta_3 \notin \Delta_1$ and $\beta_3 \in \Delta_i$ with $3 \leq i \leq \lambda$. Without loss of generality, set $\beta_3 \in \Delta_3$ for convenience, so β_3 has $(\lambda - 2)(p - 1)$ different choices.

If $\beta_4 \in \Delta_3$, then $\beta_3^{a^t} = \beta_4$ for some a^t, where $0 < t \leq p - 2$. Moreover, there has $a^t : \beta_3 \mapsto \beta_4 \mapsto \beta_5 \mapsto \cdots \mapsto \mathbf{1}$. Thus, $(t, p - 1) = 1$, $\mathbf{1} \in \Delta_3$ and $G_\mathbf{0} = \langle a^t\rangle$ acting on $\Gamma(\mathbf{0})$ has only one orbit, which is a contradiction. so $\beta_4 \notin \Delta_3$.

If $\beta_4 \in \Delta_2$, then $\beta_2^{a^{t'}} = \beta_4$ for some $a^{t'}$, where $0 < t' \leq p - 2$. It follows that there are $a^{t'} : \beta_2 \mapsto \beta_4 \mapsto \beta_6 \mapsto \cdots \mapsto \beta_2$, and $a^{t'} : \beta_3 \mapsto \beta_5 \mapsto \beta_7 \mapsto \cdots \mapsto \mathbf{1}$. Thus, $G_\mathbf{0} = \langle a^{t'}\rangle$ acting on $\Gamma(\mathbf{0})$ has two orbits, which is a contradiction, so $\beta_4 \notin \Delta_2$.

If $\beta_4 \in \Delta_1$, then $\mathbf{1}^{a^{t''}} = \beta_4$ for some $a^{t''}$, where $0 < t'' \leq p-2$. Further, there are $a^{t''} : \mathbf{1} \mapsto \beta_4 \mapsto \beta_7 \mapsto \cdots \mapsto \mathbf{1}$, $a^{t''} : \beta_2 \mapsto \beta_5 \mapsto \beta_8 \mapsto \cdots \mapsto \beta_2$, and $a^{t''} : \beta_3 \mapsto \beta_6 \mapsto \beta_9 \mapsto \cdots \mapsto \beta_3$. Thus, $G_\mathbf{0} = \langle a^{t''}\rangle$ acting on $\Gamma(\mathbf{0})$ has three orbits, which is a contradiction, so $\beta_4 \notin \Delta_1$ and $\beta_4 \in \Delta_i$ with $4 \le i \le \lambda$. Without loss of generality, set $\beta_4 \in \Delta_4$ for convenience, and we deduce that β_4 has $(\lambda - 3)(p-1)$ different choices.

If $\beta_5 \in \Delta_4$, then $\beta_4^{a^k} = \beta_5$ for some a^k, where $0 < k \le p-2$. It follows that there is $a^k : \beta_4 \mapsto \beta_5 \mapsto \beta_6 \mapsto \cdots \mapsto \mathbf{1}$. Thus, $(k, p-1) = 1$, $\mathbf{1} \in \Delta_4$, and $G_{\mathbf{0}} = \langle a^k \rangle$ acting on $\Gamma(\mathbf{0})$ has only one orbit, which is a contradiction, so $\beta_5 \notin \Delta_4$.

If $\beta_5 \in \Delta_3$, then $\beta_3^{a^{k'}} = \beta_5$ for some $a^{k'}$, where $0 < k' \le p-2$. Correspondingly, there are $a^{k'} : \beta_2 \mapsto \beta_4 \mapsto \beta_6 \mapsto \cdots \mapsto \beta_2$ and $a^{t'} : \mathbf{1} \mapsto \beta_3 \mapsto \beta_5 \mapsto \cdots \mapsto \mathbf{1}$. Thus, $G_{\mathbf{0}} = \langle a^{k'} \rangle$ acting on $\Gamma(\mathbf{0})$ has two orbits, which is a contradiction, so $\beta_5 \notin \Delta_3$.

If $\beta_5 \in \Delta_2$, then $\beta_2^{a^{k''}} = \beta_5$ for some $a^{k''}$, where $0 < k'' \le p-2$. Further, there are $a^{k''} : \mathbf{1} \mapsto \beta_4 \mapsto \beta_7 \mapsto \cdots \mapsto \mathbf{1}$, $a^{k''} : \beta_2 \mapsto \beta_5 \mapsto \beta_8 \mapsto \cdots \mapsto \beta_2$ and $a^{k''} : \beta_3 \mapsto \beta_6 \mapsto \beta_9 \mapsto \cdots \mapsto \beta_3$. Thus, $G_{\mathbf{0}} = \langle a^{k''} \rangle$ acting on $\Gamma(\mathbf{0})$ has three orbits, which is a contradiction, so $\beta_5 \notin \Delta_2$.

If $\beta_5 \in \Delta_1$, then $\mathbf{1}^{a^m} = \beta_5$ for some a^m, where $0 < m \le p-2$. It follows that there are $a^m : \mathbf{1} \mapsto \beta_5 \mapsto \beta_9 \mapsto \cdots \mapsto \mathbf{1}$, $a^m : \beta_2 \mapsto \beta_6 \mapsto \beta_{10} \mapsto \cdots \mapsto \beta_2$, $a^m : \beta_3 \mapsto \beta_7 \mapsto \beta_{11} \mapsto \cdots \mapsto \beta_3$, and $a^m : \beta_4 \mapsto \beta_8 \mapsto \beta_{12} \mapsto \cdots \mapsto \beta_4$. Thus, $G_{\mathbf{0}} = \langle a^m \rangle = \langle a \rangle$ acting on $\Gamma(\mathbf{0})$ has four orbits, namely, $\lambda = 4$. Since the number of generators of $G_{\mathbf{0}}$ is $\phi(p-1)$, we obtain that a^m has $\phi(p-1)$ different choices. Noting that $G_{\mathbf{0}}$ is cyclic, besides $\mathbf{1}, \beta_2, \beta_3$, and β_4, the remaining vertices of Δ_1, Δ_2, Δ_3, and Δ_4 can be obtained by $\mathbf{1}, \beta_2, \beta_3$, and β_4 through the conjugate action of $a, a^2, a^3, \cdots, a^{p-2}$, respectively. So

$$\rho|_{\lambda=4} := (\mathbf{1}, \beta_2, \beta_3, \beta_4, \mathbf{1}^a, \beta_2^a, \beta_3^a, \beta_4^a, \cdots, \mathbf{1}^{a^{p-2}}, \beta_2^{a^{p-2}}, \beta_3^{a^{p-2}}, \beta_4^{a^{p-2}})$$

is a cyclic permutation of $F^{\#}$. It follows that the number of ρ is determined by the choices of $\mathbf{1}, \beta_2, \beta_3$, β_4, and a. Further, the number of $\rho|_{\lambda=4}$ equals

$$\begin{aligned}(\lambda - 1)(p-1) \cdot (\lambda - 2)(p-1) \cdot (\lambda - 3)(p-1) \cdot \phi(p-1) \\ = (4-1)!(p-1)^3\phi(p-1).\end{aligned}$$

Let the corresponding maps generated by $\rho|_{\lambda=4}$ be

$$\mathcal{M}_4 := \mathcal{M}_4(\mathbf{1}, \beta_2, \beta_3, \beta_4, a).$$

Thus, the number of different orientable vertex imprimitive complete maps equals $3!(p-1)^3\phi(p-1)$ if $\lambda = 4$.

Next, suppose that $\lambda \ge 6$ as $\lambda \ne 5$. According to the above derivation, we can obtain that $\beta_5 \notin \Delta_i$ with $1 \le i \le 4$, and $\beta_5 \in \Delta_j$ with $5 \le j \le \lambda$. Without loss of generality, set $\beta_5 \in \Delta_5$ for convenience, and further, β_5 has $(\lambda - 4)(p-1)$ different choices. Similarly, $\beta_6 \in \Delta_i$ with $6 \le i \le \lambda$. It follows that $\lambda = 6$ and β_6 has $(\lambda - 5)(p-1)$ different choices. Note that a has $\phi(p-1)$ different choices, then

$$\rho|_{\lambda=6} := (\mathbf{1}, \beta_2, \cdots, \beta_6, \mathbf{1}^a, \beta_2^a, \cdots, \beta_6^a, \cdots, \mathbf{1}^{a^{p-2}}, \beta_2^{a^{p-2}}, \cdots, \beta_6^{a^{p-2}})$$

is a cyclic permutation of $F^{\#}$. Thus, the number of $\rho|_{\lambda=6}$ equals

$$\begin{aligned}(\lambda - 1)(p-1) \cdot (\lambda - 2)(p-1) \cdots (\lambda - 5)(p-1) \cdot \phi(p-1) \\ = (6-1)!(p-1)^5\phi(p-1).\end{aligned}$$

Let the corresponding maps generated by $\rho|_{\lambda=6}$ be

$$\mathcal{M}_6 := \mathcal{M}_6(\mathbf{1}, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, a).$$

Thus the number of different orientable vertex imprimitive complete maps equals $5!(p-1)^5\phi(p-1)$ if $\lambda = 6$.

In fact, λ can be generalized. If $G_\alpha = \langle a \rangle$ acting on $\Gamma(\alpha)$ has λ orbits, then $\beta_i \in \Delta_i$ such that $\beta_1 = \mathbf{1}$ and $1 \leq i \leq \lambda$, and

$$\rho|_\lambda := (\mathbf{1}, \beta_2, \cdots, \beta_\lambda, \mathbf{1}^a, \beta_2^a, \cdots, \beta_\lambda^a, \cdots, \mathbf{1}^{a^{p-2}}, \beta_2^{a^{p-2}}, \cdots, \beta_\lambda^{a^{p-2}})$$

is a cyclic permutation of $F^\#$ (see Figure 1). Since β_i has $(\lambda - i + 1)(p-1)$ different choices with $2 \leq i \leq \lambda$, and a has $\phi(p-1)$ different choices, it follows that the number of $\rho|_\lambda$ equals

$$(\lambda-1)(p-1)\cdot(\lambda-2)(p-1)\cdots(\lambda-\lambda+1)(p-1)\cdot\phi(p-1)$$

$$= (\lambda-1)!(p-1)^{\lambda-1}\phi(p-1).$$

Let the corresponding maps generated by $\rho|_\lambda$ be

$$\mathcal{M}_\lambda := \mathcal{M}_\lambda(\mathbf{1}, \beta_2, \beta_3, \cdots, \beta_\lambda, a).$$

Hence if $G_\alpha \cong \mathbb{Z}_{p-1}$, then there are exactly $(\lambda-1)!(p-1)^{\lambda-1}\phi(p-1)$ different orientable vertex imprimitive complete maps. $\square$

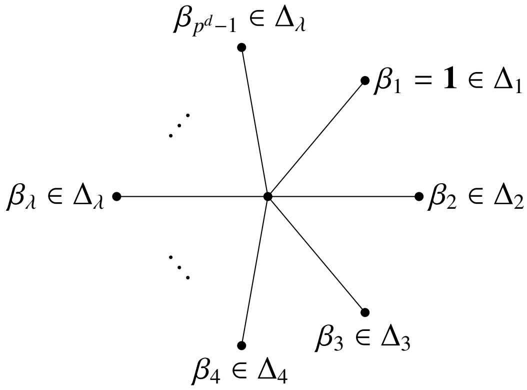

Figure 1. Cyclic permutations of $F^\#$.

Remark. The proof of Lemma 3.3 provides a general construction for orientable vertex imprimitive embeddings of complete graphs.

Recall that a Cayley map $\mathrm{CayM}(G, S)$ is called *balanced* if s and $-s$ are placed on the antipodal points for all elements $s \in S$, see [7]. Let η be the unique involution of $\mathrm{GL}(1, p^d)$. Since the map $\mathcal{M}_\lambda$ is a Cayley map of the group F^+ by Lemma 3.2, then

$$\eta: \; x \mapsto -x, \text{ for all } x \in F^+$$

is an automorphism of $\mathcal{M}_\lambda$.

Lemma 3.4. *A map $\mathcal{M}_\lambda$ is balanced if and only if $\beta_i^{-1} = \beta_{i+\frac{p^d-1}{2}}$ with $1 \leq i \leq \lambda$.*

Proof. Assume that $\mathcal{M}_\lambda$ is balanced, then the vertex β_i^{-1} is placed at the antipodal position of the vertex β_i with p as an odd prime and $1 \le i \le \lambda$. Thus, $\beta_i^{-1} = \beta_{i+\frac{p^d-1}{2}}$.

Conversely, assume that $\beta_i^{-1} = \beta_{i+\frac{p^d-1}{2}}$, then for any $1 \le l \le p-2$, we can obtain that

$$\beta_{i+l\lambda}^{-1} = (\beta_i^{-1})^{a^l} = (\beta_{i+\frac{p^d-1}{2}})^{a^l} = \beta_{\frac{p^d-1}{2}+i+l\lambda},$$

reading the subscripts modulo (p^d-1). So, $\beta_j^{-1} = \beta_{\frac{p^d-1}{2}+j}$ is at the antipodal position of β_j for all j with $1 \le j \le \frac{p^d-1}{2}$, and, therefore, $\mathcal{M}_\lambda$ is balanced. □

4. Enumeration of non-isomorphic embeddings

We notice that many different orientable vertex imprimitive complete maps may be isomorphic. To determine the number of non-isomorphic complete maps, we prepare the following lemmas, and we first give the well-known Clifford's theorem.

Lemma 4.1. ([25, Theorem 5.9]) *Let V be an irreducible FH-module and let N be a normal subgroup of H. Then the following statements are true:*

(i) V is a completely reducible FN-module, and

$$V = W_1^n \oplus W_2^n \oplus \cdots \oplus W_r^n,$$

where $W_i (i = 1, 2, ..., r)$ are all non-isomorphic irreducible FN-submodules of V.

(ii) H permutes $\{W_1^n, W_2^n, \cdots, W_r^n\}$ transitively.

(iii) If K is the stabilizer of H on W_1^n, then H is irreducible on W_1^n.

According to Lemma 4.1, we can get the next following lemma which will determine the normalizer of $\mathbb{Z}_{p-1}$ in $\mathrm{GL}(d,p)$.

Lemma 4.2. *Let $G \cong \mathbb{Z}_p^d{:}\mathbb{Z}_{p-1}$. Then $N_{\mathrm{GL}(d,p)}(\mathbb{Z}_{p-1}) \cong GL(d,p)$ and $Aut(G) \cong \mathbb{Z}_p^d{:}GL(d,p)$.*

Proof. Let $V = \mathbb{Z}_p^d$, namely, consider $\mathbb{Z}_p^d$ as a d-dimensional linear space over field F_p. By [26, Theorem 7.3 of Chapter 2], we can obtain that $\mathrm{GL}(d,p)$ has a cyclic subgroup $\mathbb{Z}_{p^d-1}$. Since $\mathbb{Z}_{p-1}$ is a normal subgroup of $\mathbb{Z}_{p^d-1}$, and $\mathbb{Z}_{p-1}$ acting on $\mathbb{Z}_p^d$ is reducible, equivalently, $p-1$ is not a primitive divisor of p^d, then due to Lemma 4.1, we have

$$\mathbb{Z}_p^d = V_1 \oplus V_2,$$

where $dim(V_i) = \frac{d}{2}$ and V_i is a faithful irreducible $F_p\mathbb{Z}_{p-1}$-module with $i = 1, 2$. By [27, 27.14(3)], it follows that $C_{\mathrm{GL}(d,p)}(\mathbb{Z}_{p-1}) = \mathrm{GL}(d,p)$. Since

$$C_{\mathrm{GL}(d,p)}(\mathbb{Z}_{p-1}) \le N_{\mathrm{GL}(d,p)}(\mathbb{Z}_{p-1}) \le \mathrm{GL}(d,p),$$

it is easy to see that $N_{\mathrm{GL}(d,p)}(\mathbb{Z}_{p-1}) \cong \mathrm{GL}(d,p)$. Furthermore, by [28, Lemma 4.5], we have

$$\mathrm{Aut}(G) \cong \mathbb{Z}_p^d{:}N_{\mathrm{GL}(d,p)}(\mathbb{Z}_{p-1}) \cong \mathbb{Z}_p^d{:}\mathrm{GL}(d,p).$$

□

Two maps $\mathcal{M}_1$ and $\mathcal{M}_2$ are *isomorphic* if there is a one-to-one correspondence from the vertices of $\mathcal{M}_1$ to the vertices of $\mathcal{M}_2$ that maps flags to flags, and it is denoted by $\mathcal{M}_1 \cong \mathcal{M}_2$. It follows that $\mathrm{Aut}\mathcal{M}_1 \cong \mathrm{Aut}\mathcal{M}_2$ if $\mathcal{M}_1 \cong \mathcal{M}_2$. A complete map is a *complete Cayley map* if its automorphism group is regular on the vertices. Moreover, the complete Cayley maps of non-isomorphic groups are not isomorphic, and we have the following lemma.

Lemma 4.3. *$\mathcal{M}_\lambda^\sigma \cong \mathcal{M}_\lambda$ for each $\sigma \in Aut(G)$. On the contrary, $\sigma \in Aut(G)$ if $\mathcal{M}_\lambda^\sigma \cong \mathcal{M}_\lambda$.*

Proof. Since $G = \mathrm{Aut}(\mathcal{M}_\lambda) \cong \mathbb{Z}_p^d{:}\mathbb{Z}_{p-1}$, then $G_\alpha \cong \mathbb{Z}_{p-1}$ is a cyclic group for any $\alpha \in V$. Suppose that σ fixes α for each $\sigma \in \mathrm{Aut}(G) \cong \mathbb{Z}_p^d{:}\mathrm{GL}(d,p)$. Note that $\mathbb{Z}_p^d$ is a regular and normal subgroup of Aut(G), then for any $1 \neq x \in \mathbb{Z}_p^d$, we have $\alpha^x = \alpha x$ or $x^{-1}\alpha$ if x is the right or left multiplication, respectively. Thus, $\alpha^x \neq \alpha$, namely, x does not fix α. Hence, $\sigma \in \mathrm{GL}(d,p)$. It follows that $\mathbb{Z}_{p-1}^\sigma \cong \mathbb{Z}_{p-1}$ since $\mathbb{Z}_{p-1} \lhd \mathrm{GL}(d,p)$. Note that for each $\tau \in \mathrm{Aut}(G)$,

$$\mathcal{M}_\lambda^\tau = \mathcal{M}_\lambda^{x\sigma} = \mathcal{M}_\lambda^\sigma \cong \mathcal{M}_\lambda$$

such that $\tau = x\sigma$, where $x \in \mathbb{Z}_p^d$ and $\sigma \in \mathrm{GL}(d,p)$. Moreover, we can obtain $\mathcal{M}_\lambda^\sigma \cong \mathcal{M}_\lambda$ for each $\sigma \in$ Aut(G) by arbitrariness of x.

On the contrary, if $\mathcal{M}_\lambda^\sigma \cong \mathcal{M}_\lambda$, then $\mathrm{Aut}(\mathcal{M}_\lambda^\sigma) \cong \mathrm{Aut}(\mathcal{M}_\lambda) = G$. It follows that for each $\sigma \in \mathrm{Aut}(\mathcal{M}_\lambda)$,

$$(\mathrm{Aut}(\mathcal{M}_\lambda))^\sigma = G^\sigma = G \cong \mathrm{Aut}(\mathcal{M}_\lambda^\sigma).$$

Since G is a Frobenius group, it is easy to see that the center $\mathrm{Z}(G) = 1$, then $G \cong G/\mathrm{Z}(G) \cong \mathrm{Inn}(G) \lhd \mathrm{Aut}(G)$. Hence, $\sigma \in \mathrm{Aut}(G)$. □

Now, we determine the number of non-isomorphic orientable vertex imprimitive complete maps if $G_\alpha \cong \mathbb{Z}_{p-1}$.

Let

$$\mathcal{A}_\lambda = \{\mathcal{M}_\lambda(\mathbf{1},\beta_2,\beta_3,\cdots,\beta_\lambda,a) | \beta_1 = \mathbf{1}, \beta_i \in \Delta_i, \beta_i^a = \beta_{i+\lambda},$$

$$where\ 1 \leq i \leq \lambda, o(a) = p-1 \geq 2, and\ read\ the\ subscripts\ modulo\ p^d - 1\}.$$

Then $\mathcal{A}_\lambda$ is a finite nonempty set and $|\mathcal{A}_\lambda| = (\lambda-1)!(p-1)^{\lambda-1}\phi(p-1)$. Let $X = \mathrm{Aut}(G)$. Thus, by Lemma 4.2, we have

$$X \cong \mathbb{Z}_p^d{:}N_{\mathrm{GL}(d,p)}(\mathbb{Z}_{p-1}) \cong \mathbb{Z}_p^d{:}\mathrm{GL}(d,p).$$

Note that $\mathbb{Z}_p^d \lhd X$ and $\mathbb{Z}_p^d$ acts on V regularly, and, thus, by [29, Exercise 1.4.1], we have $X_\alpha \cong \mathrm{GL}(d,p)$, $G \lhd X$, and $G_\alpha \lhd X_\alpha$. Further, by [26, Theorem 7.3 of Chapter 2], we can obtain that $\mathrm{GL}(d,p)$ has a cyclic subgroup $\mathbb{Z}_{p^d-1}$ and, for convenience, $\langle z\rangle := \mathbb{Z}_{p^d-1}$.

Lemma 4.4. *If $G \cong F^+{:}\mathbb{Z}_{p-1}$ and λ is a prime, then there are $\frac{(\lambda-1)!(p-1)^{\lambda-1}\phi(p-1)-\phi(p^d-1)}{|\mathrm{SL}(d,p)|}$ non-isomorphic orientable vertex imprimitive complete maps.*

Proof. Since $o(a) = p-1 = \frac{p^d-1}{\lambda}$, it follows that $z^\lambda = a$. Let

$$(\mathbf{1},\beta_2,\beta_3,\cdots,\beta_\lambda,\mathbf{1}^a,\beta_2^a,\beta_3^a,\cdots,\beta_\lambda^a,\cdots,\mathbf{1}^{a^{p-2}},\beta_2^{a^{p-2}},\beta_3^{a^{p-2}},\cdots,\beta_\lambda^{a^{p-2}})$$

be a cyclic permutation of $F^\#$ such that $\beta_i = \mathbf{1}^{z^i}$ and $2 \leq i < \lambda$.

Thus, we have

$$z^i : \mathbf{1} \mapsto \beta_i \mapsto \beta_{2i-1} \mapsto \cdots \mapsto \mathbf{1},$$

$$z^i : \beta_2 \mapsto \beta_{i+1} \mapsto \beta_{2i} \mapsto \cdots \mapsto \beta_2,$$

$$\cdots$$

$$z^i : \beta_{i-1} \mapsto \beta_{2i-2} \mapsto \beta_{3i-3} \mapsto \cdots \mapsto \beta_{i-1}.$$

That is, z^i can be identified with the permutation

$$z^i = (\mathbf{1}\beta_i\beta_{2i-1}\cdots\beta_{p^d-i+1})(\beta_2\beta_{i+1}\beta_{2i}\cdots\beta_{p^d-i+2})\cdots(\beta_{i-1}\beta_{2i-2}\beta_{3i-3}\cdots\beta_{p^d-1}).$$

Thus, $G_\alpha = \langle z^i \rangle$ and $a = z^\lambda \in \langle z^i \rangle$. Further, $i|\lambda$, which is a contradiction, as λ is a prime.

Let $(\mathbf{1},\beta_2,\beta_3,\cdots,\beta_\lambda,\mathbf{1}^a,\beta_2^a,\beta_3^a,\cdots,\beta_\lambda^a,\cdots,\mathbf{1}^{a^{p-2}},\beta_2^{a^{p-2}},\beta_3^{a^{p-2}},\cdots,\beta_\lambda^{a^{p-2}})$ be a cyclic permutation of $F^\#$ such that $\beta_2 = \mathbf{1}^z$, and it gives rise to a unique orientable complete map $\mathcal{M}_1$. We have

$$z : \mathbf{1} \mapsto \beta_2 \mapsto \beta_3 \mapsto \cdots \mapsto \beta_\lambda \mapsto \mathbf{1}^a \mapsto \beta_2^a \mapsto \cdots \mapsto \beta_\lambda^a \mapsto \cdots \mapsto \mathbf{1},$$

namely, z can be identified with the permutation

$$z = (\mathbf{1},\beta_2,\cdots,\beta_\lambda,\mathbf{1}^a,\beta_2^a,\cdots,\beta_\lambda^a,\cdots,\mathbf{1}^{a^{p-2}},\beta_2^{a^{p-2}},\cdots,\beta_\lambda^{a^{p-2}}).$$

It follows that $\mathrm{Aut}(\mathcal{M}_1) = F^+{:}\langle z \rangle = G.\lambda > G$ and $\mathcal{M}_1$ is arc transitive. Thus, $\mathcal{M}_1 \in \mathcal{A}_1$, $\mathcal{A}_1 \subset \mathcal{A}_\lambda$ and

$$|\mathcal{A}_\lambda \setminus \mathcal{A}_1| = |\mathcal{A}_\lambda| - |\mathcal{A}_1| = (\lambda-1)!(p-1)^{\lambda-1}\phi(p-1) - \phi(p^d-1).$$

So, $X \setminus G$ contains no element, which is an automorphism of $\mathcal{M}'_\lambda$ for $\mathcal{M}'_\lambda \in \mathcal{A}_\lambda \setminus \mathcal{A}_1$. Since $G \lhd X$ and

$$(X/G)_{\mathcal{M}'_\lambda} = \{xG \in X/G | (\mathcal{M}'_\lambda)^{xG} = (\mathcal{M}'_\lambda)^{Gx} = (\mathcal{M}'_\lambda)^x = \mathcal{M}'_\lambda\} = G,$$

then we have that X/G acting on $\mathcal{A}_\lambda \setminus \mathcal{A}_1$ is semi-regular.

Let X act on $\mathcal{A}_\lambda \setminus \mathcal{A}_1$. It follows that $(\mathcal{M}'_\lambda)^X$ is an orbit of this action, and the length of this orbit equals

$$|(\mathcal{M}'_\lambda)^X| = \frac{|X|}{|X_{\mathcal{M}'_\lambda}|} = \frac{|X|}{|\mathrm{Aut}(\mathcal{M}'_\lambda)|} = \frac{|X|}{|G|} = \frac{|\mathbb{Z}_p^d{:}\mathrm{GL}(d,p)|}{|\mathbb{Z}_p^d{:}\mathbb{Z}_{p-1}|} = |\mathrm{SL}(d,p)|.$$

Thus, by Lemma 4.3, there are

$$\frac{|\mathcal{A}_\lambda \setminus \mathcal{A}_1|}{|\mathrm{SL}(d,p)|} = \frac{|\mathcal{A}_\lambda| - |\mathcal{A}_1|}{|\mathrm{SL}(d,p)|} = \frac{(\lambda-1)!(p-1)^{\lambda-1}\phi(p-1) - \phi(p^d-1)}{|\mathrm{SL}(d,p)|}$$

non-isomorphic orientable vertex imprimitive complete maps. □

Below, we give an example to show existence of the orientable vertex imprimitive complete maps.

Example 4.5. *Let $\mathcal{M}$ be a Cayley map of $\mathbb{Z}_3^3$ with $p = 3$ and $d = 3$. Then $G \cong \mathbb{Z}_3^3{:}\mathbb{Z}_2$ is a Frobenius group and $G_\alpha \cong \mathbb{Z}_2$ acting on $\mathbb{Z}_3^3$ is half-transitive. Since 2 is not a primitive divisor of $3^3 - 1$, it follows that $\mathcal{M}$ is an orientable vertex imprimitive embedding of K_{27}, and G_α acting on $\Gamma(\alpha)$ has $\lambda = 13$ orbits. Furthermore, by Lemma 4.4, we can obtain that there are $\frac{|\mathcal{A}_{13}|-|\mathcal{A}_1|}{|SL(3,3)|}$ non-isomorphic such maps.*

Next, we can obtain the following results by the proof of Lemma 4.4.

Corollary 4.6. *If $\lambda = p_1 p_2$ with p_i different primes and $i = 1, 2$, then the number of non-isomorphic orientable vertex imprimitive complete maps equals*

$$\frac{|\mathcal{A}_{p_1 p_2} \setminus (\mathcal{A}_{p_1} \cup \mathcal{A}_{p_2})|}{|SL(d, p)|} = \frac{|\mathcal{A}_{p_1 p_2}| - |\mathcal{A}_{p_1}| - |\mathcal{A}_{p_2}| + |\mathcal{A}_1|}{|SL(d, p)|}.$$

Corollary 4.7. *If $\lambda = p_1^2 p_2$ with p_i different primes and $i = 1, 2$, then the number of non-isomorphic orientable vertex imprimitive complete maps equals*

$$\frac{|\mathcal{A}_{p_1^2 p_2} \setminus (\mathcal{A}_{p_1 p_2} \cup \mathcal{A}_{p_1^2})|}{|SL(d, p)|} = \frac{|\mathcal{A}_{p_1^2 p_2}| - |\mathcal{A}_{p_1 p_2}| - |\mathcal{A}_{p_1^2}| + |\mathcal{A}_{p_1}|}{|SL(d, p)|}.$$

It is an open problem to generalize the above corollaries for the general case in which $\lambda = p_1^{l_1} p_2^{l_2} \cdots p_t^{l_t}$, where $p_i (1 \leq i \leq t)$ are pairwise different primes and $l_i \geq 1$ are arbitrary positive integers.

5. Proof of the main results

In this section, we complete the proof of Theorem 1.1 in view of the above series of results.

Proof of Theorem 1.1. Let $\mathcal{M} = (V, E, F)$ be an orientable vertex imprimitive complete map. Let $G = \mathrm{Aut}(\mathcal{M})$. By Lemma 3.2, we have that $\mathcal{M}$ is a Cayley map of $\mathbb{Z}_p^d$ and $G \cong \mathbb{Z}_p^d{:}G_\alpha$ is a Frobenius group, where G_α is a cyclic group for each $\alpha \in V$, p is an odd prime, and $d \geq 2$. Further, if $G_\alpha \cong \mathbb{Z}_{p-1}$ acting on the neighborhood of α has λ orbits with $\lambda(p-1) = p^d - 1$ and $\lambda \geq 4$ is a prime, then by Lemma 3.3 and Lemma 4.4, there are exactly

$$[(\lambda - 1)!(p-1)^{\lambda - 1}\phi(p-1) - \phi(p^d - 1)]/|\mathrm{SL}(d, p)|$$

non-isomorphic orientable vertex imprimitive complete maps. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declare there is no conflict of interest.

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