Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 32(4): 2363–2379.
DOI: 10.3934/era.2024108
Received: 19 January 2024
Revised: 13 March 2024
Accepted: 17 March 2024
Published: 25 March 2024

Research article

On solutions for a class of Klein–Gordon equations coupled with Born–Infeld theory with Berestycki–Lions conditions on $\mathbb{R}^3$

Jiayi Fei[1,2] **and Qiongfen Zhang**[1,2,*]

[1] School of Mathematics and Statistics, Guilin University of Technology, Guangxi 541004, China

[2] Guangxi Colleges and Universities Key Laboratory of Applied Statistics, Guangxi 541004, China

* **Correspondence:** Email: qfzhangcsu@163.com.

Abstract: In this paper, the existence of multiple solutions for a class of Klein–Gordon equations coupled with Born–Infeld theory was investigated. The potential and the primitive of the nonlinearity in this kind of elliptic equations are both allowed to be sign-changing. Besides, we assumed that the nonlinearity satisfies the Berestycki–Lions type conditions. By employing Ekeland's variational principle, mountain pass theorem, Pohožaev identity, and various other techniques, two nontrivial solutions were obtained under some suitable conditions.

Keywords: Klein–Gordon equation; Born–Infeld theory; nontrivial solutions; Berestycki-Lions conditions; cut-off function

---

1. Introduction

This paper deals with the Klein–Gordon equation coupled with Born–Infeld theory

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = g(u) + h(x), & x \in \mathbb{R}^3; \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\Delta_4 = \mathrm{div}(|\nabla\phi|^2\nabla\phi)$, $\omega > 0, \beta > 0$, $u, V, \phi : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$. The application of the Klein–Gordon equation extends to the development of electrically charged field theory [1]. The energy of the functional related to a point-charge source is infinite in the original Maxwell theory. To overcome the problem of infinity, Born introduced the Born–Infeld (BI) electromagnetic theory [2–4]. The fundamental concept behind this theory is the principle of finiteness [5], where the conventional theory is modified to eliminate physical quantities involving infinities. Ensuring the finiteness of electric fields, a square root form with a parameter replaced the original Lagrangian density for Maxwell electrodynamics. Given its correlation in the realm of superstrings and membranes [6, 7], the Born–Infeld nonlinear electromagnetism has attracted significant focus from both theoretical physicists and mathematicians. For a more detailed exploration of the physical aspects, we recommend referring to [8–11]. To explore numerical techniques for constructing and approximating real solutions, reference [12] developed a Haar wavelet collocation method for solving first-order and second-order nonlinear hyperbolic equations. Recent advancements and outcomes concerning elliptic equations governed by a differential operator are succinctly summarized in reference [13].

First, when $h \equiv 0$, which is the homogeneous case, problem (1.1) has been widely analyzed. The seminal work by d'Avenia and Pisani [14] investigates first the existence of an infinite number of radially symmetric solutions for the following problem

$$\begin{cases} -\Delta u + [m_0^2 - (\omega+\phi)^2]u = |u|^{p-2}u, & \text{in } \mathbb{R}^3; \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega+\phi)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where m_0 is the mass of a particle (i.e., a physical constant), and $4 < p < 6$, $|\omega| < |m_0|$. When $2 < p \leq 4$ and $0 < \omega < \sqrt{\frac{1}{2}p - 1}|m_0|$, Mugnai [11] achieved an identical outcome. Subsequently, through the application of Pohožaev identity, Wang [15] improved the results of [11, 12] and derived the solitary wave solution by one of the following conditions:

(i) $3 < p < 6$ and $m_0 > \omega > 0$; (ii) $2 < p \leq 3$ and $(p-2)(4-p)m_0^2 > \omega^2 > 0$.

Yu [16] obtained the existence of the least-action solitary wave. Later, Chen and Song [17] studied the following Klein–Gordon equation with concave and convex nonlinearities coupled with BI theory:

$$\begin{cases} -\Delta u + V(x)u - (2\omega+\phi)\phi u = \lambda k(x)|u|^{q-2}u + g(x)|u|^{p-2}u, & x \in \mathbb{R}^3; \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega+\phi)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $1 < q < 2 < p < 6$. By Ekeland's variational principle and mountain pass theorem within the use of critical point theory, the existence of multiple nontrivial solutions for Eq (1.3) was demonstrated by imposing suitable assumptions on λ, $V(x)$, $k(x)$, and $g(x)$.

By replacing $|u|^{p-2}u$ with $|u|^{p-2}u + |u|^{2^*-2}u$, a nontrivial solution for Eq (1.2) was obtained by Teng and Zhang [18] under the conditions $4 \leq q < 6$ and $m_0 > \omega$. In this direction, He et al. [19] also enhanced the existence findings of equation in [18] and investigated the presence of a ground state solution for the system (1.2). For elliptic equations involving subcritical term and critical term, we can refer to [20–23]; references [24–26] provide other relevant results concerning homogeneous Klein–Gordon equations with Born–Infeld equations.

In this paper, we consider $h \not\equiv 0$, which is the nonhomogeneous case. Liu and Wu [27] recently investigated a kind of Klein–Gordon–Maxwell systems when the nonlinearity $g \in C(\mathbb{R}, \mathbb{R})$ and satisfies the following Berestycki–Lions conditions:

(g_1) $-\infty < \liminf_{s\to 0^+} \frac{g(s)}{s} \leq \limsup_{s\to 0^+} \frac{g(s)}{s} = -m < 0$;
(g_2) $\lim_{|s|\to+\infty} \frac{g(s)}{s^5} = 0$;
(g_3) there exists $\zeta > 0$ such that $G(\zeta) = \int_0^\zeta g(s)\,\mathrm{d}s > 0$, where $G(s) = \int_0^s g(t)\,\mathrm{d}t$.

Berestycki and Lions [28] introduced the above assumptions, highlighting the near indispensability of (g_1) and the necessity of (g_2) and (g_3) for proving the existence of nontrivial solutions. Liu et al. [29] delved into the existence of positive solution and multiple solutions of the Klein–Gordon–Maxwell system with Berestycki–Lions conditions. Within [30], the authors investigated standing waves for the pseudo-relativistic Hartree equation with Berestycki–Lions nonlinearity. Importantly, the Berestycki–Lions conditions are less restrictive compared to the conditions associated with g in [31–35]. Luo and Ahmed [36] concerned the Cauchy problem of nonlinear Klein–Gordon equations with general nonlinearities, establishing the global existence and finite-time blow-up of solutions with low and critical initial energy levels. It provides us some methods and insights. The assumptions mentioned earlier for the function g are utilized in this paper.

Through the application of variational methods, numerous solutions have been discovered for problem (1.4) with a constant potential $V(x) = m_0^2 - \omega^2$,

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = f(x,u) + h(x), & x \in \mathbb{R}^3; \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega + \phi)u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.4}$$

When $V(x) = m_0^2 - \omega^2$, $f(x,u) = |u|^{p-2}u$ and $h(x)$ exhibits radial symmetry, by utilizing the mountain pass theorem and the Ekeland's variational principle, Chen and Li [37] obtained two nontrivial solutions with radial symmetry for the nonhomogeneous problem (1.4), under one of the following conditions:

(i) $4 < p < 6$ and $|m_0| > \omega$; (ii) $2 < p \le 4$ and $\sqrt{\frac{1}{2}p - 1}|m_0| > \omega$.

By applying the variant fountain theorem, Wang and Xiong [38] were able to demonstrate the existence of two solutions, when considering the specified assumptions on V:

(V) $V \in C(\mathbb{R}^3, \mathbb{R})$ and $V_0 = \inf_{x\in\mathbb{R}^3} V(x) > 0$;
(V_2) there exists a constant $r > 0$ such that

$$\lim_{|y|\to+\infty} \text{meas}(\{x \in \mathbb{R}^3 : |x - y| \le r, V(x) \le M\}) = 0, \ \forall M > 0.$$

In order to ensure the compactness of Sobolev embedding, condition (V) was introduced in [39]. Wen and Tang [26] recently investigated system (1.4) with a sign-changing potential, while simultaneously setting $h(x) \equiv 0$. Apart from the given conditions (V_2), they further assumed the following condition holds:

(V_0) $V \in C(\mathbb{R}^3, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^3} V(x) > -\infty$.

By considering condition (V_0), it is implied that the potential V can be sign-changing. Inspired by [37–40], our current research focuses on investigating system (1.1), which has non-constant external potential and exhibits generalized superlinear growth conditions. Specifically, we are intrigued by the double sign-changing case, where both the primitive of g and the potential V change sign. However, this scenario poses a challenge as it prevents us from employing a conventional variational approach directly. Due to these reasons, the investigation of the double sign-changing case for the problem (1.1) has been limited in academic literature. Hence, the principal objective of this article is to discover a new result about the existence of multiple solutions based on comparatively weaker conditions. To express our conclusion, the following conditions on h and V are needed:

(V_1) $V \in C(\mathbb{R}^3, \mathbb{R})$ is radial and $\inf_{x\in\mathbb{R}^3} V(x) > -\infty$;
(V_3) $(x \cdot \nabla V(x)) \in L^2(\mathbb{R}^3)$, and there exists a constant $\varrho > 0$ such that $(x \cdot \nabla V(x)) \le \varrho$;
(h_1) $(x \cdot \nabla h) \in L^{\frac{6}{5}}(\mathbb{R}^3)$, where ∇h denotes the derivative of h and is in the weak sense;
(h_2) $h \in L^2(\mathbb{R}^3)$ is a radial function and $h \not\equiv 0$.

The main conclusion is stated here.

Theorem 1.1. *Suppose that* (V_1)–(V_3), (g_1)–(g_3) *and* (h_1)–(h_2) *hold. Then system* (1.1) *has at least two nontrivial solutions for* ω *and h, satisfying* $0 < \omega \le \omega_0$ *and* $|h|_2 < \Lambda$ *for some* ω_0, $\Lambda > 0$*, respectively.*

Remark 1.1. *In this paper, one of the two obtained solutions is negative energy and the other is positive energy. Moreover, under our assumptions, it seems difficult to obtain the second positive energy solution by the mountain pass theorem. It should be noted that problem* (1.1) *does not have a positive energy solution when* $\omega > 0$ *is sufficiently large. To overcome the difficulty, we introduce the cut-off function* η *and consider the modified function* I_T *to ensure boundedness of (PS) sequences with an additional property related to Pohožaev identity. Due to the appearance of the potential, the modified functional* $I_T(u)$ *is more complicate compared with the* $I_T(u)$ *in [26]. To prove the bounded (PS) sequence, we need more computations on the assignment of* $b_T(u)$*. Additionally, the (PS) sequence converges to a solution of problem* (1.1)*. Finally, we get that problem* (1.1) *has a positive energy solution with* $\omega > 0$ *small enough.*

2. The variational setting and preliminary results

In consideration of (V_1), the potential $V(x)$ is sign-changing in $\mathbb{R}^3$. As a result, the energy functional associated with the system (1.1) becomes quite intricate, for the quadratic form

$$B(u,u) := \int_{\mathbb{R}^3} [|\nabla u|^2 + V(x)u^2]dx$$

occurring in the energy functional lacks definiteness. To address the issue of the quadratic form's indefiniteness, we take an indirect approach by considering an equivalent system instead of directly dealing with the original system (1.1). In fact, it follows from (V_1) that there exists a constant $V' > 0$, so that $\tilde{V}(x) := V(x) + V' > 0$ for all $x \in \mathbb{R}^3$, and the quadratic form

$$\tilde{B}(u,u) := \int_{\mathbb{R}^3} [|\nabla u|^2 + \tilde{V}(x)u^2]dx$$

is positive definite. Consequently, with the assumption that $\tilde{g}(u) := g(u) + V'u$ and labeling the primitive function as $\tilde{G}(u)$, so that $\tilde{g}(u)$ and $\tilde{G}(u)$ can still meet the Berestycki-Lions conditions. We proceed to investigate the following alternative system:

$$\begin{cases} -\Delta u + \tilde{V}(x)u - (2\omega + \phi)\phi u = \tilde{g}(u) + h(x), & x \in \mathbb{R}^3; \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega + \phi)u^2, & x \in \mathbb{R}^3. \end{cases} \tag{2.1}$$

Clearly, system (1.1) is equivalent to system (2.1). Moreover, conditions (V)–(V_3) still hold for $\tilde{V}$, and we still apply conditions (g_1)–(g_3) to $\tilde{g}$ and $\tilde{G}(u)$, but the value of m is replaced by $\tilde{m} = m + V'$. Henceforth, the subsequent analysis will be focused on the study of system (2.1). For this reason, we will use (V) instead of (V_1) and assume that V is radial. So, in order to prove Theorem 1.1, we only need to prove system (2.1) has at least two nontrivial solution with the conditions (V_1)–(V_3), (g_1)–(g_3), and (h_1)–(h_2).

Given the assumption (V), the lack of compactness of the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ in our problem poses a challenge in establishing the satisfaction of the $(PS)_c$ condition for the functional I. Consequently, the work space of the functional I is the following radial space

$$H_r^1(\mathbb{R}^3) := \{u \in H^1(\mathbb{R}^3) : u(x) = u(|x|)\}, \tag{2.2}$$

and its norm is defined by

$$\|u\| = \left[\int_{\mathbb{R}^3} (|\nabla u|^2 + \tilde{V}(x)u^2)dx\right]^{\frac{1}{2}}.$$

Let E be defined by

$$E := \left\{u \in H_r^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} [|\nabla u|^2 + \tilde{V}(x)u^2]dx < \infty\right\},$$

then E is a Hilbert space. For $1 \le s < \infty$, we denote the following $|.|_s$ as the norm of the usual Lebesgue space $L^s(\mathbb{R}^3)$

$$|u|_s = \left(\int_{\mathbb{R}^3} |u|^s dx\right)^{\frac{1}{s}}.$$

The embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ is compact for any $s \in [2, 6]$ by the assumption (V_2); and is continuous for any $s \in [2, 6]$ as $\tilde{V}(x)$ is bounded from below. The embedding inequality

$$|u|_s \le \tau_s \|u\|, \quad \forall u \in E,\ s \in [2, 6],$$

holds for some $\tau_s > 0$. Let $D^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$ with the norm

$$\|u\|_{D^{1,2}} = \left[\int_{\mathbb{R}^3} |\nabla u|^2 dx\right]^{\frac{1}{2}}.$$

The radial space of $D^{1,2}(\mathbb{R}^3)$ is $D_r^{1,2}(\mathbb{R}^3) := \{u \in D^{1,2}(\mathbb{R}^3) : u = u(|x|)\}$. The completion of $C_0^\infty(\mathbb{R}^3, \mathbb{R})$ is $D(\mathbb{R}^3)$, whose norm is the following form

$$\|\phi\|_{D(\mathbb{R}^3)} = |\nabla\phi|_2 + |\nabla\phi|_4.$$

As well, $D^{1,2}(\mathbb{R}^3)$ is continuously embedded in $L^6(\mathbb{R}^3)$ by Sobolev inequality and $D(\mathbb{R}^3)$ is continuously embedded in $L^\infty(\mathbb{R}^3)$. Set $B_r := \{x \in \mathbb{R}^3 : |x| \le r\}$ and let C be a positive constant having different values in what follows.

Certainly, the energy functional of problem (2.1) is $F(u, \phi) : H^1(\mathbb{R}^3) \times D(\mathbb{R}^3) \to \mathbb{R}$, which is defined by

$$\begin{aligned} F(u, \phi) =& \frac{1}{2} \int_{\mathbb{R}^3} \left[|\nabla u|^2 + \tilde{V}(x)u^2 - (2\omega + \phi)\phi u^2\right] dx - \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla\phi|^2 dx \\ & - \frac{\beta}{16\pi} \int_{\mathbb{R}^3} |\nabla\phi|^4 dx - \int_{\mathbb{R}^3} \tilde{G}(u)dx - \int_{\mathbb{R}^3} hu dx, \end{aligned} \tag{2.3}$$

whose critical points are solutions of problem (2.1).

Lemma 2.1. *([11, 12]) For every $u \in E$, we have*
(i) there exists a unique $\phi = \phi_u \in D^{1,2}(\mathbb{R}^3)$, which solves

$$\Delta\phi + \beta\Delta_4\phi = 4\pi(\omega + \phi)u^2; \tag{2.4}$$

(ii) if u is radially symmetric, then ϕ_u is radially symmetric, too;
(iii) if $u(x) \neq 0$, then $-\omega \leq \phi_u \leq 0$;
(iv) $\|\phi_u\|_{D^{1,2}} \leq C|u|^2$ and $\int_{\mathbb{R}^3} |\phi_u|u^2 \leq C|u|^4_{\frac{12}{5}}$.

Proof. (i)–(iii) were proved in [11, 12]. Here, we establish (iv). From (2.4), we have

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla\phi_u|^2 dx + \int_{\mathbb{R}^3} |\nabla\phi_u|^4 dx &= -\int_{\mathbb{R}^3} 4\pi\omega\phi_u u^2 dx - \int_{\mathbb{R}^3} 4\pi\phi_u^2 u^2 dx \\ &\leq 4\pi\omega \int_{\mathbb{R}^3} |\phi_u|u^2 dx \leq 4\pi\omega\|\phi_u\|_{D^{1,2}}|u|^2_{\frac{12}{5}}.\end{aligned} \tag{2.5}$$

Then, from (2.5), we have $\|\phi_u\|_{D^{1,2}} \leq C|u|^2$ and $\int_{\mathbb{R}^3} |\nabla\phi_u|u^2 dx \leq C|u|^4_{\frac{12}{5}}$.

By Lemma 2.1 and the second equation in (2.1), we obtain

$$\frac{1}{4\pi}\int_{\mathbb{R}^3} |\nabla\phi_u|^2 dx + \frac{\beta}{4\pi}\int_{\mathbb{R}^3} |\nabla\phi_u|^4 = -\int_{\mathbb{R}^3} (\omega\phi_u + \phi_u^2)u^2 dx. \tag{2.6}$$

Lemma 2.2. *([11]) If $u_n \to u$ in E, then $\phi_{u_n} \to \phi_u$ in $D_r^{1,2}(\mathbb{R}^3)$ and $\phi_{u_n} \to \phi_u$ in $L_r^q(\mathbb{R}^3), 2 < q \leq 6$. Consequently, $I'(u_n) \to I'(u)$ in the sense of distributions.*

Define $I(u) = F(u, \phi_u)$, the functional $I : H_r^1(\mathbb{R}^3) \to \mathbb{R}$ for system (2.1) is as follows:

$$\begin{aligned}I(u) =& \frac{1}{2}\int_{\mathbb{R}^3} \left[|\nabla u|^2 + \tilde{V}(x)u^2 - (2\omega + \phi_u)\phi_u u^2\right] dx - \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla\phi_u|^2 dx \\ & - \frac{\beta}{16\pi}\int_{\mathbb{R}^3} |\nabla\phi_u|^4 dx - \int_{\mathbb{R}^3} \tilde{G}(u) dx - \int_{\mathbb{R}^3} hu dx.\end{aligned} \tag{2.7}$$

Under the assumptions (V), (V_2), (V_3), (g_1)–(g_3), and (h_1)–(h_2), one has $I \in C^1(E, \mathbb{R})$ and

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^3} \left[\nabla u \cdot \nabla v + \tilde{V}(x)uv - (2\omega + \phi_u)\phi_u uv\right] dx - \int_{\mathbb{R}^3} \tilde{g}(u)v dx - \int_{\mathbb{R}^3} hv dx, \tag{2.8}$$

for all $u, v \in E$.

Lemma 2.3. *(Mountain pass theorem, [41]) Let X be a Banach space, $I \in C^1(X, \mathbb{R}), e \in X$ and $r > 0$ be such that $\|e\| > r$ and*

$$b := \inf_{\|u\|=r} I(u) > I(0) \geq I(e).$$

If I satisfies the $(PS)_c$ condition with

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)), \quad \Gamma := \{\gamma \in C([0, 1], X) | \gamma(0) = 0, \gamma(1) = e\},$$

then c is a critical value of I.

3. Proof of the main result

Lemma 3.1. *Assume that* (g_1)–(g_3) *hold and let* $\{u_n\}$ *be a bounded (PS) sequence. Then,* $\{u_n\}$ *contains a convergent subsequence in* E.

Proof. Let $\{u_n\} \subset E$ be a bounded (PS) sequence in E, then up to a subsequence, it has

$$\begin{aligned} u_n &\rightharpoonup u, \quad \text{in } E; \\ u_n &\to u, \quad \text{in } L^q(\mathbb{R}^3),\ 2 \le q < 6; \\ u_n &\to u, \quad a.e. \text{ in } \mathbb{R}^3. \end{aligned}$$

Set

$$g_1(t) = \begin{cases} (\tilde{g}(t) + mt)^+, & \text{if } t \ge 0, \\ (\tilde{g}(t) + mt)^-, & \text{if } t < 0, \end{cases}$$

and $g_2(t) = g_1(t) - [\tilde{g}(t) + mt]$, $\forall\, t \in \mathbb{R}$. We obtain $\lim_{t\to 0} \frac{g_1(t)}{t} = 0$, $\lim_{t\to\pm\infty} \frac{g_1(t)}{t^5} = 0$ and $g_2(t)t \ge 0$, and $|g_2(t)| \le C(|t| + |t|^5)$, $\forall\, t \in \mathbb{R}$. Using this with Strauss's lemma (see [28]), we have

$$\int_{\mathbb{R}^3} [g_1(u_n) - g_1(u)](u_n - u)dx \to 0. \tag{3.1}$$

With Fatou's lemma, we have

$$\liminf_{n\to\infty} \int_{\mathbb{R}^3} g_2(u_n)u_n dx \ge \int_{\mathbb{R}^3} g_2(u)udx. \tag{3.2}$$

By the definition of $g_2(t)$, one sees that

$$|g_2(t)| \le C(|t| + |t|^5),\ \forall\, t \in \mathbb{R}. \tag{3.3}$$

Being $C_0^\infty(\mathbb{R}^3)$ dense in E, one knows that for any $\varepsilon > 0$, there exists $\xi \in C_0^\infty(\mathbb{R}^3)$ such that $\|\xi - u\| < \varepsilon$. Then, by the Hölder and Sobolev inequalities, we obtain

$$\begin{aligned} \left|\int_{\mathbb{R}^3} (g_2(u_n) - g_2(u))udx\right| &\le \left|\int_{\mathbb{R}^3} (g_2(u_n) - g_2(u))\xi dx\right| + \left|\int_{\mathbb{R}^3} (g_2(u_n) - g_2(u))(u - \xi)dx\right| \\ &\le o_n(1) + C(\|u_n\| + \|u\| + \|u_n\|^5 + \|u\|^5)\|\xi - u\| \le o_n(1) + C\varepsilon. \end{aligned} \tag{3.4}$$

Then, by (3.2), (3.4) and $u_n \rightharpoonup u$ in E, it has

$$\begin{aligned} &\int_{\mathbb{R}^3} [g_2(u_n) - g_2(u)](u_n - u)dx \\ = &\int_{\mathbb{R}^3} g_2(u_n)u_n dx - \int_{\mathbb{R}^3} g_2(u)u - \int_{\mathbb{R}^3} [g_2(u_n) - g_2(u)]udx - \int_{\mathbb{R}^3} g_2(u)(u_n - u)dx \ge 0. \end{aligned} \tag{3.5}$$

With the Hölder inequality and the Sobolev inequality, one sees

$$\begin{aligned} \left|\int_{\mathbb{R}^3} (\phi_{u_n} - \phi_u)u_n(u_n - u)dx\right| &\le |(\phi_{u_n} - \phi_u)(u_n - u)|_2 |u_n|_2 \\ &\le |\phi_{u_n} - \phi_u|_6 |u_n - u|_3 |u_n|_2 \\ &\le C\|\phi_{u_n} - \phi_u\|_{D^{1,2}} |u_n - u|_3 |u_n|_2, \end{aligned}$$

where $C > 0$ is a constant. As $u_n \to u$ in $L^s(\mathbb{R}^3)$ for any $2 \le s < 6$, we obtain

$$\int_{\mathbb{R}^3} (\phi_{u_n} - \phi_u) u_n (u_n - u) dx \to 0, \text{ as } n \to \infty, \tag{3.6}$$

and

$$\int_{\mathbb{R}^3} \phi_u (u_n - u)(u_n - u) dx \le |\phi_u|_6 |u_n - u|_3 |u_n - u|_2 \to 0, \text{ as } n \to \infty. \tag{3.7}$$

Consequently, from (3.6) and (3.7), we have

$$\int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u) dx = \int_{\mathbb{R}^3} (\phi_{u_n} - \phi_u) u_n (u_n - u) dx + \int_{\mathbb{R}^3} \phi_u (u_n - u)(u_n - u) dx \to 0. \tag{3.8}$$

Notice that the sequence $\{\phi_{u_n}^2 u_n\}$ is bounded in $L^{\frac{3}{2}}$ and $|\phi_{u_n}^2 u_n|_{\frac{3}{2}} \le |\phi_{u_n}|_6^2 |u_n|_3$, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^3} (\phi_{u_n}^2 - \phi_u^2)(u_n - u) dx \right| &\le |\phi_{u_n}^2 - \phi_u^2|_{\frac{3}{2}} |u_n - u|_3 \\ &\le (|\phi_{u_n}^2|_{\frac{3}{2}} + |\phi_u^2|_{\frac{3}{2}}) |u_n - u|_3 \\ &\to 0. \end{aligned} \tag{3.9}$$

Then, by (3.8) and (3.9), one has

$$\begin{aligned} &\int_{\mathbb{R}^3} [(2\omega + \phi_{u_n})\phi_{u_n} u_n - (2\omega + \phi_u)\phi_u u](u_n - u) dx \\ =& 2\omega \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u) + \int_{\mathbb{R}^3} (\phi_{u_n}^2 - \phi_u^2)(u_n - u) dx \to 0. \end{aligned} \tag{3.10}$$

It follows from (3.1), (3.5), and (3.10) that

$$\begin{aligned} \langle I'(u_n) - I'(u), u_n - u \rangle =& \int_{\mathbb{R}^3} \left[|\nabla (u_n - u)|^2 + \tilde{V}(x)(u_n - u)^2 + m(u_n - u)^2 \right] dx \\ &- \int_{\mathbb{R}^3} [(2\omega + \phi_{u_n})\phi_{u_n} u_n - (2\omega + \phi_u)\phi_u u](u_n - u) dx \\ &- \int_{\mathbb{R}^3} [g_1(u_n) - g_1(u)](u_n - u) dx + \int_{\mathbb{R}^3} [g_2(u_n) - g_2(u)](u_n - u) dx \\ \ge& \|u_n - u\|. \end{aligned} \tag{3.11}$$

From (3.11), one obtains that $u_n \to u$ in E as $n \to \infty$. The proof is completed.

Lemma 3.2. *Assume that* (V)–(V_3), (g_1)–(g_3), *and* (h_1)–(h_2) *hold. Then, system* (2.1) *has a nontrivial solution* $u_1 \in E$ *satisfying* $I(u_1) < 0$.

Proof. It follows from (g_1) and (g_2) that for some constants $D > 0$ and $C > 0$

$$\tilde{G}(t) \le -Dt^2 + Ct^6 \quad \text{for all} \quad t \in \mathbb{R}. \tag{3.12}$$

From (2.6), the functional I can be simplified to the form

$$\begin{aligned}I(u) =&\frac{1}{2}\int_{\mathbb{R}^3}\left[|\nabla u|^2+\tilde{V}(x)u^2-(2\omega+\phi_u)\phi_u u^2\right]dx-\frac{1}{8\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^2dx\\&-\frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^4dx-\int_{\mathbb{R}^3}\tilde{G}(u)dx-\int_{\mathbb{R}^3}hudx\\=&\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2+\tilde{V}(x)u^2)dx+\frac{1}{2}\int_{\mathbb{R}^3}\phi_u^2u^2dx+\frac{1}{8\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^2dx\\&+\frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^4dx-\int_{\mathbb{R}^3}\tilde{G}(u)dx-\int_{\mathbb{R}^3}hudx.\end{aligned}\tag{3.13}$$

From $-\omega\le\phi_u\le 0$, Sobolev inequality, (3.12) and (3.13), for some $C>0$, we have

$$\begin{aligned}I(u)&\ge\frac{1}{2}\|u\|^2+D\int_{\mathbb{R}^3}u^2dx-C\|u\|^6-|h|_2\|u\|\\&\ge\|u\|\left[\frac{1}{2}\|u\|-C\|u\|^5-|h|_2\right].\end{aligned}\tag{3.14}$$

Set $p(t)=\frac{1}{2}t-Ct^5$, for $t\ge 0$. It is easy to see that $\max_{t\ge 0}p(t)=\frac{1}{2}(\frac{1}{10C})^{\frac{1}{4}}-C(\frac{1}{10C})^{\frac{5}{4}}:=\Lambda>0$. Then, from (3.14), we have $I|_{\partial B_r}\ge\alpha$ for $|h|_2<\Lambda$, where $r=(\frac{1}{10C})^{\frac{1}{4}}$. By (g_1), for some $C_0>0$ and $k>0$, we have

$$|\tilde{G}(t)|\le C_0t^2\quad\text{for all}\quad|t|\le k.\tag{3.15}$$

With (g_2), for any $\sigma>0$, there exists $K_\sigma>k$ such that

$$|\tilde{G}(t)|\le\sigma t^6\quad\text{for all}\quad|t|\ge K_\sigma.\tag{3.16}$$

Since $\tilde{g}\in C(\mathbb{R},\mathbb{R})$, for some constant $C'_\sigma>0$, one has

$$|\tilde{G}(t)|\le C'_\sigma\le\frac{C'_\sigma}{k^2}t^2\quad\text{for all}\quad k\le|t|\le K_\sigma.\tag{3.17}$$

Set $C_\sigma=\max\{C_0,\frac{C'_\sigma}{k^2}\}$. Then, combining (3.15)–(3.17), for any $\sigma>0$, it yields that

$$|\tilde{G}(t)|\le C_\sigma t^2+\sigma t^6\quad\text{for all}\quad t\in\mathbb{R}.$$

We let $\sigma=\frac{1}{2}$ and let $u_0\in E$ satisfying $\int_{\mathbb{R}^3}h(x)u_0dx>0$, together with $-\omega\le\phi_u\le 0$, for $t_0>0$, one has

$$\begin{aligned}I(tu_0)=&\frac{t^2}{2}\int_{\mathbb{R}^3}(|\nabla u_0|^2+\tilde{V}(x)u_0^2)dx-\frac{t^2}{2}\int_{\mathbb{R}^3}(2\omega+\phi_{tu_0})\phi_{tu_0}u_0^2dx-\frac{1}{8\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^2dx\\&-\frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_{u_0}|^4dx-\int_{\mathbb{R}^3}\tilde{G}(tu_0)dx-\int_{\mathbb{R}^3}htu_0dx\\\le&\frac{t^2}{2}\int_{\mathbb{R}^3}(|\nabla u_0|^2+\tilde{V}(x)u_0^2)dx+t^2\omega^2\int_{\mathbb{R}^3}u_0^2dx\end{aligned}$$

$$+ C_{\frac{1}{2}} t^2 \int_{\mathbb{R}^3} u_0^2 dx + \frac{t^6}{2} \int_{\mathbb{R}^3} u_0^6 dx - t \int_{\mathbb{R}^3} h u_0 dx$$
$$<0, \ \ \forall\, 0 < t < t_0. \tag{3.18}$$

From (3.18), we know that there exists $tu_0 := u \in E$ small enough such that $I(u) < 0$. Together with $I|_{\partial B_r} \geq \alpha$, we have $c_0 := \inf_{u \in \bar{B}_r} I(u) < 0 < \inf_{u \in \partial \bar{B}_r} I(u)$. Using Ekeland's variational principle (see [42]), we obtained that a minimizing sequence $\{u_n\} \subset \bar{B}_r$ satisfying

$$c_0 \geq I(u_n) \geq c_0 + \frac{1}{n} \quad \text{and} \quad I(v) \geq I(u_n) - \frac{1}{n} \|u_n - v\|, \ \forall\, v \in \bar{B}_r. \tag{3.19}$$

So $\{u_n\}$ is a bounded (PS) sequence, combining with Lemma 3.1 and $\bar{B}_r$ is a closed set, there exists $\bar{u}_0 \in E$ such that $I'(\bar{u}_0) = 0$ and $I(\bar{u}_0) = c_0 < 0$.

Similar to the proof of Theorem 5.1 of [35], we can find a solution with positive energy to system (2.1) only when $\omega > 0$ small enough. To overcome the difficulty in obtaining bounded $(PS)_c$ sequence for the functional I, we define a cut-off function $\eta \in C^\infty(\mathbb{R}^+, \mathbb{R}^+)$ satisfying

$$\eta(t) = 1, \quad \text{for} \quad t \in [0, 1];$$
$$0 \leq \eta(t) \leq 1, \quad \text{for} \quad t \in (1, 2);$$
$$\eta(t) = 0, \quad \text{for} \quad t \in [2, +\infty);$$
$$|\eta'|_\infty \leq 2.$$

Consider the following modified functional:

$$I_T(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + \tilde{V}(x) u^2) dx - \frac{1}{2} a_T(u) \int_{\mathbb{R}^3} (2\omega + \phi_u) \phi_u u^2 dx - \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla \phi_u|^2 dx$$
$$- \frac{\beta}{16\pi} \int_{\mathbb{R}^3} |\nabla \phi_u|^4 dx - \int_{\mathbb{R}^3} \tilde{G}(u) dx - \int_{\mathbb{R}^3} h u dx,$$

and then for all $u, v \in E$,

$$\langle (I_T)'(u), v \rangle = (1 + b_T(u)) \int_{\mathbb{R}^3} \nabla u \nabla v dx + (1 + b_T(u)) \int_{\mathbb{R}^3} \tilde{V}(x) u v dx$$
$$- a_T(u) \int_{\mathbb{R}^3} (2\omega + \phi_u) \phi_u u v dx - \int_{\mathbb{R}^3} \tilde{g}(u) v dx - \int_{\mathbb{R}^3} h v dx,$$

where $T > 0$, $a_T(u) = \eta(\frac{\|u\|^2}{T^2})$ and $b_T(u) = -\frac{1}{T^2} \eta'(\frac{\|u\|^2}{T^2}) \int_{\mathbb{R}^3} (2\omega + \phi_u) \phi_u u^2 dx$. Consequently, we can obtain that $I(u) = I_T(u)$ when $\|u\| \leq T$.

Lemma 3.3. *Assume that (g_1)–(g_3) and (h_1)–(h_2) hold. Then,*
(a) $I_T|_{\partial B_r} \geq \alpha \geq 0$, where r and α are the same as that in Lemma 3.2;
(b) there exists $e_0 \in E$ such that $I_T(e_0) < 0$.

Proof. The proof of (a) is the same as that in Lemma 3.2. For (b), similar to [28], for any constant $M > 1$, one can define

$$y(x) = \zeta \text{ for } |x| \leq M; \ y(x) = \zeta(M + 1 - |x|) \text{ for } M \leq |x| \leq M + 1; \ y(x) = 0 \text{ for } |x| \geq M.$$

Hence, y is radial. Here, we work on the ball $B_M = \{x \in \mathbb{R}^N, |x| < M\}$; and let $M \to +\infty$. There exist $\bar{C}_i (i = 1, 2, 3, 4)$ such that

$$\int_{\mathbb{R}^3} |\nabla y|^2 dx \le \bar{C}_1 M^2, \quad \int_{\mathbb{R}^3} \tilde{G}(y) dx \ge \bar{C}_2 M^3 - \bar{C}_3 M^2, \quad \int_{\mathbb{R}^3} |hy| dx \le \bar{C}_4 (M+1)^{\frac{3}{2}}. \tag{3.20}$$

We define $\mathring{y} = y(\frac{\cdot}{\theta})$ for any θ, and let

$$\gamma(t) = \mathring{y}(\frac{x}{\theta}) \quad \text{for} \quad 0 < t \le 1; \quad \gamma(t) = 0 \quad \text{for} \quad t = 0.$$

By (3.20) and $-\omega \le \phi_u \le 0$, one has

$$\begin{aligned} I_T(\gamma(1)) =& \frac{1}{2} \int_{\mathbb{R}^3} [|\nabla y(\frac{x}{\theta})|^2 + \tilde{V}(x) y^2(\frac{x}{\theta})] dx - \frac{1}{2} a_T(y(\frac{x}{\theta})) \int_{\mathbb{R}^3} (2\omega + \phi_{y(\frac{x}{\theta})}) \phi_{y(\frac{x}{\theta})} y^2(\frac{x}{\theta}) dx \\ & - \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla \phi_{y(\frac{x}{\theta})}|^2 dx - \frac{\beta}{16\pi} \int_{\mathbb{R}^3} |\nabla \phi_{y(\frac{x}{\theta})}|^4 dx - \int_{\mathbb{R}^3} \tilde{G}(y(\frac{x}{\theta})) - \int_{\mathbb{R}^3} hy(\frac{x}{\theta}) dx \\ \le& \frac{\theta}{2} \bar{C}_1 M^2 + \frac{\theta^3}{2} \int_{\mathbb{R}^3} \tilde{V}(x) y^2 dx - \theta^3 (\bar{C}_2 M^3 - \bar{C}_3 M^2) + \theta^{\frac{3}{2}} \bar{C}_4 (M+1)^{\frac{3}{2}} \\ & + \omega^2 \theta^3 \eta \left(\frac{\theta \int_{\mathbb{R}^3} |\nabla y|^2 dx + \theta^3 \int_{\mathbb{R}^3} \tilde{V}(x) y^2 dx}{T^2} \right) \int_{\mathbb{R}^3} y^2 dx. \end{aligned} \tag{3.21}$$

Since θ can be large enough and $M > 1$, we can choose a y small enough such that

$$\eta\left(\frac{\theta \int_{\mathbb{R}^3} |\nabla y|^2 dx + \theta^3 \int_{\mathbb{R}^3} y^2 dx}{T^2}\right) = 0 \ \text{and} \ I_T(\gamma(1)) < 0.$$

Let $\theta_0 > 0$ sufficiently large, then the proof of Lemma 3.3 is completed by letting $e_0 = y(\frac{x}{\theta_0})$.

From Lemmas 2.3 and 3.3, we can define the mountain pass value

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_T(\gamma(t)) > 0,$$

where $\Gamma := \{\gamma \in C([0, 1], E)| : \gamma(0) = 0, I_T(\gamma(1)) < 0\}$. Similar to the proof in Appendix A of [43], we know that u satisfying $(I_T)'(u) = 0$ solves

$$\begin{cases} -(1 + b_T(u))\Delta u + (1 + b_T(u))\tilde{V}(x)u - a_T(u)(2\omega + \phi)\phi u = \tilde{g}(u) + h(x), & \text{in } \mathbb{R}^3; \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega + \phi)u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{3.22}$$

Hence, the following Pohožaev identity holds:

$$\begin{aligned} P_T(u) =& \frac{(1 + b_T(u))}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{2}(1 + b_T(u)) \int_{\mathbb{R}^3} [3\tilde{V}(x) + (x \cdot \nabla \tilde{V}(x))] u^2 dx + \frac{3\beta}{16\pi} \int_{\mathbb{R}^3} |\nabla \phi_u|^4 dx \\ & - \frac{1}{2} a_T(u) \int_{\mathbb{R}^3} (5\omega + 2\phi_u)\phi_u u^2 dx - \int_{\mathbb{R}^3} [3\tilde{G}(u) + 3hu + (x \cdot \nabla h)u] dx. \end{aligned}$$

Similar to the Lemma 3.3 of [44], one can obtain that there exists a sequence $\{u_n\} \subset E$ such that

$$I_T(u_n) \to c, \ (I_T)'(u_n) \to 0 \text{ and } P_T(u_n) \to 0, \ as \ n \to \infty. \tag{3.23}$$

Lemma 3.4. *Assume that (V)–(V_3), (g_1)–(g_3), and (h_1)–(h_2) hold. Let $\{u_n\}$ be a sequence given by (3.23). Then, for $T > 0$ large enough, there exists a positive constant ω_0 such that $\|u_n\| \leq T$ for any $0 < \omega < \omega_0$, which implies that $\{u_n\}$ is a bounded sequence for both I and I_T.*

Proof. We will argue by contradiction, from $-\omega \leq \phi_u \leq 0$ of Lemma 2.1, assumption (V_3) and the definition of η, we obtain

$$\frac{1}{2}a_T(u)\int_{\mathbb{R}^3}\phi_u^2u^2dx \leq C_1\omega^2T^2, \tag{3.24}$$

$$\frac{b_T(u)}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + 3\tilde{V}(x)u^2)dx \leq \frac{b_T(u)}{2}\cdot 3\|u\|^2 \leq C_2\omega^2T^2, \tag{3.25}$$

and

$$\frac{1+b_T(u)}{2}\int_{\mathbb{R}^3}(x\cdot\nabla\tilde{V}(x))u^2dx \leq \frac{1+b_T(u)}{2}\cdot\varrho\|u\|^2 \leq C_3\omega^2T^2, \tag{3.26}$$

where $C_1, C_2, C_3 > 0$ are constants. By (h_1) and Sobolev inequality, there exists $\kappa \in L^{\frac{6}{5}}(\mathbb{R}^3)$ and a constant $A_1 > 0$ such that

$$\left|\int_{\mathbb{R}^3}(x\cdot\nabla h)udx\right| \leq |\kappa|_{\frac{6}{5}}|u|_6 \leq A_1\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^{\frac{1}{2}}. \tag{3.27}$$

By (3.21), for some constant $A_2 > 0$, we have

$$\begin{aligned} c \leq \max_\theta I_T(y(\frac{x}{\theta})) &\leq \max_\theta\{\frac{\theta}{2}\bar{C}_1M^2 - \theta^3(\bar{C}_2M^3 - \bar{C}_3M^2) + \theta^{\frac{3}{2}}\bar{C}_4(M+1)^{\frac{3}{2}}\} \\ &\quad + \max_\theta \omega^2\theta^3a_T(y(\frac{x}{\theta}))\int_{\mathbb{R}^3}y^2(\frac{x}{\theta})dx \\ &\leq A_2 + C_1\omega^2T^2. \end{aligned} \tag{3.28}$$

From (3.24)–(3.28), we have

$$\begin{aligned} 3c + o_n(1) =& 3I_T - P_T(u_n) \\ =& \int_{\mathbb{R}^3}|\nabla u_n|^2x - \frac{b_T(u_n)}{2}\int_{\mathbb{R}^3}(|\nabla u_n|^2 + 3\tilde{V}(x)u_n^2)dx - \frac{1+b_T(u_n)}{2}\int_{\mathbb{R}^3}(x\cdot\nabla\tilde{V}(x))u_n^2dx \\ &- \frac{1}{2}a_T(u_n)\int_{\mathbb{R}^3}(\omega+\phi_u)\phi_uu_n^2dx + \int_{\mathbb{R}^3}(x\cdot\nabla h)u_ndx - \frac{3}{8\pi}\int_{\mathbb{R}^3}|\nabla\phi_{u_n}|^2dx - \frac{3\beta}{8\pi}\int_{\mathbb{R}^3}|\nabla\phi_{u_n}|^4dx. \end{aligned} \tag{3.29}$$

By (3.29), there exists $C > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}^3}|\nabla u_n|^2dx \leq& 3c + \frac{b_T(u_n)}{2}\int_{\mathbb{R}^3}(|\nabla u_n|^2 + 3\tilde{V}(x)u_n^2)dx + \frac{1+b_T(u_n)}{2}\int_{\mathbb{R}^3}(x\cdot\nabla\tilde{V}(x))u_n^2dx \\ &+ \frac{1}{2}a_T(u_n)\int_{\mathbb{R}^3}\phi_{u_n}^2u_n^2dx - \int_{\mathbb{R}^3}(x\cdot\nabla h)u_ndx + \frac{3}{8\pi}\int_{\mathbb{R}^3}|\nabla\phi_{u_n}|^2dx + \frac{3\beta}{8\pi}\int_{\mathbb{R}^3}|\nabla\phi_{u_n}|^4dx \end{aligned}$$

$$\leq 3A_2 + C\omega^2 T^2 + A_1 \left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^{\frac{1}{2}}$$
$$+ \frac{3}{8\pi} \left(\int_{\mathbb{R}^3} |\nabla \phi_{u_n}|^2 dx + \beta \int_{\mathbb{R}^3} |\nabla \phi_{u_n}|^4 dx \right) + o_n(1),$$

which implies

$$\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^{\frac{1}{2}} \leq \frac{A_1}{2} + \sqrt{\frac{A_1^2}{4} + 3A_2 + C\omega^2 T^2 + \frac{3}{8\pi}(\int_{\mathbb{R}^3} |\nabla \phi_{u_n}|^2 dx + \beta \int_{\mathbb{R}^3} |\nabla \phi_{u_n}|^4 dx) + o_n(1)}. \quad (3.30)$$

Since $\{u_n\}$ satisfies $\langle (I_T)', u_n \rangle = o_n(1)$, by (g_1) and (g_2), we obtain

$$(1 + b_T(u_n)) \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + (1 + b_T(u_n)) \int_{\mathbb{R}^3} \tilde{V}(x) u_n^2 dx$$
$$+ m \int_{\mathbb{R}^3} u_n^2 dx - a_T(u_n) \int_{\mathbb{R}^3} (2\omega + \phi_{u_n}) \phi_{u_n} u_n^2 dx - \int_{\mathbb{R}^3} h u_n dx$$
$$\leq \int_{\mathbb{R}^3} (\tilde{g}(u_n) u_n + m u_n^2) dx$$
$$\leq \frac{1}{2} \int_{\mathbb{R}^3} u_n^2 dx + C \int_{\mathbb{R}^3} u_n^6 dx + o_n(1), \quad (3.31)$$

where C is a positive constant. By Sobolev inequality and (3.31), we have

$$\min\{\frac{m}{2}, 1\} \|u_n\|^2 - C|h|_2 \|u_n\| \leq C \left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^3 + o_n(1). \quad (3.32)$$

We suppose by contradiction that $\|u_n\| > T$. Using (3.30) and (3.32), one has

$$\min\{\frac{m}{2}, 1\} T^2 - CT$$
$$\leq C \left(\frac{A_1}{2} + C \sqrt{\frac{A_1^2}{4} + 3A_2 + C\omega^2 T^2 + \frac{3}{8\pi} \left(\int_{\mathbb{R}^3} |\nabla \phi_{u_n}|^2 dx + \beta \int_{\mathbb{R}^3} |\nabla \phi_{u_n}|^4 dx \right) + o_n(1)} \right)^6 + o_n(1). \quad (3.33)$$

We can choose $\omega < \omega_0 < T^{-1}$ such that $\omega^2 T^2 < 1$, then from (3.33), we know that this is impossible for T is large enough. Thus, we complete the proof.

4. Conclusions

Proof of Theorem 1.1.

Proof. From Lemma 3.4, we know that I has a bounded (PS) sequence $\{u_n\}$ with $\|u_n\| \leq T$ and $I(u_n) \to c > 0$ for any $\omega \in (0, \omega_0]$. It follows from Lemma 3.1 that there exists $\tilde{u}_0 \in E$ such that $u_n \to \tilde{u}_0$. Then, we have $I'(\tilde{u}_0) = 0$ and $I(\tilde{u}_0) = c > 0$. Together with Lemma 3.2, the proof of Theorem 1.1 is completed.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to thank the referees for their useful suggestions, which have significantly improved the paper. This work is supported by the National Natural Science Foundation of China (No. 11961014) and Guangxi Natural Science Foundation (No. 2021GXNSFAA196040).

Conflicts of interest

The authors declare there is no conflict of interest.

References

1. B. Felsager, B. R. Holstein, Geometry, particles and fields, *Am. J. Phys.*, **52** (1984), 573. https://doi.org/10.1119/1.13608
2. M. Born, Modified field equations with a finite radius of the electron, *Nature*, **132** (1933), 282. https://doi.org/10.1038/132282a0
3. M. Born, On the quantum theory of the electromagnetic field, *Proc. R. Soc. A*, **143** (1934), 410–437. https://doi.org/10.1098/rspa.1934.0010
4. M. Born, L. Infeld, Foundations of the new field theory, *Proc. R. Soc. A*, **144** (1934), 425–451. https://doi.org/10.1098/rspa.1934.0059
5. M. E. Peskin, *An Introduction to Quantum Field Theory*, 1st edition, CRC press, 1995. https://doi.org/10.1201/9780429503559
6. N. Seiberg, E. Witten, String theory and noncommutative geometry, *J. High Energy Phys.*, **1999** (1999), 032. https://doi.org/10.1088/1126-6708/1999/09/032
7. Y. S. Yang, Classical solutions in the Born-Infeld theory, *Proc. R. Soc. A*, **456** (2000), 615–640. https://doi.org/10.1098/rspa.2000.0533
8. V. Benci, D. Fortunato, A. Masiello, L. Pisani, Solitons and the electromagnetic field, *Math. Z.*, **232** (1999), 73–102. https://doi.org/10.1007/PL00004759
9. M. Carmeli, Field theory on R×S 3 topology. I: the Klein-Gordon and Schrödinger equations, *Found. Phys.*, **15** (1985), 175—184. https://doi.org/10.1007/BF00735289
10. D. Fortunato, L. Orsina, L. Pisani, Born-Infeld type equations for electrostatic fields, *J. Math. Phys.*, **43** (2002), 5698–5706. https://doi.org/10.1063/1.1508433
11. D. Mugnai, Coupled Klein-Gorndon and Born-Infeld type equations: looking for solitary waves, *Proc. R. Soc. A*, **460** (2004), 1519–1527. https://doi.org/10.1098/rspa.2003.1267
12. W. Lei, M. Ahsan, W. Khan, Z. Uddin, M. Ahmad, A numerical Haar wavelet-finite difference hybrid method and its convergence for nonlinear hyperbolic partial differential equation, *Demonstratio Math.*, **56** (2023), 20220203. https://doi.org/10.1515/dema-2022-0203

13. N. S. Papageorgiou, Double phase problems: a survey of some recent results, *Opuscula Math.*, **42** (2022), 257–278. https://doi.org/10.7494/OpMath.2022.42.2.257
14. P. d'Avenia, L. Pisani, Nonlinear Klein-Gordon equations coupled with Born-Infeld type equations, *Electron. J. Differ. Equations*, **2002** (2002), 1–13.
15. F. Z. Wang, Solitary waves for the coupled nonlinear Klein-Gordon and Born-Infeld type equations, *Electron. J. Differ. Equations*, **2012** (2012), 1–12.
16. Y. Yu, Solitary waves for nonlinear Klein-Gordon equations coupled with Born-Infeld theory, *Ann. Inst. Henri Poincare*, **27** (2010), 351–376. https://doi.org/10.1016/j.anihpc.2009.11.001
17. S. J. Chen, S. Z. Song, The existence of multiple solutions for the Klein-Gordon equation with concave and convex nonlinearities coupled with Born-Infeld theory on $\mathbb{R}^3$, *Nonlinear Anal. Real World Appl.*, **38** (2017), 78–95. https://doi.org/10.1016/j.nonrwa.2017.04.008
18. K. M. Teng, K. J. Zhang, Existence of solitary wave solutions for the nonlinear Klein-Gordon equation coupled with BornInfeld theory with critical Sobolev exponent, *Nonlinear Anal. Theory Methods Appl.*, **74** (2011), 4241–4251. https://doi.org/10.1016/j.na.2011.04.002
19. C. M. He, L. Li, S. J. Chen, D. O'Regan, Ground state solution for the nonlinear Klein-Gordon equation coupled with Born-Infeld theory with critical exponents, *Anal. Math. Phys.*, **12** (2022), 48. https://doi.org/10.1007/s13324-022-00661-1
20. C. M. He, L. Li, S. J. Chen, Nontrivial solution for Klein-Gordon equation coupled with Born-Infeld theory with critical growth, *Adv. Nonlinear Anal.*, **12** (2023), 20220282. https://doi.org/10.1515/anona-2022-0282
21. L. Baldelli, R. Filippucci, Singular quasilinear critical Schrödinger equations in $\mathbb{R}^N$, *Commun. Pure Appl. Anal.*, **21** (2022), 2561–2586. https://doi.org/10.3934/cpaa.2022060
22. L. Baldelli, Y. Brizi, R. Filippucci, On symmetric solutions for (p,q)-Laplacian equations in $\mathbb{R}^N$ with critical terms, *J. Geom. Anal.*, **32** (2022), 120. https://doi.org/10.1007/s12220-021-00846-3
23. Z. Feng, Y. Su, Lions-type properties for the p-Laplacian and applications to quasilinear elliptic equations, *J. Geom. Anal.*, **33** (2023), 99. https://doi.org/10.1007/s12220-022-01150-4
24. F. Albuquerque, S. J. Chen, L. Li, Solitary wave of ground state type for a nonlinear Klein-Gordon equation coupled with Born-Infeld theory in $\mathbb{R}^2$, *Electron. J. Qual. Theory Differ. Equations*, **12** (2020), 1–18. https://doi.org/10.14232/ejqtde.2020.1.12
25. K. M. Teng, Existence and multiple of the solutions for the nonlinear Klein-Gordon equation coupled with Born-Infeld theory on boundary domain, *Differ. Equations Appl.*, **4** (2012), 445–457. https://doi.org/10.7153/dea-04-26
26. L. X. Wen, X. H. Tang, S. T. Chen, Infinitely many solutions and least energy solutions for Klein-Gordon equation coupled with Born-Infeld theory, *Complex Var. Elliptic Equations*, **64** (2019), 2077–2090. https://doi.org/10.1080/17476933.2019.1572124
27. X. Q. Liu, X. P. Wu, Multiple solutions for nonhomogeneous Klein-Gordon-Maxwell system with Berestycki-Lions conditions, *Appl. Math. Lett.*, **137** (2023), 108505. https://doi.org/10.1016/j.aml.2022.108505

28. H. Berestycki, P. L. Lions, Nonlinear scalar field equations. I. existence of a ground state, *Arch. Ration. Mech. Anal.*, **82** (1983), 313–345. https://doi.org/10.1007/BF00250555

29. X. Q. Liu, G. D. Li, C. L. Tang, Existence of nontrivial solutions for the Klein-Gordon-Maxwell system with Berestycki-Lions conditions, *Adv. Nonlinear Anal.*, **12** (2023), 20220294. https://doi.org/10.1515/anona-2022-0294

30. F. S. Gao, V. D. Radulescu, M. B. Yang, Y. Zhang, Standing waves for the pseudo-relativistic Hartree equation with Berestycki-Lions nonlinearity, *J. Differ. Equations*, **295** (2021), 70–112. https://doi.org/10.1016/j.jde.2021.05.047

31. S. J. Chen, S. Z. Song, Multiple solutions for nonhomogeneous Klein-Gordon-Maxwell equations on $\mathbb{R}^3$, *Nonlinear Anal. Real World Appl.*, **22** (2015), 259–271. https://doi.org/10.1016/j.nonrwa.2014.09.006

32. H. X. Shi, H. B. Chen, Multiple positive solutions for nonhomogeneous Klein-Gordon-Maxwell equations, *Appl. Math. Comput.*, **337** (2018), 504–513. https://doi.org/10.1016/j.amc.2018.05.052

33. L. Wang, Two solutions for a nonhomogeneous Klein-Gordon-Maxwell system, *Electron. J. Qual. Theory Differ. Equations*, **40** (2019), 1–12. https://doi.org/10.14232/ejqtde.2019.1.40

34. D. L. Wu, H. Lin, Multiple solutions for superlinear Klein-Gordon-Maxwell equations, *Math. Nachr.*, **293** (2020), 1827–1835. https://doi.org/10.1002/mana.201900129

35. L. Xu, H. Chen, Existence and multiplicity of solutions for nonhomogeneous Klein-Gordon-Maxwell equations, *Electron. J. Differ. Equations*, **2015** (2015), 1–12.

36. Y. Luo, M. S. Ahmed, Cauchy problem of nonlinear Klein-Gordon equations with general nonlinearities, *Rend. Circ. Mat. Palermo, Ser. 2*, **71** (2022), 959–973. https://doi.org/10.1007/s12215-021-00698-4

37. S. J. Chen, L. Li, Multiple solutions for the nonhomogeneous Klein-Gordon equation coupled with Born-Infeld theory on $\mathbb{R}^3$, *J. Math. Anal. Appl.*, **400** (2013), 517–524. https://doi.org/10.1016/j.jmaa.2012.10.057

38. L. X. Wang, C. L. Xiong, P. P. Zhao, Two solutions for nonhomogeneous Klein-Gordon equations coupled with Born-Infeld type equations, *Electron. J. Differ. Equations*, **2022** (2022), 1–11.

39. T. Bartsch, Z. Q. Wang, M. Willem, The Dirichlet problem for superlinear elliptic equations, in *Handbook of Differential Equations: Stationary Partial Differential Equations*, **2** (2005), 1–5. https://doi.org/10.1016/S1874-5733(05)80009-9

40. W. M. Zou, M. Schechter, *Critical Point Theory and Its Applications,* Springer, New York, 2006. https://doi.org/10.1007/0-387-32968-4

41. A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, **14** (1973), 349–381. https://doi.org/10.1016/0022-1236(73)90051-7

42. J. Mawhin, M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer Science + Business Media, New York, 1989. https://doi.org/10.1007/978-1-4757-2061-7

43. A. Azzollini, P. d'Avenia, A. Pomponio, On the Schrödinger-Maxwell equations under the effect of a general nonlinear term, *Ann. Inst. Henri Poincare*, **27** (2010), 779–791. https://doi.org/10.1016/j.anihpc.2009.11.012

44. X. Q. Liu, C. L. Tang, Infinitely many solutions and concentration of ground state solutions for the Klein-Gordon-Maxwell system, *J. Math. Anal. Appl.*, **505** (2022), 125521. https://doi.org/10.1016/j.jmaa.2021.125521