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ERA, 32(4): 2286–2309.
DOI: 10.3934/era.2024104
Received: 13 January 2024
Revised: 19 February 2024
Accepted: 11 March 2024
Published: 21 March 2024

Research article

The existence of solutions of Hadamard fractional differential equations with integral and discrete boundary conditions on infinite interval

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Abstract: In this article, the properties of solutions of Hadamard fractional differential equations are investigated on an infinite interval. The equations are subject to integral and discrete boundary conditions. A new proper compactness criterion is introduced in a unique space. By applying the monotone iterative technique, we have obtained two positive solutions. And, an error estimate is also shown at the end. This study innovatively uses a monotonic iterative approach to study Hadamard fractional boundary-value problems containing multiple fractional derivative terms on infinite intervals, and it enriches some of the existing conclusions. Meanwhile, it is potentially of practical significance in the research field of computational fluid dynamics related to blood flow problems and in the direction of the development of viscoelastic fluids.

Keywords: fractional derivative; infinite interval; monotone iterative; error estimate

1. Introduction

The two most important aspects for studying differential equations are fitting and solving. First, for a practical problem, we want to determine whether we can fit a suitable equation that can realistically portray the practical problem. Second, for a practical problem, we want to determine whether we can find suitable solutions to verify the reasonableness of the equation, thus reflecting the practicability of the practical problem.

Fractional differential equations (FDEs) are popular in the fields of physics, engineering, and biology because they can well characterize complex processes such as heritability and memory properties; see the related literature [1–5]. In order to better fit the constructed equations, we can add corresponding boundary-value problems (BVPs) to the equations, such as the integral BVP and the multipoint BVP; we can also regard the problem as a system of equations by means of coupling, or we can add the semilinear Laplace operators. In these ways, different practical problems can be better transformed into equations [6–8]. After we obtain the equation, we need to verify the practical significance of the solutions. And, there are a number of ways to determine the properties of solutions, such as the monotonic iterative methods [9, 10], and others. More detailed studies are as follows.

Hadamard [11] established the concept of fractional derivatives in 1892. Hadamard derivatives differ significantly from Riemann-Liouville and Caputo derivatives in terms of fractional powers; specifically, the kernel of the integral contains the logarithmic function of an arbitrary exponent. Hadamard derivatives have stable characterizations in terms of expansion and well matched the problems on the half-open interval; see [4]. At the same time, the Hadamard FDE has an important role in the mechanical behavior of viscoelastic materials and turbulence phenomena in fluid dynamics; see [12, 13]. Since the problem we discuss in this paper is restricted to half-open intervals, we consider the use of Hadamard fractional derivatives.

Integral boundary conditions are instrumental in computational fluid dynamics studies related to blood flow problems. When dealing with these problems, the usual approach is to assume that the cross-section of the blood vessels is circular, which is not always reasonable. In order to optimize this detailed problem and make the results more detailed and convincing, the integral boundary conditions can be included to develop an efficient and applicable method. More details can be found in [14]. In addition, integral boundary conditions have other uses in physics and biology; see [15].

The authors of [16] found that the study of fractional BVPs for m-points on infinite intervals is almost non-existent, so they studied the related problem by referencing [17] for the first time. The authors of [17] found that the intrinsic equations of viscoelastic fluids in the models of physics and biology are closely connected with the FDEs; see [18]. Therefore, multipoint boundary problems are beginning to be studied.

In recent years, Hao et al. [19] considered a Hadamard FDE with integral boundary conditions. By applying Schauder's fixed-point theorem and Banach's contraction principle, they obtained the unique solution of the equation. Li et al. [20] considered the two integral boundary conditions of the Riemann-Liouville FDE. By employing Krasnoselskii's fixed-point theorem and Banach's contraction principle, they obtained the existence of the solution.

Zhang and Liu [21] applied Banach's contraction mapping principle, the monotone iterative method, and the Avery-Peterson fixed-point theorem to show the existence, uniqueness, and multiplicity results of solutions:

$$\begin{cases} {}^{H}D_{1+}^{\alpha}x(t) + a(t)f(t, x(t)) = 0, 2 < \alpha < 3, t \in (1, +\infty), \\ x(1) = x'(1) = 0, \quad {}^{H}D_{1+}^{\alpha-1}x(+\infty) = \sum_{i=1}^{m} \alpha_i {}^{H}I_{1+}^{\beta_i}x(\eta) + b\sum_{j=1}^{n}\sigma_j x(\xi_j), \end{cases}$$

where ${}^{H}D_{1+}^{\alpha}$ is the Hadamard-type fractional derivative of order α; $1 < \eta < \xi_i < +\infty$; $b, \alpha_i, \sigma_j \geq 0$ $(i, j = 1, 2, \ldots, n)$.

In [22], the authors investigated the fractional BVP with

$$\begin{cases} {}^{H}D_{1+}^{\alpha}u(t) + p(t)f(t, u(t), {}^{H}D_{1+}^{\alpha-1}u(t)) = 0, \quad t \in (1, +\infty), \\ u^{(k)}(1) = 0, \quad {}^{H}D_{1+}^{\alpha-1}u(+\infty) = \int_{1}^{+\infty} g(t)u(t)\frac{dt}{t} + \sum_{i=1}^{m}\lambda_i^{H} I_{1+}^{\beta_i}x(\eta), \end{cases}$$

where $n - 1 < \alpha \leq n$; $0 \leq k \leq n - 2$, $\eta \in (1, +\infty)$; $\lambda_i, \beta_i > 0$ $(i = 1, 2, \ldots, m)$; $g \in C([1, +\infty), (0, +\infty))$. By employing the Bai-Ge fixed-point theorem, they obtained the solutions of the Hadamard-type FDE.

By considering the above two boundary conditions, research questions and the need for practical problem-solving, this article investigates the existence of positive solutions to the following Hadamard-type FDE with an integral and multipoint discrete BVP:

$$\begin{cases} {}^{H}D_{1+}^{\theta}x(t) + r(t)f(t, x(t), {}^{H}D_{1+}^{\theta-2}x(t), {}^{H}D_{1+}^{\theta-1}x(t)) = 0, \quad t \in (1, +\infty), \\ x(1) = x'(1) = 0, \quad {}^{H}D_{1+}^{\theta-1}x(+\infty) = \displaystyle\int_{1}^{+\infty} \omega(t)x(t)\frac{dt}{t} + \sum_{i=1}^{m} \beta_i x(\eta_i), \end{cases} \tag{1.1}$$

where ${}^{H}D_{1+}^{\theta}$ is the Hadamard-type fractional derivative of order θ, $2 < \theta \leq 3$; $f \in C\ ([1, +\infty) \times [0, +\infty) \times [0, +\infty), [0, +\infty))$; $\omega \in L^1\ [1, +\infty)$, where $\omega \geq 0$; $1 < \eta_1 < \eta_2 < \cdots < \eta_m < +\infty$; β_i denotes positive real constants ($i = 1, 2, \ldots, m$).

In previous works, the fixed-point theorem is usually used to determine the existence of the solutions of the equations, while the innovation of this paper is the use of the monotone iterative method to solve Hadamard FDEs containing multiple lower-order derivative terms, which results in not only obtaining the existence of the two positive solutions, but also deriving the error estimation formula for the unique positive solution. This paper enriches the use of monotone iterative methods and has potential application to the development of blood flow modeling and properties of viscoelastic fluids for practical applications.

2. Preliminaries and lemmas

In this section, it is essential to present some important lemmas.

2.1. Basic concepts and properties

Definition 1. ([23]) Let $\varphi > 0$; the Hadamard fractional integral of order φ for a function $f : [1, +\infty) \to \mathbb{R}$ is defined as

$${}^{H}I_{1+}^{\varphi}f(t) = \frac{1}{\Gamma(\varphi)} \int_{1}^{t} \left(\log\frac{t}{s}\right)^{\varphi-1} f(s)\frac{ds}{s}, \quad (t > 1).$$

Definition 2. ([23]) Let $\varphi > 0$; the Hadamard fractional derivative of order φ for a function $f : [1, +\infty) \to \mathbb{R}$ is defined as

$${}^{H}D_{1+}^{\varphi}f(t) = \frac{1}{\Gamma(n-\varphi)}\left(t\frac{d}{dt}\right)^{n} \int_{1}^{t} \left(\log\frac{t}{s}\right)^{n-\varphi-1} f(s)\frac{ds}{s}, \quad (t > 1),$$

where $n = [\varphi] + 1$ and $[\varphi]$ is the integer part of φ.

Lemma 1. *([23]) If* $\varphi, \psi > 0$*, then*

$${}^{H}I_{1+}^{\varphi}(\log t)^{\psi-1} = \frac{\Gamma(\psi)}{\Gamma(\varphi+\psi)}(\log t)^{\varphi+\psi-1}, \quad {}^{H}D_{1+}^{\varphi}(\log t)^{\psi-1} = \frac{\Gamma(\psi)}{\Gamma(\psi-\varphi)}(\log t)^{\psi-\varphi-1}.$$

Lemma 2. *([23]) Let $\varphi > 0$; the solution of ${}^{H}D^{\varphi}_{1+}x(t) = 0$ with $x \in C[1,\infty) \cap L^1[1,\infty)$ is valid if, and only if,*

$$x(t) = \sum_{i=1}^{n} c_i (\log t)^{\varphi - i},$$

and the following formula holds:

$${}^{H}I^{\varphi}_{1+}\,{}^{H}D^{\varphi}_{1+}x(t) = x(t) + \sum_{i=1}^{n} c_i (\log t)^{\varphi - i},$$

where $c_i \in \mathbb{R},\ i = 1, 2, \dots, n,\ n-1 < \varphi < n.$

Lemma 3. *We define $h(t) \in L^1(1,+\infty)$, $0 < \int_1^{+\infty} h(s)\frac{ds}{s} < +\infty$, and*

$$\Upsilon - \int_1^{+\infty} \omega(t)(\log t)^{\theta-1}\frac{dt}{t} =: \Upsilon_1 > 0, \tag{2.1}$$

where

$$\Upsilon = \Gamma(\theta) - \sum_{i=1}^{m} \beta_i (\log \eta_i)^{\theta-1}. \tag{2.2}$$

Then, the solution of the Hadamard-type FDE given by

$$\begin{cases} {}^{H}D^{\theta}_{1+}x(t) + h(t) = 0, \quad t \in (1,+\infty), \\ x(1) = x'(1) = 0, \quad {}^{H}D^{\theta-1}_{1+}x(+\infty) = \displaystyle\int_1^{+\infty} \omega(t)x(t)\frac{dt}{t} + \sum_{i=1}^{m}\beta_i x(\eta_i), \end{cases} \tag{2.3}$$

can be expressed as

$$x(t) = \int_1^{+\infty} K(t,s)h(s)ds, \quad t \in (1,+\infty), \tag{2.4}$$

where

$$K(t,s) = K_1(t,s) + K_2(t,s), \tag{2.5}$$

$$K_1(t,s) = k(t,s) + \sum_{i=1}^{m} \frac{\beta_i(\log t)^{\theta-1}}{\Upsilon\Gamma(\theta)} k_i(\eta_i, s), \tag{2.6}$$

$$K_2(t,s) = \frac{(\log t)^{\theta-1}}{\Upsilon_1}\int_1^{+\infty} K_1(t,s)\omega(t)\frac{dt}{t}, \tag{2.7}$$

$$k(t,s) = \frac{1}{\Gamma(\theta)} \begin{cases} (\log t)^{\theta-1} - \left(\log \frac{t}{s}\right)^{\theta-1}, & 1 \le s \le t < \infty, \\ (\log t)^{\theta-1}, & 1 \le t \le s < \infty, \end{cases}$$

$$k_i(\eta_i, s) = \begin{cases} (\log \eta_i)^{\theta-1} - \left(\log \frac{\eta_i}{s}\right)^{\theta-1}, & 1 \le s \le \eta_i < \infty, \\ (\log \eta_i)^{\theta-1}, & 1 \le \eta_i \le s < \infty. \end{cases}$$

Proof. Because of Lemma 2, (2.3) has a solution:

$$x(t) = -{}^{H}I_{1+}^{\theta}h(t) + c_1(\log t)^{\theta-1} + c_2(\log t)^{\theta-2} + c_3(\log t)^{\theta-3}. \tag{2.8}$$

From $x(1) = x'(1) = 0$, we know that $c_2 = c_3 = 0$. From Lemma 1, we have

$$^{H}D_{1+}^{\theta-1}x(t) = -{}^{H}I_{1+}^{1}h(t) + c_1\frac{\Gamma(\theta)}{\Gamma(1)}.$$

Considering the boundary condition ${}^{H}D_{1+}^{\theta-1}x(+\infty) = \int_1^{+\infty}\omega(t)x(t)\frac{dt}{t} + \sum_{i=1}^{m}\beta_i x(\eta_i)$, we conclude that

$$c_1 = \frac{1}{\Upsilon}\left\{\int_1^{+\infty}h(s)\frac{ds}{s} + \int_1^{+\infty}\omega(t)x(t)\frac{dt}{t} - \sum_{i=1}^{m}\frac{\beta_i}{\Gamma(\theta)}\int_1^{\eta_i}(\log\frac{\eta_i}{s})^{\theta-1}h(s)\frac{ds}{s}\right\}.$$

Consequently, substituting c_1, c_2, and c_3 into (2.8), we have

$$\begin{aligned} x(t) =& \frac{(\log t)^{\theta-1}}{\Upsilon}\int_1^{\infty}h(s)\frac{ds}{s} + \frac{(\log t)^{\theta-1}}{\Upsilon}\int_1^{\infty}\omega(t)x(t)\frac{dt}{t} - \frac{1}{\Gamma(\theta)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\theta-1}h(s)\frac{ds}{s} \\ &- \sum_{i=1}^{m}\frac{\beta_i(\log t)^{\theta-1}}{\Upsilon\Gamma(\theta)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\theta-1}h(s)\frac{ds}{s}. \end{aligned}$$

Next, after piecing together and organizing the first and third terms of the above equation, we get

$$\begin{aligned} x(t) =& \frac{(\log t)^{\theta-1}}{\Gamma(\theta)}\int_1^{\infty}h(s)\frac{ds}{s} + \frac{(\Gamma(\theta)-\Upsilon)(\log t)^{\theta-1}}{\Upsilon\Gamma(\theta)}\int_1^{\infty}h(s)\frac{ds}{s} + \frac{(\log t)^{\theta-1}}{\Upsilon}\int_1^{\infty}\omega(t)x(t)\frac{dt}{t} \\ &- \frac{1}{\Gamma(\theta)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\theta-1}h(s)\frac{ds}{s} - \sum_{i=1}^{m}\frac{\beta_i(\log t)^{\theta-1}}{\Upsilon\Gamma(\theta)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\theta-1}h(s)\frac{ds}{s}, \end{aligned}$$

with the help of (2.2) and some arrangement, we obtain

$$\begin{aligned} x(t) =& \frac{(\log t)^{\theta-1}}{\Gamma(\theta)}\int_1^{\infty}h(s)\frac{ds}{s} + \sum_{i=1}^{m}\frac{\beta_i(\log t)^{\theta-1}}{\Upsilon\Gamma(\theta)}\int_1^{\infty}(\log\eta_i)^{\theta-1}h(s)\frac{ds}{s} \\ &+ \frac{(\log t)^{\theta-1}}{\Upsilon}\int_1^{\infty}\omega(t)x(t)\frac{dt}{t} - \frac{1}{\Gamma(\theta)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\theta-1}h(s)\frac{ds}{s} \\ &- \sum_{i=1}^{m}\frac{\beta_i(\log t)^{\theta-1}}{\Upsilon\Gamma(\theta)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\theta-1}h(s)\frac{ds}{s} \\ =& \int_1^{\infty}k(t,s)h(s)\frac{ds}{s} + \sum_{i=1}^{m}\frac{\beta_i(\log t)^{\theta-1}}{\Upsilon\Gamma(\theta)}\int_1^{\infty}k_i(\eta_i,s)h(s)\frac{ds}{s} + \frac{(\log t)^{\theta-1}}{\Upsilon}\int_1^{\infty}\omega(t)x(t)\frac{dt}{t} \\ =& \int_1^{\infty}K_1(t,s)h(s)\frac{ds}{s} + \frac{(\log t)^{\theta-1}}{\Upsilon}\int_1^{\infty}\omega(t)x(t)\frac{dt}{t}. \end{aligned}$$

Then,

$$\begin{aligned}\int_1^{+\infty}\omega(t)x(t)\frac{dt}{t} &= \int_1^{\infty}\omega(t)\left(\int_1^{\infty}K_1(t,s)h(s)\frac{ds}{s}+\frac{(\log t)^{\theta-1}}{\Upsilon}\int_1^{\infty}\omega(t)x(t)\frac{dt}{t}\right)\frac{dt}{t}\\ &= \int_1^{\infty}\omega(t)\int_1^{\infty}K_1(t,s)h(s)\frac{ds}{s}\frac{dt}{t}\\ &\quad +\frac{1}{\Upsilon}\int_1^{\infty}\omega(t)(\log t)^{\theta-1}\frac{dt}{t}\int_1^{\infty}\omega(t)x(t)\frac{dt}{t}.\end{aligned}$$

Given that $\Upsilon_1 = \Upsilon - \int_1^{\infty}\omega(t)(\log t)^{\theta-1}\frac{dt}{t}$, we have

$$\begin{aligned}\int_1^{+\infty}\omega(t)x(t)\frac{dt}{t} &= \frac{\Upsilon}{\Upsilon_1}\int_1^{+\infty}\omega(t)\int_1^{+\infty}K_1(t,s)h(s)\frac{ds}{s}\frac{dt}{t}\\ &= \frac{\Upsilon}{\Upsilon_1}\int_1^{+\infty}h(s)\int_1^{+\infty}K_1(t,s)\omega(t)\frac{dt}{t}\frac{ds}{s}.\end{aligned}$$

Thus,

$$\begin{aligned}x(t) &= \int_1^{\infty}K_1(t,s)h(s)\frac{ds}{s}+\frac{(\log t)^{\theta-1}}{\Upsilon}\int_1^{\infty}\omega(t)x(t)\frac{dt}{t}\\ &= \int_1^{\infty}K_1(t,s)h(s)\frac{ds}{s}+\frac{(\log t)^{\theta-1}}{\Upsilon}\left(\frac{\Upsilon}{\Upsilon_1}\int_1^{\infty}h(s)\int_1^{\infty}K_1(t,s)\omega(t)\frac{dt}{t}\frac{ds}{s}\right)\\ &= \int_1^{\infty}K_1(t,s)h(s)\frac{ds}{s}+\int_1^{\infty}\frac{(\log t)^{\theta-1}}{\Upsilon_1}\int_1^{\infty}K_1(t,s)\omega(t)\frac{dt}{t}h(s)\frac{ds}{s}\\ &= \int_1^{+\infty}K_1(t,s)h(s)\frac{ds}{s}+\int_1^{+\infty}K_2(t,s)h(s)\frac{ds}{s}\\ &= \int_1^{+\infty}K(t,s)h(s)\frac{ds}{s}.\end{aligned}$$

Lemma 4. *The function $K(t,s)$ defined in (2.5) satisfies the following conditions:*
1) $K(t,s)$ is continuous for $(t,s)\in[1,+\infty)\times[1,+\infty)$;
2) $K(t,s)$ is nonnegative on $[1,+\infty)\times[1,+\infty)$;
3) $\frac{K(t,s)}{1+(\log t)^{\theta-1}} \le \frac{K(t,s)}{(\log t)^{\theta-1}} \le \frac{1}{\Upsilon_1}$ for all $(t,s)\in[1,+\infty)\times[1,+\infty)$.

Proof. We obviously get that conditions 1) and 2) above. Next, we show that condition 3) holds. For all $(t,s)\in[1,+\infty)\times[1,+\infty)$, we deduce the following:

$$
\begin{aligned}
\frac{K(t,s)}{(\log t)^{\theta-1}} =& \frac{K_1(t,s)}{(\log t)^{\theta-1}} + \frac{K_2(t,s)}{(\log t)^{\theta-1}} \\
=& \frac{k(t,s)}{(\log t)^{\theta-1}} + \sum_{i=1}^{m} \frac{\beta_i k_i(\eta_i,s)}{\Upsilon\Gamma(\theta)} + \frac{1}{\Upsilon_1}\int_1^{+\infty} K_1(t,s)\omega(t)\frac{dt}{t} \\
\leq& \frac{1}{\Gamma(\theta)} + \sum_{i=1}^{m} \frac{\beta_i(\log\eta_i)^{\theta-1}}{\Upsilon\Gamma(\theta)} + \frac{1}{\Upsilon_1\Gamma(\theta)}\int_1^{+\infty} \omega(t)(\log t)^{\theta-1}\frac{dt}{t} \\
&+ \frac{1}{\Upsilon_1}\sum_{i=1}^{m} \frac{\beta_i(\log\eta_i)^{\theta-1}}{\Upsilon\Gamma(\theta)}\int_1^{+\infty} \omega(t)(\log t)^{\theta-1}\frac{dt}{t} \\
=& \frac{1}{\Gamma(\theta)} + \sum_{i=1}^{m} \frac{\beta_i(\log\eta_i)^{\theta-1}}{\Upsilon\Gamma(\theta)} \\
&+ \frac{1}{\Upsilon\Upsilon_1\Gamma(\theta)}\int_1^{+\infty} \omega(t)(\log t)^{\theta-1}\frac{dt}{t}\left(\Upsilon + \sum_{i=1}^{m}\beta_i(\log\eta_i)^{\theta-1}\right) \\
=& \frac{1}{\Gamma(\theta)} + \frac{1}{\Gamma(\theta)}\sum_{i=1}^{m} \frac{\Gamma(\theta)\beta_i(\log\eta_i)^{\theta-1}}{\Upsilon\Gamma(\theta)} + \frac{1}{\Upsilon\Upsilon_1}\int_1^{+\infty} \omega(t)(\log t)^{\theta-1}\frac{dt}{t} \\
=& \frac{\Upsilon\Upsilon_1 + \Upsilon_1(\Gamma(\theta)-\Upsilon) + \Gamma(\theta)\Upsilon - \Gamma(\theta)\Upsilon_1}{\Gamma(\theta)\Upsilon\Upsilon_1} \\
=& \frac{1}{\Upsilon_1}.
\end{aligned}
$$

The proof is completed.

By Lemma 3 and (2.4), we get

$$
{}^{H}D_{1^+}^{\theta-1}x(t) = \int_1^{+\infty} K^{\star}(t,s)h(s)\frac{ds}{s}, \qquad t \in (1,+\infty),
$$

where

$$
K^{\star}(t,s) = k^{\star}(t,s) + \sum_{i=1}^{m} \frac{\beta_i k_i(\eta_i,s)}{\Upsilon} + \frac{\Gamma(\theta)}{\Upsilon_1}\int_1^{+\infty} K_1(t,s)\omega(t)\frac{dt}{t}, \tag{2.9}
$$

$$
k^{\star}(t,s) = \begin{cases} 0, & 1 \leq s \leq t < +\infty, \\ 1, & 1 \leq t \leq s < +\infty. \end{cases}
$$

From Lemma 3 and (2.4), we have

$$
{}^{H}D_{1^+}^{\theta-2}x(t) = \int_1^{+\infty} K_{\star}(t,s)h(s)\frac{ds}{s}, \qquad t \in (1,+\infty),
$$

where

$$
K_{\star}(t,s) = k_{\star}(t,s) + \sum_{i=1}^{m} \frac{\beta_i \log t}{\Upsilon} k_i(\eta_i,s) + \frac{\Gamma(\theta)\log t}{\Upsilon_1}\int_1^{+\infty} K_1(t,s)\omega(t)\frac{dt}{t}, \tag{2.10}
$$

$$
k_{\star}(t,s) = \begin{cases} \log t - \log\dfrac{t}{s}, & 1 \leq s \leq t < +\infty, \\ \log t, & 1 \leq t \leq s < +\infty. \end{cases}
$$

Lemma 5. *The functions $K^\star(t,s)$ and $K_\star(t,s)$, defined in (2.9) and (2.10), ensure that*

$$0 \le K^\star(t,s) \le \frac{\Gamma(\theta)}{\Upsilon_1}, \quad 0 \le \frac{K_\star(t,s)}{1+\log t} \le \frac{K_\star(t,s)}{\log t} \le \frac{\Gamma(\theta)}{\Upsilon_1}, \quad t,s \in [1,+\infty). \tag{2.11}$$

Proof. According to (2.9) and (2.10), we can easily get that $K^\star(t,s) \ge 0$ and $\frac{K_\star(t,s)}{1+\log t} \ge 0$. Furthermore, for all $t, s \in [1,+\infty)$,

$$\begin{aligned}
K^\star(t,s) =& k^\star(t,s) + \sum_{i=1}^{m} \frac{\beta_i k_i(\eta_i,s)}{\Upsilon} + \frac{\Gamma(\theta)}{\Upsilon_1}\int_1^{+\infty} K_1(t,s)\omega(t)\frac{dt}{t} \\
\le& 1 + \sum_{i=1}^{m} \frac{\beta_i(\log\eta_i)^{\theta-1}}{\Upsilon} + \frac{1}{\Upsilon_1}\int_1^{\infty}(\log t)^{\theta-1}\omega(t)\frac{dt}{t} \\
&+ \frac{\Gamma(\theta)}{\Upsilon_1}\sum_{i=1}^{m}\frac{\beta_i(\log\eta_i)^{\theta-1}}{\Upsilon\Gamma(\theta)}\int_1^{\infty}(\log t)^{\theta-1}\omega(t)\frac{dt}{t} \\
=& 1 + \frac{\Gamma(\theta)-\Upsilon}{\Upsilon} + \frac{1}{\Upsilon\Upsilon_1}\int_1^{\infty}(\log t)^{\theta-1}\omega(t)\frac{dt}{t}\left(\Upsilon + \sum_{i=1}^{m}\beta_i(\log\eta_i)^{\theta-1}\right) \\
=& 1 + \frac{\Gamma(\theta)-\Upsilon}{\Upsilon} + \frac{\Gamma(\theta)(\Upsilon-\Upsilon_1)}{\Upsilon\Upsilon_1} \\
=& \frac{\Gamma(\theta)}{\Upsilon_1}.
\end{aligned}$$

By the same steps, we have

$$\begin{aligned}
\frac{K_\star(t,s)}{\log t} =& \frac{k_\star(t,s)}{\log t} + \sum_{i=1}^{m} \frac{\beta_i k_i(\eta_i,s)}{\Upsilon} + \frac{\Gamma(\theta)}{\Upsilon_1}\int_1^{+\infty} K_1(t,s)\omega(t)\frac{dt}{t} \\
\le& 1 + \frac{\Gamma(\theta)-\Upsilon}{\Upsilon} + \frac{1}{\Upsilon\Upsilon_1}\int_1^{\infty}(\log t)^{\theta-1}\omega(t)\frac{dt}{t}\left(\Upsilon + \sum_{i=1}^{m}\beta_i(\log\eta_i)^{\theta-1}\right) \\
=& 1 + \frac{\Gamma(\theta)-\Upsilon}{\Upsilon} + \frac{\Gamma(\theta)(\Upsilon-\Upsilon_1)}{\Upsilon\Upsilon_1} \\
=& \frac{\Gamma(\theta)}{\Upsilon_1}.
\end{aligned}$$

The proof is completed.

2.2. Compactness of the operator

Next, let

$$F = \left\{ x, {}^H D_{1+}^{\theta-2} x, {}^H D_{1+}^{\theta-1} x \in C[1,+\infty) : \sup_{t\in[1,+\infty)} \frac{|x(t)|}{1+(\log t)^{\theta-1}} < +\infty, \right.$$
$$\left. \sup_{t\in[1,+\infty)} \frac{|{}^H D_{1+}^{\theta-2} x(t)|}{1+\log t} < +\infty, \sup_{t\in[1,+\infty)} |{}^H D_{1+}^{\theta-1} x(t)| < +\infty \right\},$$

$$\|x\| = \max\left\{ \sup_{t\in[1,+\infty)} \frac{|x(t)|}{1+(\log t)^{\theta-1}}, \sup_{t\in[1,+\infty)} \frac{|{}^H D_{1+}^{\theta-2} x(t)|}{1+\log t}, \sup_{t\in[1,+\infty)} |{}^H D_{1+}^{\theta-1} x(t)| \right\}.$$

Similar to [19], $(F, \|\cdot\|)$ is a Banach space.

Lemma 6. *Let $U \subset F$ be a bounded set. Then, U is relatively compact in F if the following conditions hold:*

1) For all $x(t) \in U$, $\frac{x(t)}{1+(\log t)^{\theta-1}}$, $\frac{{}^H D_{1+}^{\theta-2} x(t)}{1+\log t}$, and ${}^H D_{1+}^{\theta-1} x(t)$ are equicontinuous on any compact interval of $[1,+\infty)$.

2) For all $\varepsilon > 0$, there is a constant $T = T(\varepsilon) > 1$, which satifies

$$\left| \frac{x(t_1)}{1+(\log t_1)^{\theta-1}} - \frac{x(t_2)}{1+(\log t_2)^{\theta-1}} \right| < \varepsilon, \quad \left| \frac{{}^H D_{1+}^{\theta-2} x(t_1)}{1+\log t_1} - \frac{{}^H D_{1+}^{\theta-2} x(t_2)}{1+\log t_2} \right| < \varepsilon,$$

and

$$\left| {}^H D_{1+}^{\theta-1} x(t_1) - {}^H D_{1+}^{\theta-1} x(t_2) \right| < \varepsilon,$$

for any t_1, $t_2 > T$ and $x \in U$.

Proof. This proof has been proved by Lemma 4 in [19], so it is omitted here.

Define a cone P :

$$P = \{x \in F : x(t) \geq 0, \ {}^H D_{1+}^{\theta-2} x(t) \geq 0, \ {}^H D_{1+}^{\theta-1} x(t) \geq 0, \ t \in [1,+\infty)\}.$$

Next, we give two circumstances:

(H_1) $f \in C([1,+\infty) \times [0,+\infty)^3, [0,+\infty))$ and $f(t,0,0,0) \not\equiv 0$ on any subinterval of $[1,+\infty)$; when $|x| \leq M$, $|y| \leq M$ for any $M > 0$, $f(t, (1+(\log t)^{\theta-1})x, (1+\log t)y, z)$ is bounded on $[1,+\infty)$.

(H_2) $r(t) : [1,+\infty) \to [0,+\infty)$, $r(t) \not\equiv 0$ on any subinterval of $[1,+\infty)$, and $0 < \int_1^{+\infty} r(s)\frac{ds}{s} < +\infty$.

For the convenience of description, let

$$F(s) = f(s, x(s), {}^H D_{1+}^{\theta-2} x(s), {}^H D_{1+}^{\theta-1} x(s)),$$

$$F_k(s) = f(s, x_k(s), {}^H D_{1+}^{\theta-2} x_k(s), {}^H D_{1+}^{\theta-1} x_k(s)), \quad k = 0,1,2,\ldots,n.$$

According to Lemma 3, let $T : P \to F$ be an operator with

$$Tx(t) = \int_1^{+\infty} K(t,s) r(s) F(s) \frac{ds}{s}, \quad t \in [1,+\infty), \tag{2.12}$$

Thus, we can get the solution of (1.1) to be the fixed point of the operator T.

Lemma 7. *Assuming that* (H_1) *and* (H_2) *are valid, then* $T : P \to F$ *is a continuous operator.*

Proof. $Tx(t) \geq 0$ is obvious for all $t \in [1, +\infty)$ and $x \in P$. Then, we have the evidence that T is continuous:

$$\begin{aligned}
&\sup_{t\in[1,+\infty)} \frac{|Tx(t)|}{1 + (\log t)^{\theta-1}}\\
&= \sup_{t\in[1,+\infty)} \int_1^{+\infty} \frac{K(t, s)}{1 + (\log t)^{\theta-1}} |r(s)F(s)| \frac{ds}{s}\\
&\leq \frac{1}{\Upsilon_1} \int_1^{+\infty} |r(s)f[s, \frac{(1 + (\log s)^{\theta-1})x(s)}{1 + (\log s)^{\theta-1}}, \frac{(1 + \log s)^H D_{1^+}^{\theta-2} x(s)}{1 + \log s}, {}^H D_{1^+}^{\theta-1} x(s)]| \frac{ds}{s}\\
&< +\infty,
\end{aligned}$$

$$\begin{aligned}
&\sup_{t\in[1,+\infty)} \frac{|{}^H D_{1^+}^{\theta-2} Tx(t)|}{1 + \log t}\\
&= \sup_{t\in[1,+\infty)} \int_1^{+\infty} \frac{K_\star(t, s)}{1 + \log t} |r(s)F(s)| \frac{ds}{s}\\
&\leq \frac{\Gamma(\theta)}{\Upsilon_1} \int_1^{+\infty} |r(s)f[s, \frac{(1 + (\log s)^{\theta-1})x(s)}{1 + (\log s)^{\theta-1}}, \frac{(1 + \log s)^H D_{1^+}^{\theta-2} x(s)}{1 + \log s}, {}^H D_{1^+}^{\theta-1} x(s)]| \frac{ds}{s}\\
&< +\infty,
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{t\in[1,+\infty)} |{}^H D_{1^+}^{\theta-1} Tx(t)|\\
&= \sup_{t\in[1,+\infty)} \int_1^{+\infty} K^\star(t, s) |r(s)F(s)| \frac{ds}{s}\\
&\leq \frac{\Gamma(\theta)}{\Upsilon_1} \int_1^{+\infty} |r(s)f[s, \frac{(1 + (\log s)^{\theta-1})x(s)}{1 + (\log s)^{\theta-1}}, \frac{(1 + \log s)^H D_{1^+}^{\theta-2} x(s)}{1 + \log s}, {}^H D_{1^+}^{\theta-1} x(s)]| \frac{ds}{s}\\
&< +\infty.
\end{aligned}$$

Thus, we know that $T : P \to F$.

Next, we give the proof that $T : P \to F$ is a continuous operator. Let $x_n \to x_0$ as $n \to \infty$ in P; there is a constant $k_0 > 0$ such that $\sup_{n\in\mathbb{N}} \|x_n\| \leq k_0$, $\|x_0\| \leq k_0$. Let $B_{k_0} = \sup\{f(t, (1 + (\log t)^{\theta-1})x, (1 + \log t)y, z) | (t, x, y, z) \in [1, +\infty) \times [0, k_0]^3\}$. Next, we aim to prove that $\|Tx_n - Tx_0\| \to 0$ as $n \to \infty$. By (H_1), (H_2), the Lebesgue dominated convergence theorem, and continuity of f, we can get

$$\lim_{n\to\infty}\int_1^{+\infty}\frac{K(t,s)}{1+(\log t)^{\theta-1}}r(s)F_n(s)\frac{ds}{s}=\int_1^{+\infty}\frac{K(t,s)}{1+(\log t)^{\theta-1}}r(s)F_0(s)\frac{ds}{s},$$

$$\lim_{n\to\infty}\int_1^{+\infty}\frac{K_\star(t,s)}{1+\log t}r(s)F_n(s)\frac{ds}{s}=\int_1^{+\infty}\frac{K_\star(t,s)}{1+\log t}r(s)F_0(s)\frac{ds}{s},$$

$$\lim_{n\to\infty}\int_1^{+\infty}K^\star(t,s)r(s)F_n(s)\frac{ds}{s}=\int_1^{+\infty}K^\star(t,s)r(s)F_0(s)\frac{ds}{s}.$$

Therefore,

$$\begin{aligned}&\sup_{t\in[1,+\infty)}\frac{|Tx_n(t)-Tx_0(t)|}{1+(\log t)^{\theta-1}}\\ &\le\sup_{t\in[1,+\infty)}\int_1^{+\infty}\frac{K(t,s)}{1+(\log t)^{\theta-1}}r(s)\Big|F_n(s)-F_0(s)\Big|\frac{ds}{s}\to 0\quad(n\to\infty),\end{aligned}$$

$$\sup_{t\in[1,+\infty)}\frac{|{}^HD_{1+}^{\theta-2}Tx_n(t)-{}^HD_{1+}^{\theta-2}Tx_0(t)|}{1+\log t}\to 0\quad(n\to\infty),$$

and

$$\sup_{t\in[1,+\infty)}|{}^HD_{1+}^{\theta-1}Tx_n(t)-{}^HD_{1+}^{\theta-1}Tx_0(t)|\to 0\quad(n\to\infty).$$

Hence, T is continuous.

Lemma 8. *Suppose that* $\int_1^{+\infty}(\log t)^{\theta-1}\omega(t)\frac{dt}{t}<\infty$ *and* (H_1) *and* (H_2) *hold; then,* $T:P\to P$ *is a compact operator.*

Proof. We will show this lemma in the following three procedures. First, let Ω be a bounded subset of cone P to prove the truth that $T\Omega$ is a bounded set of cone P. There is a constant $k_1>0$, which guarantees that $\|x\|\le k_1$ for all $x\in\Omega$. Let $B_{k_1}=\sup\{f(t,(1+\log t)^{\theta-1})x,(1+\log t)y,z)|(t,x,y,z)\in[1,+\infty)\times[0,k_1]^3\}$. By Lemmas 4 and 5, we get

$$\begin{aligned}&\sup_{t\in[1,+\infty)}\frac{|Tx(t)|}{1+(\log t)^{\theta-1}}\\ &=\int_1^{+\infty}\frac{K(t,s)}{1+(\log t)^{\theta-1}}|r(s)F(s)|\frac{ds}{s}\\ &\le\frac{1}{\Upsilon_1}\int_1^{+\infty}|r(s)f[s,\frac{(1+(\log s)^{\theta-1})x(s)}{1+(\log s)^{\theta-1}},\frac{(1+\log s)^HD_{1+}^{\theta-2}x(s)}{1+\log s},{}^HD_{1+}^{\theta-1}x(s)]|\frac{ds}{s}\\ &\le\frac{B_{k_0}}{\Upsilon_1}\int_1^{+\infty}r(s)\frac{ds}{s}\\ &<+\infty,\end{aligned}$$

$$
\begin{aligned}
&\sup_{t\in[1,+\infty)} \frac{|{}^H D_{1+}^{\theta-2} T x(t)|}{1+\log t} \\
&= \int_1^{+\infty} \frac{K_\star(t,s)}{1+\log t} |r(s)F(s)| \frac{ds}{s} \\
&\le \frac{\Gamma(\theta)}{\Upsilon_1} \int_1^{+\infty} |r(s) f[s, \frac{(1+(\log s)^{\theta-1})x(s)}{1+(\log s)^{\theta-1}}, \frac{(1+\log s)^H D_{1+}^{\theta-2} x(s)}{1+\log s}, {}^H D_{1+}^{\theta-1} x(s)]| \frac{ds}{s} \\
&\le \frac{\Gamma(\theta) B_{k_0}}{\Upsilon_1} \int_1^{+\infty} r(s) \frac{ds}{s} \\
&< +\infty,
\end{aligned}
$$

$$
\begin{aligned}
&\sup_{t\in[1,+\infty)} |{}^H D_{1+}^{\theta-1} T x(t)| \\
&= \int_1^{+\infty} K^\star(t,s) |r(s)F(s)| \frac{ds}{s} \\
&\le \frac{\Gamma(\theta)}{\Upsilon_1} \int_1^{+\infty} |r(s) f[s, \frac{(1+(\log s)^{\theta-1})x(s)}{1+(\log s)^{\theta-1}}, \frac{(1+\log s)^H D_{1+}^{\theta-2} x(s)}{1+\log s}, {}^H D_{1+}^{\theta-1} x(s)]| \frac{ds}{s} \\
&\le \frac{\Gamma(\theta) B_{k_0}}{\Upsilon_1} \int_1^{+\infty} r(s) \frac{ds}{s} \\
&< +\infty.
\end{aligned}
$$

Hence, $T\Omega$ is bounded in P.

Second, we prove that condition 1) in Lemma 6 is established. For all $t_1, t_2 \in [L_1, L_2]$ with $t_1 < t_2$ and $x \in \Omega$ mentioned above, we get

$$
\begin{aligned}
&\left| \frac{Tx(t_2)}{1+(\log t_2)^{\theta-1}} - \frac{Tx(t_1)}{1+(\log t_1)^{\theta-1}} \right| \\
\le\ & B_{k_1} \int_1^{t_2} \left| \frac{K(t_2,s)}{1+(\log t_2)^{\theta-1}} - \frac{K(t_1,s)}{1+(\log t_1)^{\theta-1}} \right| r(s) \frac{ds}{s} \\
&+ B_{k_1} \int_{t_2}^{+\infty} \left| \frac{K(t_2,s)}{1+(\log t_2)^{\theta-1}} - \frac{K(t_1,s)}{1+(\log t_1)^{\theta-1}} \right| r(s) \frac{ds}{s} \\
\le\ & B_{k_1} \int_1^{t_2} \left| \frac{K(t_2,s)}{1+(\log t_2)^{\theta-1}} - \frac{K(t_1,s)}{1+(\log t_1)^{\theta-1}} \right| r(s) \frac{ds}{s} \\
&+ \frac{B_{k_1}}{\Upsilon_1} \int_{t_2}^{+\infty} \left| \frac{(\log t_2)^{\theta-1}}{1+(\log t_2)^{\theta-1}} - \frac{(\log t_1)^{\theta-1}}{1+(\log t_1)^{\theta-1}} \right| r(s) \frac{ds}{s},
\end{aligned}
$$

$$\begin{aligned}
&\left|\frac{{}^{H}D_{1^+}^{\theta-2}Tx(t_2)}{1+\log t_2}-\frac{{}^{H}D_{1^+}^{\theta-2}Tx(t_1)}{1+\log t_1}\right| \\
\le\ & B_{k_1}\int_1^{t_2}\left|\frac{K_\star(t_2,s)}{1+\log t_2}-\frac{K_\star(t_1,s)}{1+\log t_1}\right| r(s)\frac{ds}{s} \\
&+\frac{B_{k_1}\Gamma(\theta)}{\Upsilon_1}\int_{t_2}^{+\infty}\left|\frac{\log t_2}{1+\log t_2}-\frac{\log t_1}{1+\log t_1}\right| r(s)\frac{ds}{s}.
\end{aligned}$$

Because $\frac{K(t,s)}{1+(\log t)^{\theta-1}}$, $\frac{(\log t)^{\theta-1}}{1+(\log t)^{\theta-1}}$, and $\frac{\log t}{1+\log t}$ are uniformly continuous on any compact set $[L_1, L_2]\times[L_1, L_2]$, $[L_1, L_2]$, and $[L_1, L_2]$, respectively, we have

$$\left|\frac{Tx(t_2)}{1+(\log t_2)^{\theta-1}}-\frac{Tx(t_1)}{1+(\log t_1)^{\theta-1}}\right|\to 0,\qquad t_1\to t_2,$$

$$\left|\frac{{}^{H}D_{1^+}^{\theta-2}Tx(t_2)}{1+\log t_2}-\frac{{}^{H}D_{1^+}^{\theta-2}Tx(t_1)}{1+\log t_1}\right|\to 0,\qquad t_1\to t_2.$$

Furthermore, in view of

$$ {}^{H}D_{1^+}^{\theta-1}Tx(t)=\int_1^{+\infty}K^\star(t,s)r(s)F(s)\frac{ds}{s}, \tag{2.13}$$

the function $K^\star(t,s)\in C([1,+\infty)\times[1,+\infty))$ does not depend on t. Hence, we know that condition 1) in Lemma 6 is established.

Finally, we prove that condition 2) in Lemma 6 is established. For any $\varepsilon>0$, since $0<\int_1^{+\infty}r(s)\frac{ds}{s}<+\infty$, we can get that there exists $M_1>1$ such that

$$0<\int_{M_1}^{+\infty}r(s)\frac{ds}{s}<\varepsilon.$$

In view of

$$\lim_{t\to+\infty}\frac{(\log t)^{\theta-1}}{1+(\log t)^{\theta-1}}=1,\quad \lim_{t\to+\infty}\frac{k(t,M_1)}{1+(\log t)^{\theta-1}}=1,$$

for all $\varepsilon>0$, there exist two constants $M_2>0$, $M_3>M_1$ such that, for all $t_1, t_2>M_2$,

$$\left|\frac{(\log t_2)^{\theta-1}}{1+(\log t_2)^{\theta-1}}-\frac{(\log t_1)^{\theta-1}}{1+(\log t_1)^{\theta-1}}\right|<\varepsilon,$$

for all $t_1, t_2>M_3$ and $1\le s\le M_1$,

$$\left|\frac{k(t_2,s)}{1+(\log t_2)^{\theta-1}}-\frac{k(t_1,s)}{1+(\log t_1)^{\theta-1}}\right|\le\left|1-\frac{k(t_2,M_1)}{1+(\log t_2)^{\theta-1}}\right|+\left|1-\frac{k(t_1,M_1)}{1+(\log t_1)^{\theta-1}}\right|<\varepsilon.$$

Since

$$\lim_{t\to+\infty}\frac{\log t}{1+\log t}=1,\quad \lim_{t\to+\infty}\frac{k_\star(t,M_1)}{1+\log t}=1,$$

for all $\varepsilon > 0$, there exist constants $M_4 > 0$ and $M_5 > M_1$ such that, for all $t_1, t_2 > M_4$,

$$\left|\frac{\log t_2}{1+\log t_2} - \frac{\log t_1}{1+\log t_1}\right| < \varepsilon,$$

for all $t_1, t_2 > M_5$ and $1 \le s \le M_1$,

$$\left|\frac{k_\star(t_2, s)}{1+\log t_1} - \frac{k_\star(t_1, s)}{1+\log t_2}\right| \le \left|1 - \frac{k_\star(t_2, M_1)}{1+\log t_2}\right| + \left|1 - \frac{k_\star(t_1, M_1)}{1+\log t_1}\right| < \varepsilon.$$

Let $M > \max\{M_2, M_3, M_4, M_5\}$; for all t_1, $t_2 > M$, we deduce the following result by substituting (2.5)–(2.7) and applying Lemma 4:

$$\begin{aligned}
&\left|\frac{Tx(t_2)}{1+(\log t_2)^{\theta-1}} - \frac{Tx(t_1)}{1+(\log t_1)^{\theta-1}}\right| \\
\le& \int_1^{+\infty} \left|\frac{K(t_2, s)}{1+(\log t_2)^{\theta-1}} - \frac{K(t_1, s)}{1+(\log t_1)^{\theta-1}}\right| r(s)F(s)\frac{ds}{s} \\
\le& \int_1^{M_1} B_{k_1}\left|\frac{K_1(t_2, s)}{1+(\log t_2)^{\theta-1}} - \frac{K_1(t_1, s)}{1+(\log t_1)^{\theta-1}}\right| r(s)\frac{ds}{s} \\
&+ \int_{M_1}^{+\infty} B_{k_1}\left|\frac{K(t_2, s)}{1+(\log t_2)^{\theta-1}} - \frac{K(t_1, s)}{1+(\log t_1)^{\theta-1}}\right| r(s)\frac{ds}{s} \\
&+ \int_1^{M_1} B_{k_1}\left|\frac{K_2(t_2, s)}{1+(\log t_2)^{\theta-1}} - \frac{K_2(t_1, s)}{1+(\log t_1)^{\theta-1}}\right| r(s)\frac{ds}{s} \\
<& \varepsilon B_{k_1}\int_1^{M_1} r(s)\frac{ds}{s} + \frac{2B_{k_1}}{\Upsilon_1}\int_{M_1}^{+\infty} r(s)\frac{ds}{s} + \frac{\varepsilon B_{k_1}}{\Upsilon_1}\left(\int_1^{M_1} k(t, s)\omega(t)\frac{dt}{t}\right)\int_1^{M_1} r(s)\frac{ds}{s} \\
&+ \frac{\varepsilon B_{k_1}}{\Upsilon_1}\left(\int_1^{M_1} (\log t)^{\theta-1}\omega(t)\frac{dt}{t}\right)\int_1^{M_1} r(s)\frac{ds}{s} \\
<& \frac{B_{k_1}\varepsilon}{\Upsilon_1}\left(\Upsilon_1 + 2 + \int_1^{M_1} k(t, s)\omega(t)\frac{dt}{t} + \int_1^{M_1} (\log t)^{\theta-1}\omega(t)\frac{dt}{t}\right)\int_1^{M_1} r(s)\frac{ds}{s}.
\end{aligned}$$

Next, we substitute the specific form of $K_\star(t, s)$ and apply the properties of $K_\star(t, s)$. We have

$$\begin{aligned}
&\left|\frac{{}^H D_{1+}^{\theta-2}Tx(t_2)}{1+\log t_2} - \frac{{}^H D_{1+}^{\theta-2}Tx(t_1)}{1+\log t_1}\right| \\
\le& \int_1^{+\infty} \left|\frac{K_\star(t_2, s)}{1+(\log t_2)^{\theta-1}} - \frac{K_\star(t_1, s)}{1+(\log t_1)^{\theta-1}}\right| r(s)F(s)\frac{ds}{s} \\
<& \varepsilon B_{k_1}\int_1^{M_1} r(s)\frac{ds}{s} + \frac{2B_{k_1}\Gamma(\theta)}{\Upsilon_1}\int_{M_1}^{+\infty} r(s)\frac{ds}{s} + \frac{\varepsilon B_{k_1}}{\Upsilon_1}\left(\int_1^{M_1} k(t, s)\omega(t)\frac{dt}{t}\right)\int_1^{M_1} r(s)\frac{ds}{s} \\
&+ \frac{\varepsilon B_{k_1}}{\Upsilon_1}\left(\int_1^{M_1} (\log t)^{\theta-1}\omega(t)\frac{dt}{t}\right)\int_1^{M_1} r(s)\frac{ds}{s} \\
<& \frac{B_{k_1}\varepsilon}{\Upsilon_1}\left(\Upsilon_1 + 2\Gamma(\theta) + \int_1^{M_1} k(t, s)\omega(t)\frac{dt}{t} + \int_1^{M_1} (\log t)^{\theta-1}\omega(t)\frac{dt}{t}\right)\int_1^{M_1} r(s)\frac{ds}{s}.
\end{aligned}$$

In the same way, considering the the properties of $K^\star(t,s)$, we obtain

$$\begin{aligned}
&\left|{}^H D_{1+}^{\theta-1} T x(t_2) - {}^H D_{1+}^{\theta-1} T x(t_1)\right| \\
<&\frac{2B_{k_1}\Gamma(\theta)}{\Upsilon_1}\int_{M_1}^{+\infty} r(s)\frac{ds}{s} + \frac{\varepsilon B_{k_1}}{\Upsilon_1}\left(\int_1^{M_1} k(t,s)\omega(t)\frac{dt}{t}\right)\int_1^{M_1} r(s)\frac{ds}{s} \\
&+ \frac{\varepsilon B_{k_1}}{\Upsilon_1}\left(\int_1^{M_1} (\log t)^{\theta-1}\omega(t)\frac{dt}{t}\right)\int_1^{M_1} r(s)\frac{ds}{s} \\
<&\frac{B_{k_1}\varepsilon}{\Upsilon_1}\left(2\Gamma(\theta) + \int_1^{M_1} k(t,s)\omega(t)\frac{dt}{t} + \int_1^{M_1} (\log t)^{\theta-1}\omega(t)\frac{dt}{t}\right)\int_1^{M_1} r(s)\frac{ds}{s}.
\end{aligned}$$

Hence, we know that condition 2) in Lemma 6 is established. By Lemma 6, T is a compact operator. The proof is completed.

In consequence, according to Lemmas 7 and 8, the operator T is completely continuous.

3. Existence of solutions

(H_3) Let $\zeta > 0$ be a constant, which satisfies

$$f(t,(1+(\log t)^{\theta-1})x,(1+\log t)y,z) \le \frac{\Upsilon_1 \cdot \zeta}{\Gamma(\theta)\int_1^{+\infty} r(t)\frac{dt}{t}},$$

for all $(t,x,y,z) \in [1,+\infty) \times [0,\zeta]^3$, where Υ_1 is shown by Lemma 3.

(H_4) Regardless of the values of $t \in [1,+\infty)$ and $x,y,z \in [0,+\infty)$, $f(t,x,y,z)$ is nondecreasing and continuous, which satisfies

$$f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2) \ge 0,$$

where $x_1 > x_2$, $y_1 > y_2$, $z_1 > z_2$.

Theorem 1. *([24]) Assume that (H_1)–(H_4) hold; there are two positive solutions x^*, y^* of (1.1), where $\|x^*\|$, $\|y^*\| \in (0,\zeta]$. Actually, the solutions can be established by applying the sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$, which satisfy*

$$y_0 \le y_1 \le \cdots \le y_n \le \cdots \le y^* \le \cdots \le x^* \le \cdots \le x_n \le \cdots \le x_1 \le x_0, \tag{3.1}$$

where

$$x_{n+1}(t) = \int_1^{+\infty} K(t,s)r(s)f(s,x_n(s),{}^H D_{1+}^{\theta-2}x_n(s),{}^H D_{1+}^{\theta-1}x_n(s))\frac{ds}{s}, \tag{3.2}$$

$$y_{n+1}(t) = \int_1^{+\infty} K(t,s)r(s)f(s,y_n(s),{}^H D_{1+}^{\theta-2}y_n(s),{}^H D_{1+}^{\theta-1}y_n(s))\frac{ds}{s}, \tag{3.3}$$

with the initial values $x_0(t) = \frac{\zeta(\log t)^{\theta-1}}{\Gamma(\theta)}$ and $y_0(t) = 0$. At the same time, for all $t \in (1,+\infty)$, $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ converge to x^*, y^*, separately.

Proof. Set $P_\zeta = \{x \in P,\ \|x\| \le \zeta\}$ if $x \in P_\zeta$, where $\|x\| \le \zeta$. According to (H_3) and Lemmas 4 and 5, we have

$$\begin{aligned}
&\sup_{t\in[1,+\infty)} \frac{|Tx(t)|}{1+(\log t)^{\theta-1}}\\
&= \sup_{t\in[1,+\infty)} \int_1^{+\infty} \frac{K(t,s)}{1+(\log t)^{\theta-1}}|r(s)F(s)|\frac{ds}{s}\\
&\le \frac{1}{\Upsilon_1}\int_1^{+\infty} |r(s)f[s,\frac{(1+(\log s)^{\theta-1})x(s)}{1+(\log s)^{\theta-1}},\frac{(1+\log s)^H D_{1^+}^{\theta-2}x(s)}{1+\log s},{}^H D_{1^+}^{\theta-1}x(s)]|\frac{ds}{s}\\
&\le \frac{1}{\Upsilon_1}\frac{\Upsilon_1\cdot\zeta}{\Gamma(\theta)\int_1^{+\infty} r(s)\frac{ds}{s}}\int_1^{+\infty} r(s)\frac{ds}{s}\\
&= \frac{\zeta}{\Gamma(\theta)}\\
&< \zeta,
\end{aligned}$$

$$\begin{aligned}
&\sup_{t\in[1,+\infty)} \frac{|^H D_{1^+}^{\theta-2}Tx(t)|}{1+\log t}\\
&= \sup_{t\in[1,+\infty)} \int_1^{+\infty} \frac{K_\star(t,s)}{1+\log t}|r(s)F(s)|\frac{ds}{s}\\
&\le \frac{\Gamma(\theta)}{\Upsilon_1}\int_1^{+\infty} |r(s)f[s,\frac{(1+(\log s)^{\theta-1})x(s)}{1+(\log s)^{\theta-1}},\frac{(1+\log s)^H D_{1^+}^{\theta-2}x(s)}{1+\log s},{}^H D_{1^+}^{\theta-1}x(s)]|\frac{ds}{s}\\
&\le \frac{\Gamma(\theta)}{\Upsilon_1}\frac{\Upsilon_1\cdot\zeta}{\Gamma(\theta)\int_1^{+\infty} r(s)\frac{ds}{s}}\int_1^{+\infty} r(s)\frac{ds}{s}\\
&= \zeta,
\end{aligned}$$

$$\begin{aligned}
&\sup_{t\in[1,+\infty)} |^H D_{1^+}^{\theta-1}Tx(t)|\\
&= \sup_{t\in[1,+\infty)} \int_1^{+\infty} K^\star(t,s)|r(s)F(s)|\frac{ds}{s}\\
&\le \frac{\Gamma(\theta)}{\Upsilon_1}\int_1^{+\infty} |r(s)f[s,\frac{(1+(\log s)^{\theta-1})x(s)}{1+(\log s)^{\theta-1}},\frac{(1+\log s)^H D_{1^+}^{\theta-2}x(s)}{1+\log s},{}^H D_{1^+}^{\theta-1}x(s)]|\frac{ds}{s}\\
&\le \frac{\Gamma(\theta)}{\Upsilon_1}\frac{\Upsilon_1\cdot\zeta}{\Gamma(\theta)\int_1^{+\infty} r(s)\frac{ds}{s}}\int_1^{+\infty} r(s)\frac{ds}{s}\\
&= \zeta.
\end{aligned}$$

From the above inequalities, we prove that $\|Tx\| \le \zeta$. Thus, $T(P_\zeta) \subset P_\zeta$, $T : P_\zeta \to P_\zeta$.

First, let $x_0(t) = \frac{\zeta(\log t)^{\theta-1}}{\Gamma(\theta)}$, $x_1 = Tx_0$, and $x_2 = T^2x_0 = Tx_1$. It is clear that $\|x_0\| \le \zeta$, so $x_0(t) \in P_\zeta$. Furthermore, we know that $T : P_\zeta \to P_\zeta$, which means that $x_1 \in T(P_\zeta) \subset P_\zeta$ and $x_2 \in T(P_\zeta) \subset P_\zeta$. By

(H_3) and Lemmas 4 and 5, we have

$$\begin{aligned}
x_1(t) &= \int_1^{+\infty} K(t,s)r(s)F_0(s)\frac{ds}{s} \\
&\leq \int_1^{+\infty} \frac{(\log t)^{\theta-1}}{\Upsilon_1} r(s)\frac{ds}{s} \cdot \frac{\Upsilon_1 \cdot \zeta}{\Gamma(\theta)\int_1^{+\infty} r(s)\frac{ds}{s}} \\
&= \frac{\zeta(\log t)^{\theta-1}}{\Gamma(\theta)} \\
&= x_0(t),
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
{}^{H}D_{1+}^{\theta-2}x_1(t) = {}^{H}D_{1+}^{\theta-2}Tx_0(t) &= \int_1^{+\infty} K_\star(t,s)r(s)F_0(s)\frac{ds}{s} \\
&\leq \int_1^{+\infty} \frac{\Gamma(\theta)\log t}{\Upsilon_1} r(s)\frac{ds}{s} \cdot \frac{\Upsilon_1 \cdot \zeta}{\Gamma(\theta)\int_1^{+\infty} r(s)\frac{ds}{s}} \\
&= \zeta \cdot \log t \\
&= {}^{H}D_{1+}^{\theta-2}x_0(t),
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
{}^{H}D_{1+}^{\theta-1}x_1(t) = {}^{H}D_{1+}^{\theta-1}Tx_0(t) &= \int_1^{+\infty} K^\star(t,s)r(s)F_0(s)\frac{ds}{s} \\
&\leq \int_1^{+\infty} \frac{\Gamma(\theta)}{\Upsilon_1} r(s)\frac{ds}{s} \cdot \frac{\Upsilon_1 \cdot \zeta}{\Gamma(\theta)\int_1^{+\infty} r(s)\frac{ds}{s}} \\
&= \zeta \\
&= {}^{H}D_{1+}^{\theta-1}x_0(t).
\end{aligned} \tag{3.6}$$

Hence,

$$\begin{aligned}
&x_1(t) \leq x_0(t), \\
&{}^{H}D_{1+}^{\theta-2}x_1(t) \leq {}^{H}D_{1+}^{\theta-2}x_0(t), \\
&{}^{H}D_{1+}^{\theta-1}x_1(t) \leq {}^{H}D_{1+}^{\theta-1}x_0(t).
\end{aligned}$$

Suppose that the following holds:

$$\begin{aligned}
&x_k(t) \leq x_{k-1}(t), \\
&{}^{H}D_{1+}^{\theta-2}x_k(t) \leq {}^{H}D_{1+}^{\theta-2}x_{k-1}(t), \\
&{}^{H}D_{1+}^{\theta-1}x_k(t) \leq {}^{H}D_{1+}^{\theta-1}x_{k-1}(t).
\end{aligned}$$

Then, we show that

$$\begin{aligned}
&x_{k+1}(t) \leq x_k(t), \\
&{}^{H}D_{1+}^{\theta-2}x_{k+1}(t) \leq {}^{H}D_{1+}^{\theta-2}x_k(t), \\
&{}^{H}D_{1+}^{\theta-1}x_{k+1}(t) \leq {}^{H}D_{1+}^{\theta-1}x_k(t).
\end{aligned}$$

By (H_4), we can get

$$\begin{aligned}x_k(t)-x_{k+1}(t)=&Tx_{k-1}(t)-Tx_k(t)\\=&\int_1^{+\infty}K(t,s)r(s)f(s,x_{k-1}(s),{}^HD_{1^+}^{\theta-2}x_{k-1}(s),{}^HD_{1^+}^{\theta-1}x_{k-1}(s))\frac{ds}{s}\\&-\int_1^{+\infty}K(t,s)r(s)f(s,x_k(s),{}^HD_{1^+}^{\theta-2}x_k(s),{}^HD_{1^+}^{\theta-1}x_k(s))\frac{ds}{s}\\\geq&0.\end{aligned}$$

In the same way, according to (H_4), (2.9) and (2.10), we get

$${}^HD_{1^+}^{\theta-2}x_k(t)-{}^HD_{1^+}^{\theta-2}x_{k+1}(t)={}^HD_{1^+}^{\theta-2}Tx_{k-1}(t)-{}^HD_{1^+}^{\theta-2}Tx_k(t)\geq 0,$$

$${}^HD_{1^+}^{\theta-1}x_k(t)-{}^HD_{1^+}^{\theta-1}x_{k+1}(t)={}^HD_{1^+}^{\theta-1}Tx_{k-1}(t)-{}^HD_{1^+}^{\theta-1}Tx_k(t)\geq 0.$$

By induction, we obtain that $\{x_n\}_{n=1}^{\infty}\subset T(P_\zeta)\subset P_\zeta$ and $x_{n+1}=Tx_n$. For all $t\in[1,\infty)$, we derive the following:

$$x_{n+1}(t)\leq x_n(t), \tag{3.7}$$
$${}^HD_{1^+}^{\theta-2}x_{n+1}(t)\leq {}^HD_{1^+}^{\theta-2}x_n(t), \tag{3.8}$$
$${}^HD_{1^+}^{\theta-1}x_{n+1}(t)\leq {}^HD_{1^+}^{\theta-1}x_n(t). \tag{3.9}$$

Due to the complete continuity of T and the existence of x^* in P_ζ, we obtain that $x_{n_k}\to x^*$ as $n\to\infty$. So, we can get that there is a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$. This demonstrates that $\lim_{n\to\infty}x_n=x^*$. Given that $x_{n+1}=Tx_n$ and T is continuous, we prove that $Tx^*=x^*$, which demonstrates that x^* is a fixed point of T.

Since $y_0(t)=0$ and $\|y_0\|\leq\zeta$, we can easily get that $y_0(t)\in P_\zeta$, which implies that $T:P_\zeta\to P_\zeta$. Similarly, we can get that $y_{n+1}=Ty_n$ and $\{y_n\}_{n=1}^{\infty}\subset T(P_\zeta)\subset P_\zeta$. By induction, for any $t\in[1,\infty)$, it can be seen that

$$y_{n+1}(t)\leq y_n(t), \tag{3.10}$$
$${}^HD_{1^+}^{\theta-2}y_{n+1}(t)\leq {}^HD_{1^+}^{\theta-2}y_n(t), \tag{3.11}$$
$${}^HD_{1^+}^{\theta-1}y_{n+1}(t)\leq {}^HD_{1^+}^{\theta-1}y_n(t). \tag{3.12}$$

Because of the properties of T, there exists a $y^*\in P_\zeta$, which satifies that $y_{n_k}\to y^*$ as $n\to\infty$. So, we can get that there is a convergent subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$. This demonstrates that $\lim_{n\to\infty}y_n=y^*$. Given that $y_{n+1}=Ty_n$ and T is continuous, we have that $Ty^*=y^*$, which implies that y^* is a fixed point of T. By induction, we obtain

$$y_n(t)\leq x_n(t),\quad t\in[1,\infty), n=0,1,2,\dots \tag{3.13}$$

According to (3.7), (3.10), and (3.13), we have

$$y_0\leq y_1\leq\cdots\leq y_n\leq\cdots\leq x_n\leq\cdots\leq x_1\leq x_0.$$

This, together with $x^*=\lim_{n\to\infty}x_n$ and $y^*=\lim_{n\to\infty}y_n$, yields that

$$y_0\leq y_1\leq\cdots\leq y_n\leq\cdots\leq y^*\leq x^*\leq\cdots\leq x_n\leq\cdots\leq x_1\leq x_0.$$

Because $f(t,0,0,0)\not\equiv 0$, it implies that $x=0$ is not a solution of (1.1). Therefore, x^* and y^* are the solutions of (1.1).

4. Error estimate

(H_5) There exist constants $L_i > 0$, $i = 1, 2, 3$, $\forall t \in [1, +\infty)$, as well as $x, y, z \in [0, +\infty)$. Thus, $f(t, x, y, z)$ satisfies the conditions of the following inequality:

$$|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)| \leq L_1(x_1 - x_2) + L_2(y_1 - y_2) + L_3(z_1 - z_2),$$

where $x_1 > x_2$, $y_1 > y_2$, $z_1 > z_2$.

Suppose that (H_1)–(H_5) hold and (1.1) has a unique solution $z^* \in (0, \frac{\zeta}{\Gamma(\theta)}(\log t)^{\theta-1}]$, which relies on the following sequence:

$$z_{n+1}(t) = \int_1^{+\infty} K(t, s)r(s)f(s, z_n(s), {}^H D_{1+}^{\theta-2} z_n(s), {}^H D_{1+}^{\theta-1} z_n(s))\frac{ds}{s}. \tag{4.1}$$

The initial value of (4.1) is $z_0(t) = 0$ or $z_0(t) = \frac{\zeta}{\Gamma(\theta)}(\log t)^{\theta-1}$. Furthermore, the error estimate can be defined as

$$\|z_n - z^*\| \leq Z^n \zeta, \tag{4.2}$$

where $Z = \frac{\Gamma(\theta)}{\Upsilon_1}(L_1 + L_2 + L_3)\int_1^{+\infty} r(s)(1 + \log s)^{\theta-1}\frac{ds}{s} < 1$.

Proof. According to Theorem 1, we prove that (1.1) has two positive solutions, which can be established by $x_{n+1} = Tx_n$ and $y_{n+1} = Ty_n$, with the initial values $x_0(t) = \frac{\zeta(\log t)^{\theta-1}}{\Gamma(\theta)}$ and $y_0(t) = 0$, $t \in [1, +\infty)$.

If $\|x_n - y_n\| = \sup_{t\in[1,+\infty)} \frac{|x_n(t)-y_n(t)|}{1+(\log t)^{\theta-1}}$, by (H_5) and (3.1), we derive the following:

$$\begin{aligned}
(x_n - y_n)(t) =& Tx_{n-1}(t) - Ty_{n-1}(t) \\
=& \int_1^{+\infty} K(t, s)r(s)[f(s, x_{n-1}(s), {}^H D_{1+}^{\theta-2} x_{n-1}(s), {}^H D_{1+}^{\theta-1} x_{n-1}(s)) \\
& - f(s, y_{n-1}(s), {}^H D_{1+}^{\theta-2} y_{n-1}(s), {}^H D_{1+}^{\theta-1} y_{n-1}(s))]\frac{ds}{s} \\
\leq & \quad L_1 \int_1^{+\infty} K(t, s)r(s)[x_{n-1}(s) - y_{n-1}(s)]\frac{ds}{s} \\
& + L_2 \int_1^{+\infty} K(t, s)r(s)[{}^H D_{1+}^{\theta-2} x_{n-1}(s) - {}^H D_{1+}^{\theta-2} y_{n-1}(s)]\frac{ds}{s} \\
& + L_3 \int_1^{+\infty} K(t, s)r(s)[{}^H D_{1+}^{\theta-1} x_{n-1}(s) - {}^H D_{1+}^{\theta-1} y_{n-1}(s)]\frac{ds}{s}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|x_n - y_n\| &< \frac{1}{\Upsilon_1}(L_1 + L_2 + L_3)\int_1^{+\infty} r(s)(1 + \log s)^{\theta-1}\frac{ds}{s}\|x_{n-1} - y_{n-1}\| \\
&< \frac{\Gamma(\theta)}{\Upsilon_1}(L_1 + L_2 + L_3)\int_1^{+\infty} r(s)(1 + \log s)^{\theta-1}\frac{ds}{s}\|x_{n-1} - y_{n-1}\| \\
&= Z\|x_{n-1} - y_{n-1}\|.
\end{aligned}$$

By induction, we can get

$$\|x_n - y_n\| < Z^n\|x_0 - y_0\| < Z^n \zeta. \tag{4.3}$$

If $\|x_n - y_n\| = \sup_{t\in[1,+\infty)} \frac{|^H D_{1+}^{\theta-2} x_n(t) - ^H D_{1+}^{\theta-2} y_n(t)|}{1+\log t}$, by (H_5) and (3.1), we have

$$\begin{aligned}({}^H D_{1+}^{\theta-2} x_n - {}^H D_{1+}^{\theta-2} y_n)(t) &= {}^H D_{1+}^{\theta-2} T x_{n-1}(t) - {}^H D_{1+}^{\theta-2} T y_{n-1}(t)\\ &\le L_1 \int_1^{+\infty} K_\star(t,s) r(s)[x_{n-1}(s) - y_{n-1}(s)]\frac{ds}{s}\\ &+ L_2 \int_1^{+\infty} K_\star(t,s) r(s)[{}^H D_{1+}^{\theta-2} x_{n-1}(s) - {}^H D_{1+}^{\theta-2} y_{n-1}(s)]\frac{ds}{s}\\ &+ L_3 \int_1^{+\infty} K_\star(t,s) r(s)[{}^H D_{1+}^{\theta-1} x_{n-1}(s) - {}^H D_{1+}^{\theta-1} y_{n-1}(s)]\frac{ds}{s}.\end{aligned}$$

Thus, by the same arrangement and induction, we obtain

$$\|x_n - y_n\| < Z\|x_{n-1} - y_{n-1}\| < Z^n\|x_0 - y_0\| < Z^n\zeta.$$

If $\|x_n - y_n\| = \sup_{t\in[1,+\infty)} |^H D_{1+}^{\theta-1} x_n(t) - ^H D_{1+}^{\theta-1} y_n(t)|$, by (H_5) and (3.1), we have

$$\begin{aligned}({}^H D_{1+}^{\theta-1} x_n - {}^H D_{1+}^{\theta-1} y_n)(t) &= {}^H D_{1+}^{\theta-1} T x_{n-1}(t) - {}^H D_{1+}^{\theta-1} T y_{n-1}(t)\\ &\le L_1 \int_1^{+\infty} K^\star(t,s) r(s)[x_{n-1}(s) - y_{n-1}(s)]\frac{ds}{s}\\ &+ L_2 \int_1^{+\infty} K^\star(t,s) r(s)[{}^H D_{1+}^{\theta-2} x_{n-1}(s) - {}^H D_{1+}^{\theta-2} y_{n-1}(s)]\frac{ds}{s}\\ &+ L_3 \int_1^{+\infty} K^\star(t,s) r(s)[{}^H D_{1+}^{\theta-1} x_{n-1}(s) - {}^H D_{1+}^{\theta-1} y_{n-1}(s)]\frac{ds}{s}.\end{aligned}$$

Thus, by using the same method, we get

$$\|x_n - y_n\| < Z\|x_{n-1} - y_{n-1}\| < Z^n\|x_0 - y_0\| < Z^n\zeta.$$

From the above steps, we have

$$\|x_n - y_n\| < Z^n\|x_0 - y_0\| < Z^n\zeta. \tag{4.4}$$

Consequently, we show that

$$\begin{cases}\|y_{n+m} - y_n\| \le \|x_n - y_n\| < Z^n\zeta,\\ \|x_n - x_{n+m}\| \le \|x_n - y_n\| < Z^n\zeta.\end{cases} \tag{4.5}$$

We suppose that (1.1) has a solution $z \in [y_0, x_0]$. It is apparent that $Tz = z$ and $y_0(t) \le z(t) \le x_0(t)$. Since T is continuous and nondecreasing, together with (3.1), we can get

$$y^* \le z \le x^*, \ (n \to \infty).$$

Furthermore, by (4.4), we know that

$$\|x^* - y^*\| < Z^n\zeta \to 0, \ (n \to +\infty). \tag{4.6}$$

This shows that $x^* = y^* \triangleq z^*$. Hence, (1.1) has a unique solution $z^* \in [y_0, x_0]$. In consequence, in view of (4.5), it is obvious that (4.2) holds. The proof is completed.

5. Example

$$\begin{cases} {}^{H}D_{1+}^{5/2}x(t) + \dfrac{1}{e^{\log t}} f[1 + \sin\left(\dfrac{\pi}{2} \cdot \dfrac{x(t)}{1 + (\log t)^{3/2} + x(t)}\right) + \sin\left(\dfrac{\pi}{2} \cdot \dfrac{{}^{H}D_{1+}^{1/2}x(t)}{1 + \log t + {}^{H}D_{1+}^{1/2}x(t)}\right) \\ + \sin\left(\dfrac{\pi}{2} \cdot \dfrac{{}^{H}D_{1+}^{3/2}x(t)}{1 + {}^{H}D_{1+}^{3/2}x(t)}\right)] = 0, \quad t \in (1, +\infty), \\ x(1) = x'(1) = 0, \ {}^{H}D_{1+}^{3/2}x(+\infty) = \displaystyle\int_1^{+\infty} \frac{1}{3e^{\log t^2}} x(t) \frac{dt}{t} + \sum_{i=1}^{m} \beta_i x(\eta_i), \end{cases} \tag{5.1}$$

where $\theta = \frac{5}{2}$, $r(t) = \frac{1}{e^{\log t}}$, $\omega(t) = \frac{1}{3e^{\log t^2}}$, $\theta_1 = \frac{1}{5}$, $\theta_2 = \frac{2}{5}$, $\theta_3 = \frac{3}{5}$, $\theta_4 = \frac{4}{5}$, $\eta_1 = \frac{8}{7}$, $\eta_2 = \frac{7}{6}$, $\eta_3 = \frac{6}{5}$, $\eta_4 = \frac{5}{4}$,

$$f(t, x, y, z) = \begin{cases} 1 + \dfrac{e^{-\log t}}{1000} + \sin\left(\dfrac{\pi}{2} \cdot \dfrac{x(t)}{1 + (\log t)^{3/2} + x(t)}\right) + \sin\left(\dfrac{\pi}{2} \cdot \dfrac{y(t)}{1 + \log t + y(t)}\right) \\ \quad + \sin\left(\dfrac{\pi}{2} \cdot \dfrac{z(t)}{1 + z(t)}\right), \qquad 0 \le x \le e, \\ 1 + \dfrac{e^{-\log t}}{1000} + \sin\left(\dfrac{\pi}{2} \cdot \dfrac{e}{1 + (\log t)^{3/2} + e}\right) + \sin\left(\dfrac{\pi}{2} \cdot \dfrac{y(t)}{1 + \log t + y(t)}\right) \\ \quad + \sin\left(\dfrac{\pi}{2} \cdot \dfrac{z(t)}{1 + z(t)}\right), \qquad x \ge e. \end{cases}$$

$$\Upsilon_1 = \Gamma(\theta) - \int_1^{+\infty} \omega(t)(\log t)^{\theta-1} \frac{dt}{t} - \sum_{i=1}^{m} \beta_i x(\eta_i) = \frac{1}{3}\Gamma(\frac{5}{2}) \approx 0.72122.$$

$$\zeta = 8.$$

$$\frac{\Upsilon_1 \cdot \zeta}{\Gamma(\theta) \int_1^{+\infty} r(t) \frac{dt}{t}} \approx 4.34031.$$

$$\begin{aligned} f(t, (1 + (\log t)^{3/2})x, (1 + \log t)y, z) =& 1 + \sin\left(\frac{\pi}{2} \cdot \frac{1 + (\log t)^{3/2} x(t)}{1 + (\log t)^{3/2} + (1 + (\log t)^{3/2})x(t)}\right) \\ & + \sin\left(\frac{\pi}{2} \cdot \frac{(1 + \log t)^{H}D_{1+}^{1/2}x(t)}{1 + \log t + (1 + \log t)^{H}D_{1+}^{1/2}x(t)}\right) \\ & + \sin\left(\frac{\pi}{2} \cdot \frac{{}^{H}D_{1+}^{3/2}x(t)}{1 + {}^{H}D_{1+}^{3/2}x(t)}\right). \\ \le& 1 + 1 + 1 + 1 \\ \le& 4. \end{aligned}$$

Thus,

$$f(t, (1 + (\log t)^{\theta-1})x, (1 + \log t)y, z) \le \frac{\Upsilon_1 \cdot \zeta}{\Gamma(\theta) \int_1^{+\infty} r(t) \frac{dt}{t}}.$$

From the above steps, the conditions (H_1)–(H_5) hold; according to Theorem 1, the BVP (1.1) has twin positive solutions x^* and y^* such that $\|x^*\|, \|y^*\| \in (0, 8]$.

6. Conclusions

In this article, we first transformed the solutions of the equation into fixed points of the operator by means of a nonlinear alternative, while proving that the operator is completely continuous. Then, we applied two iterative sequences to find two positive solutions of the equation via the monotone iterative method. Finally, we derived the unique solution of the Hadamard fractional BVP. The significance of this article lies in the fact that we can use the monotone iterative method to discuss the existence and uniqueness of the positive solution to the equation containing multiple fractional-order derivative terms.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under grant (62073186), the Taishan Scholar Project of Shandong Province of China, and the NSF of Shandong Province (ZR2021MA097).

Conflict of interest

The authors declare there are no conflicts of interest.

References

1. M. Nategh, A novel approach to an impulsive feedback control with and without memory involvement, *J. Differ. Equations*, **263** (2017), 2661–2671. https://doi.org/10.1016/j.jde.2017.04.008
2. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, 2006.
3. V. Lakshmikantham, S. Leela, J. V. Devi, *Theory of Fractional Dynamic Systems*, Cambridge: Cambridge Academic Publishers, 2009.
4. S. G. Samko, A. A. Kilbas, O. I. Maricev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, 1993.
5. I. Podlubny, *Fractional Differential Equations*, MDPI, 1999. https://doi.org/10.3390/books978-3-03921-733-5
6. X. Q. Zhang, Q. Y. Zhong, Uniqueness of solution for higher-order fractional differential equations with conjugate type integral conditions, *Fract. Calc. Appl. Anal.*, **20** (2017), 1471–1484. https://doi.org/10.1515/fca-2017-0077

7. G. T. Wang, A. Cabada, L. H. Zhang, An integral boundary value problem for nonlinear differential equations of fractional order on an unbounded domain, *J. Integr. Equations Appl.*, **26** (2014), 117–129. https://doi.org/10.1216/jie-2014-26-1-117

8. J. K. He, M. Jia, X. P. Liu, H. Chen, Existence of positive solutions for a high order fractional differential equation integral boundary value problem with changing sign nonlinearity, *Adv. Differ. Equations*, **2018** (2018), 49. https://doi.org/10.1186/s13662-018-1465-6

9. F. A. McRae, Monotone iterative technique and existence results for fractional differential equations, *Nonlinear Anal. Theory Methods Appl.*, **71** (2009), 6093–6096. https://doi.org/10.1016/j.na.2009.05.074

10. C. Z. Hu, B. Liu, S. F. Xie, Monotone iterative solutions for nonlinear boundary value problems of fractional differential equation with deviating arguments, *Appl. Math. Comput.*, **222** (2013), 72–81. https://doi.org/10.1016/j.amc.2013.07.048

11. J. Hadamard, Essai sur l'étude des fonctions données par leur développement de Taylor, *J. Math. Pures Appl.*, **8** (1892), 101–186.

12. F. Mainard, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*, Imperial College Press, 2010. https://doi.org/10.1142/9781848163300

13. H. G. Sun, Y. Zhang, D. Baleanu, W. Chen, Y. Q. Chen, A new collection of real world applications of fractional calculus in science and engineering. *J. Nonlinear Sci. Appl.*, **64** (2018), 213–231. https://doi.org/10.1016/j.cnsns.2018.04.019

14. B. Ahmad, A. Alsaedi, B. S. Alghamdi, Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions, *Nonlinear Anal. Real World Appl.*, **9** (2008), 1727–1740. https://doi.org/10.1016/j.nonrwa.2007.05.005

15. R. Čiegis, A. Bugajev, Numerical approximation of one model of the bacterial self-organization, *J. Appl. Math. Comput.*, **17** (2012), 253–270. https://doi.org/10.15388/NA.17.3.14054

16. S. H. Liang, S. Y. Shi, Existence of multiple positive solutions for m-point fractional boundary value problems with p-Laplacian operator on infinite interval, *J. Appl. Math. Comput.*, **38** (2012), 687–707. https://doi.org/10.1007/s12190-011-0505-0

17. X. K. Zhao, W. G. Ge, Unbounded solutions for a fractional boundary value problem on the infinite interval, *Acta Appl. Math.*, **109** (2010), 495–505. https://doi.org/10.1007/s10440-008-9329-9

18. J. H. He, Some applications of nonlinear fractional differential equations and their approximations, *Bull. Sci. Technol.*, **15** (1999), 86–90.

19. X. A. Hao, H. Sun, L. S. Liu, Existence results for fractional integral boundary value problem involving fractional derivatives on an infinite interval, *Math. Methods Appl. Sci.*, **41** (2018), 6984–6996. https://doi.org/10.1002/mma.5210

20. X. C. Li, X. P. Liu, M. Jia, Y. Li, S. Zhang, Existence of positive solutions for integral boundary value problems of fractional differential equations on infinite interval, *Math. Methods Appl. Sci.*, **36** (2017), 1892–1904. https://doi.org/10.1002/mma.4106

21. W. Zhang, W. B. Liu, Existence, uniqueness, and multiplicity results on positive solutions for a class of Hadamard-type fractional boundary value problem on an infinite interval, *Math. Methods Appl. Sci.*, **43** (2020), 2251–2275. https://doi.org/10.1002/mma.6038

22. T. S. Cerdik, F. Y. Deren, New results for higher-order hadamard-type fractional differential equations on the half-line, *Math. Methods Appl. Sci.*, **45** (2022), 2315–2330. https://doi.org/10.1002/mma.7926

23. B. Ahmad, A. Alsaedi, S. K. Ntouyas, J. Tariboon, *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, Springer Cham, 2017. https://doi.org/10.1007/978-3-319-52141-1

24. G. T. Wang, K. Pei, R. P. Agarwal, L. H. Zhang, B. Ahmad, Nonlocal Hadamard fractional boundary value problem with Hadamard integral and discrete boundary conditions on a half-line, *J. Comput. Appl. Math.*, **343** (2018), 230–239. https://doi.org/10.1016/j.cam.2018.04.062

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