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Research article

The short interval results for power moments of the Riesz mean error term

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Abstract: Let $\Delta_1(x;\varphi)$ denote the error term in the classical Rankin-Selberg problem. In this paper, our main results are getting the k-th $(3 \le k \le 5)$ power moments of $\Delta_1(x;\varphi)$ in short intervals and its asymptotic formula by using large value arguments.

Keywords: the Rankin-Selberg problem; power moment; integral mean value; Voronoï formula

1. Introduction

Let $\varphi(z)$ be a holomorphic form of weight κ with respect to the full modular group $SL_2(\mathbb{Z})$ and denote by $a(n)$ the n-th Fourier coefficient of $\varphi(z)$. We assume that $\varphi(z)$ is normalized such that $a(1) = 1$ and $T(n)\varphi = a(n)\varphi$ for every $n \in \mathbb{N}$, where $T(n)$ is the Hecke operator of order n. Let c_n be the convolution function defined by

$$c_n = n^{1-\kappa} \sum_{m^2|n} m^{2(\kappa-1)} \left| a\left(\frac{n}{m^2}\right)\right|^2 .$$

In 1974, Deligne [1] proved the estimate $|a(n)| \le n^{\frac{\kappa-1}{2}} d(n)$, where $d(n)$ is the Dirichlet divisor function, which implies $c_n \ll_\varepsilon n^\varepsilon$. Here and in what follows, ε denotes an arbitrarily small positive number which is not necessarily the same at each occurrence. The classical Rankin-Selberg problem is to estimate the upper bound of the error term

$$\Delta(x;\varphi) := \sum_{n\le x} c_n - Cx, \tag{1.1}$$

where C is an explicit constant. In 1939, Rankin [2] proved that

$$\Delta(x;\varphi) = O(x^{\frac{3}{5}}), \tag{1.2}$$

which was stated by Selberg [3] again without proof. However, no improvement of (1.2) has been obtained after Rankin and Selberg. In [4], Ivić obtained that $\Delta(x;\varphi) = \Omega_\pm(x^{3/8})$ and conjectured that $\Delta(x;\varphi) = O(x^{3/8+\varepsilon})$.

Ivić, Matsumoto and Tanigawa [5] considered the Riesz mean of the type

$$D_\rho(x;\varphi) := \frac{1}{\Gamma(\rho+1)}\sum_{n\le x}(x-n)^\rho c_n$$

for any fixed $\rho \ge 0$ and define the error term $\Delta_\rho(c;\varphi)$ by

$$D_\rho(x;\varphi) = \frac{\pi^2\kappa R_0}{6\Gamma(\rho+2)}x^{\rho+1} + \frac{Z(0)}{\Gamma(\rho+1)}x^\rho + \Delta_\rho(x;\varphi), \tag{1.3}$$

where

$$R_0 = \frac{12(4\pi)^{\kappa-1}}{\Gamma(\kappa+1)}\iint_{\mathfrak{F}} y^{\kappa-2}|\varphi(z)|^2 dxdy,$$
$$Z(s) = \sum_{n=1}^{\infty} c_n n^{-s},\ \Re s > 1,$$

where $Z(s)$ can be continued to the whole plane and the integral being taken over a fundamental domain $\mathfrak{F}$ of $SL_2(\mathbb{Z})$. They considered the relation between $\Delta(x;\varphi)$ and $\Delta_1(x;\varphi)$ and proved that $\Delta(x;\varphi) = O(x^{\alpha/2})$ if $\Delta_1(x;\varphi) = O(x^\alpha)$ holds for some $\alpha \ge 0$. They also proved that

$$\Delta_1(x;\varphi) = O(x^{\frac{6}{5}})$$

and

$$\int_1^T \Delta_1^2(x;\varphi)dx = \frac{2}{13}(2\pi)^{-4}\left(\sum_{n=1}^{\infty} c_n^2 n^{-7/4}\right)T^{13/4} + O(T^{3+\varepsilon}).$$

Since this kind sums play a very important role in the study of analytic number theory, many number theorists and scholars have obtained a series of meaningful research results (for example see [6–9, 11–13], etc.). In particular, in [9], Tanigawa, Zhai and Zhang studied the third, fourth and fifth power moments of $\Delta_1(x;\varphi)$ and proved that

$$\begin{aligned}\int_1^T \Delta_1^3(x;\varphi)dx &= \frac{B_3(c)}{1120\pi^6}T^{\frac{35}{8}} + O\left(T^{\frac{35}{8}-\frac{1}{36}+\varepsilon}\right),\\ \int_1^T \Delta_1^4(x;\varphi)dx &= \frac{B_4(c)}{11264\pi^8}T^{\frac{11}{2}} + O\left(T^{\frac{11}{2}-\frac{1}{221}+\varepsilon}\right),\\ \int_1^T \Delta_1^5(x;\varphi)dx &= \frac{B_5(c)}{108544\pi^{10}}T^{\frac{53}{8}} + O\left(T^{\frac{53}{8}-\frac{1}{1731}+\varepsilon}\right),\end{aligned} \tag{1.4}$$

where

$$B_k(f) := \sum_{l=1}^{k-1}\binom{k-1}{l}s_{k;l}(f)\cos\frac{\pi(k-2l)}{4},$$
$$s_{k;l} := \sum_{\sqrt[4]{n_1}+\cdots+\sqrt[4]{n_l}=\sqrt[4]{n_{l+1}}+\cdots+\sqrt[4]{n_k}} \frac{f(n_1)\cdots f(n_k)}{(n_1\cdots n_k)^{7/8}},\ 1\le l\le k.$$

In this paper we shall prove that the k-th power moment of $\Delta_1(x;\varphi)$ in short intervals for $k = 3, 4, 5$, the theorem is as follows.

Theorem 1. *Let $k \geq 3$ be a fixed integer. For any sufficiently small $\varepsilon > 0$, let $\delta_k := (k-1)\left(4^{k-1} + \frac{k-10}{2}\right) + 3$, $0 < \delta < 1$ be a fixed constant, which satisfies $\frac{8}{3}\delta_k\varepsilon < \delta$, T and H are two large positive real number, which satisfies*

$$\int_{T-H}^{T+H} |\Delta_1(x;\varphi)|^{k+\delta} dx \ll HT^{9(k+\delta)/8+\varepsilon} \tag{1.5}$$

and $T^{3/4+2\delta_k\varepsilon/(3\delta)} \leq H \leq T$. Then, we have

$$\int_{T-H}^{T+H} \Delta_1^k(x;\varphi)dx = \frac{B_k(c)}{2^{k-1}\cdot(2\pi)^{2k}} \int_{T-H}^{T+H} x^{9k/8}dx + O\left(HT^{\frac{9k}{8}+\varepsilon}(HT^{-3/4})^{-\frac{3\delta}{\delta_k}}\right), \tag{1.6}$$

where

$$B_k(c) := \sum_{l=1}^{k-1} \binom{k-1}{l} s_{k;l}(c) \cos\frac{\pi(k-2l)}{4}$$
$$s_{k;l}(c) := \sum_{\sqrt[4]{n_1}+\cdots+\sqrt[4]{n_l}=\sqrt[4]{n_{l+1}}+\cdots+\sqrt[4]{n_k}} \frac{c_{n_1}\cdots c_{n_k}}{(n_1\cdots n_k)^{7/8}},\ 1 \leq l < k.$$

Remark 1. *If we take $H = T$, δ is larger (for example $\delta = \frac{1}{2}$), then Theorem 1 implies asymptotic formula* (1.4).

As corollaries, we have the following Theorems 2 and 3. Theorem 3 implies the best possible result.

Theorem 2. *Suppose $3 \leq k \leq 5$, $1/8 < \theta < 1/5$ is a real number. Let $\Delta_1(x;\varphi) \ll x^{\theta+1}$. Then, we have asymptotic formula*

$$\int_{T-H}^{T+H} \Delta_1^k(x;\varphi)dx = \frac{B_k(c)}{2^{k-1}\cdot(2\pi)^{2k}} \int_{T-H}^{T+H} x^{9k/8}dx + O\left(HT^{\frac{9k}{8}-\varepsilon}\right), \tag{1.7}$$

when $T^{1+(k-2)\theta-k/8+\sqrt{\varepsilon}} \leq H \leq T$.

Corollary 1. *For $3 \leq k \leq 5$, if $T^{(k-2)/5+1-k/8+\sqrt{\varepsilon}} \leq H \leq T$, asymptotic formula* (1.7) *is true.*

Theorem 3. *Suppose $k \geq 3$ be a any fixed integer and conjecture $\Delta_1(x;\varphi) = O(x^{9/8+\varepsilon})$ is true. Then, asymptotic formula* (1.7) *is true if $T^{3/4+\sqrt{\varepsilon}/2} \leq H \leq T$.*

Remark 2. *By variable substitution, it is easy to see that*

$$\int_{T-H}^{T+H} \Delta_1^k(x;\varphi)dx = 4\int_{T'-H'}^{T'+H'} \Delta_1^k(x^4;\varphi)x^3dx,$$

here

$$T' := \frac{(T+H)^{1/4}+(T-H)^{1/4}}{4} \asymp T^{1/4},\ H' := \frac{(T+H)^{1/4}-(T-H)^{1/4}}{4} \asymp H/T^{3/4}.$$

If the conjecture $\Delta_1(x;\varphi) = O(X^{9/8+\varepsilon})$ is true and $H = T^{3/4+\sqrt{\varepsilon}/2}$, then we have $H' \asymp T^{\sqrt{\varepsilon}/2}$. Thus, Theorem 3 contains the integral $\int_{T-G}^{T+G} \Delta_1^k(x^4;\varphi)dx$ has asymptotic formula for $G = T^{\sqrt{\varepsilon}/2}$. Thus, the constant $3/4$ in Theorem 3 is probably the best.

2. Some Preliminary Lemmas

Lemma 1. *Suppose $x > 1$ is a real number. For $1 \ll N \ll x^2$ a parameter we have*

$$\Delta_1(x;\varphi) = \frac{1}{(2\pi)^2}\mathcal{R}(x;N) + O(x^{1+\varepsilon} + x^{3/2+\varepsilon}N^{-1/2}), \tag{2.1}$$

where

$$\mathcal{R} := \mathcal{R}(x,N) = x^{9/8}\sum_{n\le N}\frac{c_n}{n^{7/8}}\cos\left(8\pi\sqrt[4]{nx} - \frac{\pi}{4}\right).$$

Proof. This is [9, Lemma 2.1].

Lemma 2. *Suppose $k \ge 3$, $(i_1,\cdots,i_{k-1}) \in \{0,1\}^{k-1}$ such that*

$$\sqrt[4]{n_1} + (-1)^{i_1}\sqrt[4]{n_2} + (-1)^{i_2}\sqrt[4]{n_3} + \cdots + (-1)^{i_{k-1}}\sqrt[4]{n_k} \ne 0.$$

Then, we have

$$|\sqrt[4]{n_1} + (-1)^{i_1}\sqrt[4]{n_2} + (-1)^{i_2}\sqrt[4]{n_3} + \cdots + (-1)^{i_{k-1}}\sqrt[4]{n_k}| \gg \max(n_1,\cdots,n_k)^{-(4^{k-2}-4^{-1})}.$$

Proof. This is [9, Lemma 2.3].

Lemma 3. *If $f(x)$ and $g(x)$ are continuous real-valued functions of x and $f(x)$ is monotonic, then we have*

$$\int_a^b f(x)g(x)dx \ll \left(\max_{a\le x\le b}|f(x)|\right)\left(\max_{a\le u<v\le b}\left|\int_u^v g(x)dx\right|\right).$$

Proof. This follows from the second mean value theorem.

For any real numbers $p(\ne 0)$ and q, by using this lemma we can obtain

$$\begin{aligned}\int_T^{2T} x^{27/8}\cos(p\sqrt[4]{x}+q)dx &= \int_T^{2T} 4p^{-1}x^{33/8}\left(\frac{p}{4x^{3/4}}\cos(p\sqrt[4]{x}+q)\right)dx\\ &\ll T^{33/8}|p|^{-1}\left|\int_u^v \frac{p}{4x^{3/4}}\cos(p\sqrt[4]{x}+q)dx\right|\\ &\ll T^{33/8}|p|^{-1}.\end{aligned}$$

Lemma 4. *Suppose $k \ge 3$, $(i_1,\cdots,i_{k-1}) \in \{0,1\}^{k-1}$, $(i_1,\cdots,i_{k-1}) \ne (0,\cdots,0)$, $N_1,\cdots,N_k > 1$, $0 < \Delta \ll E^{1/4}$, $E = \max(N_1,\cdots,N_k)$. Let $\mathcal{A}$ denote the number of solutions of the inequality*

$$|\sqrt[4]{n_1} + (-1)^{i_1}\sqrt[4]{n_2} + (-1)^{i_2}\sqrt[4]{n_3} + \cdots + (-1)^{i_{k-1}}\sqrt[4]{n_k}| < \Delta \tag{2.2}$$

with $N_j < n_j \le 2N_j$, $1 \le j \le k$, where

$$\mathcal{A} = \mathcal{A}(N_1,\cdots,N_k;i_1,\cdots,i_{k-1};\Delta).$$

Then, we have

$$\mathcal{A} \ll \Delta E^{-1/4}N_1\cdots N_k + E^{-1}N_1\cdots N_k.$$

Proof. The proof of this lemma is similar to the proof of [10, Lemma 2.4]. Suppose $E = N_k$. If $(n_1, \cdots, n_k)$ satisfies (2.2), then for some $|\theta| < 1$, we can obtain

$$\sqrt[4]{n_1} + (-1)^{i_1}\sqrt[4]{n_2} + (-1)^{i_2}\sqrt[4]{n_3} + \cdots + (-1)^{i_{k-2}}\sqrt[4]{n_{k-1}} = (-1)^{i_{k-1}}\sqrt[4]{n_k} + \theta\Delta.$$

Thus, we have

$$\sqrt[4]{n_1} + (-1)^{i_1}\sqrt[4]{n_2} + (-1)^{i_2}\sqrt[4]{n_3} + \cdots + (-1)^{i_{k-2}}\sqrt[4]{n_{k-1}} = (-1)^{i_{k-1}}\sqrt[4]{n_k} + \theta\Delta,$$

$$\left(\sqrt[4]{n_1} + (-1)^{i_1}\sqrt[4]{n_2} + (-1)^{i_2}\sqrt[4]{n_3} + \cdots + (-1)^{i_{k-2}}\sqrt[4]{n_{k-1}}\right)^4 = n_k + O(\Delta N_k^{3/4}).$$

Therefore, for fixed $(n_1, \cdots, n_{k-1})$, the number of n_k is $\ll 1 + \Delta N_k^{3/4}$ and so

$$\mathcal{A} \ll \Delta N_k^{3/4} N_1 \cdots N_{k-1} + N_1 \cdots N_{k-1}.$$

3. Proof of Theorem 1

If $T/2 \le H \le T$, it is easily to get Theorem 1. Suppose $H \le \frac{T}{2}$ and y is a parameter such that $T^{\varepsilon} < y \le T^{1/3}$. For any $T \le x \le 2T$, we define

$$\mathcal{R} = \frac{x^{9/8}}{(2\pi)^2}\sum_{n\le y}\frac{c_n}{n^{7/8}}\cos\left(8\pi\sqrt[4]{nx} - \frac{\pi}{4}\right),$$
$$\mathcal{R}_1 = \mathcal{R}_1(x, y) := \Delta_1(x;\varphi) - \mathcal{R}.$$

We will prove that the higher-power moment of $\mathcal{R}_1$ is small, so the integral $\int_{T-H}^{T+H}\Delta_1^k(x;\varphi)dx$ can be well approximated by $\int_{T-H}^{T+H}\mathcal{R}^k dx$, which is easy to evaluate.

3.1. *Evaluation of the integral* $\int_{T-H}^{T+H}\mathcal{R}^k dx$

First, by the elementary formula

$$\cos b_1 \cdots \cos b_k = \frac{1}{2^{k-1}}\sum_{(i_1,\cdots,i_{k-1})\in\{0,1\}^{k-1}}\cos(b_1 + (-1)^{i_1}b_2 + (-1)^{i_2}b_3 + \cdots + (-1)^{i_{k-1}}b_k),$$

we can write

$$\begin{aligned}\mathcal{R}^k =& (2\pi)^{-2k}x^{9k/8}\sum_{n_1\le y}\cdots\sum_{n_k\le y}\frac{c_{n_1}\cdots c_{n_k}}{(n_1\cdots n_k)^{7/8}}\prod_{j=1}^{k}\cos(8\pi\sqrt[4]{n_j x} - \pi/4)\\ =& (2\pi)^{-2k}x^{9k/8}\sum_{(i_1,\cdots,i_{k-1})\in\{0,1\}^{k-1}}\sum_{n_1\le y}\cdots\sum_{n_k\le y}\frac{c_{n_1}\cdots c_{n_k}}{(n_1\cdots n_k)^{7/8}}\\ &\times\cos\left(8\pi\sqrt[4]{x}\alpha(n_1,\cdots,n_k;i_1,\cdots,i_{k-1}) - \pi/4\beta(i_1,\cdots,i_{k-1})\right),\end{aligned}$$

where

$$\alpha(n_1,\cdots,n_k;i_1,\cdots,i_{k-1}) := \sqrt[4]{n_1}+(-1)^{i_1}\sqrt[4]{n_2}+(-1)^{i_2}\sqrt[4]{n_3}+\cdots+(-1)^{i_{k-1}}\sqrt[4]{n_k},$$
$$\beta(i_1,\cdots,i_{k-1}) := 1+(-1)^{i_1}+(-1)^{i_2}+\cdots+(-1)^{i_{k-1}}.$$

Therefore, we have

$$\begin{aligned}\mathcal{R}^k =&\frac{1}{2^{k-1}(2\pi)^{2k}}x^{9k/8}\sum_{(i_1,\cdots,i_{k-1})\in\{0,1\}^{k-1}}\cos\Big(-\frac{\pi\beta}{4}\Big)\sum_{\substack{n_j\le y,1\le j\le k\\ \alpha=0}}\frac{c_{n_1}\cdots c_{n_k}}{(n_1\cdots n_k)^{7/8}}\\ &+\frac{1}{2^{k-1}(2\pi)^{2k}}x^{9k/8}\sum_{(i_1,\cdots,i_{k-1})\in\{0,1\}^{k-1}}\sum_{\substack{n_j\le y,1\le j\le k\\ \alpha\ne0}}\frac{c_{n_1}\cdots c_{n_k}}{(n_1\cdots n_k)^{7/8}}\cos\Big(8\pi\alpha\sqrt[4]{x}-\frac{\pi\beta}{4}\Big)\\ =&\frac{1}{2^{k-1}(2\pi)^{2k}}\left(S_3(x,k)+S_4(x,k)\right),\end{aligned}\tag{3.1}$$

where

$$\alpha := \alpha(n_1,\cdots,n_k;i_1,\cdots,i_{k-1}),\ \beta := \beta(i_1,\cdots,i_{k-1}).$$

Consider $S_3(x,k)$. We have

$$\int_{T-H}^{T+H}S_3(x,k)dx = \sum_{(i_1,\cdots,i_{k-1})\in\{0,1\}^{k-1}}\cos\Big(-\frac{\pi\beta}{4}\Big)\sum_{\substack{n_j\le y,1\le j\le k\\ \alpha=0}}\frac{c_{n_1}\cdots c_{n_k}}{(n_1\cdots n_k)^{7/8}}\int_{T-H}^{T+H}x^{9k/8}dx.\tag{3.2}$$

By (4.3) in [9], we know that

$$\frac{1}{2^{k-1}\cdot(2\pi)^{2k}}\int_{T-H}^{T+H}S_3(x,k)dx = \frac{B_k(c)}{2^{k-1}\cdot(2\pi)^{2k}}\int_{T-H}^{T+H}x^{9k/8}dx+O\Big(HT^{9k/8+\varepsilon}y^{-3/4}\Big).\tag{3.3}$$

Now we consider $S_4(x,k)$. By the first derivative of van der Corput method, one has

$$\int_{T-H}^{T+H}S_4(x,k)dx \ll T^{3/4+9k/8}\sum_{(i_1,\cdots,i_{k-1})\in\{0,1\}^{k-1}}\sum_{\substack{n_j\le y,1\le j\le k\\ \alpha\ne0}}\frac{c_{n_1}\cdots c_{n_k}}{(n_1\cdots n_k)^{7/8}|\alpha|}.\tag{3.4}$$

For fixed $(i_1,\cdots,i_{k-1})\in\{0,1\}^{k-1}$, we write

$$\sum(y;i_1,\cdots,i_{k-1}) = \sum_{\substack{n_j\le y,1\le j\le k\\ \alpha\ne0}}\frac{c_{n_1}\cdots c_{n_k}}{(n_1\cdots n_k)^{7/8}|\alpha|}.$$

If $(i_1,\cdots,i_{k-1})=(0,\cdots,0)$, then we have

$$\begin{aligned}\sum(y;0,\cdots,0) &\ll \sum_{n_j\le y,1\le j\le k}\frac{c_{n_1}\cdots c_{n_k}}{(n_1\cdots n_k)^{7/8}(\sqrt[4]{n_1}+\cdots+\sqrt[4]{n_k})}\\ &\ll \sum_{n_j\le y,1\le j\le k}\frac{c_{n_1}\cdots c_{n_k}}{(n_1\cdots n_k)^{7/8+1/(4k)}}\\ &\ll y^{(k-2)/8}\log^k y.\end{aligned}$$

For $(i_1, \cdots, i_{k-1}) \neq (0, \cdots, 0)$, by a splitting argument we deduce that there exist a collection of numbers $1 < N_1, \cdots, N_k < y$ such that

$$\sum(y; i_1, \cdots, i_{k-1}) \ll \sum_{\substack{N_j < n_j \le 2N_j, 1 \le j \le k \\ \alpha \neq 0}} \frac{c_{n_1} \cdots c_{n_k}}{(n_1 \cdots n_k)^{7/8} |\alpha|} \log^k y.$$

Suppose $N_1 \le \cdots \le N_k \le y$. According to Lemma 2, we know that $|\alpha| \gg N_k^{-(4^{k-2}-4^{-1})}$. For some $N_k^{-(4^{k-2}-4^{-1})} \ll \Delta < y^{1/4}$, by using Lemma 4 we can get

$$\begin{aligned}
\sum_{\substack{N_j < n_j \le 2N_j, 1 \le j \le k \\ \alpha \neq 0}} \frac{c_{n_1} \cdots c_{n_k}}{(n_1 \cdots n_k)^{7/8} |\alpha|} &\ll \frac{y^{\varepsilon}}{(N_1 \cdots N_k)^{7/8} \Delta} \mathcal{A}(N_1, \cdots, N_k; i_1, \cdots, i_{h-1}; \Delta) \\
&\ll \frac{y^{\varepsilon}}{(N_1 \cdots N_k)^{7/8} \Delta} \left(\Delta N_k^{3/4} N_1 \cdots N_{k-1} + N_1 \cdots N_{k-1} \right) \\
&\ll y^{\varepsilon} \left(N_k^{\frac{k-2}{8}} + N_k^{4^{k-2} + \frac{k-10}{8}} \right) \\
&\ll y^{b(k)+\varepsilon},
\end{aligned} \tag{3.5}$$

where $b(k) = 4^{k-2} + \frac{k-10}{8}$. Therefore we have

$$\int_{T-H}^{T+H} S_4(x, k) dx \ll H^{3/4} T^{9k/8+\varepsilon} y^{b(k)}. \tag{3.6}$$

According to (3.1)-(3.6), we have

$$\int_{T-H}^{T+H} |\mathcal{R}|^k dx = \frac{B_k(c)}{2^{k-1} \cdot (2\pi)^{2k}} \int_{T-H}^{T+H} x^{9k/8} dx + O\left(HT^{9k/8+\varepsilon} y^{-3/4} + T^{3/4+9k/8+\varepsilon} y^{b(k)} \right) \tag{3.7}$$

3.2. Higher-power moments of $\mathcal{R}_1$

From the definition of K_0 ($K_0 := \min\{n \in \mathbb{N} : n \ge A_0, 2|n\}$ in [9] we write

$$K_0 = \begin{cases} k+1, & k \text{ is odd}, \\ k+2, & \text{otherwise}. \end{cases}$$

Suppose that $y \le (HT^{-3/4})^{1/b(K_0)}$. Let $N = T$ in the formula (2.1) of Lemma 1. Then, we can obtain

$$\begin{aligned}
\mathcal{R}_1 =& (2\pi)^{-2} x^{9/8} \sum_{y < n \le T} \frac{c_n}{n^{7/8}} \cos(8\pi \sqrt[4]{nx} - \pi/4) + O(T^{1+\varepsilon}) \\
\ll& \left| x^{9/8} \sum_{y < n \le T} \frac{c_n}{n^{7/8}} e(4\sqrt[4]{nx}) \right| + T^{1+\varepsilon}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}\int_{T-H}^{T+H}\mathcal{R}_1^2dx &\ll HT^{2+\varepsilon}+\int_{T-H}^{T+H}\left(x^{9/8}\sum_{y<n\le T}\frac{c_n}{n^{7/8}}e(4\sqrt[4]{nx})\right)^2dx\\ &\ll HT^{2+\varepsilon}+HT^{9/4}\sum_{y<n\le T}\frac{c_n^2}{n^{7/4}}+T^3\sum_{y<m<n\le T}\frac{c_nc_m}{(mn)^{7/8}(\sqrt[4]{n}-\sqrt[4]{m})}\\ &\ll HT^{2+\varepsilon}+T^{3+\varepsilon}+\frac{HT^{9/4}\log^3 T}{y^{3/4}}\\ &\ll \frac{HT^{9/4}\log^3 T}{y^{3/4}}.\end{aligned} \tag{3.8}$$

From (3.7) we can obtain that

$$\int_{T-H}^{T+H}|\mathcal{R}|^{K_0}dx \ll HT^{9K_0/8+\varepsilon}.$$

By Hölder's inequality and the above formula we have

$$\int_{T-H}^{T+H}|\mathcal{R}|^{A_0}dx \ll HT^{9A_0/8+\varepsilon} \tag{3.9}$$

for $A_0=k+\delta$, here $0<\delta<1$ is a fixed constant. From (1.5) and (3.9) we get

$$\int_{T-H}^{T+H}|\mathcal{R}_1|^{A_0}dx \ll \int_{T-H}^{T+H}\left(|\Delta(x;\varphi)|^{A_0}+|\mathcal{R}|^{A_0}\right)dx \ll HT^{9A_0/8+\varepsilon}. \tag{3.10}$$

For any $2<A<A_0$, by (3.8), (3.10) and Hölder inequality we have

$$\begin{aligned}\int_{T-H}^{T+H}|\mathcal{R}_1|^{A}dx &= \int_{T-H}^{T+H}|\mathcal{R}_1|^{\frac{2(A_0-A)}{A_0-2}+\frac{A_0(A-2)}{A_0-2}}dx\\ &\ll \left(\int_{T-H}^{T+H}\mathcal{R}_1^2dx\right)^{\frac{(A_0-A)}{A_0-2}}\left(\int_{T-H}^{T+H}|\mathcal{R}_1|^{A_0}dx\right)^{\frac{(A-2)}{A_0-2}}\\ &\ll HT^{\frac{9A}{8}+\varepsilon}y^{-\frac{3(A_0-A)}{4(A_0-2)}}.\end{aligned} \tag{3.11}$$

Thus we have

Lemma 5. *Suppose $T^{\varepsilon}\le y\le (HT^{-3/4})^{1/b(K_0)}$, $2<A<A_0$. Then, we have*

$$\int_{T-H}^{T+H}|\mathcal{R}_1|^{A}dx \ll HT^{\frac{9A}{8}+\varepsilon}y^{-\frac{3(A_0-A)}{4(A_0-2)}}. \tag{3.12}$$

3.3. Proof of Theorem 1

Suppose $3\le k<A_0$. By the elementary formula $(a+b)^k-a^k\ll |b|^k+|a^{k-1}b|$ we have

$$\int_{T-H}^{T+H}\Delta_1^k(x;\varphi)dx=\int_{T-H}^{T+H}\mathcal{R}^kdx+O\left(\int_{T-H}^{T+H}|\mathcal{R}^{k-1}\mathcal{R}_1|dx\right)+O\left(\int_{T-H}^{T+H}|\mathcal{R}_1|^kdx\right). \tag{3.13}$$

By (3.9), Lemma 5 and Hölder inequality we have

$$\begin{aligned}\int_{T-H}^{T+H}|\mathcal{R}^{k-1}\mathcal{R}_1|dx &\ll \left(\int_{T-H}^{T+H}|\mathcal{R}^{A_0}|dx\right)^{\frac{k-1}{A_0}}\left(\int_{T-H}^{T+H}|\mathcal{R}_1^{\frac{A_0}{A_0-k+1}}|dx\right)^{\frac{A_0-k+1}{A_0}}\\ &\ll HT^{\frac{9k}{8}+\varepsilon}y^{-\frac{3(A_0-k)}{4(A_0-2)}}.\end{aligned} \tag{3.14}$$

Taking $y=(HT^{-3/4})^{\frac{1}{b(K_0)+\frac{3\delta}{4(k+\delta-2)}}}$. From (3.13), (3.14), (3.7) and Lemma 5, we can obtain

$$\begin{aligned}\int_{T-H}^{T+H}\Delta_1^k(x;\varphi)dx =&\frac{B_k(c)}{2^{k-1}\cdot(2\pi)^{2k}}\int_{T-H}^{T+H}x^{9k/8}dx+O\Big(T^{3/4+9k/8+\varepsilon}y^{b(K_0)}+HT^{\frac{9k}{8}+\varepsilon}y^{-\frac{3(A_0-k)}{4(A_0-2)}}\Big)\\ =&\frac{B_k(c)}{2^{k-1}\cdot(2\pi)^{2k}}\int_{T-H}^{T+H}x^{9k/8}dx+O\Big(HT^{\frac{9k}{8}+\varepsilon}(HT^{-3/4})^{-\frac{3\delta}{4b(K_0)(k+\delta-2)+3\delta}}\Big)\\ =&\frac{B_k(c)}{2^{k-1}\cdot(2\pi)^{2k}}\int_{T-H}^{T+H}x^{9k/8}dx+O\Big(HT^{\frac{9k}{8}+\varepsilon}(HT^{-3/4})^{-\frac{3\delta}{\delta_k}}\Big).\end{aligned} \tag{3.15}$$

From this we can easily get Theorem 1.

4. Proof of Theorem 2

In this section, we will give the proof of Theorem 2. Let $N=T^{3/5}$. By (2.1) we can obtain

$$\Delta_1(x;\varphi)\ll x^{6/5}.$$

Suppose $\frac{1}{8}<\theta<\frac{1}{5}$ be a fixed real number. Let $\Delta_1(x;\varphi)\ll x^{\theta+1}$.

Suppose $T-H\le x_1<\cdots<x_R\le T+H$ satisfies $|x_r-x_s|\ge V$ $(r\ne s\le R)$, $T^{1/8}\ll V\ll T^{\theta}$ and $|\Delta_1(x_r;\varphi)|\gg VT$ $(r=1,\cdots,R)$. Dividing interval $[T-H,T+H]$ into the subinterval of length not exceeding T_0 $(T_0\ge V)$. Let R_0 denotes the number of x_r which exists in the subinterval of length not exceeding T_0. Then, we have

$$R\ll R_0(1+H/T_0).$$

Tanigawa, Zhai and Zhang [9] proved that

$$R\ll TV^{-3}\mathcal{L}^5+HT^3V^{-25}\mathcal{L}^{39}, \tag{4.1}$$

if $T_0=V^{22}T^{-2}\mathcal{L}^{-34}$, $R_0\ll T^{1+4\varepsilon}V^{-2}$.

Now we consider $\int_{T-H}^{T+H}|\Delta_1(x;\varphi)|^A dx$, where $2<A<24$ be a fixed real number. According to [9], we have

$$\int_{T-H}^{T+H}|\Delta_1(x;\varphi)|^A dx\ll HT^{9A/8+\varepsilon}+\sum_V V\sum_{r\le R_V}|\Delta_1(x_r;\varphi)|^A, \tag{4.2}$$

where $T^{1/8}\le V=2^m\le T^{\theta}$, $VT<|\Delta_1(x_r;\varphi)|\le 2VT$ $(r=1,\cdots,R_V)$ and $|x_r-x_s|\ge V$ for $r\ne s\le R\le R_V$. By (4.1) we have

$$\begin{aligned}V\sum_{r\le R_V}|\Delta_1(x_r;\varphi)|^A &\ll R_VT^AV^{A+1}\ll \mathcal{L}^5T^{1+A}V^{A-2}+\mathcal{L}^{39}HT^{3+A}V^{A-24}\\ &\ll T^{1+A}V^{A-2}\mathcal{L}^5+HT^{9A/8}\mathcal{L}^{39}.\end{aligned} \tag{4.3}$$

Combining (4.2) and (4.3), we get

$$\int_{T-H}^{T+H} |\Delta_1(x;\varphi)|^A dx \ll HT^{9A/8+\varepsilon} + T^{1+A+\theta(A-2)+\varepsilon}.$$

Now suppose $2 < A < 2\theta/(\theta - 1/8)$. If $H \geq T^{1+\theta(A-2)-A/8}$, then we have

$$\int_{T-H}^{T+H} |\Delta_1(x;\varphi)|^A dx \ll HT^{9A/8+\varepsilon}.$$

We notice that $2\theta/(\theta - 1/8) = 16/3 = 5.33\cdots$, if $\theta = 1/5$. In Theorem 1, taking $\delta = 4/3\delta_k\sqrt{\varepsilon}$ for $3 \leq k \leq 5$, we can get Theorem 2.

5. Proof of Theorem 3

Suppose $\Delta_1(x;\varphi) \ll x^{9/8+\varepsilon}$. Then,

$$\int_{T-H}^{T+H} |\Delta_1(x;\varphi)|^{k+\delta} dx \ll HT^{9(k+\delta)/8+\varepsilon'},$$

where $\varepsilon' = (k+1)\varepsilon$. From Theorem 1, we can get asymptotic formula

$$\int_{T-H}^{T+H} \Delta_1^k(x;\varphi)dx = \frac{B_k(c)}{2^{k-1}\cdot(2\pi)^{2k}}\int_{T-H}^{T+H} x^{9k/8}dx + O\left(HT^{\frac{9k}{8}+\varepsilon'}(HT^{-3/4})^{-3\delta/\delta_k}\right)$$

for $T^{\frac{2\delta_k\varepsilon'}{3\delta}+\frac{3}{4}} \leq H \leq T$. Taking $\delta = \frac{4}{3}\delta_k(k+1)\sqrt{\varepsilon}$ in Theorem 1, we can get Theorem 3.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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