



Research article

# Characterizations of matrix equalities involving the sums and products of multiple matrices and their generalized inverse

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**Abstract:** It is common knowledge that matrix equalities involving ordinary algebraic operations of inverses or generalized inverses of given matrices can be constructed arbitrarily from theoretical and applied points of view because of the noncommutativity of the matrix algebra and singularity of given matrices. Two of such matrix equality examples are given by  $A_1 B_1^{g_1} C_1 + A_2 B_2^{g_2} C_2 + \dots + A_k B_k^{g_k} C_k = D$  and  $A_1 B_1^{g_1} A_2 B_2^{g_2} \dots A_k B_k^{g_k} A_{k+1} = A$ , where  $A_1, A_2, \dots, A_{k+1}, C_1, C_2, \dots, C_k$  and  $A$  and  $D$  are given, and  $B_1^{g_1}, B_2^{g_2}, \dots, B_k^{g_k}$  are generalized inverses of matrices  $B_1, B_2, \dots, B_k$ . These two matrix equalities include many concrete cases for different choices of the generalized inverses, and they have been attractive research topics in the area of generalized inverse theory. As an ongoing investigation of this subject, the present author presents in this article several groups of new results and facts on constructing and characterizing the above matrix equalities for the mixed combinations of {1}- and {1, 2}-generalized inverses of matrices with  $k = 2, 3$  by using some elementary methods, including a series of explicit rank equalities for block matrices.

**Keywords:** block matrix; equality; generalized inverse; invariance; product; rank; sum

## 1. Introduction

Throughout this article, let  $\mathbb{C}^{m \times n}$  stand for the set of all  $m \times n$  matrices over the field of complex numbers;  $A^*$  stands for the conjugate transpose of  $A \in \mathbb{C}^{m \times n}$ ;  $r(A)$  stands for the rank of  $A \in \mathbb{C}^{m \times n}$ ;  $I_m$  stands for the identity matrix of order  $m$  and  $[A, B]$  stands for a columnwise partitioned matrix consisting of two submatrices  $A$  and  $B$ . We introduce the concept of generalized inverses of a matrix. For an  $A \in \mathbb{C}^{m \times n}$ , the Moore–Penrose generalized inverse of  $A$  is defined to be the unique matrix  $X \in \mathbb{C}^{n \times m}$  that satisfies the four Penrose equations:

$$(1) AXA = A, \quad (2) XAX = X, \quad (3) (AX)^* = AX, \quad (4) (XA)^* = XA. \quad (1.1)$$

A matrix  $X \in \mathbb{C}^{n \times m}$  is called an  $\{i, \dots, j\}$ -generalized inverse of  $A \in \mathbb{C}^{m \times n}$ , denoted by  $A^{(i, \dots, j)}$ , if it satisfies the  $i$ th,  $\dots$ ,  $j$ th matrix equations in (1.1). Notice that solutions of each of the four equations in (1.1) are not necessarily unique. Hence, the set of all  $\{i, \dots, j\}$ -generalized inverses of the matrix  $A$  is usually denoted by the symbol  $\{A^{(i, \dots, j)}\}$ . There are altogether 15 sorts of  $\{i, \dots, j\}$ -generalized inverses of  $A$  by the above definition, but the following

$$A^\dagger, A^{(1,3,4)}, A^{(1,2,4)}, A^{(1,2,3)}, A^{(1,4)}, A^{(1,3)}, A^{(1,2)}, A^{(1)} \quad (1.2)$$

are usually called the eight commonly-used types of generalized inverses of  $A$  in the literature. Admittedly, the Moore–Penrose generalized inverse of a matrix is a natural extension of the standard inverse of an invertible matrix, which has become one of the basic and classic concepts in matrix theory and its applications due to its usefulness in dealing with singular matrices. For more basic results and facts concerning generalized inverses of matrices and their properties, we refer the reader to the three references [1–3].

Let us consider the following two general matrix equalities:

$$A_1 B_1^{(i_1, \dots, j_1)} C_1 + A_2 B_2^{(i_2, \dots, j_2)} C_2 + \dots + A_k B_k^{(i_k, \dots, j_k)} C_k = D, \quad (1.3)$$

$$A_1 B_1^{(i_1, \dots, j_1)} A_2 B_2^{(i_2, \dots, j_2)} \dots A_k B_k^{(i_k, \dots, j_k)} A_{k+1} = A, \quad (1.4)$$

where it is assumed that  $A_1, A_2, \dots, A_{k+1}, B_1, B_2, \dots, B_k, C_1, C_2, \dots, C_k$  and  $A$  and  $D$  are matrices of appropriate sizes. Clearly, (1.3) and (1.4) are generalizations of various concrete matrix equalities that involve mixed operations of ordinary sums and products of several matrices and their generalized inverses. For the sake of exposition, let us take the following two well-known examples:

$$B^{(i, \dots, j)} = B_1^{(i_1, \dots, j_1)} + B_2^{(i_2, \dots, j_2)} + \dots + B_k^{(i_k, \dots, j_k)}, \quad (1.5)$$

$$(B_k \dots B_2 B_1)^{(i, \dots, j)} = B_1^{(i_1, \dots, j_1)} B_2^{(i_2, \dots, j_2)} \dots B_k^{(i_k, \dots, j_k)}, \quad (1.6)$$

which are called the parallel sum of matrices and the reverse-order law for generalized inverses of multiple matrix products, respectively. As demonstrated in Lemma 2.2 below, generalized inverses of a matrix can be written as certain linear and bilinear matrix expressions involving two arbitrary matrices. Hence, the two matrix equalities in (1.3) and (1.4) can be equivalently represented as two kinds of linear and nonlinear matrix equations involving multiple arbitrary matrices (variable matrices).

As we know, in algebra theory, or a given algebraic expression or equality that involves variable entries, algebraists can propose many kinds of questions relevant to its properties and performances, but of particular interest regarding the expression or equality is to determine explicit conditions under which the expression is invariant with respect to the variable entries or the equality holds for some/all values of the variable entries included. In view of this consideration, it is natural to ask the questions related to determining necessary and sufficient conditions for the matrix equalities in the context of (1.3) and (1.4) to hold respectively for some/all of the generalized inverses of the matrices in them. In fact, this sort of research problems were proposed and investigated by lots of authors focused generalized inverses of matrices and its various applications [4–7]. Prompted by this kind of work, the present author and his collaborators considered a selection of linear and nonlinear matrix expressions or matrix equalities that involve one or more arbitrary matrices, established a fundamental theory related to the invariance properties of different types of linear and nonlinear matrix expressions and presented their

essential utilizations in the characterizations of many kinds of matrix equalities that involve algebraic operations of generalized inverses of matrices [8–14]. Particularly in two recent papers [15, 16], the present author considered (1.4) for  $\{1\}$ -generalized inverses of matrices of  $B_1, B_2, \dots, B_k$  with  $k = 2, 3$ , deriving different types of identifying conditions under which the given matrix equalities always hold with respect to all possible  $\{1\}$ -generalized inverses of the given matrices by using a diversity of precise rank equalities associated with certain specified block matrices that are constructed from the given matrices in (1.4). Obviously, characterization problems of (1.3) and (1.4) involve lots of difficult and divisive issues because they are quite inclusive for various possible choices of  $\{i, \dots, j\}$ -generalized inverses of the given matrices, thus, there exist many concrete cases of (1.3) and (1.4) that we need to give further deep explorations.

As an ongoing study of the above algebraic equality characterization problems on matrices, the main objective of this article is to consider several cases of the matrix equalities in (1.3) and (1.4) for the mixed combinations of  $\{1\}$ - and  $\{1, 2\}$ -generalized inverses of matrices with  $k = 2, 3$ , and to give identifying conditions under which these algebraic matrix equalities always hold by using a series of ordinary but skillful calculations of ranks of matrices. The rest of this paper is organized in five sections. In Section 2, the present author introduces a sequence of well-known or existing basic formulas, facts and results related to ranks, ranges and generalized inverses of matrices that we shall utilize in the derivations of our main results regarding (1.3) and (1.4). In Section 3, the author derives necessary and sufficient conditions for the matrix equalities in (1.3) to hold with mixed combinations of  $\{1\}$ - and  $\{1, 2\}$ -generalized inverses of matrices for  $k = 2, 3$  by means of the elementary but the powerful block matrix representation method (BMRM) and the matrix rank method (MRM). In Section 4, the author gives identifying conditions for the matrix equalities in (1.4) to always hold with mixed combinations of  $\{1\}$ - and  $\{1, 2\}$ -generalized inverses of matrices for  $k = 2, 3$ . In Section 5, the author presents a variety of parallel conclusions related to several kinds of algebraic matrix equalities that involve different mixed algebraic operations of given matrices and their weighted  $\{1\}$ - and  $\{1, 2\}$ -generalized inverses. An application is presented in Section 6 for a matrix equality involving generalized inverses of a  $2 \times 2$  block matrix. Concluding remarks are given in Section 7.

## 2. Some preliminaries

In this section, the author presents some well-known or existing results and conclusions regarding a two-sided linear matrix equation, the general expressions of generalized inverses of a matrix, as well as some equalities and inequalities associated with ranks of block matrices, which we shall adopt as fundamental and feasible matrix analysis tools in the examinations of the algebraic matrix equality problems described in Section 1.

**Lemma 2.1** ([17]). *Let*

$$AXB = C \tag{2.1}$$

*be a two-sided linear matrix equation, where  $A \in \mathbb{C}^{m \times n}$ ,  $B \in \mathbb{C}^{p \times q}$  and  $C \in \mathbb{C}^{m \times q}$  are given. Then, the following four statements are equivalent:*

- (a) *Equation (2.1) is solvable for  $X$ .*

(b)  $\mathcal{R}(A) \supseteq \mathcal{R}(C)$  and  $\mathcal{R}(B^*) \supseteq \mathcal{R}(C^*)$ .

(c)  $r[A, C] = r(A)$  and  $r \begin{bmatrix} B \\ C \end{bmatrix} = r(B)$ .

(d)  $AA^\dagger CB^\dagger B = C$ .

In this case, the general solution of (2.1) can be written in the following parametric expression

$$X = A^\dagger CB^\dagger + F_A U + V E_B, \quad (2.2)$$

where  $F_A = I_n - A^\dagger A$ ,  $E_B = I_p - BB^\dagger$  and  $U, V \in \mathbb{C}^{n \times p}$  are arbitrary matrices. Particularly, (2.1) holds for all matrices  $X \in \mathbb{C}^{n \times p}$  if and only if

$$[A, C] = 0 \text{ or } \begin{bmatrix} B \\ C \end{bmatrix} = 0. \quad (2.3)$$

**Lemma 2.2** ([17]). Let  $A \in \mathbb{C}^{m \times n}$ . Then, the general expressions of  $A^{(1)}$  and  $A^{(1,2)}$  can be written as

$$A^{(1)} = A^\dagger + F_A U + V E_A, \quad (2.4)$$

$$A^{(1,2)} = (A^\dagger + F_A U)A(A^\dagger + V E_A), \quad (2.5)$$

where  $E_A = I_m - AA^\dagger$ ,  $F_A = I_n - A^\dagger A$  and  $U, V \in \mathbb{C}^{n \times m}$  are arbitrary.

Recall a simple fact in linear algebra that a matrix is null if and only if its rank is zero. As a direct consequence of this fact, we see that two matrices  $A$  and  $B$  of the same size are equal if and only if the matrix rank equality  $r(A - B) = 0$  holds. In other words, these obvious two equivalent statements connect the matrix equality  $A = B$  alternatively to the matrix rank equality  $r(A - B) = 0$ . In view of this assertion, we can see that if certain nontrivial analytical formulas for calculating the rank of  $A - B$  are obtained, then we can directly employ them to reveal essential connections between the two matrices. Of course, the usefulness of this proposed methodology is based on the fact that we can precisely calculate ranks of matrix expressions by using elementary operations of matrices. As a matter of fact, algebraists realized that numerous matrix rank formulas can be established in reasonable ways, while these matrix rank formulas have been recognized as powerful tools to construct, characterize and understand hidden structures and properties of various algebraic matrix expressions and matrix equalities.

**Lemma 2.3** ([18]). Let  $A \in \mathbb{C}^{m \times n}$ ,  $B \in \mathbb{C}^{m \times k}$  and  $C \in \mathbb{C}^{l \times n}$ . Then,

$$r[A, B] \geq r(A), \quad (2.6)$$

$$r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \geq r(B) + r(C). \quad (2.7)$$

**Lemma 2.4** ([19, 20]). Let  $A \in \mathbb{C}^{m \times n}$ ,  $B \in \mathbb{C}^{m \times k}$ ,  $C \in \mathbb{C}^{l \times n}$  and  $D \in \mathbb{C}^{l \times k}$  be given. Then,

$$\max_{A^{(1)}} r(D - CA^{(1)}B) = \min \left\{ r[C, D], r \begin{bmatrix} B \\ D \end{bmatrix}, r \begin{bmatrix} A & B \\ C & D \end{bmatrix} - r(A) \right\}, \quad (2.8)$$

$$\max_{A^{(1,2)}} r(D - CA^{(1,2)}B) = \min \left\{ r(A) + r(D), r[C, D], r \begin{bmatrix} B \\ D \end{bmatrix}, r \begin{bmatrix} A & B \\ C & D \end{bmatrix} - r(A) \right\}. \quad (2.9)$$

Therefore,

$$CA^{(1)}B = D \text{ holds for all } A^{(1)} \Leftrightarrow [C, D] = 0, \text{ or } \begin{bmatrix} B \\ D \end{bmatrix} = 0, \text{ or } r \begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A); \quad (2.10)$$

$$CA^{(1,2)}B = D \text{ holds for all } A^{(1,2)} \Leftrightarrow A = 0 \text{ and } D = 0, \text{ or } [C, D] = 0, \text{ or } \begin{bmatrix} B \\ D \end{bmatrix} = 0, \text{ or } r \begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A). \quad (2.11)$$

### 3. Additive matrix equalities that involve two generalized inverses

The cases of the matrix equalities in (1.3) for  $k = 2$  and  $\{1\}$ - and  $\{1, 2\}$ -generalized inverses of  $B_1$  and  $B_2$  are given by

$$A_1B_1^{(1)}C_1 + A_2B_2^{(1)}C_2 = D, \quad (3.1)$$

$$A_1B_1^{(1)}C_1 + A_2B_2^{(1,2)}C_2 = D, \quad (3.2)$$

$$A_1B_1^{(1,2)}C_1 + A_2B_2^{(1)}C_2 = D, \quad (3.3)$$

$$A_1B_1^{(1,2)}C_1 + A_2B_2^{(1,2)}C_2 = D, \quad (3.4)$$

where  $A_1 \in \mathbb{C}^{m \times p_1}$ ,  $B_1 \in \mathbb{C}^{q_1 \times p_1}$ ,  $C_1 \in \mathbb{C}^{q_1 \times n}$ ,  $A_2 \in \mathbb{C}^{m \times p_2}$ ,  $B_2 \in \mathbb{C}^{q_2 \times p_2}$ ,  $C_2 \in \mathbb{C}^{q_2 \times n}$  and  $D \in \mathbb{C}^{m \times n}$ . In this section, we first give conditions under which the four matrix equalities in (3.1)–(3.4) always hold for all  $\{1\}$ - and  $\{1, 2\}$ -generalized inverses of  $B_1$  and  $B_2$ .

**Theorem 3.1.** *Let  $A_1, B_1, C_1, A_2, B_2, C_2$  and  $D$  be as given in (3.1). Then, the following three statements are equivalent:*

- (a) Equation (3.1) holds for all  $B_1^{(1)}$  and  $B_2^{(1)}$ .
- (b)  $A_1B_1^{(1)}C_1 + A_2B_2^{(1)}C_2$  is invariant with respect to all  $B_1^{(1)}$  and  $B_2^{(1)}$  and  $A_1B_1^\dagger C_1 + A_2B_2^\dagger C_2 = D$  holds.
- (c) Any one of the following 9 conditions holds:
  - (i)  $A_1 = 0, A_2 = 0$  and  $D = 0$ .
  - (ii)  $A_1 = 0, C_2 = 0$  and  $D = 0$ .
  - (iii)  $C_1 = 0, A_2 = 0$  and  $D = 0$ .
  - (iv)  $C_1 = 0, C_2 = 0$  and  $D = 0$ .
  - (v)  $A_1 = 0$  and  $r \begin{bmatrix} B_2 & C_2 \\ A_2 & D \end{bmatrix} = r(B_2)$ .
  - (vi)  $C_1 = 0$  and  $r \begin{bmatrix} B_2 & C_2 \\ A_2 & D \end{bmatrix} = r(B_2)$ .
  - (vii)  $A_2 = 0$  and  $r \begin{bmatrix} B_1 & C_1 \\ A_1 & D \end{bmatrix} = r(B_1)$ .

$$(viii) \ C_2 = 0 \text{ and } r \begin{bmatrix} B_1 & C_1 \\ A_1 & D \end{bmatrix} = r(B_1).$$

$$(ix) \ r \begin{bmatrix} B_1 & 0 & C_1 \\ 0 & B_2 & C_2 \\ A_1 & A_2 & D \end{bmatrix} = r(B_1) + r(B_2).$$

*Proof.* By (2.10), we first obtain that (2.10) holds for all generalized inverses  $B_1^{(1)}$  if and only if

$$[A_1, D - A_2 B_2^{(1)} C_2] = 0 \text{ or } \begin{bmatrix} C_1 \\ D - A_2 B_2^{(1)} C_2 \end{bmatrix} = 0 \text{ or } r \begin{bmatrix} B_1 & C_1 \\ A_1 & D - A_2 B_2^{(1)} C_2 \end{bmatrix} = r(B_1). \quad (3.5)$$

Subsequently, we see from (2.10) that the first equality in (3.5) holds for all generalized inverses  $B_2^{(1)}$  if and only if

$$A_1 = 0 \text{ and any one of } [A_2, D] = 0, \begin{bmatrix} C_2 \\ D \end{bmatrix} = 0, \ r \begin{bmatrix} B_2 & A_2 \\ C_2 & D \end{bmatrix} = r(B_2), \quad (3.6)$$

as required for (i), (ii) and (v) in (c). Meanwhile by (2.10), the second equality in (3.5) holds for all generalized inverses  $B_2^{(1)}$  if and only if

$$C_1 = 0 \text{ and any one of the three conditions } [A_2, D] = 0, \begin{bmatrix} C_2 \\ D \end{bmatrix} = 0, \ r \begin{bmatrix} B_2 & A_2 \\ C_2 & D \end{bmatrix} = r(B_2), \quad (3.7)$$

as required for (iii), (iv) and (vi) in (c). Further, by (2.8),

$$\begin{aligned} \max_{B_2^{(1)}} r \begin{bmatrix} B_1 & C_1 \\ A_1 & D - A_2 B_2^{(1)} C_2 \end{bmatrix} &= \max_{B_2^{(1)}} r \left( \begin{bmatrix} B_1 & C_1 \\ A_1 & D \end{bmatrix} - \begin{bmatrix} 0 \\ A_2 \end{bmatrix} B_2^{(1)} [0, C_2] \right) \\ &= \min \left\{ r \begin{bmatrix} B_1 & C_1 & 0 \\ A_1 & 0 & A_2 \end{bmatrix}, r \begin{bmatrix} B_1 & C_1 \\ A_1 & 0 \\ 0 & C_2 \end{bmatrix}, r \begin{bmatrix} B_1 & 0 & C_1 \\ 0 & B_2 & C_2 \\ A_1 & A_2 & D \end{bmatrix} - r(B_2) \right\}. \end{aligned} \quad (3.8)$$

Hence, the third equality in (3.5) holds for all generalized inverses  $B_2^{(1)}$  if and only if

$$r \begin{bmatrix} B_1 & C_1 & 0 \\ A_1 & 0 & A_2 \end{bmatrix} = r(B_1) \text{ or } r \begin{bmatrix} B_1 & C_1 \\ A_1 & 0 \\ 0 & C_2 \end{bmatrix} = r(B_1) \text{ or } r \begin{bmatrix} B_1 & 0 & C_1 \\ 0 & B_2 & C_2 \\ A_1 & A_2 & D \end{bmatrix} = r(B_1) + r(B_2). \quad (3.9)$$

Also by (2.6) and (2.7), we see that

$$r \begin{bmatrix} B_1 & C_1 & 0 \\ A_1 & 0 & A_2 \end{bmatrix} \geq r[B_1, C_1] + r(A_2) \geq r(B_1) \text{ and } r \begin{bmatrix} B_1 & C_1 \\ A_1 & 0 \\ 0 & C_2 \end{bmatrix} \geq r \begin{bmatrix} B_1 \\ A_1 \end{bmatrix} + r(C_2) \geq r(B_1).$$

Combining the rank inequalities with the first two equalities in (3.9) yields (vii) and (viii) in (c). The third rank inequality in (3.9) is just (ix) in (c).

It can be seen from the nine conditions in Theorem 3.1(c) that if the matrices  $A_i \neq 0$  and  $C_i \neq 0$  in (3.1) for  $i = 1, 2$ , then (3.1) holds for all  $B_1^{(1)}$  and  $B_2^{(1)}$  if and only if Condition (ix) in Theorem 3.1(c) holds. This special fact brings convenience of using the assertions in Theorem 3.1.

By a similar approach, we can show that the following two theorems hold, and we omit the details of their proofs.

**Theorem 3.2.** *Let  $A_1, B_1, C_1, A_2, B_2, C_2$  and  $D$  be as given in (3.3). Then, the following three statements are equivalent:*

- (a) Equation (3.2) holds for all  $B_1^{(1,2)}$  and  $B_2^{(1)}$ .
- (b)  $A_1 B_1^{(1,2)} C_1 + A_2 B_2^{(1)} C_2$  is invariant with respect to all  $B_1^{(1,2)}$  and  $B_2^{(1)}$  and  $A_1 B_1^\dagger C_1 + A_2 B_2^\dagger C_2 = D$  holds.
- (c) Any one of the following 12 conditions holds:
- (i)  $A_1 = 0, A_2 = 0$  and  $D = 0$ .
  - (ii)  $A_1 = 0, C_2 = 0$  and  $D = 0$ .
  - (iii)  $B_1 = 0, A_2 = 0$  and  $D = 0$ .
  - (iv)  $B_1 = 0, C_2 = 0$  and  $D = 0$ .
  - (v)  $C_1 = 0, A_2 = 0$  and  $D = 0$ .
  - (vi)  $C_1 = 0, C_2 = 0$  and  $D = 0$ .
  - (vii)  $A_1 = 0$  and  $r \begin{bmatrix} B_2 & C_2 \\ A_2 & D \end{bmatrix} = r(B_2)$ .
  - (viii)  $B_1 = 0$  and  $r \begin{bmatrix} B_2 & C_2 \\ A_2 & D \end{bmatrix} = r(B_2)$ .
  - (ix)  $C_1 = 0$  and  $r \begin{bmatrix} B_2 & C_2 \\ A_2 & D \end{bmatrix} = r(B_2)$ .
  - (x)  $A_2 = 0$  and  $r \begin{bmatrix} B_1 & C_1 \\ A_1 & D \end{bmatrix} = r(B_1)$ .
  - (xi)  $C_2 = 0$  and  $r \begin{bmatrix} B_1 & C_1 \\ A_1 & D \end{bmatrix} = r(B_1)$ .
  - (xii)  $r \begin{bmatrix} B_1 & 0 & C_1 \\ 0 & B_2 & C_2 \\ A_1 & A_2 & D \end{bmatrix} = r(B_1) + r(B_2)$ .

In particular, assume that the matrices  $A_i \neq 0$  and  $C_i \neq 0$  in (3.4) for  $i = 1, 2$ . Then (3.3) holds for all  $B_1^{(1,2)}$  and  $B_2^{(1)}$  if and only if Condition (xii) in (c) holds.

**Theorem 3.3.** *Let  $A_1, B_1, C_1, A_2, B_2, C_2$  and  $D$  be as given in (3.4). Then, the following three statements are equivalent:*

- (a) Equation (3.4) holds for all  $B_1^{(1,2)}$  and  $B_2^{(1,2)}$ .

(b)  $A_1 B_1^{(1,2)} C_1 + A_2 B_2^{(1,2)} C_2$  is invariant with respect to all  $B_1^{(1,2)}$  and  $B_2^{(1,2)}$  and  $A_1 B_1^\dagger C_1 + A_2 B_2^\dagger C_2 = D$  holds.

(c) Any one of the following 16 conditions holds:

- (i)  $A_1 = 0, A_2 = 0$  and  $D = 0$ .
- (ii)  $A_1 = 0, B_2 = 0$  and  $D = 0$ .
- (iii)  $A_1 = 0, C_2 = 0$  and  $D = 0$ .
- (vi)  $B_1 = 0, A_2 = 0$  and  $D = 0$ .
- (v)  $B_1 = 0, B_2 = 0$  and  $D = 0$ .
- (vi)  $B_1 = 0, C_2 = 0$  and  $D = 0$ .
- (vii)  $C_1 = 0, A_2 = 0$  and  $D = 0$ .
- (viii)  $C_1 = 0, B_2 = 0$  and  $D = 0$ .
- (ix)  $C_1 = 0, C_2 = 0$  and  $D = 0$ .
- (x)  $A_1 = 0$  and  $r \begin{bmatrix} B_2 & C_2 \\ A_2 & D \end{bmatrix} = r(B_2)$ .
- (xi)  $B_1 = 0$  and  $r \begin{bmatrix} B_2 & C_2 \\ A_2 & D \end{bmatrix} = r(B_2)$ .
- (xii)  $C_1 = 0$  and  $r \begin{bmatrix} B_2 & C_2 \\ A_2 & D \end{bmatrix} = r(B_2)$ .
- (xiii)  $A_2 = 0$  and  $r \begin{bmatrix} B_1 & C_1 \\ A_1 & D \end{bmatrix} = r(B_1)$ .
- (xiv)  $B_2 = 0$  and  $r \begin{bmatrix} B_1 & C_1 \\ A_1 & D \end{bmatrix} = r(B_1)$ .
- (xv)  $C_2 = 0$  and  $r \begin{bmatrix} B_1 & C_1 \\ A_1 & D \end{bmatrix} = r(B_1)$ .
- (xvi)  $r \begin{bmatrix} B_1 & 0 & C_1 \\ 0 & B_2 & C_2 \\ A_1 & A_2 & D \end{bmatrix} = r(B_1) + r(B_2)$ .

In particular, assume that the matrices  $A_i \neq 0$  and  $C_i \neq 0$  in (3.4) for  $i = 1, 2$ . Then (3.4) holds for all  $B_1^{(1,2)}$  and  $B_2^{(1,2)}$  if and only if Condition (xvi) in (c) holds.

The case of (1.3) for  $k = 3$  is given by

$$A_1 B_1^{(i_1, \dots, j_1)} C_1 + A_2 B_2^{(i_2, \dots, j_2)} C_2 + A_3 B_3^{(i_3, \dots, j_3)} C_3 = D, \quad (3.10)$$

where  $A_1 \in \mathbb{C}^{m \times p_1}$ ,  $B_1 \in \mathbb{C}^{q_1 \times p_1}$ ,  $C_1 \in \mathbb{C}^{q_1 \times n}$ ,  $A_2 \in \mathbb{C}^{m \times p_2}$ ,  $B_2 \in \mathbb{C}^{q_2 \times p_2}$ ,  $C_2 \in \mathbb{C}^{q_2 \times n}$ ,  $A_3 \in \mathbb{C}^{m \times p_3}$ ,  $B_3 \in \mathbb{C}^{q_3 \times p_3}$ ,  $C_3 \in \mathbb{C}^{q_3 \times n}$  and  $D \in \mathbb{C}^{m \times n}$ . Clearly, there are eight possible expressions in (3.10) for  $\{1\}$ - and  $\{1, 2\}$ -generalized inverses of  $B_1$ ,  $B_2$  and  $B_3$ . In the remaining part of this section, we only give two groups of conditions under which (3.10) always holds for all  $\{1\}$ - and  $\{1, 2\}$ -generalized inverses of  $B_1$ ,  $B_2$  and  $B_3$ . It is easy to see that, if any one  $A_i$  and  $C_i$  in (3.10) is zero for  $i = 1, 2, 3$ , (3.10) reduces to two-term cases on the left-hand side. So, we assume that  $A_i \neq 0$  and  $C_i \neq 0$  in (3.10) for  $i = 1, 2, 3$  without loss of generality.



**Theorem 3.4.** Let  $A_i, B_i, C_i$  and  $D$  be as given in (3.10) and assume that  $A_i \neq 0$  and  $C_i \neq 0$  for  $i = 1, 2, 3$ . Then, the following three statements are equivalent:

- (a)  $A_1B_1^{(1)}C_1 + A_2B_2^{(1)}C_2 + A_3B_3^{(1)}C_3 = D$  holds for all  $B_1^{(1)}, B_2^{(1)}$  and  $B_3^{(1)}$ .
- (b)  $A_1B_1^{(1)}C_1 + A_2B_2^{(1)}C_2 + A_3B_3^{(1)}C_3$  is invariant with respect to all  $B_1^{(1)}, B_2^{(1)}$  and  $B_3^{(1)}$  and  $A_1B_1^\dagger C_1 + A_2B_2^\dagger C_2 + A_3B_3^\dagger C_3 = D$  holds.
- (c)  $r \begin{bmatrix} B_1 & 0 & 0 & C_1 \\ 0 & B_2 & 0 & C_2 \\ 0 & 0 & B_3 & C_3 \\ A_1 & A_2 & A_3 & D \end{bmatrix} = r(B_1) + r(B_2) + r(B_3)$ .

*Proof.* Rewriting  $A_1B_1^{(1)}C_1 + A_2B_2^{(1)}C_2 + A_3B_3^{(1)}C_3 = D$  as  $A_1B_1^{(1)}C_1 + A_2B_2^{(1)}C_2 = D - A_3B_3^{(1)}C_3$  and applying Theorem 3.1(c), we obtain that it holds for all  $B_1^{(1)}$  and  $B_2^{(1)}$  if and only if  $r \begin{bmatrix} B_1 & 0 & C_1 \\ 0 & B_2 & C_2 \\ A_1 & A_2 & D - A_3B_3^{(1)}C_3 \end{bmatrix} = r(B_1) + r(B_2)$ . Consequently, applying (2.8) and (2.10) to the matrix equality yields (c).

By a similar approach, we can show the following result, so we omit the details of its proof.

**Theorem 3.5.** Let  $A_i, B_i, C_i$  and  $D$  be as given in (3.10) and assume that  $A_i \neq 0, B_i \neq 0$  and  $C_i \neq 0$  for  $i = 1, 2, 3$ . Then, the following three statements are equivalent:

- (a)  $A_1B_1^{(1,2)}C_1 + A_2B_2^{(1,2)}C_2 + A_3B_3^{(1,2)}C_3 = D$  holds for all  $B_1^{(1,2)}, B_2^{(1,2)}$  and  $B_3^{(1,2)}$ .
- (b)  $A_1B_1^{(1,2)}C_1 + A_2B_2^{(1,2)}C_2 + A_3B_3^{(1,2)}C_3$  is invariant with respect to all  $B_1^{(1,2)}, B_2^{(1,2)}$  and  $B_3^{(1,2)}$  and  $A_1B_1^\dagger C_1 + A_2B_2^\dagger C_2 + A_3B_3^\dagger C_3 = D$  holds.

(c)  $r \begin{bmatrix} B_1 & 0 & 0 & C_1 \\ 0 & B_2 & 0 & C_2 \\ 0 & 0 & B_3 & C_3 \\ A_1 & A_2 & A_3 & D \end{bmatrix} = r(B_1) + r(B_2) + r(B_3)$ .

By induction, we are able to derive the following result.

**Theorem 3.6.** Assume that  $A_i \neq 0, B_i \neq 0$  and  $C_i \neq 0$  in (1.3) are for  $i = 1, 2, \dots, k$ . Then, (1.3) holds for all mixed combinations of  $\{1\}$ - and  $\{1, 2\}$ -generalized inverses of  $B_1, B_2, \dots, B_k$  if and only if

$$r \begin{bmatrix} B_1 & 0 & 0 & 0 & C_1 \\ 0 & B_2 & 0 & 0 & C_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_k & C_k \\ A_1 & A_2 & \cdots & A_k & D \end{bmatrix} = r(B_1) + r(B_2) + \cdots + r(B_k).$$

#### 4. Matrix equalities that involve mixed products of two generalized inverses

The cases of the matrix equalities in (1.4) for  $k = 2$  and  $\{1\}$ - and  $\{1, 2\}$ -generalized inverses of  $B_1$  and  $B_2$  are given by

$$A_1 B_1^{(1)} A_2 B_2^{(1)} A_3 = A, \quad (4.1)$$

$$A_1 B_1^{(1)} A_2 B_2^{(1,2)} A_3 = A, \quad (4.2)$$

$$A_1 B_1^{(1,2)} A_2 B_2^{(1)} A_3 = A, \quad (4.3)$$

$$A_1 B_1^{(1,2)} A_2 B_2^{(1,2)} A_3 = A, \quad (4.4)$$

where  $A_1 \in \mathbb{C}^{m_1 \times m_2}$ ,  $B_1 \in \mathbb{C}^{m_3 \times m_2}$ ,  $A_2 \in \mathbb{C}^{m_3 \times m_4}$ ,  $B_2 \in \mathbb{C}^{m_5 \times m_4}$ ,  $A_3 \in \mathbb{C}^{m_5 \times m_6}$  and  $A \in \mathbb{C}^{m_1 \times m_6}$ . Particularly, the case of (4.1) for  $\{1\}$ -generalized inverses of  $B_1$  and  $B_2$  was studied in [11, 15], and the following results were obtained.

**Theorem 4.1** ([11, 15]). *Let  $A_1, B_1, A_2, B_2, A_3$  and  $A$  be as given in (4.1). Then, the following three statements are equivalent:*

- (a) Equation (4.1) holds for all  $B_1^{(1)}$  and  $B_2^{(1)}$ .
- (b)  $A_1 B_1^{(1)} A_2 B_2^{(1)} A_3$  is invariant with respect to  $B_1^{(1)}$  and  $B_2^{(1)}$  and  $A_1 B_1^\dagger A_2 B_2^\dagger A_3 = A$  holds.
- (c) Any one of the following 6 conditions holds:
  - (i)  $A = 0$  and  $A_1 = 0$ .
  - (ii)  $A = 0$  and  $A_2 = 0$ .
  - (iii)  $A = 0$  and  $A_3 = 0$ .
  - (iv)  $A = 0$  and  $r \begin{bmatrix} 0 & A_1 \\ A_2 & B_1 \end{bmatrix} = r(B_1)$ .
  - (v)  $A = 0$  and  $r \begin{bmatrix} 0 & A_2 \\ A_3 & B_2 \end{bmatrix} = r(B_2)$ .
  - (vi)  $r \begin{bmatrix} -A & 0 & A_1 \\ 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \end{bmatrix} = r \begin{bmatrix} A_2 & B_1 \\ B_2 & 0 \end{bmatrix} = r(B_1) + r(B_2)$ .

In what follows, we give the results on the other three cases for  $\{1\}$ - and  $\{1, 2\}$ -generalized inverses of  $B_1$  and  $B_2$  in (4.1).

**Theorem 4.2.** *Let  $A_1, B_1, A_2, B_2, A_3$  and  $A$  be as given in (4.2). Then, the following three statements are equivalent:*

- (a) Equation (4.2) holds for all  $B_1^{(1)}$  and  $B_2^{(1,2)}$ .
- (b)  $A_1 B_1^{(1)} A_2 B_2^{(1,2)} A_3$  is invariant with respect to all  $B_1^{(1)}$  and  $B_2^{(1,2)}$  and  $A_1 B_1^\dagger A_2 B_2^\dagger A_3 = A$  holds.
- (c) Any one of the following 7 conditions holds:
  - (i)  $A = 0$  and  $A_1 = 0$ .

- (ii)  $A = 0$  and  $A_2 = 0$ .  
 (iii)  $A = 0$  and  $B_2 = 0$ .  
 (iv)  $A = 0$  and  $A_3 = 0$ .  
 (v)  $A = 0$  and  $r \begin{bmatrix} 0 & A_1 \\ A_2 & B_1 \end{bmatrix} = r(B_1)$ .  
 (vi)  $A = 0$  and  $r \begin{bmatrix} 0 & A_2 \\ A_3 & B_2 \end{bmatrix} = r(B_2)$ .  
 (vii)  $r \begin{bmatrix} -A & 0 & A_1 \\ 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \end{bmatrix} = r \begin{bmatrix} A_2 & B_1 \\ B_2 & 0 \end{bmatrix} = r(B_1) + r(B_2)$ .

*Proof.* The equivalence of (a) and (b) follows from the definition of the Moore–Penrose inverse of a matrix and its uniqueness.

By Lemma 4.1(c), we first see that the product  $A_1 B_1^{(1)} A_2 B_2^{(1,2)} A_3 = A$  holds for all generalized inverses  $B_1^{(1)}$  if and only if any one of the following three conditions holds:

- (i)  $A = 0$  and  $A_1 = 0$ .  
 (ii)  $A = 0$  and  $A_2 B_2^{(1,2)} A_3 = 0$ .  
 (iii)  $r \begin{bmatrix} A & A_1 \\ A_2 B_2^{(1,2)} A_3 & B_1 \end{bmatrix} = r(B_1)$ .

Consequently, we derive the conditions under which (ii) and (iii) hold, respectively. By (2.11), the equality  $A_2 B_2^{(1,2)} A_3 = 0$  in (ii) of the above three conditions holds for all generalized inverses  $B_2^{(1,2)}$  if and only if

$$A_2 = 0 \text{ or } B_2 = 0 \text{ or } A_3 = 0 \text{ or } r \begin{bmatrix} 0 & A_2 \\ A_3 & B_2 \end{bmatrix} = r(B_2) \quad (4.5)$$

holds. So, the above condition (ii) is equivalent to (ii), (iii), (iv) and (vi) in (c). By (2.9),

$$\begin{aligned} \max_{B_2^{(1,2)}} r \begin{bmatrix} A & A_1 \\ A_2 B_2^{(1,2)} A_3 & B_1 \end{bmatrix} &= \max_{B_2^{(1,2)}} r \left( \begin{bmatrix} A & A_1 \\ 0 & B_1 \end{bmatrix} + \begin{bmatrix} 0 \\ A_2 \end{bmatrix} B_2^{(1,2)} [A_3, 0] \right) \\ &= \min \left\{ r \begin{bmatrix} A & A_1 \\ 0 & B_1 \end{bmatrix} + r(B_2), r \begin{bmatrix} A & A_1 & 0 \\ 0 & B_1 & A_2 \end{bmatrix}, r \begin{bmatrix} A & A_1 \\ 0 & B_1 \\ A_3 & 0 \end{bmatrix}, r \begin{bmatrix} -A & 0 & A_1 \\ 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \end{bmatrix} - r(B_2) \right\}. \end{aligned}$$

Hence, the rank equality in (iii) of the above three conditions holds for all generalized inverses  $B_2^{(1,2)}$  if and only if

$$r \begin{bmatrix} A & A_1 \\ 0 & B_1 \end{bmatrix} + r(B_2) = r(B_1) \text{ or } r \begin{bmatrix} A & A_1 & 0 \\ 0 & B_1 & A_2 \end{bmatrix} = r(B_1) \text{ or } r \begin{bmatrix} A & A_1 \\ 0 & B_1 \\ A_3 & 0 \end{bmatrix} = r(B_1) \quad (4.6)$$

$$\text{or } r \begin{bmatrix} -A & 0 & A_1 \\ 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \end{bmatrix} = r(B_1) + r(B_2). \quad (4.7)$$

By (2.7),

$$r \begin{bmatrix} A & A_1 \\ 0 & B_1 \end{bmatrix} + r(B_2) \geq r(A) + r(B_1) + r(B_2) \geq r(B_1)$$

holds. Combining it with the first equality in (4.6) leads to  $r(A) = r(B_2) = 0$ ; thus, the first equality in (4.6) is equivalent to (iii) in (c). By (2.6) and (2.7),

$$r \begin{bmatrix} A & A_1 & 0 \\ 0 & B_1 & A_2 \end{bmatrix} \geq r(A) + r[B_1, A_2] \geq r(B_1)$$

holds. Combining it with the second equality in (4.6) leads to  $A = 0$  and  $r \begin{bmatrix} 0 & A_1 \\ A_2 & B_1 \end{bmatrix} = r(B_1)$ ; thus, the second equality in (4.6) is equivalent to (v) in (c). By (2.7),

$$r \begin{bmatrix} A & A_1 \\ 0 & B_1 \\ A_3 & 0 \end{bmatrix} \geq r \begin{bmatrix} A \\ A_3 \end{bmatrix} + r(B_1) \geq r(B_1)$$

holds. Combining it with the third equality in (4.6) leads to  $A = 0$  and  $A_3 = 0$ ; thus the third equality in (4.6) is involved in (iv) in (c). The rank equality in (4.7) is equivalent to (vii) in (c).

We also declare that the following theorems can be shown by a similar approach, so the details of the proofs are omitted.

**Theorem 4.3.** *Let  $A_1, B_1, A_2, B_2, A_3$  and  $A$  be as given in (4.4). Then, the following three statements are equivalent:*

- (a) Equation (4.4) holds for all  $B_1^{(1,2)}$  and  $B_2^{(1,2)}$ .
- (b)  $A_1 B_1^{(1,2)} A_2 B_2^{(1,2)} A_3$  is invariant with respect to all  $B_1^{(1,2)}$  and  $B_2^{(1,2)}$  and  $A_1 B_1^\dagger A_2 B_2^\dagger A_3 = A$  holds.
- (c) Any one of the following 8 conditions holds:
  - (i)  $A = 0$  and  $A_1 = 0$ .
  - (ii)  $A = 0$  and  $B_1 = 0$ .
  - (iii)  $A = 0$  and  $A_2 = 0$ .
  - (iv)  $A = 0$  and  $B_2 = 0$ .
  - (v)  $A = 0$  and  $A_3 = 0$ .
  - (vi)  $A = 0$  and  $r \begin{bmatrix} 0 & A_1 \\ A_2 & B_1 \end{bmatrix} = r(B_1)$ .
  - (vii)  $A = 0$  and  $r \begin{bmatrix} 0 & A_2 \\ A_3 & B_2 \end{bmatrix} = r(B_2)$ .

$$(viii) \ r \begin{bmatrix} -A & 0 & A_1 \\ 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \end{bmatrix} = r \begin{bmatrix} A_2 & B_1 \\ B_2 & 0 \end{bmatrix} = r(B_1) + r(B_2).$$

The case of (1.4) for  $k = 3$  is given by

$$A_1 B_1^{(i_1, \dots, j_1)} A_2 B_2^{(i_2, \dots, j_2)} A_3 B_3^{(i_3, \dots, j_3)} A_4 = A, \quad (4.8)$$

where  $A_1 \in \mathbb{C}^{m_1 \times m_2}$ ,  $B_1 \in \mathbb{C}^{m_3 \times m_2}$ ,  $A_2 \in \mathbb{C}^{m_3 \times m_4}$ ,  $B_2 \in \mathbb{C}^{m_5 \times m_4}$ ,  $A_3 \in \mathbb{C}^{m_5 \times m_6}$ ,  $B_3 \in \mathbb{C}^{m_7 \times m_6}$ ,  $A_4 \in \mathbb{C}^{m_7 \times m_8}$  and  $A \in \mathbb{C}^{m_1 \times m_8}$ . Particularly, the case of (4.8) for  $\{1\}$ -generalized inverses of  $B_1$ ,  $B_2$  and  $B_3$  was studied in [16], and the following results were obtained.

**Theorem 4.4** ([16]). *Let  $A_1, B_1, A_2, B_2, A_3, B_3, A_4$  and  $A$  be as given in (4.8). Then, the following three statements are equivalent:*

- (a)  $A_1 B_1^{(1)} A_2 B_2^{(1)} A_3 B_3^{(1)} A_4 = A$  holds for all  $B_1^{(1)}$ ,  $B_2^{(1)}$  and  $B_3^{(1)}$ .  
 (b)  $A_1 B_1^{(1)} A_2 B_2^{(1)} A_3 B_3^{(1)} A_4$  is invariant with respect to all  $B_1^{(1)}$ ,  $B_2^{(1)}$  and  $B_3^{(1)}$  and  $A_1 B_1^\dagger A_2 B_2^\dagger A_3 B_3^\dagger A_4 = A$  holds.

(c) Any one of the following 10 conditions holds:

- (i)  $A = 0$  and  $A_1 = 0$ .  
 (ii)  $A = 0$  and  $A_2 = 0$ .  
 (iii)  $A = 0$  and  $A_3 = 0$ .  
 (iv)  $A = 0$  and  $A_4 = 0$ .  
 (v)  $A = 0$  and  $r \begin{bmatrix} 0 & A_1 \\ A_2 & B_1 \end{bmatrix} = r(B_1)$ .  
 (vi)  $A = 0$  and  $r \begin{bmatrix} 0 & A_2 \\ A_3 & B_2 \end{bmatrix} = r(B_2)$ .  
 (vii)  $A = 0$  and  $r \begin{bmatrix} 0 & A_3 \\ A_4 & B_3 \end{bmatrix} = r(B_3)$ .  
 (viii)  $A = 0$  and  $r \begin{bmatrix} 0 & 0 & A_1 \\ 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \end{bmatrix} = r \begin{bmatrix} A_2 & B_1 \\ B_2 & 0 \end{bmatrix} = r(B_1) + r(B_2)$ .  
 (ix)  $A = 0$  and  $r \begin{bmatrix} 0 & 0 & A_2 \\ 0 & A_3 & B_2 \\ A_4 & B_3 & 0 \end{bmatrix} = r \begin{bmatrix} A_3 & B_2 \\ B_3 & 0 \end{bmatrix} = r(B_2) + r(B_3)$ .  
 (x)  $r \begin{bmatrix} A & 0 & 0 & A_1 \\ 0 & 0 & A_2 & B_1 \\ 0 & A_3 & B_2 & 0 \\ A_4 & B_3 & 0 & 0 \end{bmatrix} = r \begin{bmatrix} 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \\ B_3 & 0 & 0 \end{bmatrix} = r(B_1) + r(B_2) + r(B_3)$ .

For the other seven matrix equalities in (4.8) with mixed combinations of  $\{1\}$ - and  $\{1, 2\}$ -generalized inverses of matrices, we have the following results.

**Theorem 4.5.** Let  $A_1, B_1, A_2, B_2, A_3, B_3, A_4$  and  $A$  be as given in (4.8). Then, the following three statements are equivalent:

- (a)  $A_1 B_1^{(1)} A_2 B_2^{(1)} A_3 B_3^{(1,2)} A_4 = A$  holds for all  $B_1^{(1)}, B_2^{(1)}$  and  $B_3^{(1,2)}$ .
- (b)  $A_1 B_1^{(1)} A_2 B_2^{(1)} A_3 B_3^{(1,2)} A_4$  is invariant with respect to all  $B_1^{(1)}, B_2^{(1)}$  and  $B_3^{(1,2)}$  and  $A_1 B_1^\dagger A_2 B_2^\dagger A_3 B_3^\dagger A_4 = A$  holds.
- (c) Any one of the following 11 conditions holds:
- (i)  $A = 0$  and  $A_1 = 0$ .
  - (ii)  $A = 0$  and  $A_2 = 0$ .
  - (iii)  $A = 0$  and  $A_3 = 0$ .
  - (iii)  $A = 0$  and  $B_3 = 0$ .
  - (v)  $A = 0$  and  $A_4 = 0$ .
  - (vi)  $A = 0$  and  $r \begin{bmatrix} 0 & A_1 \\ A_2 & B_1 \end{bmatrix} = r(B_1)$ .
  - (vii)  $A = 0$  and  $r \begin{bmatrix} 0 & A_2 \\ A_3 & B_2 \end{bmatrix} = r(B_2)$ .
  - (viii)  $A = 0$  and  $r \begin{bmatrix} 0 & A_3 \\ A_4 & B_3 \end{bmatrix} = r(B_3)$ .
  - (ix)  $A = 0$  and  $r \begin{bmatrix} 0 & 0 & A_1 \\ 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \end{bmatrix} = r \begin{bmatrix} A_2 & B_1 \\ B_2 & 0 \end{bmatrix} = r(B_1) + r(B_2)$ .
  - (x)  $A = 0$  and  $r \begin{bmatrix} 0 & 0 & A_2 \\ 0 & A_3 & B_2 \\ A_4 & B_3 & 0 \end{bmatrix} = r \begin{bmatrix} A_3 & B_2 \\ B_3 & 0 \end{bmatrix} = r(B_2) + r(B_3)$ .
  - (xi)  $r \begin{bmatrix} A & 0 & 0 & A_1 \\ 0 & 0 & A_2 & B_1 \\ 0 & A_3 & B_2 & 0 \\ A_4 & B_3 & 0 & 0 \end{bmatrix} = r \begin{bmatrix} 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \\ B_3 & 0 & 0 \end{bmatrix} = r(B_1) + r(B_2) + r(B_3)$ .

*Proof.* The equivalence of (a) and (b) follows from the definition of the Moore–Penrose inverse of a matrix and its uniqueness.

By Lemma 4.1(c), we first see that the matrix equality  $A_1 B_1^{(1)} A_2 B_2^{(1)} A_3 B_3^{(1,2)} A_4 = A$  holds for all generalized inverses  $B_1^{(1)}$  and  $B_2^{(1)}$  if and only if any one of the following 6 conditions holds:

- (i)  $A = 0$  and  $A_1 = 0$ .
- (ii)  $A = 0$  and  $A_2 = 0$ .
- (iii)  $A = 0$  and  $A_3 B_3^{(1,2)} A_4 = 0$ .
- (iv)  $A = 0$  and  $r \begin{bmatrix} 0 & A_1 \\ A_2 & B_1 \end{bmatrix} = r(B_1)$ .

$$(v) \ A = 0 \text{ and } r \begin{bmatrix} 0 & A_2 \\ A_3 B_3^{(1,2)} A_4 & B_2 \end{bmatrix} = r(B_2).$$

$$(vi) \ r \begin{bmatrix} -A & 0 & A_1 \\ 0 & A_2 & B_1 \\ A_3 B_3^{(1,2)} A_4 & B_2 & 0 \end{bmatrix} = r \begin{bmatrix} A_2 & B_1 \\ B_2 & 0 \end{bmatrix} = r(B_1) + r(B_2).$$

We next derive the conditions under which (iii), (v) and (vi) hold, respectively. By (2.10), the equality  $A_3 B_3^{(1,2)} A_4 = 0$  in (iii) of the above 6 conditions holds for all generalized inverses  $B_3^{(1,2)}$  if and only if

$$A_3 = 0 \text{ or } B_3 = 0 \text{ or } A_4 = 0 \text{ or } r \begin{bmatrix} 0 & A_3 \\ A_4 & B_3 \end{bmatrix} = r(B_3) \quad (4.9)$$

holds. By (2.8),

$$\begin{aligned} \max_{B_3^{(1,2)}} r \begin{bmatrix} 0 & A_2 \\ A_3 B_3^{(1,2)} A_4 & B_2 \end{bmatrix} &= \max_{B_3^{(1,2)}} r \left( \begin{bmatrix} 0 & A_2 \\ 0 & B_2 \end{bmatrix} + \begin{bmatrix} 0 \\ A_3 \end{bmatrix} B_3^{(1,2)} [A_4, 0] \right) \\ &= \min \left\{ r \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} + r(B_3), r \begin{bmatrix} 0 & A_2 \\ A_3 & B_2 \end{bmatrix}, r \begin{bmatrix} 0 & A_2 \\ 0 & B_2 \\ A_4 & 0 \end{bmatrix}, r \begin{bmatrix} 0 & 0 & A_2 \\ 0 & A_3 & B_2 \\ A_4 & -B_3 & 0 \end{bmatrix} - r(B_3) \right\} \\ &= \min \left\{ r \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} + r(B_3), r \begin{bmatrix} 0 & A_2 \\ A_3 & B_2 \end{bmatrix}, r \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} + r(A_4), r \begin{bmatrix} 0 & 0 & A_2 \\ 0 & A_3 & B_2 \\ A_4 & B_3 & 0 \end{bmatrix} - r(B_3) \right\}. \end{aligned}$$

So, the rank equality  $r \begin{bmatrix} 0 & A_2 \\ A_3 B_3^{(1,2)} A_4 & B_2 \end{bmatrix} = r(B_2)$  in (v) of the above 6 conditions holds for all  $B_3^{(1,2)}$  if and only if

$$r \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} + r(B_3) = r(B_2) \text{ or } r \begin{bmatrix} 0 & A_2 \\ A_3 & B_2 \end{bmatrix} = r(B_2) \text{ or } r \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} + r(A_4) = r(B_2) \quad (4.10)$$

or

$$r \begin{bmatrix} 0 & 0 & A_2 \\ 0 & A_3 & B_2 \\ A_4 & B_3 & 0 \end{bmatrix} = r(B_2) + r(B_3), \quad (4.11)$$

where the first and third rank equalities in (4.10) mean that  $B_3 = 0$  and  $A_4 = 0$  by noting that  $r \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} \geq r(B_2)$ . By (2.9),

$$\begin{aligned} \max_{B_3^{(1,2)}} r \begin{bmatrix} -A & 0 & A_1 \\ 0 & A_2 & B_1 \\ A_3 B_3^{(1,2)} A_4 & B_2 & 0 \end{bmatrix} &= \max_{B_3^{(1,2)}} r \left( \begin{bmatrix} -A & 0 & A_1 \\ 0 & A_2 & B_1 \\ 0 & B_2 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ A_3 \end{bmatrix} B_3^{(1,2)} [A_4, 0, 0] \right) \\ &= \min \left\{ r \begin{bmatrix} -A & 0 & A_1 \\ 0 & A_2 & B_1 \\ 0 & B_2 & 0 \end{bmatrix} + r(B_3), r \begin{bmatrix} -A & 0 & 0 & A_1 \\ 0 & 0 & A_2 & B_1 \\ 0 & A_3 & B_2 & 0 \end{bmatrix}, r \begin{bmatrix} -A & 0 & A_1 \\ 0 & A_2 & B_1 \\ 0 & B_2 & 0 \\ A_4 & 0 & 0 \end{bmatrix}, r \begin{bmatrix} -A & 0 & 0 & A_1 \\ 0 & 0 & A_2 & B_1 \\ 0 & A_3 & B_2 & 0 \\ A_4 & -B_3 & 0 & 0 \end{bmatrix} - r(B_3) \right\} \end{aligned}$$

$$= \min \left\{ r \begin{bmatrix} A & 0 & A_1 \\ 0 & A_2 & B_1 \\ 0 & B_2 & 0 \end{bmatrix} + r(B_3), r \begin{bmatrix} A & 0 & 0 & A_1 \\ 0 & 0 & A_2 & B_1 \\ 0 & A_3 & B_2 & 0 \end{bmatrix}, r \begin{bmatrix} A & 0 & A_1 \\ 0 & A_2 & B_1 \\ 0 & B_2 & 0 \\ A_4 & 0 & 0 \end{bmatrix}, r \begin{bmatrix} A & 0 & 0 & A_1 \\ 0 & 0 & A_2 & B_1 \\ 0 & A_3 & B_2 & 0 \\ A_4 & B_3 & 0 & 0 \end{bmatrix} - r(B_3) \right\}.$$

Hence, the rank equality in (vi) of the above 6 conditions holds for all  $B_3^{(1,2)}$  if and only if

$$r \begin{bmatrix} A & 0 & A_1 \\ 0 & A_2 & B_1 \\ 0 & B_2 & 0 \end{bmatrix} + r(B_3) = r(B_1) + r(B_2) \text{ or } r \begin{bmatrix} A & 0 & 0 & A_1 \\ 0 & 0 & A_2 & B_1 \\ 0 & A_3 & B_2 & 0 \end{bmatrix} = r(B_1) + r(B_2) \quad (4.12)$$

$$\text{or } r \begin{bmatrix} A & 0 & A_1 \\ 0 & A_2 & B_1 \\ 0 & B_2 & 0 \\ A_4 & 0 & 0 \end{bmatrix} = r(B_1) + r(B_2) \text{ or } r \begin{bmatrix} A & 0 & 0 & A_1 \\ 0 & 0 & A_2 & B_1 \\ 0 & A_3 & B_2 & 0 \\ A_4 & B_3 & 0 & 0 \end{bmatrix} = r(B_1) + r(B_2) + r(B_3). \quad (4.13)$$

Note from (2.6) and (2.7) that the following four groups of matrix rank inequalities

$$r \begin{bmatrix} A & 0 & A_1 \\ 0 & A_2 & B_1 \\ 0 & B_2 & 0 \end{bmatrix} + r(B_3) \geq r \begin{bmatrix} A & 0 & A_1 \\ 0 & A_2 & B_1 \\ 0 & B_2 & 0 \end{bmatrix} \geq r(A) + r \begin{bmatrix} A_2 & B_1 \\ B_2 & 0 \end{bmatrix} \geq r(B_1) + r(B_2), \quad (4.14)$$

$$r \begin{bmatrix} A & 0 & 0 & A_1 \\ 0 & 0 & A_2 & B_1 \\ 0 & A_3 & B_2 & 0 \end{bmatrix} \geq r(A) + r \begin{bmatrix} 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \end{bmatrix} \geq r \begin{bmatrix} A_2 & B_1 \\ B_2 & 0 \end{bmatrix} \geq r(B_1) + r(B_2), \quad (4.15)$$

$$r \begin{bmatrix} A & 0 & A_1 \\ 0 & A_2 & B_1 \\ 0 & B_2 & 0 \\ A_4 & 0 & 0 \end{bmatrix} \geq r \begin{bmatrix} A & 0 & A_1 \\ 0 & A_2 & B_1 \\ 0 & B_2 & 0 \end{bmatrix} \geq r(A) + r \begin{bmatrix} A_2 & B_1 \\ B_2 & 0 \end{bmatrix} \geq r(B_1) + r(B_2), \quad (4.16)$$

$$r \begin{bmatrix} A & 0 & A_1 \\ 0 & A_2 & B_1 \\ 0 & B_2 & 0 \\ A_4 & 0 & 0 \end{bmatrix} \geq r \begin{bmatrix} 0 & A_2 & B_1 \\ 0 & B_2 & 0 \\ A_4 & 0 & 0 \end{bmatrix} = r(A_4) + r \begin{bmatrix} A_2 & B_1 \\ B_2 & 0 \end{bmatrix} \geq r(B_1) + r(B_2) \quad (4.17)$$

hold. Combining (4.14) with the first equality in (4.12) leads to

$$A = 0, \quad B_3 = 0 \text{ and } r \begin{bmatrix} 0 & 0 & A_1 \\ 0 & A_2 & B_1 \\ 0 & B_2 & 0 \end{bmatrix} = r \begin{bmatrix} A_2 & B_1 \\ B_2 & 0 \end{bmatrix} = r(B_1) + r(B_2). \quad (4.18)$$

This condition, however, is included in (4.9). Combining (4.15) with the second equality in (4.12) leads to

$$A = 0 \text{ and } r \begin{bmatrix} 0 & 0 & A_1 \\ 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \end{bmatrix} = r(B_1) + r(B_2). \quad (4.19)$$



Combining (4.16) and (4.17) with the first equality in (4.13) leads to

$$A = 0 \text{ and } A_4 = 0. \quad (4.20)$$

Combining the conditions in (i), (ii) and (iv) of the proof with (4.12)–(4.20) leads to the 11 statements in (c).

The following 6 theorems can be established by a similar approach; therefore, the details of the proofs are omitted.

**Theorem 4.6.** *Let  $A_1, B_1, A_2, B_2, A_3, B_3, A_4$  and  $A$  be as given in (4.8). Then, the following three statements are equivalent:*

- (a)  $A_1 B_1^{(1,2)} A_2 B_2^{(1,2)} A_3 B_3^{(1)} A_4 = A$  holds for all  $B_1^{(1,2)}, B_2^{(1,2)}$  and  $B_3^{(1)}$ .
- (b)  $A_1 B_1^{(1,2)} A_2 B_2^{(1,2)} A_3 B_3^{(1)} A_4$  is invariant with respect to all  $B_1^{(1,2)}, B_2^{(1,2)}$  and  $B_3^{(1)}$  and  $A_1 B_1^\dagger A_2 B_2^\dagger A_3 B_3^\dagger A_4 = A$  holds.
- (c) Any one of the following 12 conditions holds:
- (i)  $A = 0$  and  $A_1 = 0$ .
- (ii)  $A = 0$  and  $B_1 = 0$ .
- (iii)  $A = 0$  and  $A_2 = 0$ .
- (iv)  $A = 0$  and  $B_2 = 0$ .
- (v)  $A = 0$  and  $A_3 = 0$ .
- (vi)  $A = 0$  and  $A_4 = 0$ .
- (vii)  $A = 0$  and  $r \begin{bmatrix} 0 & A_1 \\ A_2 & B_1 \end{bmatrix} = r(B_1)$ .
- (viii)  $A = 0$  and  $r \begin{bmatrix} 0 & A_2 \\ A_3 & B_2 \end{bmatrix} = r(B_2)$ .
- (ix)  $A = 0$  and  $r \begin{bmatrix} 0 & A_3 \\ A_4 & B_3 \end{bmatrix} = r(B_3)$ .
- (x)  $A = 0$  and  $r \begin{bmatrix} 0 & 0 & A_1 \\ 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \end{bmatrix} = r \begin{bmatrix} A_2 & B_1 \\ B_2 & 0 \end{bmatrix} = r(B_1) + r(B_2)$ .
- (xi)  $A = 0$  and  $r \begin{bmatrix} 0 & 0 & A_2 \\ 0 & A_3 & B_2 \\ A_4 & B_3 & 0 \end{bmatrix} = r \begin{bmatrix} A_3 & B_2 \\ B_3 & 0 \end{bmatrix} = r(B_2) + r(B_3)$ .
- (xii)  $r \begin{bmatrix} A & 0 & 0 & A_1 \\ 0 & 0 & A_2 & B_1 \\ 0 & A_3 & B_2 & 0 \\ A_4 & B_3 & 0 & 0 \end{bmatrix} = r \begin{bmatrix} 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \\ B_3 & 0 & 0 \end{bmatrix} = r(B_1) + r(B_2) + r(B_3)$ .

**Theorem 4.7.** *Let  $A_1, B_1, A_2, B_2, A_3, B_3, A_4$  and  $A$  be as given in (4.8). Then, the following three statements are equivalent:*

- (a)  $A_1 B_1^{(1,2)} A_2 B_2^{(1,2)} A_3 B_3^{(1,2)} A_4 = A$  holds for all  $B_1^{(1,2)}$ ,  $B_2^{(1,2)}$  and  $B_3^{(1,2)}$ .
- (b)  $A_1 B_1^{(1,2)} A_2 B_2^{(1,2)} A_3 B_3^{(1,2)} A_4$  is invariant with respect to all  $B_1^{(1,2)}$ ,  $B_2^{(1,2)}$  and  $B_3^{(1,2)}$  and  $A_1 B_1^\dagger A_2 B_2^\dagger A_3 B_3^\dagger A_4 = A$  holds.

(c) Any one of the following 13 conditions holds:

- (i)  $A = 0$  and  $A_1 = 0$ .
- (ii)  $A = 0$  and  $B_1 = 0$ .
- (iii)  $A = 0$  and  $A_2 = 0$ .
- (iv)  $A = 0$  and  $B_2 = 0$ .
- (v)  $A = 0$  and  $A_3 = 0$ .
- (vi)  $A = 0$  and  $B_3 = 0$ .
- (vii)  $A = 0$  and  $A_4 = 0$ .
- (viii)  $A = 0$  and  $r \begin{bmatrix} 0 & A_1 \\ A_2 & B_1 \end{bmatrix} = r(B_1)$ .
- (ix)  $A = 0$  and  $r \begin{bmatrix} 0 & A_2 \\ A_3 & B_2 \end{bmatrix} = r(B_2)$ .
- (x)  $A = 0$  and  $r \begin{bmatrix} 0 & A_3 \\ A_4 & B_3 \end{bmatrix} = r(B_3)$ .
- (xi)  $A = 0$  and  $r \begin{bmatrix} 0 & 0 & A_1 \\ 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \end{bmatrix} = r \begin{bmatrix} A_2 & B_1 \\ B_2 & 0 \end{bmatrix} = r(B_1) + r(B_2)$ .
- (xii)  $A = 0$  and  $r \begin{bmatrix} 0 & 0 & A_2 \\ 0 & A_3 & B_2 \\ A_4 & B_3 & 0 \end{bmatrix} = r \begin{bmatrix} A_3 & B_2 \\ B_3 & 0 \end{bmatrix} = r(B_2) + r(B_3)$ .
- (xiii)  $r \begin{bmatrix} A & 0 & 0 & A_1 \\ 0 & 0 & A_2 & B_1 \\ 0 & A_3 & B_2 & 0 \\ A_4 & B_3 & 0 & 0 \end{bmatrix} = r \begin{bmatrix} 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \\ B_3 & 0 & 0 \end{bmatrix} = r(B_1) + r(B_2) + r(B_3)$ .

## 5. Matrix equalities that involve mixed products of two or three weighted $\{1\}$ - and $\{1, 2\}$ -inverses

In addition to the classical definition in (1.1), some other weighted generalized inverses of a matrix can be defined from common solutions of given matrix equations derived from theory and applied points of view, alternatively there are many approaches and discussions concerning different kinds of weighted generalized inverses of matrices in the literature. Here, we mention such an example of weighted generalized inverses: given three matrices  $A \in \mathbb{C}^{m \times n}$ ,  $P \in \mathbb{C}^{p \times m}$  and  $Q \in \mathbb{C}^{n \times q}$ ,  $A$  is said to be weighted Moore–Penrose invertible with respect to the two matrices  $(P, Q)$  if and only if there exists a matrix  $X \in \mathbb{C}^{q \times p}$  that satisfies the following four matrix equations

$$(1) AQPXPA = A, \quad (2) XPAQX = X, \quad (3) (PAQX)^* = PAQX, \quad (4) (XPAQ)^* = XPAQ. \quad (5.1)$$

In this case, the matrix  $X$  is denoted by  $X = A^{(\dagger, P, Q)}$ . In addition, a matrix  $X$  is said to be a  $\{i, \dots, j, P, Q\}$ -generalized inverse of  $A$ , denoted by  $A^{(i, \dots, j, P, Q)}$ , if it satisfies the  $i$ th,  $\dots$ ,  $j$ th matrix equations in (5.1), while the set of all  $\{i, \dots, j, P, Q\}$ -generalized inverses of the matrix  $A$  is, as usual, denoted by  $\{A^{(i, \dots, j, P, Q)}\}$ .

Concerning the existence of  $X$  satisfying (5.1) and the corresponding expressions of  $X$ , we have the following results.

**Lemma 5.1.** *Let  $A \in \mathbb{C}^{m \times n}$ ,  $P \in \mathbb{C}^{p \times m}$  and  $Q \in \mathbb{C}^{n \times q}$ . Then,  $A$  is weighted Moore–Penrose invertible with respect to the two matrices  $(P, Q)$  if and only if the two range equalities  $\mathcal{R}((PA)^*) = \mathcal{R}(A^*)$  and  $\mathcal{R}(AQ) = \mathcal{R}(A)$  hold, or equivalently, the matrix rank equality  $r(PAQ) = r(A)$  holds. In such a case, the matrix equality*

$$A^{(i, \dots, j, P, Q)} = (PAQ)^{(i, \dots, j)} \quad (5.2)$$

always holds. In particular, the general expressions of  $A^{(1, P, Q)}$  and  $A^{(1, 2, P, Q)}$  are given by

$$A^{(1, P, Q)} = (PAQ)^{(1)} = (PAQ)^\dagger + (I_q - (AQ)^\dagger AQ)G + H(I_p - PA(PA)^\dagger), \quad (5.3)$$

$$A^{(1, 2, P, Q)} = (PAQ)^{(1, 2)} = ((PAQ)^\dagger + (I_q - (AQ)^\dagger AQ)G)PAQ((PAQ)^\dagger + H(I_p - PA(PA)^\dagger)), \quad (5.4)$$

and the following matrix equalities

$$AQA^{(1, P, Q)} = AQA^{(1, 2, P, Q)} = A(PA)^\dagger + AQH(I_p - PA(PA)^\dagger), \quad (5.5)$$

$$A^{(1, P, Q)}PA = A^{(1, 2, P, Q)}PA = (AQ)^\dagger A + (I_q - (AQ)^\dagger AQ)GPA, \quad (5.6)$$

$$PAQA^{(1, P, Q)} = PAQA^{(1, 2, P, Q)} = PA(PA)^\dagger + PAQH(I_p - PA(PA)^\dagger), \quad (5.7)$$

$$A^{(1, P, Q)}PAQ = A^{(1, 2, P, Q)}PAQ = (AQ)^\dagger AQ + (I_q - (AQ)^\dagger AQ)GPAQ \quad (5.8)$$

hold, where  $G, H \in \mathbb{C}^{q \times p}$  are two arbitrary matrices.

*Proof.* By Lemma 2.1, the first equation in (5.1) is solvable for the matrix  $X$  if and only if  $\mathcal{R}(AQ) \supseteq \mathcal{R}(A)$  and  $\mathcal{R}((PA)^*) \supseteq \mathcal{R}(A^*)$  hold, both of which are equivalent to the matrix range equalities  $\mathcal{R}((PA)^*) = \mathcal{R}(A^*)$  and  $\mathcal{R}(AQ) = \mathcal{R}(A)$ , or equivalently, the matrix rank equality  $r(PAQ) = r(A)$ . In this case, the first equation in (5.1) is equivalent to  $PAQXPAQ = PAQ$ . In this case, we obtain (5.3) and (5.4) from Lemma 2.2.

Analogous to (4.8), it is possible to formulate four additive matrix equalities that involve weighted  $\{1\}$ - and  $\{1, 2\}$ -generalized inverses as follows:

$$A_1 B_1^{(1, P_1, Q_1)} C_1 + A_2 B_2^{(1, P_2, Q_2)} C_2 = D, \quad (5.9)$$

$$A_1 B_1^{(1, 2, P_1, Q_1)} C_1 + A_2 B_2^{(1, 2, P_2, Q_2)} C_2 = D, \quad (5.10)$$

$$A_1 B_1^{(1, P_1, Q_1)} C_1 + A_2 B_2^{(1, 2, P_2, Q_2)} C_2 = D, \quad (5.11)$$

$$A_1 B_1^{(1, 2, P_1, Q_1)} C_1 + A_2 B_2^{(1, 2, P_2, Q_2)} C_2 = D, \quad (5.12)$$

where  $A_1 \in \mathbb{C}^{m \times q_1}$ ,  $P_1 \in \mathbb{C}^{p_1 \times s_1}$ ,  $B_1 \in \mathbb{C}^{s_1 \times t_1}$ ,  $Q_1 \in \mathbb{C}^{t_1 \times q_1}$ ,  $C_1 \in \mathbb{C}^{p_1 \times n}$ ,  $A_2 \in \mathbb{C}^{m \times q_2}$ ,  $P_2 \in \mathbb{C}^{p_2 \times s_2}$ ,  $B_2 \in \mathbb{C}^{s_2 \times t_2}$ ,  $Q_2 \in \mathbb{C}^{t_2 \times q_2}$ ,  $C_2 \in \mathbb{C}^{p_2 \times n}$  and  $D \in \mathbb{C}^{m \times n}$  are given.

**Theorem 5.2.** Assume that  $B_1^{(1,P_1,Q_1)}$  and  $B_2^{(1,P_2,Q_2)}$  exist. Then, the following three statements are equivalent:

- (a) Equation (5.9) holds for all  $B_1^{(1,P_1,Q_1)}$  and  $B_2^{(1,P_2,Q_2)}$ .
- (b)  $A_1 B_1^{(1,P_1,Q_1)} C_1 + A_2 B_2^{(1,P_2,Q_2)} C_2$  is invariant with respect to all  $B_1^{(1,P_1,Q_1)}$  and  $B_2^{(1,P_2,Q_2)}$  and  $A_1(P_1 B_1 Q_1)^\dagger C_1 + A_2(P_2 B_2 Q_2)^\dagger C_2 = D$  holds.
- (c) Any one of the following 9 conditions holds:
- (i)  $A_1 = 0, A_2 = 0$  and  $D = 0$ .
  - (ii)  $A_1 = 0, C_2 = 0$  and  $D = 0$ .
  - (iii)  $C_1 = 0, A_2 = 0$  and  $D = 0$ .
  - (iv)  $C_1 = 0, C_2 = 0$  and  $D = 0$ .
  - (v)  $A_1 = 0$  and  $r \begin{bmatrix} P_2 B_2 Q_2 & C_2 \\ A_2 & D \end{bmatrix} = r(B_2)$ .
  - (vi)  $C_1 = 0$  and  $r \begin{bmatrix} P_2 B_2 Q_2 & C_2 \\ A_2 & D \end{bmatrix} = r(B_2)$ .
  - (vii)  $A_2 = 0$  and  $r \begin{bmatrix} P_1 B_1 Q_1 & C_1 \\ A_1 & D \end{bmatrix} = r(B_1)$ .
  - (viii)  $C_2 = 0$  and  $r \begin{bmatrix} P_1 B_1 Q_1 & C_1 \\ A_1 & D \end{bmatrix} = r(B_1)$ .
  - (ix)  $r \begin{bmatrix} P_1 B_1 Q_1 & 0 & C_1 \\ 0 & P_2 B_2 Q_2 & C_2 \\ A_1 & A_2 & D \end{bmatrix} = r(B_1) + r(B_2)$ .

*Proof.* By (5.3), the matrix equality in (5.9) is equivalent to

$$A_1(P_1 B_1 Q_1)^{(1)} C_1 + A_2(P_2 B_2 Q_2)^{(1)} C_2 = D.$$

In this case, applying Theorem 3.1 to the matrix equality yields the results in (a), (b) and (c).

The following theorems can be shown by a similar approach.

**Theorem 5.3.** Assume that  $B_1^{(1,P_1,Q_1)}$  and  $B_2^{(1,P_2,Q_2)}$  exist. Then, the following three statements are equivalent:

- (a) Equation (5.10) holds for all  $B_1^{(1,2,P_1,Q_1)}$  and  $B_2^{(1,P_1,Q_1)}$ .
- (b)  $A_1 B_1^{(1,2,P_1,Q_1)} C_1 + A_2 B_2^{(1,P_2,Q_2)} C_2$  is invariant with respect to all  $B_1^{(1,2,P_1,Q_1)}$  and  $B_2^{(1,P_2,Q_2)}$  and  $A_1(P_1 B_1 Q_1)^\dagger C_1 + A_2(P_2 B_2 Q_2)^\dagger C_2 = D$  holds.
- (c) Any one of the following 12 conditions holds:
- (i)  $A_1 = 0, A_2 = 0$  and  $D = 0$ .
  - (ii)  $A_1 = 0, C_2 = 0$  and  $D = 0$ .

- (iii)  $B_1 = 0, A_2 = 0$  and  $D = 0$ .
- (iv)  $B_1 = 0, C_2 = 0$  and  $D = 0$ .
- (v)  $C_1 = 0, A_2 = 0$  and  $D = 0$ .
- (vi)  $C_1 = 0, C_2 = 0$  and  $D = 0$ .
- (vii)  $A_1 = 0$  and  $r \begin{bmatrix} P_2 B_2 Q_2 & C_2 \\ A_2 & D \end{bmatrix} = r(B_2)$ .
- (viii)  $B_1 = 0$  and  $r \begin{bmatrix} P_2 B_2 Q_2 & C_2 \\ A_2 & D \end{bmatrix} = r(B_2)$ .
- (ix)  $C_1 = 0$  and  $r \begin{bmatrix} P_2 B_2 Q_2 & C_2 \\ A_2 & D \end{bmatrix} = r(B_2)$ .
- (x)  $A_2 = 0$  and  $r \begin{bmatrix} P_1 B_1 Q_1 & C_1 \\ A_1 & D \end{bmatrix} = r(B_1)$ .
- (xi)  $C_2 = 0$  and  $r \begin{bmatrix} P_1 B_1 Q_1 & C_1 \\ A_1 & D \end{bmatrix} = r(B_1)$ .
- (xii)  $r \begin{bmatrix} P_1 B_1 Q_1 & 0 & C_1 \\ 0 & P_2 B_2 Q_2 & C_2 \\ A_1 & A_2 & D \end{bmatrix} = r(B_1) + r(B_2)$ .

**Theorem 5.4.** Assume that  $B_1^{(1,P_1,Q_1)}$  and  $B_2^{(1,P_2,Q_2)}$  exist. Then, the following three statements are equivalent:

- (a) Equation (5.12) holds for all  $B_1^{(1,2,P_1,Q_1)}$  and  $B_2^{(1,2,P_1,Q_1)}$ .
- (b)  $A_1 B_1^{(1,2,P_1,Q_1)} C_1 + A_2 B_2^{(1,2,P_2,Q_2)} C_2$  is invariant with respect to all  $B_1^{(1,2,P_1,Q_1)}$  and  $B_2^{(1,2,P_2,Q_2)}$  and  $A_1(P_1 B_1 Q_1)^\dagger C_1 + A_2(P_2 B_2 Q_2)^\dagger C_2 = D$  holds.
- (b) Any one of the following 16 conditions holds:
- (i)  $A_1 = 0, A_2 = 0$  and  $D = 0$ .
- (ii)  $A_1 = 0, B_2 = 0$  and  $D = 0$ .
- (iii)  $A_1 = 0, C_2 = 0$  and  $D = 0$ .
- (vi)  $B_1 = 0, A_2 = 0$  and  $D = 0$ .
- (v)  $B_1 = 0, B_2 = 0$  and  $D = 0$ .
- (vi)  $B_1 = 0, C_2 = 0$  and  $D = 0$ .
- (vii)  $C_1 = 0, A_2 = 0$  and  $D = 0$ .
- (viii)  $C_1 = 0, B_2 = 0$  and  $D = 0$ .
- (ix)  $C_1 = 0, C_2 = 0$  and  $D = 0$ .
- (x)  $A_1 = 0$  and  $r \begin{bmatrix} P_2 B_2 Q_2 & C_2 \\ A_2 & D \end{bmatrix} = r(B_2)$ .
- (xi)  $B_1 = 0$  and  $r \begin{bmatrix} P_2 B_2 Q_2 & C_2 \\ A_2 & D \end{bmatrix} = r(B_2)$ .

$$(xii) \ C_1 = 0 \text{ and } r \begin{bmatrix} P_2 B_2 Q_2 & C_2 \\ A_2 & D \end{bmatrix} = r(B_2).$$

$$(xiii) \ A_2 = 0 \text{ and } r \begin{bmatrix} P_1 B_1 Q_1 & C_1 \\ A_1 & D \end{bmatrix} = r(B_1).$$

$$(xiv) \ B_2 = 0 \text{ and } r \begin{bmatrix} P_1 B_1 Q_1 & C_1 \\ A_1 & D \end{bmatrix} = r(B_1).$$

$$(xv) \ C_2 = 0 \text{ and } r \begin{bmatrix} P_1 B_1 Q_1 & C_1 \\ A_1 & D \end{bmatrix} = r(B_1).$$

$$(xvi) \ r \begin{bmatrix} P_1 B_1 Q_1 & 0 & C_1 \\ 0 & P_2 B_2 Q_2 & C_2 \\ A_1 & A_2 & D \end{bmatrix} = r(B_1) + r(B_2).$$

Analogous to (1.4), it is possible to formulate some matrix equalities that involve alternative products of matrices and their weighted  $\{1\}$ - and  $\{1, 2\}$ -generalized inverses. In what follows, we present a sequence of extensions of the results in the preceding sections to matrix equalities that involve mixed products of two or three weighted  $\{1\}$ - and  $\{1, 2\}$ -inverses. In particular, we are able to construct four matrix equalities that involves weighted  $\{1\}$ - and  $\{1, 2\}$ -generalized inverses as follows:

$$A_1 B_1^{(1, P_1, Q_1)} A_2 B_2^{(1, P_2, Q_2)} A_3 = A, \quad (5.13)$$

$$A_1 B_1^{(1, 2, P_1, Q_1)} A_2 B_2^{(1, P_2, Q_2)} A_3 = A, \quad (5.14)$$

$$A_1 B_1^{(1, P_1, Q_1)} A_2 B_2^{(1, 2, P_2, Q_2)} A_3 = A, \quad (5.15)$$

$$A_1 B_1^{(1, 2, P_1, Q_1)} A_2 B_2^{(1, 2, P_2, Q_2)} A_3 = A, \quad (5.16)$$

where  $A_1 \in \mathbb{C}^{m_1 \times m_2}$ ,  $P_1 \in \mathbb{C}^{m_3 \times q_1}$ ,  $B_1 \in \mathbb{C}^{q_1 \times p_1}$ ,  $Q_1 \in \mathbb{C}^{p_1 \times m_2}$ ,  $A_2 \in \mathbb{C}^{m_3 \times m_4}$ ,  $P_2 \in \mathbb{C}^{m_5 \times q_2}$ ,  $B_2 \in \mathbb{C}^{q_2 \times p_2}$ ,  $Q_2 \in \mathbb{C}^{p_2 \times m_4}$ ,  $A_3 \in \mathbb{C}^{m_5 \times m_6}$  and  $A \in \mathbb{C}^{m_1 \times m_6}$ .

**Theorem 5.5.** Assume that  $B_1^{(1, P_1, Q_1)}$  and  $B_2^{(1, P_2, Q_2)}$  exist. Then, the following three statements are equivalent:

(a) Equation (5.13) holds for all  $B_1^{(1, P_1, Q_1)}$  and  $B_2^{(1, P_2, Q_2)}$ .

(b)  $A_1 B_1^{(1, P_1, Q_1)} A_2 B_2^{(1, P_2, Q_2)} A_3$  is invariant with respect to all  $B_1^{(1, P_1, Q_1)}$  and  $B_2^{(1, P_2, Q_2)}$  and  $A_1 (P_1 B_1 Q_1)^\dagger A_2 (P_2 B_2 Q_2)^\dagger A_3 = A$  holds.

(c) Any one of the following 6 conditions holds:

(i)  $A = 0$  and  $A_1 = 0$ .

(ii)  $A = 0$  and  $A_2 = 0$ .

(iii)  $A = 0$  and  $A_3 = 0$ .

(iv)  $A = 0$  and  $r \begin{bmatrix} 0 & A_1 \\ A_2 & P_1 B_1 Q_1 \end{bmatrix} = r(B_1)$ .

(v)  $A = 0$  and  $r \begin{bmatrix} 0 & A_2 \\ A_3 & P_2 B_2 Q_2 \end{bmatrix} = r(B_2)$ .

$$(vi) \ r \begin{bmatrix} -A & 0 & A_1 \\ 0 & A_2 & P_1 B_1 Q_1 \\ A_3 & P_2 B_2 Q_2 & 0 \end{bmatrix} = r \begin{bmatrix} A_2 & P_1 B_1 Q_1 \\ P_2 B_2 Q_2 & 0 \end{bmatrix} = r(B_1) + r(B_2).$$

*Proof.* Under the given assumptions, we obtain from (5.3) that  $A_1 B_1^{(1,P_1,Q_1)} A_2 B_2^{(1,P_2,Q_2)} A_3 = A$  is equivalent to

$$A_1 B_1^{(1,P_1,Q_1)} A_2 B_2^{(1,P_2,Q_2)} A_3 = A_1 (P_1 B_1 Q_1)^{(1)} A_2 (P_2 B_2 Q_2)^{(1)} A_3 = A.$$

In this case, applying Theorem 4.1 to the matrix equality yields the results in (a), (b) and (c).

By a similar approach, we obtain the following three groups of results from Theorems 4.2 and 4.3.

**Theorem 5.6.** Assume that  $B_1^{(1,P_1,Q_1)}$  and  $B_2^{(1,P_2,Q_2)}$  exist. Then, the following three statements are equivalent:

- (a) Equation (5.14) holds for all  $B_1^{(1,2,P_2,Q_2)}$  and  $B_2^{(1,P_2,Q_2)}$ .
- (b)  $A_1 B_1^{(1,2,P_2,Q_2)} A_2 B_2^{(1,P_2,Q_2)} A_3$  is invariant with respect to all  $B_1^{(1,2,P_2,Q_2)}$  and  $B_2^{(1,P_2,Q_2)}$  and  $A_1 (P_1 B_1 Q_1)^\dagger A_2 (P_2 B_2 Q_2)^\dagger A_3 = A$  holds.
- (c) Any one of the following 7 conditions holds:
- (i)  $A = 0$  and  $A_1 = 0$ .
- (ii)  $A = 0$  and  $B_1 = 0$ .
- (iii)  $A = 0$  and  $A_2 = 0$ .
- (iv)  $A = 0$  and  $A_3 = 0$ .
- (v)  $A = 0$  and  $r \begin{bmatrix} 0 & A_1 \\ A_2 & P_1 B_1 Q_1 \end{bmatrix} = r(B_1)$ .
- (vi)  $A = 0$  and  $r \begin{bmatrix} 0 & A_2 \\ A_3 & P_2 B_2 Q_2 \end{bmatrix} = r(B_2)$ .
- (vii)  $r \begin{bmatrix} -A & 0 & A_1 \\ 0 & A_2 & P_1 B_1 Q_1 \\ A_3 & P_2 B_2 Q_2 & 0 \end{bmatrix} = r \begin{bmatrix} A_2 & P_1 B_1 Q_1 \\ P_2 B_2 Q_2 & 0 \end{bmatrix} = r(B_1) + r(B_2)$ .

**Theorem 5.7.** Assume that  $B_1^{(1,P_1,Q_1)}$  and  $B_2^{(1,P_2,Q_2)}$  exist. Then, the following three statements are equivalent:

- (a) Equation (5.16) holds for all  $B_1^{(1,2,P_1,Q_1)}$  and  $B_2^{(1,2,P_2,Q_2)}$ .
- (b)  $A_1 B_1^{(1,2,P_1,Q_1)} A_2 B_2^{(1,2,P_2,Q_2)} A_3$  is invariant with respect to all  $B_1^{(1,2,P_1,Q_1)}$  and  $B_2^{(1,2,P_2,Q_2)}$  and  $A_1 (P_1 B_1 Q_1)^\dagger A_2 (P_2 B_2 Q_2)^\dagger A_3 = A$  holds.
- (c) Any one of the following 8 conditions holds:
- (i)  $A = 0$  and  $A_1 = 0$ .
- (ii)  $A = 0$  and  $B_1 = 0$ .
- (iii)  $A = 0$  and  $A_2 = 0$ .

(iv)  $A = 0$  and  $B_2 = 0$ .

(v)  $A = 0$  and  $A_3 = 0$ .

(vi)  $A = 0$  and  $r \begin{bmatrix} 0 & A_1 \\ A_2 & P_1 B_1 Q_1 \end{bmatrix} = r(B_1)$ .

(vii)  $A = 0$  and  $r \begin{bmatrix} 0 & A_2 \\ A_3 & P_2 B_2 Q_2 \end{bmatrix} = r(B_2)$ .

(viii)  $r \begin{bmatrix} -A & 0 & A_1 \\ 0 & A_2 & P_1 B_1 Q_1 \\ A_3 & P_2 B_2 Q_2 & 0 \end{bmatrix} = r \begin{bmatrix} A_2 & P_1 B_1 Q_1 \\ P_2 B_2 Q_2 & 0 \end{bmatrix} = r(B_1) + r(B_2)$ .

Finally, we present extensions of Theorems 4.4 and 4.7 to matrix equalities that involve mixed products of three weighted  $\{1\}$ - and  $\{1, 2\}$ -inverses. Let

$$A_1 \in \mathbb{C}^{m_1 \times m_2}, P_1 \in \mathbb{C}^{m_3 \times q_1}, B_1 \in \mathbb{C}^{q_1 \times p_1}, Q_1 \in \mathbb{C}^{p_1 \times m_2}, A_2 \in \mathbb{C}^{m_3 \times m_4}, \quad (5.17)$$

$$P_2 \in \mathbb{C}^{m_5 \times q_2}, B_2 \in \mathbb{C}^{q_2 \times p_2}, Q_2 \in \mathbb{C}^{p_2 \times m_4}, A_3 \in \mathbb{C}^{m_5 \times m_6}, P_3 \in \mathbb{C}^{m_7 \times q_3}, \quad (5.18)$$

$$B_3 \in \mathbb{C}^{q_3 \times p_3}, Q_3 \in \mathbb{C}^{p_3 \times m_6}, A_4 \in \mathbb{C}^{m_7 \times m_8}. \quad (5.19)$$

Then, the matrix product  $A_1 B_1^{(1, P_1, Q_1)} A_2 B_2^{(1, P_2, Q_2)} A_3 B_3^{(1, P_3, Q_3)} A_4$  is defined accordingly, and we have the following results.

**Theorem 5.8.** *Given the matrices in (5.17)–(5.19) and  $A \in \mathbb{C}^{m_1 \times m_8}$ , and assume that  $B_1^{(1, P_1, Q_1)}$ ,  $B_2^{(1, P_2, Q_2)}$  and  $B_3^{(1, P_3, Q_3)}$  exist. Then, the following three statements are equivalent:*

(a)  $A_1 B_1^{(1, P_1, Q_1)} A_2 B_2^{(1, P_2, Q_2)} A_3 B_3^{(1, P_3, Q_3)} A_4 = A$  holds for all  $B_1^{(1, P_1, Q_1)}$ ,  $B_2^{(1, P_2, Q_2)}$  and  $B_3^{(1, P_3, Q_3)}$ .

(b)  $A_1 B_1^{(1, P_1, Q_1)} A_2 B_2^{(1, P_2, Q_2)} A_3 B_3^{(1, P_3, Q_3)} A_4$  is invariant with respect to all  $B_1^{(1, P_1, Q_1)}$ ,  $B_2^{(1, P_2, Q_2)}$  and  $B_3^{(1, P_3, Q_3)}$  and  $A_1 (P_1 B_1 Q_1)^\dagger A_2 (P_2 B_2 Q_2)^\dagger A_3 (P_3 B_3 Q_3)^\dagger A_4 = A$  holds.

(c) Any one of the following 10 conditions holds:

(i)  $A = 0$  and  $A_1 = 0$ .

(ii)  $A = 0$  and  $A_2 = 0$ .

(iii)  $A = 0$  and  $A_3 = 0$ .

(iv)  $A = 0$  and  $A_4 = 0$ .

(v)  $A = 0$  and  $r \begin{bmatrix} 0 & A_1 \\ A_2 & P_1 B_1 Q_1 \end{bmatrix} = r(B_1)$ .

(vi)  $A = 0$  and  $r \begin{bmatrix} 0 & A_2 \\ A_3 & P_2 B_2 Q_2 \end{bmatrix} = r(B_2)$ .

(vii)  $A = 0$  and  $r \begin{bmatrix} 0 & A_3 \\ A_4 & P_3 B_3 Q_3 \end{bmatrix} = r(B_3)$ .

(viii)  $A = 0$  and  $r \begin{bmatrix} 0 & 0 & A_1 \\ 0 & A_2 & P_1 B_1 Q_1 \\ A_3 & P_2 B_2 Q_2 & 0 \end{bmatrix} = r \begin{bmatrix} A_2 & P_1 B_1 Q_1 \\ P_2 B_2 Q_2 & 0 \end{bmatrix} = r(B_1) + r(B_2)$ .



$$(ix) \ A = 0 \text{ and } r \begin{bmatrix} 0 & 0 & A_2 \\ 0 & A_3 & P_2 B_2 Q_2 \\ A_4 & P_3 B_3 Q_3 & 0 \end{bmatrix} = r \begin{bmatrix} A_3 & P_2 B_2 Q_2 \\ P_3 B_3 Q_3 & 0 \end{bmatrix} = r(B_2) + r(B_3).$$

$$(x) \ r \begin{bmatrix} A & 0 & 0 & A_1 \\ 0 & 0 & A_2 & P_1 B_1 Q_1 \\ 0 & A_3 & P_2 B_2 Q_2 & 0 \\ A_4 & P_3 B_3 Q_3 & 0 & 0 \end{bmatrix} = r \begin{bmatrix} 0 & A_2 & P_1 B_1 Q_1 \\ A_3 & P_2 B_2 Q_2 & 0 \\ P_3 B_3 Q_3 & 0 & 0 \end{bmatrix} = r(B_1) + r(B_2) + r(B_3).$$

**Theorem 5.9.** Given the assumptions in (5.17)–(5.19) and  $A \in \mathbb{C}^{m_1 \times m_8}$ , assume that  $B_1^{(1, P_1, Q_1)}$ ,  $B_2^{(1, P_2, Q_2)}$  and  $B_3^{(1, P_3, Q_3)}$  exist. Then, the following three statements are equivalent:

- (a)  $A_1 B_1^{(1, 2, P_1, Q_1)} A_2 B_2^{(1, 2, P_2, Q_2)} A_3 B_3^{(1, 2, P_3, Q_3)} A_4 = A$  holds for all  $B_1^{(1, 2, P_1, Q_1)}$ ,  $B_2^{(1, 2, P_2, Q_2)}$  and  $B_3^{(1, 2, P_3, Q_3)}$ .
- (b)  $A_1 B_1^{(1, 2, P_1, Q_1)} A_2 B_2^{(1, 2, P_2, Q_2)} A_3 B_3^{(1, 2, P_3, Q_3)} A_4$  is invariant with respect to all  $B_1^{(1, 2, P_1, Q_1)}$ ,  $B_2^{(1, 2, P_2, Q_2)}$  and  $B_3^{(1, 2, P_3, Q_3)}$  and  $A_1 (P_1 B_1 Q_1)^\dagger A_2 (P_2 B_2 Q_2)^\dagger A_3 (P_3 B_3 Q_3)^\dagger A_4 = A$  holds.
- (c) Any one of the following 13 conditions holds:

(i)  $A = 0$  and  $A_1 = 0$ .

(ii)  $A = 0$  and  $B_1 = 0$ .

(iii)  $A = 0$  and  $A_2 = 0$ .

(iv)  $A = 0$  and  $B_2 = 0$ .

(v)  $A = 0$  and  $A_3 = 0$ .

(vi)  $A = 0$  and  $B_3 = 0$ .

(vii)  $A = 0$  and  $A_4 = 0$ .

(viii)  $A = 0$  and  $r \begin{bmatrix} 0 & A_1 \\ A_2 & P_1 B_1 Q_1 \end{bmatrix} = r(B_1)$ .

(ix)  $A = 0$  and  $r \begin{bmatrix} 0 & A_2 \\ A_3 & P_2 B_2 Q_2 \end{bmatrix} = r(B_2)$ .

(x)  $A = 0$  and  $r \begin{bmatrix} 0 & A_3 \\ A_4 & P_3 B_3 Q_3 \end{bmatrix} = r(B_3)$ .

(xi)  $A = 0$  and  $r \begin{bmatrix} 0 & 0 & A_1 \\ 0 & A_2 & P_1 B_1 Q_1 \\ A_3 & P_2 B_2 Q_2 & 0 \end{bmatrix} = r \begin{bmatrix} A_2 & P_1 B_1 Q_1 \\ P_2 B_2 Q_2 & 0 \end{bmatrix} = r(B_1) + r(B_2)$ .

(xii)  $A = 0$  and  $r \begin{bmatrix} 0 & 0 & A_2 \\ 0 & A_3 & P_2 B_2 Q_2 \\ A_4 & P_3 B_3 Q_3 & 0 \end{bmatrix} = r \begin{bmatrix} A_3 & P_2 B_2 Q_2 \\ P_3 B_3 Q_3 & 0 \end{bmatrix} = r(B_2) + r(B_3)$ .

(xiii)  $r \begin{bmatrix} A & 0 & 0 & A_1 \\ 0 & 0 & A_2 & P_1 B_1 Q_1 \\ 0 & A_3 & P_2 B_2 Q_2 & 0 \\ A_4 & P_3 B_3 Q_3 & 0 & 0 \end{bmatrix} = r \begin{bmatrix} 0 & A_2 & P_1 B_1 Q_1 \\ A_3 & P_2 B_2 Q_2 & 0 \\ P_3 B_3 Q_3 & 0 & 0 \end{bmatrix} = r(B_1) + r(B_2) + r(B_3)$ .

## 6. An application

There exist many concrete matrix equalities, as described in (1.5) and (1.6), that can be represented in the forms of (1.3) and (1.4). As an application, we consider in this section the following partitioned matrix:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where  $A \in \mathbb{C}^{m_1 \times n_1}$ ,  $B \in \mathbb{C}^{m_1 \times n_2}$ ,  $C \in \mathbb{C}^{m_2 \times n_1}$  and  $D \in \mathbb{C}^{m_2 \times n_2}$ . Given such a partitioned matrix, we are able to construct a special matrix equality

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{(i, \dots, j)} = \begin{bmatrix} A^{(i_1, \dots, j_1)} & C^{(i_2, \dots, j_2)} \\ B^{(i_3, \dots, j_3)} & D^{(i_4, \dots, j_4)} \end{bmatrix} \quad (6.1)$$

for the generalized inverses. Because generalized inverses of matrices are not necessarily unique, (6.1) can be divided into the following four cases:

$$\left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{(i, \dots, j)} \right\} \cap \left\{ \begin{bmatrix} A^{(i_1, \dots, j_1)} & C^{(i_2, \dots, j_2)} \\ B^{(i_3, \dots, j_3)} & D^{(i_4, \dots, j_4)} \end{bmatrix} \right\} \neq \emptyset, \quad \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{(i, \dots, j)} \right\} \subseteq \left\{ \begin{bmatrix} A^{(i_1, \dots, j_1)} & C^{(i_2, \dots, j_2)} \\ B^{(i_3, \dots, j_3)} & D^{(i_4, \dots, j_4)} \end{bmatrix} \right\}, \quad (6.2)$$

$$\left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{(i, \dots, j)} \right\} \supseteq \left\{ \begin{bmatrix} A^{(i_1, \dots, j_1)} & C^{(i_2, \dots, j_2)} \\ B^{(i_3, \dots, j_3)} & D^{(i_4, \dots, j_4)} \end{bmatrix} \right\}, \quad \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{(i, \dots, j)} \right\} = \left\{ \begin{bmatrix} A^{(i_1, \dots, j_1)} & C^{(i_2, \dots, j_2)} \\ B^{(i_3, \dots, j_3)} & D^{(i_4, \dots, j_4)} \end{bmatrix} \right\}. \quad (6.3)$$

It is a tedious work to characterize these four matrix set relations for different choices of the generalized inverses. In what follows, we only consider a special case of the first matrix set inclusion in (6.3). By definition, the first matrix set inclusion in (6.3) holds for  $\{1\}$ -inverses of the matrices if and only if

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} A^{(1)} & C^{(1)} \\ B^{(1)} & D^{(1)} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad (6.4)$$

holds for all  $A^{(1)}$ ,  $B^{(1)}$ ,  $C^{(1)}$  and  $D^{(1)}$ , which in turn can be represented in the form

$$P_1 A^{(1)} Q_1 + P_2 B^{(1)} Q_1 + P_1 C^{(1)} Q_2 + P_2 D^{(1)} Q_2 = M, \quad (6.5)$$

where  $P_1 = \begin{bmatrix} A \\ C \end{bmatrix}$ ,  $P_2 = \begin{bmatrix} B \\ D \end{bmatrix}$ ,  $Q_1 = [A, B]$  and  $Q_2 = [C, D]$ . In this case, we are able to find from Theorem 3.6 the following result.

**Theorem 6.1.** *Let  $P_i$  and  $Q_i$  be as given in (6.5) and assume that  $P_i \neq 0$  and  $Q_i \neq 0$  for  $i = 1, 2$ . Then, (6.4) holds for all  $A^{(1)}$ ,  $B^{(1)}$ ,  $C^{(1)}$  and  $D^{(1)}$  if and only if*

$$r \begin{bmatrix} A & 0 & 0 & 0 & Q_1 \\ 0 & B & 0 & 0 & Q_1 \\ 0 & 0 & C & 0 & Q_2 \\ 0 & 0 & 0 & D & Q_2 \\ P_1 & P_2 & P_1 & P_2 & M \end{bmatrix} = r(A) + r(B) + r(C) + r(D),$$

or equivalently,

$$r \begin{bmatrix} A & B & A & 0 \\ C & 0 & C & D \\ A & B & 0 & B \\ 0 & D & C & D \end{bmatrix} = r(A) + r(B) + r(C) + r(D) - r(M).$$

## 7. Conclusions

We investigated the two sorts of general matrix equalities in (1.3) and (1.4) that involve ordinary sum and product operations of matrices and their generalized inverses, and we obtained a variety of necessary and sufficient conditions for them to hold. It can be seen that the corresponding results and facts are fairly easy to understand and utilize within the common discipline of matrix theory and generalized inverses of matrices since they were explicitly derived through various common algebraic calculations of matrices and their generalized inverses, as well as the skillful employments of the elementary but powerful BMRM and MRM.

In addition to the work related to  $\{1\}$ - and  $\{1, 2\}$ -generalized inverses of matrices, it would be of interest to consider other  $\{i, \dots, j\}$ -generalized inverses of the matrices in the two matrix equalities in (1.3) and (1.4); yet, it is not difficult to see that we can easily extend them by induction to general matrix equalities through certain special block matrices that are constructed from the given matrices in the equalities. Finally, we conclude that this study will reveal numerous novel and intrinsic properties and performances of matrix expressions and matrix equalities that are encountered in the theory of generalized inverses. We also believe that these results and facts will provide enough supports to approach various concrete matrix equality problems from theoretical and applied points of view.

### Use of AI tools declaration

The author declares that he has not used artificial intelligence tools in the creation of this article.

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### Conflict of interest

The author declares that there is no conflict of interest.

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