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Research article

On the reciprocal sums of products of mth-order linear recurrence sequences

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Abstract: In this paper, we use the method of error estimation to consider the reciprocal sums of products of any mth-order linear recurrence sequences $\{u_n\}$. Specifically, we find that a series of sequences are "asymptotically equivalent" to the reciprocal sums of products of any mth-order linear recurrence sequences $\{u_n\}$.

Keywords: mth-order linear recurrence sequences; reciprocal sums; products; asymptotically equivalent

1. Introduction

It is well known that if a series, $\sum\limits_{k=1}^{\infty} a_k$ is convergent, then its 'tail' $\lim\limits_{n\to\infty}\left(\sum\limits_{k=n}^{\infty} a_k\right)_n = 0$. This means $\lim\limits_{n\to\infty}\left(\sum\limits_{k=n}^{\infty} a_k\right)_n^{-1} = \infty$. To understand the values of $\sum\limits_{k=1}^{\infty} a_k$, it would be helpful to understand the values of $\sum\limits_{k=n}^{\infty} a_k$. In the past years, many scholars have been interested in studying the properties and forms of the reciprocal 'tails' $\left(\sum\limits_{k=n}^{\infty} a_k\right)^{-1}$, where $\{a_n\}$ is related to some recurrence sequences. This problem starts from the reciprocal sum of Fibonacci sequences.

For any positive integer n, the Fibonacci sequence $\{F_n\}$ is recursively defined as $F_0 = 0, F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$. In [1], Ohtsuka and Nakamura considered the partial infinite sums of reciprocal Fibonacci sequence, for $n \geq 2$, the authors proved that:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k}\right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & if\, n\, is\, even; \\ F_{n-2} - 1, & if\, n\, is\, odd, \end{cases}$$

and

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1, & if\, n\, is\, even; \\ F_{n-1}F_n, & if\, n\, is\, odd, \end{cases}$$

where $\lfloor z \rfloor$ denotes the floor function.

For any positive integer n, the Pell sequence $\{P_n\}$ is defined from the recurrence relation $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$. In [2], for $n \geq 2$, Zhang and Wang considered the partial infinite sums of reciprocal Pell sequence, and proved that:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{P_k} \right)^{-1} \right\rfloor = \begin{cases} P_n + P_{n-2}, & if\, n\, is\, even; \\ P_n + P_{n-2} - 1, & if\, n\, is\, odd. \end{cases}$$

For any positive integer n, the k-Fibonacci sequence $\{G_n\}$ is defined by the recurrence $G_0 = 0$, $G_1 = 1$, and $G_{n+2} = kG_{n+1} + G_n$. In [3], for $n \geq 2$, Holliday and Komatsu considered the partial infinite sums of reciprocal k-Fibonacci sequence, and proved that:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_k} \right)^{-1} \right\rfloor = \begin{cases} G_n - G_{n-1}, & if\, n\, is\, even; \\ G_n - G_{n-1} - 1, & if\, n\, is\, odd. \end{cases}$$

For more facts in this topic, we recommend to the reader the papers [4–9]. In addition, we found the series of papers [10–15], where the partial sums related to the Riemann zeta function $\zeta(s)$. For example, in [11], for $n \geq 1$, Lin proved that:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{k^2} \right)^{-1} \right\rfloor = n - 1,$$

and

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{k^3} \right)^{-1} \right\rfloor = 2n(n - 1).$$

For any positive integers $a_1, a_2, \ldots, a_m$, the mth-order linear recursive sequence $\{u_n\}$ is defined from the recurrence relation as follows:

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_m u_{n-m}, \quad if\, n > m, \tag{1.1}$$

where initial values $u_i \in \mathbb{N}$ for $0 \leq i < m$, at least one of them is different from zero. If $m = 2$, $a_1 = a_2 = 1$, and initial values $u_0 = 0$, $u_1 = 1$, then $\{u_n\}$ reduces to the Fibonacci sequence. If $m = 2$, $a_1 = 2$, $a_2 = 1$, and initial values $u_0 = 0$, $u_1 = 1$, then $\{u_n\}$ reduces to the Pell sequence. If $m = 2$, $a_1 = k$, $a_2 = 1$, and initial values $u_0 = 0$, $u_1 = 1$, then $\{u_n\}$ reduces to the k-Fibonacci sequence. The characteristic polynomial of the sequence $\{u_n\}$ is given by

$$f(x) = x^m - a_1 x^{m-1} - \cdots - a_{m-1}x - a_m = (x - \alpha_1)^{m_1} \cdots (x - \alpha_l)^{m_l},$$

where the α_i are distinct for $i = 1, \ldots, l$, which are called the 'roots' of the recurrence. Moreover, the recurrence $\{u_n\}$ has a 'dominant root' if one of its roots has strictly largest absolute value.

Recently many papers discussed to reciprocal sums of mth-order linear recursive sequence. In [17], for any positive integers $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$, Wu and Zhang proved that there exists a positive integer n_1 such that

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_k} \right)^{-1} \right\| = u_n - u_{n-1}, \quad n \geq n_1.$$

where $\|z\|$ denotes the nearest integer. Clearly that $\|x\| = \left\lfloor x + \frac{1}{2} \right\rfloor$.

For more discussion of the mth-order linear recursive sequence studies, see [16,18–20]. Specifically, in [19], Trojovský considered by finding a sequence which is "asymptotically equivalent" to partial infinite sums of $\{u_n\}$, and proved that

$$\left\{ \left(\sum_{k=n}^{\infty} \frac{1}{P(u_k)} \right)^{-1} \right\} \quad \text{and} \quad \{P(u_n) - P(u_{n-1})\}$$

are asymptotically equivalent, where $P(z) \in \mathbb{C}[z]$ be a non-constant polynomial. Specifically, we called that two sequences $\{G_n\}$ and $\{H_n\}$ are called asymptotically equivalent, if $\{G_n\} / \{H_n\}$ tends to 1 as $n \to \infty$.

In this paper, we continue the discussion to reciprocal sums of mth-order linear recurrence sequences. Different from previous studies, we mainly consider that a series of sequences are asymptotically equivalent to the reciprocal sums of products and the products interleaving terms of any mth-order linear recurrence sequences $\{u_n\}$. In addition, we extend these results to equidistant sub-sequences of $\{u_n\}$. We separate the main term and the error term, and in order to control the range of the error term during the specific operation, the reciprocal and the form of the general term in it should be properly constructed. The major results are as follows:

Theorem 1. Let $\{u_n\}$ be mth-order linear recurrence sequence defined by (1.1), where the restriction $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$. Then for any positive integers $p_1, p_2, \ldots, p_s$ and s.

(i) The sequences

$$\left\{ \left(\sum_{k=n}^{\infty} \frac{a_1^{(s+1)k}}{u_k \prod_{i=1}^{s} u_{k+p_i}} \right)^{-1} \right\} \quad \text{and} \quad \left\{ \frac{u_n \prod_{i=1}^{s} u_{n+p_i}}{a_1^{(s+1)n}} - \frac{u_{n-1} \prod_{i=1}^{s} u_{n+p_i-1}}{a_1^{(s+1)(n-1)}} \right\},$$

are asymptotically equivalent.

(ii) The sequences

$$\left\{ \left(\sum_{k=n}^{\infty} \frac{(-a_1)^{(s+1)k}}{u_k \prod_{i=1}^{s} u_{k+p_i}} \right)^{-1} \right\} \quad \text{and} \quad \left\{ (-1)^{(s+1)n} \left(\frac{u_n \prod_{i=1}^{s} u_{n+p_i}}{a_1^{(s+1)n}} + (-1)^s \cdot \frac{u_{n-1} \prod_{i=1}^{s} u_{n+p_i-1}}{a_1^{(s+1)(n-1)}} \right) \right\},$$

are asymptotically equivalent.

Theorem 2. Let $\{u_n\}$ be mth-order linear recurrence sequence defined by (1.1), where the restriction $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$. Then for any positive integers $q, p_1, p_2, \ldots, p_s$ with $0 \leq p_i < q$ for $0 \leq i \leq s$.

(iii) The sequences

$$\left\{\left(\sum_{k=n}^{\infty}\frac{a_1^{(s+1)qk}}{u_{qk}\prod_{i=1}^{s}u_{qk+p_i}}\right)^{-1}\right\} \quad \text{and} \quad \left\{\frac{u_{qn}\prod_{i=1}^{s}u_{qn+p_i}}{a_1^{(s+1)qn}}-\frac{u_{qn-q}\prod_{i=1}^{s}u_{qn+p_i-q}}{a_1^{q(s+1)(n-1)}}\right\},$$

are asymptotically equivalent.

(iv) The sequences

$$\left\{\left(\sum_{k=n}^{\infty}\frac{(-a_1)^{(s+1)qk}}{u_{qk}\prod_{i=1}^{s}u_{qk+p_i}}\right)^{-1}\right\} \quad \text{and}$$

$$\left\{(-1)^{(s+1)qn}\left(\frac{u_{qn}\prod_{i=1}^{s}u_{qn+p_i}}{a_1^{(s+1)qn}}+(-1)^{\delta(sq+q+1)}\cdot\frac{u_{qn-q}\prod_{i=1}^{s}u_{qn+p_i-q}}{a_1^{q(s+1)(n-1)}}\right)\right\},$$

are asymptotically equivalent, where $\delta(z)=z-2\left\lfloor\frac{z}{2}\right\rfloor$ is the parity function.

From these two theorems we may immediately deduce the following corollaries:

Corollary 1. Let $\{u_n\}$ be mth-order linear recurrence sequence defined by (1.1), where the restriction $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$. Then for any positive integers p and q with $0 \leq p < q$.

(i) The sequences

$$\left\{\left(\sum_{k=n}^{\infty}\frac{(\pm a_1)^{2k}}{u_k u_{k+p}}\right)^{-1}\right\} \quad \text{and} \quad \left\{\frac{u_n u_{n+p}}{a_1^{2n}}-\frac{u_{n-1}u_{n+p-1}}{a_1^{2n-2}}\right\},$$

are asymptotically equivalent.

(ii) The sequences

$$\left\{\left(\sum_{k=n}^{\infty}\frac{(\pm a_1)^{2qk}}{u_{qk}u_{qk+p}}\right)^{-1}\right\} \quad \text{and} \quad \left\{\frac{u_{qn}u_{qn+p}}{a_1^{2qn}}-\frac{u_{qn-q}u_{qn+p-q}}{a_1^{2q(n-1)}}\right\},$$

are asymptotically equivalent.

Corollary 2. Let $\{u_n\}$ be mth-order linear recurrence sequence defined by (1.1), where the restriction $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$. Then for any positive integer q.

(iii) The sequences

$$\left\{\left(\sum_{k=n}^{\infty}\frac{a_1^{sqk}}{u_{qk}^{s}}\right)^{-1}\right\} \quad \text{and} \quad \left\{\frac{u_{qn}^{s}}{a_1^{sqn}}-\frac{u_{qn-q}^{s}}{a_1^{sq(n-1)}}\right\},$$

are asymptotically equivalent.

(iv) The sequences

$$\left\{\left(\sum_{k=n}^{\infty}\frac{(-a_1)^{sqk}}{u_{qk}^{s}}\right)^{-1}\right\} \quad \text{and} \quad \left\{(-1)^{sqn}\left(\frac{u_{qn}^{s}}{a_1^{sqn}}+(-1)^{\delta(sq+1)}\cdot\frac{u_{qn-q}^{s}}{a_1^{sq(n-1)}}\right)\right\},$$

are asymptotically equivalent, where $\delta(z)=z-2\left\lfloor\frac{z}{2}\right\rfloor$ is the parity function.

2. Auxiliary results

In this section, we give two lemmas that are necessary in the proofs of the theorems.

Lemma 1. ([17]) Let $\{u_n\}$ be mth-order linear recursive sequence defined by (1.1). The coefficients of the characteristic polynomial $f(x)$ are satisfied that $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$. Then the closed formula of $\{u_n\}$ is given by:

$$u_n = a\alpha^n + O\left(c^{-n}\right), \quad (n \to \infty),$$

where $a > 0$ is a constant, $c > 1$, α is the positive real dominant root of $f(x)$ for $a_1 < \alpha < a_1 + 1$, and "O" (the Landau symbol) denotes if $g(x) > 0$ for all $x \geq a$, we write $f(x) = O(g(x))$ to mean that the quotient $f(x)/g(x)$ is bounded for $x \geq a$.

Lemma 2. Let $\{u_n\}$ be mth-order linear recursive sequence defined by (1.1). Then

$$u_k \prod_{i=1}^{s} u_{k+p_i} = a^{s+1}\alpha^{(s+1)k+\sum\limits_{i=1}^{s} p_i} + O\left(\alpha^{sk+\sum\limits_{i=1}^{s} p_i} c^{-k}\right). \tag{2.1}$$

where $a > 0$ is a constant, $c > 1$, α is the positive real dominant root of the characteristic polynomial $f(x)$ for $a_1 < \alpha < a_1 + 1$.

Proof. We prove (2.1) by mathematical induction. When $s = 1$, by Lemma 1,

$$\begin{aligned} u_k u_{k+p_1} &= \left(a\alpha^k + O\left(c^{-k}\right)\right)\left(a\alpha^{k+p_1} + O\left(c^{-k-p_1}\right)\right) \\ &= a^2\alpha^{2k+p_1} + O\left(\alpha^k c^{-k-p_1}\right) + O\left(\alpha^{k+p_1} c^{-k}\right) + O\left(c^{-2k-p_1}\right) \\ &= a^2\alpha^{2k+p_1} + O\left(\alpha^{k+p_1} c^{-k}\right). \end{aligned}$$

That is, (2.1) is true for $s = 1$, suppose that for any integers $s - 1$, we have

$$u_k \prod_{i=1}^{s-1} u_{k+p_i} = a^s\alpha^{sk+\sum\limits_{i=1}^{s-1} p_i} + O\left(\alpha^{(s-1)k+\sum\limits_{i=1}^{s-1} p_i} c^{-k}\right).$$

Then for s, by Lemma 1, we have

$$\begin{aligned} u_k \prod_{i=1}^{s} u_{k+p_i} &= \left(a^s\alpha^{sk+\sum\limits_{i=1}^{s-1} p_i} + O\left(\alpha^{(s-1)k+\sum\limits_{i=1}^{s-1} p_i} c^{-k}\right)\right) \cdot \left(a\alpha^{k+p_s} + O\left(c^{-(k+p_s)}\right)\right) \\ &= a^{s+1}\alpha^{(s+1)k+\sum\limits_{i=1}^{s} p_i} + O\left(\alpha^{sk+\sum\limits_{i=1}^{s} p_i} c^{-k}\right) + O\left(\alpha^{sk+\sum\limits_{i=1}^{s-1} p_i} c^{-(k+p_s)}\right) \\ &+ O\left(\alpha^{(s-1)k+\sum\limits_{i=1}^{s-1} p_i}\left(c^{-(2k+p_s)}\right)\right) \\ &= a^{s+1}\alpha^{(s+1)k+\sum\limits_{i=1}^{s} p_i} + O\left(\alpha^{sk+\sum\limits_{i=1}^{s} p_i} c^{-k}\right). \end{aligned} \tag{2.2}$$

Now (2.1) follows form (2.2) and mathematical induction, which completes the proof. □

3. Proof of the theorems

Here, we only prove that Theorem 1, and Theorem 2 are proved similarly.

Proof of Theorem 1. First, we prove (i), from the geometric series as $\epsilon \to 0$, we find:

$$\frac{1}{1 \pm \epsilon} = 1 \mp \epsilon + O\left(\epsilon^2\right) = 1 + O(\epsilon). \tag{3.1}$$

Using Lemma 2, we have

$$\begin{aligned}
\frac{a_1^{(s+1)k+\sum_{i=1}^{s} p_i}}{u_k \prod_{i=1}^{s} u_{k+p_i}} &= \frac{a_1^{(s+1)k+\sum_{i=1}^{s} p_i}}{a^{s+1}\alpha^{(s+1)k+\sum_{i=1}^{s} p_i} + O\left(\alpha^{sk+\sum_{i=1}^{s} p_i} c^{-k}\right)} \\
&= \frac{a_1^{(s+1)k+\sum_{i=1}^{s} p_i}}{a^{s+1}\alpha^{(s+1)k+\sum_{i=1}^{s} p_i} \left(1 + O\left(\alpha^{-k}c^{-k}\right)\right)} \\
&= \frac{a_1^{(s+1)k+\sum_{i=1}^{s} p_i}}{a^{s+1}\alpha^{(s+1)k+\sum_{i=1}^{s} p_i}} \left(1 + O\left(\alpha^{-k}c^{-k}\right)\right) \qquad \text{(by (3.1))} \\
&= \frac{a_1^{(s+1)k+\sum_{i=1}^{s} p_i}}{a^{s+1}\alpha^{(s+1)k+\sum_{i=1}^{s} p_i}} + O\left(\frac{a_1^{(s+1)k+\sum_{i=1}^{s} p_i}}{\alpha^{(s+2)k+\sum_{i=1}^{s} p_i} c^k}\right),
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{k=n}^{\infty} \frac{a_1^{(s+1)k+\sum_{i=1}^{s} p_i}}{u_k \prod_{i=1}^{s} u_{k+p_i}} &= \frac{a_1^{\sum_{i=1}^{s} p_i}}{a^{s+1}\alpha^{\sum_{i=1}^{s} p_i}} \sum_{k=n}^{\infty} \left(\frac{a_1^{s+1}}{\alpha^{s+1}}\right)^k + O\left(\sum_{k=n}^{\infty} \frac{a_1^{(s+1)k+\sum_{i=1}^{s} p_i}}{\alpha^{(s+2)k+\sum_{i=1}^{s} p_i} c^k}\right) \\
&= \frac{a_1^{(s+1)n+\sum_{i=1}^{s} p_i}}{a^{s+1}\alpha^{(s+1)n+\sum_{i=1}^{s} p_i}} \left(\frac{\alpha^{s+1}}{\alpha^{s+1} - a_1^{s+1}}\right) + O\left(\frac{a_1^{(s+1)n+\sum_{i=1}^{s} p_i}}{\alpha^{(s+2)n+\sum_{i=1}^{s} p_i} c^n}\right),
\end{aligned}$$

Taking reciprocal, we get

$$\left(\sum_{k=n}^{\infty}\frac{a_1^{(s+1)k+\sum_{i=1}^{s}p_i}}{u_k\prod_{i=1}^{s}u_{k+p_i}}\right)^{-1}=\frac{1}{\frac{a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}{a^{s+1}\alpha^{(s+1)n+\sum_{i=1}^{s}p_i}}\left(\frac{\alpha^{s+1}}{\alpha^{s+1}-a_1^{s+1}}\right)+O\left(\frac{a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}{\alpha^{(s+2)n+\sum_{i=1}^{s}p_i}c^n}\right)}$$

$$=\frac{1}{\frac{a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}{a^{s+1}\alpha^{(s+1)n+\sum_{i=1}^{s}p_i}}\left(\frac{\alpha^{s+1}}{\alpha^{s+1}-a_1^{s+1}}\right)\left(1+O\left(\frac{1}{\alpha^nc^n}\right)\right)}$$

$$=\frac{a^{s+1}\alpha^{(s+1)n+\sum_{i=1}^{s}p_i}}{a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}\left(\frac{\alpha^{s+1}-a_1^{s+1}}{\alpha^{s+1}}\right)\left(1+O\left(\frac{1}{\alpha^nc^n}\right)\right),\qquad \text{(by (3.1))}$$

which yields that

$$\frac{\left\{\left(\sum_{k=n}^{\infty}\frac{a_1^{(s+1)k+\sum_{i=1}^{s}p_i}}{u_k\prod_{i=1}^{s}u_{k+p_i}}\right)^{-1}\right\}}{\left\{\frac{u_n\prod_{i=1}^{s}u_{n+p_i}}{a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}-\frac{u_{n-1}\prod_{i=1}^{s}u_{n+p_i-1}}{a_1^{(s+1)(n-1)+\sum_{i=1}^{s}p_i}}\right\}}$$

$$=\frac{\frac{a^{s+1}\alpha^{(s+1)n+\sum_{i=1}^{s}p_i}}{a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}\left(\frac{\alpha^{s+1}-a_1^{s+1}}{\alpha^{s+1}}\right)\left(1+O\left(\frac{1}{\alpha^nc^n}\right)\right)}{\frac{a^{s+1}\alpha^{(s+1)n+\sum_{i=1}^{s}p_i}+O\left(\alpha^{sn+\sum_{i=1}^{s}p_i}c^{-n}\right)}{a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}-\frac{a^{s+1}\alpha^{(s+1)(n-1)+\sum_{i=1}^{s}p_i}+O\left(\alpha^{s(n-1)+\sum_{i=1}^{s}p_i}c^{-n+1}\right)}{a_1^{(s+1)(n-1)+\sum_{i=1}^{s}p_i}}}\tag{3.2}$$

$$=\frac{\frac{a^{s+1}\alpha^{(s+1)n+\sum_{i=1}^{s}p_i}}{a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}\left(\frac{\alpha^{s+1}-a_1^{s+1}}{\alpha^{s+1}}\right)\left(1+O\left(\frac{1}{\alpha^nc^n}\right)\right)}{\frac{a^{s+1}\alpha^{(s+1)n+\sum_{i=1}^{s}p_i}}{a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}\left(\frac{\alpha^{s+1}-a_1^{s+1}}{\alpha^{s+1}}\right)\left(1+O\left(\frac{1}{\alpha^nc^n}\right)+O\left(\frac{a_1^{s+1}}{\alpha^{n+s}c^n}\right)\right)}$$

We obtain that

$$\frac{\left\{\left(\sum_{k=n}^{\infty}\frac{a_1^{(s+1)k+\sum_{i=1}^{s}p_i}}{u_k\prod_{i=1}^{s}u_{k+p_i}}\right)^{-1}\right\}}{\left\{\frac{u_n\prod_{i=1}^{s}u_{n+p_i}}{a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}-\frac{u_{n-1}\prod_{i=1}^{s}u_{n+p_i-1}}{a_1^{(s+1)(n-1)+\sum_{i=1}^{s}p_i}}\right\}}\quad \text{tends to } 1, \quad \text{as} \quad n\to\infty,$$

Therefore, the sequences

$$\left\{\left(\sum_{k=n}^{\infty}\frac{a_1^{(s+1)k}}{u_k\prod_{i=1}^{s}u_{k+p_i}}\right)^{-1}\right\} \quad \text{and} \quad \left\{\frac{u_n\prod_{i=1}^{s}u_{n+p_i}}{a_1^{(s+1)n}}-\frac{u_{n-1}\prod_{i=1}^{s}u_{n+p_i-1}}{a_1^{(s+1)(n-1)}}\right\},$$

are asymptotically equivalent.

Now we prove (ii), using Lemma 2, we have

$$\begin{aligned}
\frac{(-a_1)^{(s+1)k+\sum_{i=1}^{s}p_i}}{u_k\prod_{i=1}^{s}u_{k+p_i}} &= \frac{(-a_1)^{(s+1)k+\sum_{i=1}^{s}p_i}}{a^{s+1}\alpha^{(s+1)k+\sum_{i=1}^{s}p_i}+O\left(\alpha^{sk+\sum_{i=1}^{s}p_i}c^{-k}\right)} \\
&= \frac{(-a_1)^{(s+1)k+\sum_{i=1}^{s}p_i}}{a^{s+1}\alpha^{(s+1)k+\sum_{i=1}^{s}p_i}\left(1+O\left(\alpha^{-k}c^{-k}\right)\right)} \\
&= \frac{(-a_1)^{(s+1)k+\sum_{i=1}^{s}p_i}}{a^{s+1}\alpha^{(s+1)k+\sum_{i=1}^{s}p_i}}\left(1+O\left(\alpha^{-k}c^{-k}\right)\right) \qquad (\text{by } (3.1)) \\
&= \frac{(-a_1)^{(s+1)k+\sum_{i=1}^{s}p_i}}{a^{s+1}\alpha^{(s+1)k+\sum_{i=1}^{s}p_i}}+O\left(\frac{a_1^{(s+1)k+\sum_{i=1}^{s}p_i}}{\alpha^{(s+2)k+\sum_{i=1}^{s}p_i}c^k}\right),
\end{aligned}$$

Thus,

$$\sum_{k=n}^{\infty}\frac{(-a_1)^{(s+1)k+\sum_{i=1}^{s}p_i}}{u_k\prod_{i=1}^{s}u_{k+p_i}} = \frac{(-a_1)^{\sum_{i=1}^{s}p_i}}{a^{s+1}\alpha^{\sum_{i=1}^{s}p_i}}\sum_{k=n}^{\infty}\left(\frac{(-a_1)^{s+1}}{\alpha^{s+1}}\right)^k+O\left(\sum_{k=n}^{\infty}\frac{a_1^{(s+1)k+\sum_{i=1}^{s}p_i}}{\alpha^{(s+2)k+\sum_{i=1}^{s}p_i}c^k}\right).$$

For an odd s,

$$\sum_{k=n}^{\infty}\frac{(-a_1)^{(s+1)k+\sum_{i=1}^{s}p_i}}{u_k\prod_{i=1}^{s}u_{k+p_i}} = \frac{(-1)^{\sum_{i=1}^{s}p_i}\cdot a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}{a^{s+1}\alpha^{(s+1)n+\sum_{i=1}^{s}p_i}}\left(\frac{\alpha^{s+1}}{\alpha^{s+1}-a_1^{s+1}}\right)+O\left(\frac{a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}{\alpha^{(s+2)n+\sum_{i=1}^{s}p_i}c^n}\right),$$

Taking reciprocal, we get

$$\left(\sum_{k=n}^{\infty}\frac{(-a_1)^{(s+1)k+\sum_{i=1}^{s}p_i}}{u_k\prod_{i=1}^{s}u_{k+p_i}}\right)^{-1}$$
$$=\frac{1}{\frac{(-1)^{\sum_{i=1}^{s}p_i}\cdot a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}{a^{s+1}\alpha^{(s+1)n+\sum_{i=1}^{s}p_i}}\left(\frac{\alpha^{s+1}}{\alpha^{s+1}-a_1^{s+1}}\right)+O\left(\frac{a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}{\alpha^{(s+2)n+\sum_{i=1}^{s}p_i}c^n}\right)}$$
$$=\frac{1}{\frac{(-1)^{\sum_{i=1}^{s}p_i}\cdot a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}{a^{s+1}\alpha^{(s+1)n+\sum_{i=1}^{s}p_i}}\left(\frac{\alpha^{s+1}}{\alpha^{s+1}-a_1^{s+1}}\right)\left(1+O\left(\frac{1}{\alpha^nc^n}\right)\right)}\qquad \text{(by (3.1))}$$
$$=(-1)^{\sum_{i=1}^{s}p_i}\cdot\frac{a^{s+1}\alpha^{(s+1)n+\sum_{i=1}^{s}p_i}}{a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}\left(\frac{\alpha^{s+1}-a_1^{s+1}}{\alpha^{s+1}}\right)\left(1+O\left(\frac{1}{\alpha^nc^n}\right)\right),$$

which yields that

$$\frac{\left\{\left(\sum_{k=n}^{\infty}\frac{(-a_1)^{(s+1)k+\sum_{i=1}^{s}p_i}}{u_k\prod_{i=1}^{s}u_{k+p_i}}\right)^{-1}\right\}}{\left\{(-1)^{\sum_{i=1}^{s}p_i}\cdot\left(\frac{u_n\prod_{i=1}^{s}u_{n+p_i}}{a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}-\frac{u_{n-1}\prod_{i=1}^{s}u_{n+p_i-1}}{a_1^{(s+1)(n-1)+\sum_{i=1}^{s}p_i}}\right)\right\}}$$
$$=\frac{(-1)^{\sum_{i=1}^{s}p_i}\cdot\frac{a^{s+1}\alpha^{(s+1)n+\sum_{i=1}^{s}p_i}}{a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}\left(\frac{\alpha^{s+1}-a_1^{s+1}}{\alpha^{s+1}}\right)\left(1+O\left(\frac{1}{\alpha^nc^n}\right)\right)}{(-1)^{\sum_{i=1}^{s}p_i}\cdot\left(\frac{a^{s+1}\alpha^{(s+1)n+\sum_{i=1}^{s}p_i}+O\left(\alpha^{sn+\sum_{i=1}^{s}p_i}c^{-n}\right)}{a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}-\frac{a^{s+1}\alpha^{(s+1)(n-1)+\sum_{i=1}^{s}p_i}+O\left(\alpha^{s(n-1)+\sum_{i=1}^{s}p_i}c^{-n+1}\right)}{a_1^{(s+1)(n-1)+\sum_{i=1}^{s}p_i}}\right)}$$
$$=\frac{\frac{a^{s+1}\alpha^{(s+1)n+\sum_{i=1}^{s}p_i}}{a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}\left(\frac{\alpha^{s+1}-a_1^{s+1}}{\alpha^{s+1}}\right)\left(1+O\left(\frac{1}{\alpha^nc^n}\right)\right)}{\frac{a^{s+1}\alpha^{(s+1)n+\sum_{i=1}^{s}p_i}}{a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}\left(\frac{\alpha^{s+1}-a_1^{s+1}}{\alpha^{s+1}}\right)\left(1+O\left(\frac{1}{\alpha^nc^n}\right)+O\left(\frac{a_1^{s+1}}{\alpha^{n+s}c^n}\right)\right)}.$$

We obtain that

$$\frac{\left\{\left(\sum_{k=n}^{\infty}\frac{(-a_1)^{(s+1)k+\sum_{i=1}^{s}p_i}}{u_k\prod_{i=1}^{s}u_{k+p_i}}\right)^{-1}\right\}}{\left\{(-1)^{\sum_{i=1}^{s}p_i}\cdot\left(\frac{u_n\prod_{i=1}^{s}u_{n+p_i}}{a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}-\frac{u_{n-1}\prod_{i=1}^{s}u_{n+p_i-1}}{a_1^{(s+1)(n-1)+\sum_{i=1}^{s}p_i}}\right)\right\}}\quad \text{tends}\quad \text{to}\quad 1,\quad \text{as}\quad n\to\infty,$$

For an even s,

$$\sum_{k=n}^{\infty}\frac{(-a_1)^{(s+1)k+\sum_{i=1}^{s}p_i}}{u_k\prod_{i=1}^{s}u_{k+p_i}}=\frac{(-a_1)^{(s+1)n+\sum_{i=1}^{s}p_i}}{a^{s+1}\alpha^{(s+1)n+\sum_{i=1}^{s}p_i}}\left(\frac{\alpha^{s+1}}{\alpha^{s+1}+a_1^{s+1}}\right)+O\left(\frac{a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}{\alpha^{(s+2)n+\sum_{i=1}^{s}p_i}c^n}\right),$$

Taking reciprocal, we get

$$\begin{aligned}
&\left(\sum_{k=n}^{\infty}\frac{(-a_1)^{(s+1)k+\sum_{i=1}^{s}p_i}}{u_k\prod_{i=1}^{s}u_{k+p_i}}\right)^{-1}\\
&=\frac{1}{\frac{(-1)^{(s+1)n+\sum_{i=1}^{s}p_i}\cdot a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}{a^{s+1}\alpha^{(s+1)n+\sum_{i=1}^{s}p_i}}\left(\frac{\alpha^{s+1}}{\alpha^{s+1}+a_1^{s+1}}\right)+O\left(\frac{a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}{\alpha^{(s+2)n+\sum_{i=1}^{s}p_i}c^n}\right)}\\
&=\frac{1}{\frac{(-1)^{(s+1)n+\sum_{i=1}^{s}p_i}\cdot a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}{a^{s+1}\alpha^{(s+1)n+\sum_{i=1}^{s}p_i}}\left(\frac{\alpha^{s+1}}{\alpha^{s+1}+a_1^{s+1}}\right)\left(1+O\left(\frac{1}{\alpha^nc^n}\right)\right)}\\
&=(-1)^{(s+1)n+\sum_{i=1}^{s}p_i}\cdot\left(\frac{a^{s+1}\alpha^{(s+1)n+\sum_{i=1}^{s}p_i}}{a_1^{(s+1)n+\sum_{i=1}^{s}p_i}}\left(\frac{\alpha^{s+1}+a_1^{s+1}}{\alpha^{s+1}}\right)\left(1+O\left(\frac{1}{\alpha^nc^n}\right)\right)\right),\qquad (\text{by}\quad (3.1))
\end{aligned}$$

which yields that

$$\frac{\left\{\left(\sum\limits_{k=n}^{\infty}\frac{(-a_1)^{(s+1)k+\sum\limits_{i=1}^{s}p_i}}{u_k\prod\limits_{i=1}^{s}u_{k+p_i}}\right)^{-1}\right\}}{\left\{(-1)^{(s+1)n+\sum\limits_{i=1}^{s}p_i}\cdot\left(\frac{u_n\prod\limits_{i=1}^{s}u_{n+p_i}}{a_1^{(s+1)n+\sum\limits_{i=1}^{s}p_i}}+\frac{u_{n-1}\prod\limits_{i=1}^{s}u_{n+p_i-1}}{a_1^{(s+1)(n-1)+\sum\limits_{i=1}^{s}p_i}}\right)\right\}}$$

$$=\frac{(-1)^{(s+1)n+\sum\limits_{i=1}^{s}p_i}\cdot\left(\frac{a^{s+1}\alpha^{(s+1)n+\sum\limits_{i=1}^{s}p_i}}{a_1^{(s+1)n+\sum\limits_{i=1}^{s}p_i}}\left(\frac{\alpha^{s+1}+a_1^{s+1}}{\alpha^{s+1}}\right)\left(1+O\left(\frac{1}{\alpha^nc^n}\right)\right)\right)}{(-1)^{(s+1)n+\sum\limits_{i=1}^{s}p_i}\cdot\left(\frac{a^{s+1}\alpha^{(s+1)n+\sum\limits_{i=1}^{s}p_i}+O\left(\alpha^{sn+\sum\limits_{i=1}^{s}p_i}c^{-n}\right)}{a_1^{(s+1)n+\sum\limits_{i=1}^{s}p_i}}+\frac{a^{s+1}\alpha^{(s+1)(n-1)+\sum\limits_{i=1}^{s}p_i}+O\left(\alpha^{s(n-1)+\sum\limits_{i=1}^{s}p_i}c^{-n+1}\right)}{a_1^{(s+1)(n-1)+\sum\limits_{i=1}^{s}p_i}}\right)}$$

$$=\frac{\frac{a^{s+1}\alpha^{(s+1)n+\sum\limits_{i=1}^{s}p_i}}{a_1^{(s+1)n+\sum\limits_{i=1}^{s}p_i}}\left(\frac{\alpha^{s+1}+a_1^{s+1}}{\alpha^{s+1}}\right)\left(1+O\left(\frac{1}{\alpha^nc^n}\right)\right)}{\frac{a^{s+1}\alpha^{(s+1)n+\sum\limits_{i=1}^{s}p_i}}{a_1^{(s+1)n+\sum\limits_{i=1}^{s}p_i}}\left(\frac{\alpha^{s+1}+a_1^{s+1}}{\alpha^{s+1}}\right)\left(1+O\left(\frac{1}{\alpha^nc^n}\right)+O\left(\frac{a_1^{s+1}}{\alpha^{n+s}c^n}\right)\right)}.$$

For an even s, n, we obtain that,

$$\frac{\left\{\left(\sum\limits_{k=n}^{\infty}\frac{(-a_1)^{(s+1)k+\sum\limits_{i=1}^{s}p_i}}{u_k\prod\limits_{i=1}^{s}u_{k+p_i}}\right)^{-1}\right\}}{\left\{(-1)^{\sum\limits_{i=1}^{s}p_i}\cdot\left(\frac{u_n\prod\limits_{i=1}^{s}u_{n+p_i}}{a_1^{(s+1)n+\sum\limits_{i=1}^{s}p_i}}+\frac{u_{n-1}\prod\limits_{i=1}^{s}u_{n+p_i-1}}{a_1^{(s+1)(n-1)+\sum\limits_{i=1}^{s}p_i}}\right)\right\}}\quad \text{tends to } 1,\quad \text{as}\quad n\to\infty,$$

are asymptotically equivalent.

For an even s and an odd n, we obtain that

$$\frac{\left\{\left(\sum\limits_{k=n}^{\infty}\frac{(-a_1)^{(s+1)k+\sum\limits_{i=1}^{s}p_i}}{u_k\prod\limits_{i=1}^{s}u_{k+p_i}}\right)^{-1}\right\}}{\left\{(-1)^{\sum\limits_{i=1}^{s}p_i+1}\cdot\left(\frac{u_n\prod\limits_{i=1}^{s}u_{n+p_i}}{a_1^{(s+1)n+\sum\limits_{i=1}^{s}p_i}}+\frac{u_{n-1}\prod\limits_{i=1}^{s}u_{n+p_i-1}}{a_1^{(s+1)(n-1)+\sum\limits_{i=1}^{s}p_i}}\right)\right\}}\quad \text{tends to } 1,\quad \text{as}\quad n\to\infty,$$

Therefore, the sequences

$$\left\{\left(\sum_{k=n}^{\infty}\frac{(-a_1)^{(s+1)k}}{u_k\prod\limits_{i=1}^{s}u_{k+p_i}}\right)^{-1}\right\}\quad\text{and}\quad\left\{(-1)^{(s+1)n}\left(\frac{u_n\prod\limits_{i=1}^{s}u_{n+p_i}}{a_1^{(s+1)n}}+(-1)^s\cdot\frac{u_{n-1}\prod\limits_{i=1}^{s}u_{n+p_i-1}}{a_1^{(s+1)(n-1)}}\right)\right\},$$

are asymptotically equivalent, which completes the proof. □

Remark. We will now discuss relative error of an asymptotic behavior of the result in (i) of Theorem 1. By identity (3.2) we obtain

$$\left| \frac{\left(\sum_{k=n}^{\infty} \frac{a_1^{(s+1)k+\sum_{i=1}^{s} p_i}}{u_k \prod_{i=1}^{s} u_{k+p_i}} \right)^{-1}}{\frac{u_n \prod_{i=1}^{s} u_{n+p_i}}{a_1^{(s+1)n+\sum_{i=1}^{s} p_i}} - \frac{u_{n-1} \prod_{i=1}^{s} u_{n+p_i-1}}{a_1^{(s+1)(n-1)+\sum_{i=1}^{s} p_i}}} - 1 \right| = \left| \frac{1 + O\left(\frac{1}{\alpha^n c^n}\right)}{1 + O\left(\frac{1}{\alpha^n c^n}\right) + O\left(\frac{a_1^{s+1}}{\alpha^{n+s} c^n}\right)} - 1 \right| \tag{3.3}$$

$$= O\left(\frac{1}{\alpha^n c^n}\right) + O\left(\frac{a_1^{s+1}}{\alpha^{n+s} c^n}\right) \qquad \text{(by (3.1))}$$

Determine the dominant root α well as the magnitude of the relative error of asymptotic equivalence by identity (3.3) for n= 100, for the following two sequences $\{u_n\}$ defined by linear recurrences of the fourth-order:

$$u_n = 5u_{n-1} + 3u_{n-2} + u_{n-3} + u_{n-4},$$

$$u_n = 5u_{n-1} + 3u_{n-2} + 2u_{n-3} + u_{n-4}.$$

Requested computations are in Table 1. We used software Mathematica.

Table 1. The fourth-order linear recurrences with $c = 2$, $s = 3$.

u_n	α	$O\left(\frac{1}{\alpha^n c^n}\right) + O\left(\frac{a_1^{s+1}}{\alpha^{n+s} c^n}\right)$
$u_n = 5u_{n-1} + 3u_{n-2} + u_{n-3} + u_{n-4}$	5.5760	3.02896×10^{-105}
$u_n = 5u_{n-1} + 3u_{n-2} + 2u_{n-3} + u_{n-4}$	5.6046	1.80125×10^{-105}

4. Conclusions

In this paper, we discuss the reciprocal sums of products of any mth-order linear recurrence sequences. We use the idea of "asymptotic equivalence" to consider the reciprocal infinite sums of products and the products interleaving terms of any mth-order linear recurrence sequences $\{u_n\}$. In addition, we generalize this conclusion to equidistant sub-sequences of $\{u_n\}$. Of course, as a special form of conclusion, we can obtain the reciprocal infinite sums of the products of any Fibonacci sequences, Pell sequences or k-Fibonacci sequences and other classical linear recursive sequences. Obviously, we have obtained the estimation formula of the reciprocal infinite sums of the products of any mth-order linear recursive sequences, and whether we can get more accurate results is an open problem.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

All authors declare no conflicts of interest in this paper.

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