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http://www.aimspress.com/journal/era

ERA, 31(9): 5716–5731.
DOI: 10.3934/era.2023290
Received: 11 April 2023
Revised: 28 June 2023
Accepted: 11 July 2023
Published: 17 August 2023

Research article

Resonant problems for non-local elliptic operators with unbounded nonlinearites

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Abstract: In this paper we study the existence of nontrivial solutions of a class of asymptotically resonant problems driven by a non-local integro-differential operator with homogeneous Dirichlet boundary conditions by applying Morse theory and critical groups for a C^2 functional at both isolated critical points and infinity.

Keywords: integro-differential operator; variational methods; resonance; critical group; Morse theory

1. Introduction

In this paper we will study the existence of nontrivial solutions of the following nonlocal elliptic problem

$$\begin{cases} -\mathcal{L}_K u = \lambda_\ell u + g(x,u) & x \in \Omega, \\ u = 0 & x \in \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ with a smooth boundary, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a differential function whose properties will be given later, λ_ℓ is an eigenvalue of $\mathcal{L}_K$ and $\mathcal{L}_K$ is a non-local elliptic operator formally defined as follows

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^N} (u(x+y) + u(x-y) - 2u(x))K(y)dy, \quad x \in \mathbb{R}^N, \tag{1.2}$$

where the kernel $K : \mathbb{R}^N \setminus \{0\} \to (0, \infty)$ is a function with the properties that

$$\begin{cases} mK \in L^1(\mathbb{R}^N) \text{ with } m(x) = \min\{|x|^2, 1\}, \text{ and there is } \theta > 0 \text{ such that} \\ K(x) \geqslant \theta |x|^{-(N+2s)} \text{ for any } x \in \mathbb{R}^N \setminus \{0\}, \text{ and } s \in (0,1) \text{ is fixed.} \end{cases} \tag{1.3}$$

The integro-differential operator $\mathcal{L}_K$ is a generalization of the fractional Laplacian $-(-\Delta)^s$ which is defined as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}} dy, \quad x \in \mathbb{R}^N. \tag{1.4}$$

When one takes the kernel $K(x) = |x|^{-(N+2s)}$ then $\mathcal{L}_K = -(-\Delta)^s$. In this case the problem (1.1) becomes

$$\begin{cases} (-\Delta)^s u = \lambda_\ell u + g(x,u) & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.5}$$

The problem (1.5) can be regarded as the counterpart of the semilinear elliptic boundary value problem

$$\begin{cases} -\Delta u = \lambda_\ell u + g(x,u) & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.6}$$

where λ_ℓ is an eigenvalue of $-\Delta$ with a 0-Dirichlet boundary value.

A weak solution for (1.1) is a function $u : \mathbb{R}^N \to \mathbb{R}$ such that

$$\begin{cases} \displaystyle\int_{\mathbb{R}^{2N}} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dxdy - \lambda_\ell \int_\Omega u(x)\varphi(x)dx \\ \displaystyle\qquad\qquad\qquad = \int_\Omega g(x,u(x))\varphi(x)dx \qquad \text{for all } \varphi \in X_0 \\ u \in X_0. \end{cases} \tag{1.7}$$

Here the linear space

$$X_0 = \{v \in X : \ v = 0 \ \text{ a.e. in } \mathbb{R}^N \setminus \Omega\},$$

and the functional space X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function v in X belongs to $L^2(\Omega)$ and

$$\text{the map } (x,y) \mapsto (v(x)-v(y))\sqrt{K(x-y)} \ \text{ is in } \ L^2(\mathbb{R}^{2N} \setminus (C\Omega \times C\Omega), dxdy),$$

where $C\Omega := \mathbb{R}^N \setminus \Omega$. The properties of the functional space X_0 will be introduced in the next section.

The non-local equations have been experiencing impressive applications in different subjects, such as the thin obstacle problem, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes and flame propagation, conservation laws, ultrarelativistic limits of quantum mechanics, quasigeostrophic flows, multiple scattering, minimal surfaces, materials science, water waves, elliptic problems with measured data, optimization, finance, etc. See [1] and the references therein. The non-local problems and operators have been widely studied in the literature and have attracted the attention of lot of mathematicians coming from different research areas due to the interesting analytical structure and broad applicability. Many mathematicians have applied variational methods [2] such as the mountain pass theorem [3], the saddle-point theorem [2] or other linking type of critical point theorem in the study of non-local equations with various nonlinearities that exhibit subcritical or critical growth; see [1,4–13] and references therein.

In the present paper we will apply the Morse theory to find weak solutions to (1.1). We assume, throughout the whole paper, that the nonlinear function $g \in C^1(\bar{\Omega}\times\mathbb{R},\mathbb{R})$ satisfies the following growth condition

(g) there is $C > 0$ and $p \in (2, \frac{2N}{N-2s})$ such that

$$|g_t'(x,t)| \leqslant C(1 + |t|^{p-2}) \quad \text{for all } (x,t) \in \bar{\Omega} \times \mathbb{R}. \tag{1.8}$$

We consider the situation that the problem (1.1) has the trivial solution $u \equiv 0$ and is resonant at infinity in the sense that the function g satisfies the following assumptions

$$g(x, 0) = 0 \quad \text{uniformly in } x \in \bar{\Omega}, \tag{1.9}$$

$$\lim_{|t|\to\infty} \frac{g(x,t)}{t} = 0 \quad \text{uniformaly in } x \in \bar{\Omega}. \tag{1.10}$$

We refer the reader to [12, Proposition 9 and Appendix A], [14, Propositions 2.3 and 2.4] and [11, Proposition 4] for the existence and basic properties of the eigenvalue of the linear non-local eigenvalue problem given by

$$\begin{cases} -\mathcal{L}_K u = \lambda u & x \in \Omega, \\ u = 0 & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.11}$$

that will be collected in the next section.

We make some further conditions on g.

(g_1) There are $c_1 > 0$ and $r \in (0, 1)$ such that

$$|g(x,t)| \leqslant c_1(|t|^r + 1) \quad \text{for all } t \in \mathbb{R},\ x \in \Omega.$$

$(g_2^\pm)$ There are $c_2 > 0$ and $r \in (0, 1)$ given in (g_1) such that

$$\pm g(x,t)t \geqslant 0, \quad \pm g(x,t)t \geqslant c_2(|t|^{1+r} - 1) \quad \text{for all } t \in \mathbb{R},\ x \in \Omega.$$

We note here that (g_1) implies (1.10) which characterizes the problem (1.1) as asymptotically linear resonant near infinity at the eigenvalue λ_ℓ of the non-local operator $-\mathcal{L}_K$. As an example, we can take the function $g(x,t) = \pm a(x)|t|^{r-1}t$ with $a \in L^\infty(\Omega)$, $\inf_\Omega a > 0$ and $r \in (0, 1)$.

We will prove the following theorems. We first consider the case that $g_t'(x, 0) + \lambda_\ell$ is not an eigenvalue of (1.11). We have the following conclusions.

Theorem 1.1. *Assume* (1.3), (1.9) *and* (g_1). *Then the problem* (1.1) *admits at least one nontrivial weak solution in each of the following cases:*

(i) (g_2^+), $\lambda_m < g_t'(x,0) + \lambda_\ell < \lambda_{m+1}$, $\lambda_m \neq \lambda_\ell$;

(ii) (g_2^-), $\lambda_m < g_t'(x,0) + \lambda_\ell < \lambda_{m+1}$, $\lambda_m \neq \lambda_{\ell-1} < \lambda_\ell$.

For the case that $g_t'(x,0) + \lambda_\ell = \lambda_m$, an eigenvalue of (1.11), i.e., the trivial solution $u = 0$ of (1.1), is degenerate. In this case the problem (1.1) is double resonant at both infinity and zero. We have the following conclusions.

Theorem 1.2. *Assume* (1.3), (1.9) *and* (g_1). *Then the problem* (1.1) *admits at least one nontrivial weak solution in each of the following cases:*

(i) (g_2^+), $g_t'(x,0) + \lambda_\ell = \lambda_m$, $\lambda_\ell < \lambda_{m-1} < \lambda_m$ *or* $\lambda_m < \lambda_\ell$;

(ii) (g_2^-), $g_t'(x,0) + \lambda_\ell = \lambda_m$, $\lambda_m < \lambda_{\ell-1} < \lambda_\ell$ *or* $\lambda_{\ell-1} < \lambda_{m-1}$.

Notice that in Theorem 1.2 there is a large difference between λ_ℓ and λ_m. This can be reduced by imposing on g some local sign conditions near zero. We denote $f(x,t) := \lambda_\ell t + g(x,t)$ and $F(x,t) = \int_0^t f(x,s)ds$. We assume the following

$(F_0^\pm)$ $f_t'(x,0) \equiv \lambda_m$ and there is $\delta > 0$ such that

$$\pm 2F_0(x,t) := \pm(2F(x,t) - \lambda_m t^2) \geqslant 0, \quad x \in \overline{\Omega}, \quad |t| \leqslant \delta.$$

Theorem 1.3. *Assume* (1.3)*,* (1.9) *and* (g_1)*. Then the problem* (1.1) *admits at least one nontrivial weak solution in each of the following cases:*

$$\text{(i)}\ (g_2^+),\ (F_0^+),\quad \lambda_\ell \neq \lambda_{m-1} < \lambda_m; \quad \text{(ii)}\ (g_2^-),\ (F_0^+),\quad \lambda_\ell \neq \lambda_m;$$

$$\text{(iii)}\ (g_2^+),\ (F_0^-),\quad \lambda_m < \lambda_{\ell-1} < \lambda_\ell; \quad \text{(iv)}\ (g_2^-),\ (F_0^-),\quad \lambda_{\ell-1} < \lambda_{m-1}.$$

We give some remarks and comparisons. The non-local equations with resonance at infinity have been studied in some recent works. In [6, 7], the famous saddle-point theorem [2] has been applied in the existence of solutions of the non-local problem related to (1.1) for the Landesman-Lazer resonance condition [15]. In [6], the authors treated a case in which one version of the Landesman-Lazer resonance condition [15] was formulated as follows:

$$\begin{cases} g(x,t) \text{ is bounded for all } (x,t) \in \bar{\Omega} \times \mathbb{R}; \\ G(x,t) = \int_0^t f(x,\varsigma)d\varsigma \to -\infty \text{ as } |t| \to \infty. \end{cases} \tag{1.12}$$

In [7], the authors treated an autonomous case in which another version of the Landesman-Lazer resonance condition [15] was formulated as follows:

$$\begin{cases} g \in C^1(\mathbb{R}),\ g_l := \lim\limits_{t\to-\infty} g(t) \in \mathbb{R},\ g_r := \lim\limits_{t\to+\infty} g(t) \in \mathbb{R} \text{ with } g_l > g_r; \\ g_r \int_\Omega \phi^- dx - g_l \int_\Omega \phi^+ dx < 0 < g_l \int_\Omega \phi^- dx - g_r \int_\Omega \phi^+ dx, \quad \forall\, \phi \in E(\lambda_\ell) \setminus \{0\}, \end{cases} \tag{1.13}$$

where $E(\lambda_\ell)$ is the linear space generated by the eigenfunctions corresponding to λ_ℓ. In [7], there is a crucial assumption that all functions in $E(\lambda_\ell)$ having a nodal set with the zero Lebesgue measure, which is valid for the fractional Laplacian $(-\Delta)^s$ (see [16]) and is still open for the general non-local elliptic operator $-\mathcal{L}_K$ (see [7, Equation (1.12)] and remarks therein).

We note here that the common feature in (1.12) and (1.13) is that the nonlinear term g is bounded. Motivated by previous works [6, 7], we treat, in the present paper, the completely resonant case via the application of Morse theory and critical groups. The results in this paper are new in two aspects. On one hand, the nonlinear term g is indeed unbounded and by imposing on g the global conditions (g_1) and $(g_2^\pm)$, we do not make the same assumption on the eigenfunctions of (1.11) as that in [7]. On the other hand, we explore a new application of the abstract results about critical groups at infinity that were built in [17] and modified in [18]. The conditions on g used here were first constructed in [19] for semilinear elliptic problems at resonance. Some of the above theorems may be regarded as the natural extension of local setting (1.6) to the non-local fractional setting.

We prove the main results via Morse theory [20, 21] and critical group computations. Precisely, we will work under the abstract framework built in [17] and modified in [18]. In Section 2, we collect some preliminaries about the variational formulas related to (1.1). In Section 3, we give the proofs of the main theorems including some technical lemmas.

2. Preliminaries

In this section we will give the preliminaries for the variational structure of (1.1) and preliminary results in Morse theory.

2.1. The functional setting

We first recall some basic results on the functional X_0 mentioned in Section 1. The functional space X_0 is non-empty because $C_0^2(\Omega) \subset X_0$ (see [22, Lemma 11]), and it is endowed with the norm defined as

$$\|v\|_{X_0} =: \left(\int_{\mathbb{R}^{2N}} |v(x) - v(y)|^2 K(x-y) dxdy \right)^{\frac{1}{2}}. \tag{2.1}$$

Furthermore, $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space with a scalar product (see [10, Lemmas 6 and 7]) defined by

$$\langle u, v\rangle_{X_0} = \int_{\mathbb{R}^{2N}} (u(x) - u(y))(v(x) - v(y))K(x-y)dxdy, \quad u, v \in X_0. \tag{2.2}$$

The norm (2.1) on X_0 is related to the so-called Gagliardo norm

$$\|v\|_{H^s(\Omega)} =: \|v\|_{L^2(\Omega)} + \left(\int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{\frac{1}{2}}$$

of the usual fractional Sobolev space $H^s(\Omega)$. For further details related to the fractional Sobolev spaces one can see [1, 10, 13] and the references therein.

By [10, Lemma 8] and [13, Lemma 9], we have following embedding results.

Proposition 2.1. *For each $q \in [1, \frac{2N}{N-2s}]$, the embedding $X_0 \hookrightarrow L^q(\mathbb{R}^N)$ is continuous and there is $C_q > 0$ such that*

$$\|u\|_{L^q(\mathbb{R}^N)} \leqslant C_q \|u\|_{X_0}, \quad \forall\, u \in X_0.$$

This embedding is compact whenever $q \in [1, \frac{2N}{N-2s})$.

2.2. An eigenvalue problem for $-\mathcal{L}_K$

Next, we recall some basic facts about the eigenvalue problem associated with the integro-differential operator $-\mathcal{L}_K$

$$\begin{cases} -\mathcal{L}_K u = \lambda u & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.3}$$

The number $\lambda \in \mathbb{R}$ is an eigenvalue of (2.3) if there is a nontrivial function $v : \mathbb{R}^N \to \mathbb{R}$ such that for all $\varphi \in X_0$

$$\begin{cases} \int_{\mathbb{R}^{2N}} (v(x) - v(y))(\varphi(x) - \varphi(y))K(x-y)dxdy = \int_\Omega v(x)\varphi(x)dx \\ v \in X_0. \end{cases}$$

We denote by $\{\lambda_k\}_{k\in\mathbb{N}}$ the sequence of the eigenvalue of the problem (2.3), with

$$0 < \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots \quad \text{and} \quad \lambda_k \to +\infty \;\text{ as }\; k \to +\infty. \tag{2.4}$$

We denote by ϕ_k the eigenfunction corresponding to λ_k. The sequence $\{\phi_k\}_{k\in\mathbb{N}}$ can be normalized in such a way that the sequence provides an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of X_0. By [14, Proposition 2.4] one has that all $\phi_k \in L^\infty(\Omega)$. One can refer to [12, Proposition 9 and Appendix A], [14, Proposition 2.3] and [11, Proposition 4] for a complete study of the spectrum of the integro-differential operator $-\mathcal{L}_K$.

The first eigenvalue λ_1 is simple and can be characterized as

$$\lambda_1 = \min_{u\in X_0, \|u\|_{L^2(\Omega)}=1} \int_{\mathbb{R}^{2N}} |u(x)-u(y)|^2 K(x-y)dxdy.$$

Each eigenvalue λ_k, $k \geqslant 2$, has finite multiplicity. More precisely, we say that λ_k has the finite multiplicity $\nu_k \in \mathbb{N}$ if

$$\lambda_{k-1} < \lambda_k = \lambda_{k+1} \cdots = \lambda_{k+\nu_k-1} < \lambda_{k+\nu_k}. \tag{2.5}$$

The set of all of the eigenfunctions corresponding to λ_k agrees with

$$E(\lambda_k) := \operatorname{span}\{\phi_k, \phi_{k+1}, \cdots, \phi_{k+\nu_k-1}\}, \ \ \dim E(\lambda_k) = \nu_k.$$

The eigenvalue λ_1 is achieved at a positive function ϕ_1 with $\|\phi_1\|_{L^2(\Omega)} = 1$. For each $k \geqslant 2$, the eigenvalue λ_k can be characterized as follows:

$$\lambda_k = \min_{u\in \mathbb{P}_k, \|u\|_{L^2(\Omega)}=1} \int_{\mathbb{R}^{2N}} |u(x)-u(y)|^2 K(x-y)dxdy, \tag{2.6}$$

where

$$\mathbb{P}_k := \{u \in X_0 : \langle u, \phi_j\rangle_{X_0} = 0 \ \text{ for all } \ j = 1, 2, \cdots, k-1\}.$$

Corresponding to the eigenvalue λ_k of $-\mathcal{L}_K$ with multiplicity ν_k, the space X_0 can be split as follows:

$$X_0 = W_k^- \oplus V_k \oplus W_k^+ = V_k \oplus W_k, \quad W_k = W_k^- \oplus W_k^+,$$

where

$$W_k^- = \bigoplus_{\lambda_j<\lambda_k} E(\lambda_j), \ \ V_k = E(\lambda_k), \ \ W_k^+ = (W_k^- \oplus V_k)^\perp = \overline{\bigoplus_{\lambda_j>\lambda_k} E(\lambda_j)}.$$

For each eigenvalue λ_k, we can define a linear operator $\mathcal{A}_k : X_0 \to X_0^*$ by

$$\langle \mathcal{A}_k u, v\rangle := \int_{\mathbb{R}^{2N}} (u(x)-u(y))(v(x)-v(y))K(x-y)dxdy - \lambda_k \int_\Omega uv dx. \tag{2.7}$$

By the continuous embedding from X_0 into $L^2(\Omega)$ in Proposition 2.1, one can deduce that $\mathcal{A}_k$ is a bounded self-adjoint linear operator so that $\langle \mathcal{A}_k\phi, \phi\rangle = 0$ for all $\phi \in V_k =: \ker(\mathcal{A}_k)$.

Finally, we conclude this subsection with the following variational inequalities which can be deduced by the variational characterization of the eigenvalues and the standard Fourier decomposition:

$$\langle \mathcal{A}_k u, u\rangle \leqslant \left(1 - \frac{\lambda_k}{\lambda_{k-1}}\right)\|u\|_{X_0}^2, \quad \forall\, u \in W_k^-, \tag{2.8}$$

$$\langle \mathcal{A}_k v, v\rangle \geqslant \left(1 - \frac{\lambda_k}{\lambda_{k+\nu_k}}\right)\|v\|_{X_0}^2, \quad \forall\, v \in W_k^+. \tag{2.9}$$

3. Proofs of main results

In this section we give the proofs of the main results in this paper via some abstract results on Morse theory [20, 21] for a C^2 functional $\mathcal{J}$ defined on a Hilbert space. These results come from [17, 18, 20, 21, 23–25], etc. We refer the readers to [26] for a brief summary of the concepts, definitions and the abstract results about critical groups and Morse theory.

First of all, we observe that the problem (1.1) has a variational structure; indeed, it is the Euler-Lagrange equation of the functional $\mathcal{J} : X_0 \to \mathbb{R}$ defined as

$$\mathcal{J}(u) = \frac{1}{2}\int_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 K(x-y)dxdy - \frac{1}{2}\lambda_\ell \int_\Omega |u|^2 dx - \int_\Omega G(x,u)dx,\ u \in X_0, \tag{3.1}$$

where $G(x,t) = \int_0^t g(x,\varsigma)d\varsigma$. Since the nonlinear function g satisfies the assumption (g), by Proposition 2.1, the functional $\mathcal{J}$ is well defined on X_0 and is of class C^2 (see a detailed proof in [26]) with the derivatives given by

$$\begin{aligned}\langle \mathcal{J}'(u), v\rangle = &\int_{\mathbb{R}^{2N}} (u(x)-u(y))(v(x)-v(y))K(x-y)dxdy \\ &-\lambda_\ell \int_\Omega uvdx - \int_\Omega g(x,u)vdx, \quad \forall\, u, v \in X_0,\end{aligned} \tag{3.2}$$

$$\begin{aligned}\langle \mathcal{J}''(u)v, w\rangle = &\int_{\mathbb{R}^{2N}} (v(x)-v(y))(w(x)-w(y))K(x-y)dxdy \\ &-\lambda_\ell \int_\Omega vwdx - \int_\Omega g_t'(x,u)vwdx, \quad \forall\, u, v, w \in X_0.\end{aligned} \tag{3.3}$$

From (3.2) and (1.7), one sees that critical points of $\mathcal{J}$ are exactly weak solutions to (1.1).

Define the functional $\mathcal{F} : X_0 \to \mathbb{R}$ by

$$\mathcal{F}(u) = -\int_\Omega G(x,u(x))dx, \quad u \in X_0. \tag{3.4}$$

According to (2.7) with λ_ℓ, the functional $\mathcal{J}$ can be written as

$$\mathcal{J}(u) = \frac{1}{2}\langle \mathcal{A}_\ell u, u\rangle + \mathcal{F}(u), \quad u \in X_0. \tag{3.5}$$

Using the assumption (g_1) we can deduce that

$$\|\mathcal{F}'(u)\| = o(\|u\|_{X_0}) \quad \text{as} \quad \|u\|_{X_0} \to \infty. \tag{3.6}$$

Therefore $\mathcal{J}$ fits the basic assumptions in the abstract framework required by [26, Proposition 2.5] with respect to $X_0 = V_\ell \oplus W_\ell$.

Next we prove one technical lemma that will be used to verify the angle conditions required by [26, Proposition 2.5] for computation of the critical groups at infinity.

Lemma 3.1. *Assume (g_1) and $(g_2^\pm)$. Then there exist $M > 0$, $\epsilon \in (0,1)$ and $\beta > 0$ such that*

$$\pm\int_\Omega g(x,u)vdx \geqslant \beta\|v\|_{X_0}^{1+r}. \tag{3.7}$$

for any $u = v + w \in X_0 = V_\ell \oplus W_\ell$ with $\|u\|_{X_0} \geqslant M$ and $\|w\|_{X_0} \leqslant \epsilon\|u\|_{X_0}$.

Proof. We give the proof for the case that (g_1) and (g_2^+) hold.

For $u = v + w \in X_0 = V_\ell \oplus W_\ell$, we set

$$C(M, \epsilon) = \{u = v + w : \ \|u\|_{X_0} \geqslant M, \ \|w\|_{X_0} \leqslant \epsilon \|u\|_{X_0}\},$$

where $M > 0$ and $\epsilon \in (0, 1)$ will be chosen below.

For $u \in C(M, \epsilon)$, we have

$$|v\|_{X_0} \geqslant \sqrt{1-\epsilon^2}\, \|u\|_{X_0}, \quad \|w\|_{X_0} \leqslant \frac{\epsilon}{\sqrt{1-\epsilon^2}} \|v\|_{X_0}. \tag{3.8}$$

It follows from (g_1) and (g_2^+) that

$$\begin{aligned} \int_\Omega g(x,u) v dx = & \int_\Omega g(x,u) u dx - \int_\Omega g(x,u) w dx \\ \geqslant & \int_\Omega g(x,u) u dx - \int_\Omega |g(x,u)||w| dx \\ \geqslant & \ c_2 \int_\Omega (|u|^{1+r} - 1) dx - c_1 \int_\Omega (|u|^r + 1)|w| dx. \\ \geqslant & \ c_2 \int_\Omega |u|^{1+r} dx - c_1 \int_\Omega |u|^r |w| dx - c_1 C_1 \|w\|_{X_0} - c_2 |\Omega|. \end{aligned} \tag{3.9}$$

By Proposition 2.1 and the Hölder inequality we have

$$\begin{aligned} \int_\Omega |u|^r |w| dx \leqslant & \left(\int_\Omega |u|^{1+r} \right)^{\frac{r}{1+r}} \left(\int_\Omega |w|^{1+r} \right)^{\frac{1}{1+r}} \\ \leqslant & \ C_{1+r}^{1+r} \|u\|_{X_0}^r \|w\|_{X_0} \\ \leqslant & \ C_{1+r}^{1+r} \epsilon \|u\|_{X_0}^{1+r}. \end{aligned} \tag{3.10}$$

Since V_ℓ is finite dimensional, by the elementary inequality $|a + b|^q \leqslant 2^{q-1}(|a|^q + |b|^q)$ for all $a, b \in \mathbb{R}$, we have that

$$\begin{aligned} \int_\Omega |u|^{1+r} dx = & \int_\Omega |v + w|^{1+r} dx \\ \geqslant & \ \frac{1}{2^r} \int_\Omega |v|^{1+r} dx - \int_\Omega |w|^{1+r} dx \\ \geqslant & \ \frac{1}{2^r} \|v\|_{L^{1+r}(\Omega)}^{1+r} - C_{1+r}^{1+r} \|w\|_{X_0}^{1+r} \\ \geqslant & \ \frac{1}{2^r} \hat{c}^{1+r} (1-\epsilon^2)^{\frac{1+r}{2}} \|u\|_{X_0}^{1+r} - C_{1+r}^{1+r} \epsilon^{1+r} \|u\|_{X_0}^{1+r}. \end{aligned} \tag{3.11}$$

here $\hat{c}$ is the embedding constant of $L^{1+r}(\Omega) \hookrightarrow V_\ell$. Therefore for $u \in C(M, \epsilon)$, it follows from (3.9)–

(3.11) that

$$
\begin{aligned}
&\int_\Omega g(x,u)v dx \\
\geqslant\ & \left(\frac{1}{2^r}c_2\hat{c}^{1+r}(1-\epsilon^2)^{\frac{1+r}{2}} - c_2C_{1+r}^{1+r}\epsilon^{1+r} - c_1C_{1+r}^{1+r}\epsilon\right)\|u\|_{X_0}^{1+r} \\
& -c_1C_1\epsilon\|u\|_{X_0} - c_2|\Omega| \qquad (3.12)\\
\geqslant\ & \left(\frac{c_2}{2^r}\hat{c}^{1+r}(1-\epsilon^2)^{\frac{1+r}{2}} - c_2C_{1+r}^{1+r}\epsilon^{1+r} - c_1C_{1+r}^{1+r}\epsilon - \frac{c_1C_1\epsilon}{M^r} - \frac{c_2|\Omega|}{M^{1+r}}\right)\|u\|_{X_0}^{1+r} \\
=:\ & \beta\|u\|_{X_0}^{1+r}.
\end{aligned}
$$

Now we can take $M > 0$ large enough and $0 < \epsilon < 1$ small enough so that

$$\beta = \frac{c_2}{2^r}\hat{c}^{1+r}(1-\epsilon^2)^{\frac{1+r}{2}} - c_2C_{1+r}^{1+r}\epsilon^{1+r} - c_1C_{1+r}^{1+r}\epsilon - \frac{c_1C_1\epsilon}{M} - \frac{c_2|\Omega|}{M^{1+r}} > 0;$$

hence,

$$\int_\Omega g(x,u)v dx \geqslant \beta\|u\|_{X_0}^{1+r} \geqslant \beta\|v\|_{X_0}^{1+r} \text{ for } u \in C(M,\epsilon). \tag{3.13}$$

The proof is complete. □

In order to apply [26, Proposition 2.5] and Morse theory to prove our results, we have to verify that $\mathcal{J}$ satisfies the Palais-Smale condition.

Lemma 3.2. *Assume* (g_1) *and* $(g_2^\pm)$*. Then the functional* $\mathcal{J}$ *defined by* (3.1) *satisfies the Palais-Smale condition.*

Proof. Let the sequence $\{u_n\} \subset X_0$ be such that

$$\mathcal{J}'(u_n) \to 0, \quad n \to \infty. \tag{3.14}$$

We show that $\{u_n\}$ is bounded in X_0. Suppose, by the way of contradiction, that

$$\|u_n\|_{X_0} \to \infty \text{ as } n \to \infty. \tag{3.15}$$

Write $u_n = v_n + w_n$, where $v_n \in V_\ell$ and $w_n \in W_\ell$. By the variational inequalities (2.8) and (2.9), we have

$$|\langle \mathcal{A}_\ell w_n, w_n\rangle| \geqslant \sigma\|w_n\|_{X_0}^2, \quad \forall\, n \in \mathbb{N}, \tag{3.16}$$

where

$$\sigma = \min\left\{1 - \frac{\lambda_\ell}{\lambda_{\ell+\nu_\ell}},\ \frac{\lambda_\ell}{\lambda_{\ell-1}} - 1\right\}.$$

By (3.11), there is $N_1 \in \mathbb{N}$ such that

$$|\langle \mathcal{J}'(u_n), w_n\rangle| \leqslant \|w_n\|_{X_0}, \quad \forall\, n \geqslant N_1. \tag{3.17}$$

By (3.2) and (3.5), we have

$$\langle \mathcal{J}'(u_n), w_n\rangle = \langle \mathcal{A}_\ell w_n, w_n\rangle_{X_0} + \langle \mathcal{F}'(u_n), w_n\rangle. \tag{3.18}$$

By (3.6) and (3.15) we have

$$\|\mathcal{F}'(u_n)\| = o(\|u_n\|_{X_0}), \quad n \to \infty. \tag{3.19}$$

It follows that, for any given $\delta > 0$ sufficiently small and all n sufficiently large,

$$\begin{aligned} \sigma\|w_n\|_{X_0}^2 &\leqslant |\langle \mathcal{A}_\ell w_n, w_n\rangle_{X_0}| \\ &\leqslant |\langle \mathcal{J}'(u_n), w_n\rangle| + |\langle \mathcal{F}'(u_n), w_n\rangle| \\ &\leqslant \|w_n\|_{X_0} + \delta\|u_n\|_{X_0}\|w_n\|_{X_0}. \end{aligned} \tag{3.20}$$

Since $\delta > 0$ was chosen arbitrarily, from (3.15) and (3.20) we deduce that

$$\frac{\|w_n\|_{X_0}}{\|u_n\|_{X_0}} \to 0 \ \text{ as } \ n \to \infty. \tag{3.21}$$

It follows that there is $N_2 \in \mathbb{N}$ with $N_2 \geqslant N_1$ such that

$$\|u_n\|_{X_0} \geqslant M \ \text{ and } \ \|w_n\|_{X_0} \leqslant \epsilon\|u_n\|_{X_0} \ \text{ for } \ n \geqslant N_2, \tag{3.22}$$

where $M > 0$ and $\epsilon \in (0, 1)$ was given in Lemma 3.1. Therefore by Lemma 3.1 we have that

$$\pm \int_\Omega g(x, u_n)\frac{v_n}{\|v_n\|_{X_0}}dx \geqslant \beta\|v_n\|_{X_0}^r \geqslant \beta(1-\epsilon^2)^{\frac{r}{2}}M^r > 0 \ \text{ for } \ n \geqslant N_2. \tag{3.23}$$

On the other hand, by (3.14), we get

$$\lim_{n\to\infty}\left|\int_\Omega g(x, u_n)\frac{v_n}{\|v_n\|_{X_0}}dx\right| = \lim_{n\to\infty}\left|\left\langle \mathcal{J}'(u_n), \frac{v_n}{\|v_n\|_{X_0}}\right\rangle\right| = 0. \tag{3.24}$$

This contradicts (3.23). Hence $\{u_n\}$ is bounded in X_0.

Since X_0 is a Hilbert space, there is a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and there exists $u^* \in X_0$, such that

$$u_n \rightharpoonup u^* \ \ \text{weakly in } X_0 \ \text{ as } \ n \to \infty. \tag{3.25}$$

By Proposition 2.1, up to a subsequence, it holds that

$$\begin{aligned} &u_n \to u^* && \text{in } L^q(\mathbb{R}^N) \ \ \forall\, q \in [1, \tfrac{2N}{N-2s}), \\ &u_n(x) \to u^*(x) && \text{a.e. in } \mathbb{R}^N. \end{aligned} \tag{3.26}$$

as $n \to \infty$. By (g_1), Proposition 2.1 and (3.26) we get

$$\begin{aligned} &\left|\int_\Omega (g(x, u_n) - g(x, u^*))(u_n - u^*)dx\right| \\ \leqslant\ & 2c_1\|u_n - u^*\|_{L^1(\Omega)} + c_1C_{1+r}^r(\|u^*\|_{X_0}^r + \|u_n\|_{X_0}^r)\|u_n - u^*\|_{L^{1+r}(\Omega)} \\ \to\ & 0 \ \text{ as } \ n \to \infty. \end{aligned} \tag{3.27}$$

From (3.14) we deduce that

$$\begin{aligned}\langle \mathcal{J}'(u_n), u_n\rangle = & \int_{\mathbb{R}^{2N}} |u_n(x) - u_n(y)|^2 K(|x-y|)dxdy \\ & -\lambda_\ell \int_\Omega |u_n|^2 dx - \int_\Omega g(x, u_n(x))u_n(x)dx \to 0\end{aligned} \tag{3.28}$$

as $n \to \infty$, and

$$\begin{aligned}\langle \mathcal{J}'(u_n), u^*\rangle = & \int_{\mathbb{R}^{2N}} (u_n(x) - u_n(y))(u^*(x) - u^*(y))K(|x-y|)dxdy \\ & -\lambda_\ell \int_\Omega u_n u^* dx - \int_\Omega g(x, u_n(x))u^*(x)dx \to 0\end{aligned} \tag{3.29}$$

as $n \to \infty$. Now by (3.14) and (3.26)–(3.29), we deduce from

$$\langle \mathcal{J}'(u_n) - \mathcal{J}'(u^*), u_n - u^*\rangle \to 0 \quad n \to \infty$$

that

$$\|u_n - u^*\|_{X_0}^2 \to 0, \quad n \to \infty.$$

This completes the proof for verifying the Palais-Smale condition. □

Notice here that only (3.14) is used for verifying the Palais-Smale condition, it follows that the critical point set of $\mathcal{J}$ is compact and is then bounded.

Now we are ready to give the proofs of the main results in this paper.

Proof of Theorem 1.1. We give the proof of the case (i). Since

$$\langle \mathcal{J}'(u), v\rangle = -\int_\Omega g(x, u)vdx, \quad \forall\ v \in V_\ell,$$

it follows from Lemma 3.1 that $\mathcal{J}$ satisfies the angle condition $(\mathcal{AC}_\infty^-)$ in [26, Proposition 2.5] at infinity with respect to $X_0 = V_\ell \oplus W_\ell$. Thus by [26, Proposition 2.5(ii)] we have

$$C_q(\mathcal{J}, \infty) \cong \delta_{q,\mu_\ell+\nu_\ell}\mathbb{Z}, \ q \in \mathbb{Z}, \tag{3.30}$$

where

$$\mu_\ell = \dim \bigoplus_{\lambda_k<\lambda_\ell} \ker E(\lambda_k), \qquad \nu_\ell = \dim E(\lambda_\ell).$$

Therefore $\mathcal{J}$ has a critical point u_* satisfying

$$C_{\mu_\ell+\nu_\ell}(\mathcal{J}, u_*) \ncong 0. \tag{3.31}$$

The second derivative of $\mathcal{J}$ at the trivial solution $u = 0$ can be written as

$$\langle \mathcal{J}''(0)\phi, \phi\rangle = \|\phi\|_{X_0}^2 - \int_\Omega (\lambda_\ell + g_t'(x, 0))\phi^2 dx, \quad \phi \in X_0. \tag{3.32}$$

By the condition we see that $u = 0$ is a nondegenerate critical point of $\mathcal{J}$ with the Morse index

$$\bar{\mu}_0 = \dim \bigoplus_{\lambda_k \leqslant \lambda_m} \ker E(\lambda_k). \tag{3.33}$$

Hence

$$C_q(\mathcal{J},0) \cong \delta_{q,\bar{\mu}_0}\mathbb{Z}. \tag{3.34}$$

Since $\lambda_m \neq \lambda_\ell$, we get that $\mu_\ell + \nu_\ell \neq \bar{\mu}_0$, and we see from (3.33) and (3.34) that $u_* \neq 0$. The case (ii) can be proved in the same way. The proof is complete. □

Proof of Theorem 1.2 We give the proof of the case (ii). It follows from Lemma 3.1 that $\mathcal{J}$ satisfies the angle condition $(\mathcal{AC}_\infty^+)$ in [26, Proposition 2.5] at infinity with respect to $X_0 = V_\ell \oplus W_\ell$. Thus by [26, Proposition 2.5(ii)] we have

$$C_q(\mathcal{J},\infty) \cong \delta_{q,\mu_\ell}\mathbb{Z},\ q \in \mathbb{Z}, \tag{3.35}$$

and $\mathcal{J}$ has a critical point u_* satisfying

$$C_{\mu_\ell}(\mathcal{J},u_*) \not\cong 0. \tag{3.36}$$

Now $\mathcal{J}''(0)$ takes the form

$$\langle \mathcal{J}''(0)\phi,\phi\rangle = \|\phi\|_{X_0}^2 - \lambda_m \int_\Omega \phi^2 dx, \quad \phi \in X_0. \tag{3.37}$$

It follows that 0 is a degenerate critical point of $\mathcal{J}$ with the Morse index μ_0 and the nullity ν_0 given by

$$\mu_0 = \dim \bigoplus_{\lambda_k \leqslant \lambda_{m-1}} \ker E(\lambda_k), \quad \nu_0 = \dim E(\lambda_m). \tag{3.38}$$

By the Gromoll-Meyer result [27], we have that

$$C_q(\mathcal{J},0) \cong 0, \ \text{ for } q \notin [\mu_0, \mu_0+\nu_0]. \tag{3.39}$$

It follows from $\lambda_m < \lambda_{\ell-1} < \lambda_\ell$ or $\lambda_{\ell-1} < \lambda_{m-1}$ that $\mu_0 + \nu_0 < \mu_\ell$ or $\mu_0 > \mu_\ell$, and we see from (3.36) and (3.39) that $u_* \neq 0$. The case (i) can be proved in the same way. The proof is complete. □

Lemma 3.3. *Assume* (1.3), (1.9), (g_1) *and* $(F_0^\pm)$. *Then*

(i) $C_q(\mathcal{J},0) \cong \delta_{q,\mu_0+\nu_0}\mathbb{Z}$ *for* (F_0^+) *holds,*
(ii) $C_q(\mathcal{J},0) \cong \delta_{q,\mu_0}\mathbb{Z}$ *for* (F_0^-) *holds,*

where μ_0 *and* ν_0 *are given by* (3.38).

Proof. We will apply [24, Proposition 2.3] to prove the results. We first note that by (g_1) and the last part in the proof of Lemma 3.2, the functional $\mathcal{J}$ verifies the bounded Palais-Smale condition which ensures the deformation property for computing $C_q(\mathcal{J},0)$ (see [20,21]).

We treat the case (ii) for which (F_0^-) holds. We will prove that $\mathcal{J}$ has the local linking structure at 0 as with respect to $X_0 = E^- \oplus E^+$, where $E^- = W_m^-$ and $E^+ = V_m \oplus W_m^+$. We refer the readers to [28,29] for the concept of the local linking.

1) Take $u \in E^- = W_m^-$. Since W_m^- is finite dimensional, there is $\rho > 0$ such that

$$\|u\|_{X_0} \leqslant \rho \ \Rightarrow\ |u(x)| \leqslant \delta, \ \text{ a.e. } x \in \Omega.$$

Consequently, thanks to (2.8) with λ_m and (F_0^-), for any $u \in E^-$ with $\|u\| \leqslant \rho$, we get

$$\mathcal{J}(u) \leqslant \frac{1}{2}\Big(1 - \frac{\lambda_m}{\lambda_{m-1}}\Big)\|u\|^2_{X_0} - \int_\Omega F_0(x,u)dx \leqslant \frac{1}{4}\Big(1 - \frac{\lambda_m}{\lambda_{m-1}}\Big)\|u\|^2_{X_0} \leqslant 0. \tag{3.40}$$

2) For $u \in E^+ = V_m \oplus W_m^+$, we write $u = w + z$, where $w \in W_m^+$ and $z \in V_m$. Then

$$\mathcal{J}(u) \geqslant \frac{1}{2}\Big(1 - \frac{\lambda_m}{\lambda_{m+\nu_m}}\Big)\|w\|^2_{X_0} - \int_\Omega F_0(x,u)dx. \tag{3.41}$$

Since V_m is finite dimensional, there is $\rho > 0$ such that

$$\|z\| \leqslant \|u\| \leqslant \rho \ \Rightarrow \ |z(x)| < \frac{1}{3}\delta, \ \text{ a.e. } x \in \Omega.$$

Consequently,

$$|u(x)| > \delta \ \Rightarrow \ |w(x)| = |u(x) - z(x)| \geqslant |u(x)| - |z(x)| > \frac{2}{3}|u(x)|.$$

By (F_0^-), we have

$$\int_{\{|u(x)|\leqslant\delta\}} F_0(x,u)dx \leqslant 0. \tag{3.42}$$

By (g_1), we get that, for each given $\sigma \in (2, \frac{2N}{N-2s}]$, there is $\kappa = \kappa(\sigma,\delta) > 0$ such that

$$|F_0(x,t)| \leqslant \kappa|u|^\sigma, \quad x \in \Omega, \ \ |t| > \delta. \tag{3.43}$$

Hence

$$\int_{\{|u(x)|>\delta\}} F_0(x,u)dx \leqslant \kappa\int_{\{|u(x)|>\delta\}} |u|^\sigma dx \leqslant \kappa(3/2)^\sigma \int_\Omega |w|^\sigma dx \leqslant C(\sigma,\delta)\|w\|^\sigma_{X_0}. \tag{3.44}$$

Now, by (3.41), (3.42) and (3.44) we get

$$\mathcal{J}(u) \geqslant \frac{1}{2}\Big(1 - \frac{\lambda_m}{\lambda_{m+\nu_m}}\Big)\|w\|^2_{X_0} - \int_{\{|u(x)|\leqslant\delta\}} F_0(x,u)dx - C(\kappa,\sigma)\|w\|^\sigma_{X_0}. \tag{3.45}$$

Since $\sigma > 2$, one sees from (3.42) and (3.45) that for $\rho > 0$ small enough once again, it holds that

$$\Phi(u) > 0, \qquad \forall \ \|u\| \leqslant \rho \ \text{ with } w \neq 0. \tag{3.46}$$

For $z \in V_m$ with $\|z\| \leqslant \rho$, we have by (F_0^-) that

$$2F_0(x,z(x)) = 2F(x,z(x)) - \lambda_m z(x)^2 \leqslant 0, \quad \forall \text{ a.e. } x \in \Omega.$$

Thus for all $z \in B_\rho \cap V_m$,

$$\mathcal{J}(z) = -\frac{1}{2}\int_\Omega \Big(2F(x,z(x)) - \lambda_m z(x)^2\Big)dx \geqslant 0. \tag{3.47}$$

To apply [24, Proposition 2.3], we need to show that the above inequality holds strictly for $z \neq 0$. Assume, for contradiction, that for any $0 < \epsilon \leqslant \rho$, there is $z_\epsilon \in V_m$ such that $0 < \|z_\epsilon\| < \epsilon$ and $\mathcal{J}(z_\epsilon) = 0$. Then, the following holds:

$$2F(x,z_\epsilon(x)) = \lambda_m z_\epsilon(x)^2, \quad \text{ a.e. } x \in \Omega,$$

and then

$$f(x, z_\epsilon(x)) = \lambda_m z_\epsilon(x), \quad \text{a.e. } x \in \Omega.$$

Given that $z_\epsilon \in V_m$, going back to (1.1), we see that z_ϵ is a nontrivial solution of (1.1). This contradicts the conventional assumption that 0 is an isolated solution of (1.1). In summary, we obtain by (3.46) and (3.47) that

$$\mathcal{J}(u) > 0, \qquad \forall\, 0 < \|u\| \leqslant \rho,\ u \in E^+.$$

Therefore, $\mathcal{J}$ has a local linking structure with respect to $E = E^- \oplus E^+$ with $\mu_0 = \dim E^-$. It follows from [24, Proposition 2.3] that $C_q(\mathcal{J}, 0) \cong \delta_{q,\mu_0}\mathbb{Z}$.

The case (i) is proved in a similar and simpler way. The proof is complete. □

Proof of Theorem 1.3. We give the proof of the case (iv). As in the proof of Theorem 1.1(ii), we have gotten the following conclusion that $\mathcal{J}$ satisfies the angle condition $(\mathcal{AC}_\infty^+)$ in [26, Proposition 2.5] at infinity with respect to $X_0 = V_\ell \oplus W_\ell$, and then that $\mathcal{J}$ has a critical point u_* satisfying

$$C_{\mu_\ell}(\mathcal{J}, u_*) \ncong 0. \tag{3.48}$$

By (F_0^-) and Lemma 3.3, J has a local linking at 0 with respect to $X_0 = E^- \oplus E^+$. Thus it follows from [24, Proposition 2.3] that

$$C_q(\mathcal{J}, 0) \cong \delta_{q,\mu_0}\mathbb{Z}. \tag{3.49}$$

By $\lambda_{\ell-1} < \lambda_{m-1}$, we have that $\lambda_\ell < \mu_0$. It follows from (3.48) and (3.49) that $u_* \neq 0$. The other cases can be proved in the same way. The proof is complete. □

Remark 3.4. We conclude the paper with some remarks.

1) In Theorem 1.3, the result for one nontrivial solution is valid for f that is locally Lipschitz continuous with $f'(x, 0) \equiv \lambda_m$ being replaced by satisfying $\lim_{t\to 0} \frac{f(x,t)}{t} = \lambda_m$. In this case, we have only $\mathcal{J} \in C^{2-0}(X_0, \mathbb{R})$ and no Morse index is involved. We have the critical groups at zero by applying the local linking theorem in [23] as follows:

$$C_{\mu_0+\nu_0}(\mathcal{J}, 0) \neq 0 \ \text{ for } \ (F_0^+) \ \text{ holds}; \quad C_{\mu_0}(\mathcal{J}, 0) \neq 0 \ \text{ for } \ (F_0^-) \ \text{ holds}.$$

2) In the case that $\lambda_\ell = \lambda_1$ and (g_2^-) holds, we have that $\mu_\ell = 0$ and

$$C_q(\mathcal{J}, \infty) \cong \delta_{q,0}\mathbb{Z}. \tag{3.50}$$

Thus $\mathcal{J}$ has a critical point u^* with

$$C_0(\mathcal{J}, u^*) \ncong 0. \tag{3.51}$$

It follows that

$$C_q(\mathcal{J}, u^*) \cong \delta_{q,0}\mathbb{Z}. \tag{3.52}$$

Indeed, (3.50) is equivalent to $\mathcal{J}$ being bounded from below and (3.51) is equivalent to u^* being a local minimizer of $\mathcal{J}$. Furthermore, in the case that $C_q(\mathcal{J}, 0) \neq 0$ for some $q \geqslant 1$, we can apply [30, Theorem 2.1], i.e., the most general version of the three critical point theorem, to get two nontrivial solutions of (1.1).

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

Acknowledgments

The authors appreciate the reviewers for carefully reading the manuscript and giving valuable comments to improve the exposition of the paper. This work was supported by the NSFC (12001382,12271373,12171326).

Conflict of interest

The authors declare there is no conflicts of interest.

References

1. E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.*, **136** (2012), 521–573. https://doi.org/10.1016/j.bulsci.2011.12.004
2. P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, American Mathematical Society, Providence, RI 1986.
3. A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.,* **14** (1973), 349–381. https://doi.org/10.1016/0022-1236(73)90051-7
4. G. M. Bisci, R. Servadei, A Brezis-Nirenberg splitting approach for nonlocal fractional equations, *Nonlinear Anal. Theory Methods Appl.*, **119** (2015), 341–353. https://doi.org/10.1016/j.na.2014.10.025
5. G. M. Bisci, D. Mugnai, R. Servadei, On multiple solutions for nonlocal fractional problems via ∇-theorems, *Differ. Integr. Equations*, **30** (2017), 641–666.
6. A. Fiscella, R. Servadei, E. Valdinoci, A resonance problem for non-Local elliptic operators, *Zeitschrift für Analysis und ihre Anwendungen*, **32** (2013), 411–431. https://doi.org/10.4171/ZAA/1492
7. A. Fiscella, R. Servadei, E. Valdinoci, Asymptotically linear problems driven by fractionl operators, *Math. Methods Appl. Sci.*, **38** (2015), 3551–3563. https://doi.org/10.1002/mma.3438
8. D. Mugnai, D. Pagliardini, Existence and multiplicity results for the fractional Laplacian in bounded domains, *Adv. Calc. Var.*, **10** (2017), 111–124. https://doi.org/10.1515/acv-2015-0032
9. R. Servadei, A critical fractional Laplace equation in the resonant case, *Topol. Methods Nonlinear Anal.*, **43** (2014), 251–267.
10. R. Servadei, E. Valdinoci, Mountain pass solutions for non-local elliptic operators, *J. Math. Anal. Appl.*, **389** (2012), 887–898. https://doi.org/10.1016/j.jmaa.2011.12.032
11. R. Servadei, E. Valdinoci, A Brezis-Nirenberg result for non-local critical equations in low dimension, *Commun. Pure Appl. Anal.*, **12** (2013), 2445–2464. https://doi.org/10.3934/cpaa.2013.12.2445
12. R. Servadei, E. Valdinoci, Variational methods for non-local operators of elliptic type, *Discrete Contin. Dyn. Syst.*, **33** (2013), 2105–2137. https://doi.org/10.3934/10.3934/dcds.2013.33.2105
13. R. Servadei, E. Valdinoci, The Brézis-Nirenberg result for the fractional Laplacian, *Trans. Am. Math. Soc.*, **367** (2015), 67–102. https://doi.org/10.1090/S0002-9947-2014-05884-4

14. R. Servadei, The Yamabe equation in a non-local setting, *Adv. Nonlinear Anal.*, **2** (2013), 235–270. https://doi.org/10.1515/anona-2013-0008
15. E. M. Landesman, A. C. Lazer, Nonlinear perturbations of linear elliptic boundary value problems at resonance, *J. Math. Mech.*, **19** (1970), 609–623.
16. M. M. Fall, V. Felli, Unique continuation property and local asymptotics of solutions to fractional elliptic equations, *Commun. Partial Differ. Equations*, **39** (2014), 354–397. https://doi.org/10.1080/03605302.2013.825918
17. T. Bartsch, S. Li, Critical point theory for asymptotically quadratic functionals and applications to problems with resonance, *Nonlinear Anal. Theory Methods Appl.*, **28** (1997), 419–441. https://doi.org/10.1016/0362-546X(95)00167-T
18. J. Su, L. Zhao, An elliptic resonance problem with multiple solutions, *J. Math. Appl. Appl.*, **319** (2006), 604–616. https://doi.org/10.1016/j.jmaa.2005.10.059
19. J. Su, Semilinear elliptic resonant problems at higher eigenvalue with unbounded nonlinear terms, *Acta Math. Sin.*, **14** (1998), 411–419. https://doi.org/10.1007/BF02580445
20. K. C. Chang, *Infinite Dimensional Morse Theory and Multiple Solutions Problems*, Birkhauser, Boston, 1993.
21. J. Mawhin, M. Willem, *Critical point theory and Hamiltonian systems*, Springer, Berlin, 1989.
22. R. Servadei, E. Valdinoci, Lewy-Stampacchia type estimates for variational inequalities driven by (non)local operators, *Rev. Mat. Iberoam.*, **29** (2013), 1091–1126. https://doi.org/10.4171/RMI/750
23. J. Liu, A Morse index for a saddle point, *Syst. Sci. Math. Sci.,* **2** (1989), 32–39.
24. J. Su, Semilinear elliptic boundary value problems with double resonance between two consecutive eigenvalues, *Nonlinear Anal. Theory Methods Appl.*, **48** (2002), 881–895. https://doi.org/10.1016/S0362-546X(00)00221-2
25. Z. Q. Wang, Multiple solutions for indefinite functionals and applications to asymptotically linear problems, *Acta Math. Sin.*, **5** (1989), 101–113. https://doi.org/10.1007/BF02107664
26. Y. Chen, J. Su, Resonant problems for fractional Laplacian, *Commun. Pure Appl. Anal.*, **16** (2017), 163–187. https://doi.org/10.3934/cpaa.2017008
27. D. Gromoll, M. Meyer, On differential functions with isolated point, *Topology*, **8** (1969), 361–369. https://doi.org/10.1016/0040-9383(69)90022-6
28. S. Li, J. Liu, Some existence theorems on multiple critical points and their applications, *Kexue Tongbao*, **17** (1984), 1025–1027.
29. S. Li, M. Willem, Applications of local linking to critical point theory, *J. Math. Anal. Appl.*, **189** (1995), 6–32.
30. J. Liu, J. Su, Remarks on multiple nontrivial solutions for quasi-linear resonant problems, *J. Math. Anal. Appl.*, **258** (2001), 209–222. https://doi.org/10.1006/jmaa.2000.7374