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Research article

Robust portfolio choice with limited attention

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Abstract: This paper investigates a robust portfolio selection problem with the agent's limited attention. The agent has access to a risk-free asset and a stock in a financial market. But she does not observe perfectly the expected return rate of the stock so she has to estimate this key parameter before making decisions. Besides the general observable financial information, the agent can also acquire a news signal process whose accuracy depends on the agent's attention. We assume that the agent pays limited attention on the signal and she does not trust her estimation model. So it is necessary to consider model ambiguity in this paper as well. The agent maximizes the expected utility of her terminal wealth under the worst-case scenario. Under this setting, we derive the robust optimal strategy explicitly. In the presence of the attention and ambiguity aversion, the myopic term of the strategy, the hedging term of the strategy and the worst-case scenario are all changed. We find that more attention makes the variance of the estimated return smaller. The numerical examples also show that a more attentive agent has a better estimation of the unobservable parameter and is more confident on her estimation. Consequently, the worst-case scenario deviates less from the reference model, which implies a higher expected return rate under the worst-case scenario, thus invests more in the stock.

Keywords: limited attention; robust strategy; portfolio choice; CRRA utility; HJB equation

1. Introduction

Model-based portfolio choice problem has become one important pillar in financial economics, especially when incorporating some well-documented features such as return predictability and model uncertainty. However, numerous psychological studies show that the agent does not have infinite time and effort for processing information. In this vein, some people may lack financial literacy, meaning that they do not have the necessary skills and knowledge to make the perfect estimation and to make informed and effective investment decisions as in [1] and [2]. Furthermore, given the uncertainty about the stock return, it is far from clear how the agent's ambiguity and limited attention on stock return impact investment decision. In this study, we study a dynamic portfolio choice problem of an ambiguity-averse agent with unknown stock return and limited attention under no transaction cost and information cost assumption.

We assume that the agent allocates her wealth into a risk-free asset and a stock over time. The information that the agent can acquire are the stock price and an additional information showed as a signal in the model. We firstly make transformation of the Brownian motions involved and then use Kalman-Bucy filtering method to estimate the unknown stock return. Obviously, at this stage, the problem becomes full information optimization problem.

Our agent can not infer the important parameters exactly by her limited attention on information, see [3, 4]. Such ignorance generates pervasive uncertainty for agents when they make economic and financial decisions [5–7]. More information will be more useful for her trading behavior. Based on the idea that the agent has limited attention for processing information, it is reasonable to assume that the agent does not have a perfect estimation model and does not completely trust her estimation model. She adopts the robust portfolio choice rules. Therefore, in this paper, we assume that the agent can only pay limited attention on estimating a stock return and takes into account robustness to deal with parameter ambiguity.

Next, we briefly review the relevant literature. This study is mainly related to the literature on investment decision with agents' attentions and model uncertainty. There are related literatures studying the investment decision with agents' attention about the key model parameter, for example, stock return, return predictor, the total wealth or interest rate, see as in [8–11]. The analytical tractability of our model in this paper mainly relies on the crucial assumption of the constant attention on the additional information. Because the form of the additional information and the parameter of the agent's constant attention are already clear enough to characterize the concept of attention on the signal. In this setting, we can also naturally introduce the model uncertainty and get the closed-form expressions of the investment strategy. Theoretically, we characterize this additional information by a signal [12–14]. Since information acquisition is always a one-time choice, we can consider that the agent will not change her attention dramatically in an investment and it is reasonable to take the value of the long-term mean of her attention. By convention, we have the setting that the agent keep a constant attention to maintain the precision of the additional signal. When the agent is more attentive to news, the signal is more accurate.

We apply the robust control theory where the set of alternative models are defined through density generators. Chen and Epstein [15] introduces the robust recursive utility model and Anderson et al. [16] proposes the penalty-based utility model. Indeed, there are several mainstream approaches to model robust strategies with unknown probability for the stochastic risk in the previous literature. The first one is the popular max-min expected utility approach as in [17] and the second one is the so-called smooth ambiguity preferences approach adopted by Klibanoff et al. [18]. Moreover, Anderson et al. [16] investigate a robust control approach to deal with model uncertainty in continuous time framework. They assume that an agent has a particular reference probability measure. However, the ambiguity-averse agent does not totally trust this probability measure. She naturally considers a range of alternative probability measures around the reference measure. However, these papers are not that clear on how to find the range of alternative model sets.

Growing empirical evidence suggests that uncertainty on the return rate of stock price is a primary aspect of the model ambiguity problem [19–21]. The uncertainty on the return rate has huge impacts on quantitative methods, and the problem on limited attention of the return rate also faces model uncertainty. There are also a lot of applications of robust optimal strategy in investment and reinsurance problem and DC pension investment problem. For example, Yi et al. [22] considers a robust investment and reinsurance problem with Heston's stochastic volatility. And Yi et al. [23] studies investment-reinsurance problem with model uncertainty in a mean-variance framework. Lei and Yan [24] investigates a robust optimal reinsurance and investment problem under CEV model. Wang and Li [25] proposes a robust optimal portfolio choice problem with stochastic interest rate and stochastic volatility. Zeng et al. [26] investigates a derivative-based optimal investment problem with model uncertainty. And Wang et al. [27] studies a robust optimal investment problem with inflation risk and mean-reverting risk premium in a DC pension plan. Lin et al. [28] investigates a model that investors are uncertain about the dynamics of the expected returns and the correlation between the returns of two risky assets. In these papers, robust optimization is useful for dealing with parameter ambiguity and providing robust strategy under the presence of the agent's limited distribution knowledge. In contrast to these studies, we take into account robustness in a limited attention model. The ambiguity-averse agent makes robust strategy to avoid the loss due to adverse scenarios. Rather than thinking of the only estimated model, the ambiguity-averse agent always considers a set of alternative models. The reason is that it probably contains some misspecification errors in the model given by her estimation technique.

This paper contributes to the literature on robust portfolio choice with limited attention in three folds. First, we provide the optimal robust investment strategy. We use Girsanov's theorem to change the probability measures and construe probability distortion processes by the Radon-Nikodym derivative process. Based on the previous analysis, we use dynamic programming method to find the agent's robust optimal portfolio to maximize the expected utility of her terminal wealth and an entropy penalty for alternative models. We derive the closed-form solution of the general case and that of the special case as well. Wang et al. [27] also considered similar robust optimization problems, but we introduce the classic concept of "limited attention" in behavioral finance to understand the effects of information.

Second, we estimate the risky investment share incurred by limited agent attention. In particular, we find the risky investment share increases with the agent's attention on the information about expected return. When the agent pays more attention on the investment, the variance of the estimation return becomes smaller, which means the estimation is more accurate. That is, when the agent is more attentive to the information, she can process the signal in a relative accurate way. Although Branger et al. [29] and Wang et al. [30] are also in a setting where returns are predictable, we used a news signal to depict some of the information in the financial market, making the model more realistic.

Third, we estimate the robustness with selected alternative model based on a choosing worst-case scenario. Interestingly, the more ambiguous agent will take into account more alternative models around the reference model for choosing the worst-case scenario. Therefore, the expected return under the worst-case scenario becomes lower, and the agent reduces her optimal demand for the stock. Finally, we find that both attention and ambiguity aversion of the agent impact the level and structure of the optimal investment strategy and the worst-case distortion. Inspired by [10], we extend the problem of investor attention on portfolio choice in robust optimization. Significantly, the analysis between the general case solution and the special case solution provides a consistent conclusion with numerical examples.

The rest of this paper is organized as follows. In Section 2, we introduce the basic financial market and characterize the agent's attention to information. Optimization problem is presented in Section 3. In Section 4, we derive the explicit solution by dynamic programming method. Numerical analysis is given in Section 5. Finally, Section 6 concludes the paper with further remarks.

2. The model

This paper considers a financial market where trading takes place continuously and there are no transaction costs or taxes. We define a continuous-time, fixed time horizon ($T \geq 0$) model. The uncertainty is represented by a complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ where filtration $\{\mathscr{F}_t\}_{0\leq t\leq T}$ satisfying the usual conditions, i.e., $\{\mathscr{F}_t\}_{0\leq t\leq T}$ is right continuous and $\mathbb{P}$ complete. Suppose that the financial market consists of two tradable assets: a risk-free asset and a stock. The price process of the risk-free asset satisfies the following ordinary differential equation

$$\frac{dS_{0,t}}{S_{0,t}} = rdt, \tag{2.1}$$

where the constant r represents the risk-free rate.

The price dynamics of the stock is given as follows

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_S dB_{S,t}, \tag{2.2}$$

where $\sigma_S > 0$ is the volatility rate of the stock, $B_{S,t}$ is a one-dimensional standard Brownian motion on the filtered complete probability space. The drift μ_t in (2.2) is the unobservable stock return, which is assumed to be random and follows an Ornstein-Uhlenbeck process

$$d\mu_t = \lambda_\mu(\bar{\mu} - \mu_t)dt + \sigma_\mu dB_{\mu,t}, \tag{2.3}$$

where $\lambda_\mu > 0$ denotes a mean-reversion rate, $\bar{\mu}$ is the long-run mean return rate, $\sigma_\mu > 0$ is the volatility parameter and $B_{\mu,t}$ is a standard Brownian motion which is independent of $B_{S,t}$.

In this setting, instead of having full information of the stock return, the agent needs to estimate the unobservable return μ_t under her knowledge of the economy before deciding the robust investment strategy. We allow that the agent can have opportunity to actively learn about the unknown return and get access to a signal which represents the information from market. Following Peng [13] and Kasa [12], we adopt the noisy-information specification and assume that the agent acquires a news signal y_t with the following dynamics

$$dy_t = \mu_t dt + \frac{1}{\sqrt{a}} dB_{y,t}, \tag{2.4}$$

where $B_{y,t}$ is a standard Brownian motion, which is independent of $B_{S,t}$ and $B_{\mu,t}$, and $a > 0$ represents the agent's attention on the news signal which is not infinite. Unlike the setting of a control variable attention in Andrei and Hasler [10], we consider the parameter a as a constant for tractability and ease of interpretation. However, we introduce model uncertainty in our model. Different agents have different attention on the drift estimation due to different viewpoints of financial data. In general, a huge increase in financial literacy or investment interest always takes a lot of time and efforts and is unlikely to happen in a short investment horizon. Hence we assume that this parameter takes its long-term average. When she is attentive to news, the amount of information she gets is large. Therefore, the signal is more accurate as the volatility of the signal is smaller, see as in [10]. When the agent is inattentive to news, the signal that she acquires is inaccurate. Given this, we call a the agent's limited attention to news. To make optimal decisions, the agent is required to filter the value of μ_t in the optimal way using the observed S_t and y_t.

2.1. Filter

In order to build the optimal problem and solve the optimal problem with unobservable return μ_t, we infer an estimation parameter $\hat{\mu}_t$ using the Kalman-Bucy filtering method at first. Then we use $\hat{\mu}_t$ instead of μ_t for the robust optimal investment problem in the following part of this paper.

We denote by $\hat{\mu}_t := E\left[\mu_t \middle| \mathscr{F}_t^{S,y}\right]$ the estimated stock return and by $\gamma_t := E\left[(\mu_t - \hat{\mu}_t)^2 \middle| \mathscr{F}_t^{S,y}\right]$ the posterior variance. By Girsanov's theorem, we have the following independent standard Brownian motions under the agent's filtration $\mathscr{F}_t^{S,y}$ generated by the stock price S and the signal y

$$d\hat{B}_{S,t} = \frac{(\mu_t - \hat{\mu}_t)dt}{\sigma_S} + dB_{S,t}, \tag{2.5}$$

$$d\hat{B}_{y,t} = \frac{(\mu_t - \hat{\mu}_t)dt}{\frac{1}{\sqrt{a}}} + dB_{y,t}. \tag{2.6}$$

We note that these two innovation process $\hat{B}_{S,t}$ and $\hat{B}_{y,t}$ are mutually independent Brownian motions.

Following Theorem 12.7 of Lipster and Shiryaev [31], the observable variables are S_t and y_t, the unobservable variable is μ_t. The dynamics of observable variables can be written in the matrix form as follows

$$\begin{pmatrix} \frac{dS_t}{S_t} \\ dy_t \end{pmatrix} = \left[\underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\mathcal{A}_0} + \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\mathcal{A}_1} \mu_t \right] dt + \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\mathcal{B}_1} dB_{\mu,t} + \underbrace{\begin{pmatrix} \sigma_S & 0 \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix}}_{\mathcal{B}_2} \begin{pmatrix} dB_{S,t} \\ dB_{y,t} \end{pmatrix},$$

and the dynamics of the unobservable variable

$$d\mu_t = \left[\underbrace{(\lambda_\mu \bar{\mu})}_{\mathfrak{a}_0} + \underbrace{(-\lambda_\mu)}_{\mathfrak{a}_1} \mu_t \right] dt + \underbrace{\sigma_\mu}_{\mathfrak{b}_1} dB_{\mu,t} + \underbrace{(0,0)}_{\mathfrak{b}_2} \begin{pmatrix} dB_{S,t} \\ dB_{y,t} \end{pmatrix}. \tag{2.7}$$

Using the notations of Theorem 12.7 in [31], we obtain

$$\mathfrak{b} \circ \mathfrak{b} := \mathfrak{b}_1 \mathfrak{b}_1' + \mathfrak{b}_2 \mathfrak{b}_2' = \sigma_\mu^2, \tag{2.8}$$

$$\mathcal{B} \circ \mathcal{B} := \mathcal{B}_1 \mathcal{B}_1' + \mathcal{B}_2 \mathcal{B}_2' = \begin{pmatrix} \sigma_S^2 & 0 \\ 0 & \frac{1}{a} \end{pmatrix}, \tag{2.9}$$

$$\mathfrak{b} \circ \mathcal{B} := \mathfrak{b}_1 \mathcal{B}_1' + \mathfrak{b}_2 \mathcal{B}_2' = (0,0), \tag{2.10}$$

and

$$[(\mathfrak{b} \circ \mathcal{B}) + \gamma_t \mathcal{A}_1'](\mathcal{B}_2^{-1})' = (\gamma_t, \gamma_t) \begin{pmatrix} \frac{1}{\sigma_S} & 0 \\ 0 & \sqrt{a} \end{pmatrix} = \left(\frac{\gamma_t}{\sigma_S}, \gamma_t \sqrt{a} \right), \tag{2.11}$$

where the operator $\circ$ denotes the matrix multiplication and Σ' denotes the transpose of a matrix Σ.

Then, according to Theorem 12.7 of Lipster and Shiryaev [31], the estimation of the stock return $\hat{\mu}_t$ and the conditional variance γ_t can be written as

$$d\hat{\mu}_t = \left[\mathfrak{a}_0 + \mathfrak{a}_1\hat{\mu}_t\right] dt + [(\mathfrak{b} \circ \mathcal{B}) + \gamma_t\mathcal{A}_1'](\mathcal{B} \circ \mathcal{B})^{-1} \left[\begin{pmatrix} \frac{dS_t}{S_t} \\ dy_t \end{pmatrix} - (\mathcal{A}_0 + \mathcal{A}_1\hat{\mu}_t)\, dt\right], \tag{2.12}$$

and

$$\frac{d\gamma_t}{dt} = 2\mathfrak{a}_1\gamma_t + \mathfrak{b} \circ \mathfrak{b} - [(\mathfrak{b} \circ \mathcal{B}) + \gamma_t\mathcal{A}_1'](\mathcal{B}_2^{-1})'(\mathcal{B}_2^{-1})[(\mathfrak{b} \circ \mathcal{B})' + \gamma_t\mathcal{A}_1] \tag{2.13}$$

Consequently, in our setting, the dynamics of $\hat{\mu}_t$ and γ_t are given as

$$d\hat{\mu}_t = \lambda_\mu(\bar{\mu} - \hat{\mu}_t)dt + \frac{\gamma_t}{\sigma_S} d\hat{B}_{S,t} + \gamma_t \sqrt{a} d\hat{B}_{y,t}, \tag{2.14}$$

$$\frac{d\gamma_t}{dt} = -2\lambda_\mu\gamma_t + \sigma_\mu^2 - (\frac{\gamma_t^2}{\sigma_S^2} + \gamma_t^2 a). \tag{2.15}$$

As Eq (2.14) shown, there are two sources of information: realized stock return and changes in the news signal. The agent divides stochastic weights into these two sources of information. Shocks to the stock price and signal impact the agent's estimate.

We follow Branger et al. [32] to adopt the long-run level of γ since the variance of the estimation is always a deterministic function of time. For simplicity, we assume that it has already converged to a constant. So we have $\frac{d\gamma_t}{dt} = 0$, which means *

$$\gamma = \frac{-\lambda_\mu\sigma_S^2 + \sqrt{\lambda_\mu^2\sigma_S^4 + (1 + a\sigma_S^2)\sigma_\mu^2\sigma_S^2}}{1 + a\sigma_S^2} > 0. \tag{2.16}$$

Finally the dynamics of the state variables after filter becomes

$$\frac{dS_t}{S_t} = \hat{\mu}_t dt + \sigma_S d\hat{B}_{S,t}, \tag{2.17}$$

$$d\hat{\mu}_t = \lambda_\mu(\bar{\mu} - \hat{\mu}_t)dt + \frac{\gamma}{\sigma_S} d\hat{B}_{S,t} + \gamma \sqrt{a} d\hat{B}_{y,t}, \tag{2.18}$$

2.2. Robustness

Basing on the previous inferring, the agent could have a reference model $\mathbb{P}$ of the financial market. But this paper focuses on an ambiguity-averse agent who does not have full confidence in her reference model and has to take into account a range of possible alternative models. Because of the fear of risk brought by uncertainty, she chooses to maximize her expected utility in a worst case scenario.

*We only look at the positive root due to the variable γ representing the meaning of variance.

As in literature, the agent can define the alternative models by a class of probability measures which are equivalent to $\mathbb{P}$: $\mathcal{Q} := \{\mathbb{Q} \mid \mathbb{Q} \sim \mathbb{P}\}$, such that, for each $\mathbb{Q} \in \mathcal{Q}$, there exists a progressively measurable process θ_t which can be referred as the following probability distortion process

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathscr{F}_t^{S,y}} = \Sigma_t^\theta, \tag{2.19}$$

where $\Sigma_t^\theta = \exp\{-\frac{1}{2}\int_0^t \theta_\tau^2 d\tau + \int_0^t \theta_\tau d\hat{B}_{S,\tau}\}$.

We assume θ_t satisfies Novikov's condition $E^{\mathbb{P}}\left[\exp\left\{\int_0^T \frac{1}{2}\theta_\tau^2 d\tau\right\}\right] < \infty$. Then Σ_t^θ is a $\mathbb{P}$-martingale with filtration $\{\mathscr{F}_t^{S,y}\}_{0\le t\le T}$.

Furthermore, according to Girsanov's theorem, the following stochastic process $B_{S,t}^{\mathbb{Q}}$ is a one-dimensional standard Brownian motion under the alternative measure $\mathbb{Q}$,

$$dB_{S,t}^{\mathbb{Q}} = d\hat{B}_{S,t} - \theta_t dt.$$

The dynamics of the state variables under $\mathbb{Q}$ becomes

$$\frac{dS_t}{S_t} = (\hat{\mu}_t + \sigma_S\theta_t)dt + \sigma_S dB_{S,t}^{\mathbb{Q}}, \tag{2.20}$$

$$d\hat{\mu}_t = [\lambda_\mu(\bar{\mu} - \hat{\mu}_t) + \frac{\gamma}{\sigma_S}\theta_t]dt + \frac{\gamma}{\sigma_S}dB_{S,t}^{\mathbb{Q}} + \gamma\sqrt{a}d\hat{B}_{y,t}. \tag{2.21}$$

2.3. Dynamics of wealth

We assume the portfolio choice π_t is the proportion of the agent's wealth invested on the stock at time t. So we have the following wealth process

$$\begin{aligned} dW_t^\pi &= W_t^\pi\left[(1-\pi_t)\frac{dS_{0,t}}{S_{0,t}} + \pi_t\frac{dS_t}{S_t}\right] \\ &= W_t^\pi\left[(1-\pi_t)rdt + \pi_t(\hat{\mu}_t + \sigma_S\theta_t)dt + \pi_t\sigma_S dB_{S,t}^{\mathbb{Q}}\right] \\ &= W_t^\pi rdt + W_t^\pi\pi_t(\hat{\mu}_t + \sigma_S\theta_t - r)dt + W_t^\pi\pi_t\sigma_S dB_{S,t}^{\mathbb{Q}}. \end{aligned} \tag{2.22}$$

3. Optimization problem

The agent is looking for an optimal strategy to maximize the expected utility of her terminal wealth at the given horizon T. We assume the agent has a power utility with the relative risk aversion coefficient $\alpha > 1$ as follows

$$U(w) = \frac{w^{1-\alpha}}{1-\alpha}. \tag{3.1}$$

The agent is ambiguity-averse and seeks for the robust strategy. In our setting, we assume that she makes the optimal decision after getting the worst-case probability measure. That is, the drift adjustment θ_t is firstly chosen to minimize the sum of the expected terminal payoff and an entropy penalty. The relative entropy penalty exists for describing the difference between probability measures

$\mathbb{P}$ and the alternative measures $\mathbb{Q}$. Since we only have the probability measure $\mathbb{P}$ for reference, any model deviation will be penalized. Specifically, we have the following objective function

$$V(t,w,\hat{\mu}) = \sup_{\pi}\inf_{\theta} E^{\mathbb{Q}}\left[\frac{(W_T^{\pi})^{1-\alpha}}{1-\alpha} + \int_t^T \frac{\theta_s^2}{2\Psi(s,W_s^{\pi},\hat{\mu}_s)}ds \middle| W_t^{\pi}=w,\hat{\mu}_t=\hat{\mu}\right], \tag{3.2}$$

subject to the wealth constraint (2.22) and the terminal condition $V(T,w,\hat{\mu}) = \frac{w^{1-\alpha}}{1-\alpha}$.

In optimization problem (3.2), the expected utility is measured under the alternative model $\mathbb{Q}$. The penalty term is characterized by the scaled relative entropy which penalizes the alternative models deviated from the reference model $\mathbb{P}$. For analytical tractability, we adopt the homothetic robustness motivated in [33] and [34] to capture the agent's robustness preference and make following assumption

$$\Psi(t,w,\hat{\mu}) = \frac{\beta}{(1-\alpha)V(t,w,\hat{\mu})}, \tag{3.3}$$

where $\beta > 0$ is the ambiguity-aversion coefficient describing the agent's attitude towards model uncertainty. The term $\frac{1}{V(t,w,\hat{\mu})}$ in (3.3) can be thought as a normalization factor that maintains the consistency of unit between the relative entropy and utility, which means that the entropy penalty has the same unit with the value function, see the homothetic robustness approach as in [33]. Moreover, scaling β by the value function allows us to find an explicit solution to our optimization problem. Notice that in the limiting case of $\beta = 0$, the entropy penalty term in (3.2) is equal to $+\infty$ unless $\theta \equiv 0$ and the robust utility maximization problem is reduced to the classical one, meaning the agent is not ambiguity-averse at all; in the other limiting case of $\beta = +\infty$, the entropy penalty term in (3.2) is equal to 0 and the worst case is chosen among all possible models in Q, meaning the agent is extremely ambiguity-averse. The larger the value of β, the more alternative models in Q are considered by the agent to find the worst-case scenario.

4. HJB equation

According to the dynamic programming principle, we obtain the following Hamilton-Jacobi-Bellman (HJB) equation for the value function

$$\begin{aligned} 0 = & \sup_{\pi}\inf_{\theta} \{V_t + V_w w[r + \pi(\hat{\mu} + \sigma_S\theta - r)] + V_{\hat{\mu}}[\lambda_\mu(\bar{\mu} - \hat{\mu}) + \frac{\gamma}{\sigma_S}\theta] \\ & + \frac{1}{2}V_{ww}w^2\sigma_S^2\pi^2 + \frac{1}{2}V_{\hat{\mu}\hat{\mu}}(\frac{\gamma^2}{\sigma_S^2} + \gamma^2 a) + w\gamma\pi V_{w\hat{\mu}} + \frac{1}{2\Psi}\theta^2\}, \end{aligned} \tag{4.1}$$

with the boundary condition $V(T,w,\hat{\mu}) = \frac{w^{1-\alpha}}{1-\alpha}$.

By the first order condition for the infimum, we obtain the worst-case scenario

$$\theta^* = -\Psi(V_w w\sigma_S\pi + V_{\hat{\mu}}\frac{\gamma}{\sigma_S}). \tag{4.2}$$

Substituting (4.2) into (4.1), we have

$$0 = \sup_{\pi} \Bigg\{V_t + V_w w\left[r + \pi[\hat{\mu} - \sigma_S\Psi(V_w w\sigma_S\pi + V_{\hat{\mu}}\frac{\gamma}{\sigma_S}) - r]\right]$$

$$+V_{\hat{\mu}}\left[\lambda_\mu(\bar{\mu}-\hat{\mu})-\frac{\gamma}{\sigma_S}\Psi(V_w w\sigma_S\pi+V_{\hat{\mu}}\frac{\gamma}{\sigma_S})\right]$$
$$+\frac{1}{2}V_{ww}w^2\sigma_S^2\pi^2+\frac{1}{2}V_{\hat{\mu}\hat{\mu}}(\frac{\gamma^2}{\sigma_S^2}+\gamma^2 a)+w\gamma\pi V_{w\hat{\mu}}+\frac{1}{2}\Psi(V_w w\sigma_S\pi+V_{\hat{\mu}}\frac{\gamma}{\sigma_S})^2\bigg\}. \tag{4.3}$$

Again, by the first order condition for the supremum, we get

$$\pi^* = \frac{V_w w\hat{\mu}-V_w wr+V_{w\hat{\mu}}w\gamma-V_w V_{\hat{\mu}}\Psi\gamma w}{\Psi V_w^2 w^2\sigma_S^2-V_{ww}w^2\sigma_S^2}. \tag{4.4}$$

Plugging the above formulas for θ^* and π^* into the HJB equation (4.1) and rearranging the terms, we have

$$0 = V_t+V_w wr+V_{\hat{\mu}}[\lambda_\mu(\bar{\mu}-\hat{\mu})]+\frac{1}{2}V_{\hat{\mu}\hat{\mu}}(\frac{\gamma^2}{\sigma_S^2}+\gamma^2 a)-\frac{1}{2}V_{\hat{\mu}}^2\Psi\frac{\gamma^2}{\sigma_S^2}$$
$$+\frac{[V_w\hat{\mu}-V_w r+V_{w\hat{\mu}}\gamma-V_w V_{\hat{\mu}}\Psi\gamma]^2}{-2(V_{ww}-V_w^2\Psi)\sigma_S^2}. \tag{4.5}$$

We conjecture a solution to the HJB equation (4.1), which has the following form

$$V(t,w,\hat{\mu}) = \frac{w^{1-\alpha}}{1-\alpha}g(t,\hat{\mu}),$$

where $g(t,\hat{\mu})$ will be determined later. Because of the boundary condition $V(T,w,\hat{\mu})=\frac{w^{1-\alpha}}{1-\alpha}$, we have immediately $g(T,\hat{\mu})=1$. Then taking derivatives of V with respect to the different variables yields

$$V_t=\frac{w^{1-\alpha}}{1-\alpha}g_t,\quad V_w=w^{-\alpha}g,\quad V_{\hat{\mu}}=\frac{w^{1-\alpha}}{1-\alpha}g_{\hat{\mu}},$$
$$V_{ww}=-\alpha w^{-\alpha-1}g,\quad V_{\hat{\mu}\hat{\mu}}=\frac{w^{1-\alpha}}{1-\alpha}g_{\hat{\mu}\hat{\mu}},\quad V_{w\hat{\mu}}=w^{-\alpha}g_{\hat{\mu}}.$$

Substituting the above derivatives of V in (4.5) implies

$$0 = \frac{1}{1-\alpha}g_t+gr+\frac{1}{1-\alpha}g_{\hat{\mu}}\lambda_\mu(\bar{\mu}-\hat{\mu})+\frac{1}{2(1-\alpha)}g_{\hat{\mu}\hat{\mu}}(\frac{\gamma^2}{\sigma_S^2}+a\gamma^2)$$
$$-\frac{g_{\hat{\mu}}^2\gamma^2\beta}{2(1-\alpha)^2\sigma_S^2 g}+\frac{[g\hat{\mu}-gr+g_{\hat{\mu}}\gamma-\frac{g_{\hat{\mu}}\beta\gamma}{1-\alpha}]^2}{2\sigma_S^2 g(\alpha+\beta)}. \tag{4.6}$$

Since we already have the boundary condition $g(T,\hat{\mu})=1$, we propose a further ansatz: we assume that $g(t,\hat{\mu})$ has the following exponential form

$$g(t,\hat{\mu})=e^{\frac{1}{2}g_1(t)\hat{\mu}^2+g_2(t)\hat{\mu}+g_3(t)}, \tag{4.7}$$

and the corresponding boundary conditions are $g_1(T)=0, g_2(T)=0, g_3(T)=0$.

Replacing the derivatives of g by those of g_1, g_2, g_3 in the Eq (4.6)

$$g_t=g(\frac{1}{2}g_1'\hat{\mu}^2+g_2'\hat{\mu}+g_3'),\quad g_{\hat{\mu}}=g(g_1\hat{\mu}+g_2),\quad g_{\hat{\mu}\hat{\mu}}=g(g_1\hat{\mu}+g_2)^2+gg_1,$$

we derive the following equation

$$
\begin{aligned}
0 &= \frac{1}{2}g_1'\hat{\mu}^2 + g_2'\hat{\mu} + g_3' + (1-\alpha)r + \lambda_\mu(g_1\hat{\mu} + g_2)(\bar{\mu} - \hat{\mu}) \\
&\quad + \frac{1}{2}[(g_1\hat{\mu} + g_2)^2 + g_1](\frac{\gamma^2}{\sigma_S^2} + a\gamma^2) - \frac{\gamma^2\beta(g_1\hat{\mu} + g_2)^2}{2(1-\alpha)\sigma_S^2} \\
&\quad + \frac{[\hat{\mu} - r + (g_1\hat{\mu} + g_2)\gamma - \frac{(g_1\hat{\mu}+g_2)\beta\gamma}{1-\alpha}]^2(1-\alpha)}{2\sigma_S^2(\alpha+\beta)}.
\end{aligned} \tag{4.8}
$$

For the above equation holding true, the coefficients before the quadratic term $\hat{\mu}^2$, the term $\hat{\mu}$ and the constant term 1 must be respectively equal to 0. Thus we obtain the following system of ODEs for g_1, g_2 and g_3

$$
\begin{aligned}
0 &= g_1' + 2\left[\frac{1}{2}(\frac{\gamma^2}{\sigma_S^2} + a\gamma^2) - \frac{\beta\gamma^2}{2(1-\alpha)\sigma_S^2} + \frac{(1-\alpha-\beta)^2\gamma^2}{2\sigma_S^2(\alpha+\beta)(1-\alpha)}\right]g_1^2 + 2\left[\frac{(1-\alpha-\beta)\gamma}{(\alpha+\beta)\sigma_S^2} - \lambda_\mu\right]g_1 + \frac{1-\alpha}{(\alpha+\beta)\sigma_S^2}, \\
0 &= g_2' + \left[\frac{(1-\alpha-\beta)\gamma}{(\alpha+\beta)\sigma_S^2} - \lambda_\mu + (\frac{\gamma^2}{\sigma_S^2} + a\gamma^2 - \frac{\beta\gamma^2}{(1-\alpha)\sigma_S^2} + \frac{(1-\alpha-\beta)^2\gamma^2}{\sigma_S^2(\alpha+\beta)(1-\alpha)})g_1\right]g_2 \\
&\quad + \left[\lambda_\mu\bar{\mu} - \frac{(1-\alpha-\beta)\gamma}{\sigma_S^2(\alpha+\beta)}r\right]g_1 - \frac{(1-\alpha)r}{(\alpha+\beta)\sigma_S^2}, \\
0 &= g_3' + \left[\frac{1}{2}(\frac{\gamma^2}{\sigma_S^2} + a\gamma^2) - \frac{\beta\gamma^2}{2(1-\alpha)\sigma_S^2} + \frac{(1-\alpha-\beta)^2\gamma^2}{2\sigma_S^2(\alpha+\beta)(1-\alpha)}\right]g_2^2 + \left[\lambda_\mu\bar{\mu} - \frac{(1-\alpha-\beta)\gamma}{\sigma_S^2(\alpha+\beta)}r\right]g_2 \\
&\quad + \frac{1}{2}(\frac{\gamma^2}{\sigma_S^2} + a\gamma^2)g_1 + (1-\alpha)r + \frac{(1-\alpha)r^2}{2(\alpha+\beta)\sigma_S^2}.
\end{aligned}
$$

To simplify the notation, we introduce the following auxiliary constants

$$
\begin{aligned}
a_1 &= \frac{1}{2}(\frac{\gamma^2}{\sigma_S^2} + a\gamma^2), \qquad a_2 = -\frac{\beta\gamma^2}{2(1-\alpha)\sigma_S^2} + \frac{(1-\alpha-\beta)^2\gamma^2}{2\sigma_S^2(\alpha+\beta)(1-\alpha)}, \qquad a_3 = \frac{(1-\alpha-\beta)\gamma}{(\alpha+\beta)\sigma_S^2} - \lambda_\mu, \\
a_4 &= \frac{1-\alpha}{2(\alpha+\beta)\sigma_S^2}, \qquad a_5 = \lambda_\mu\bar{\mu} - \frac{(1-\alpha-\beta)\gamma}{\sigma_S^2(\alpha+\beta)}r, \qquad a_6 = (1-\alpha)r + \frac{(1-\alpha)r^2}{2(\alpha+\beta)\sigma_S^2}, \\
\triangle &= 4a_3^2 - 16a_4(a_1 + a_2), \qquad k_1 = -a_3 - \frac{\sqrt{\triangle}}{2}, \qquad k_2 = -a_3 + \frac{\sqrt{\triangle}}{2}.
\end{aligned}
$$

Then the system of ODEs becomes

$$
\begin{aligned}
0 &= g_1' + 2(a_1 + a_2)g_1^2 + 2a_3g_1 + 2a_4, \\
0 &= g_2' + [a_3 + 2(a_1 + a_2)g_1]g_2 + a_5g_1 - 2a_4r, \\
0 &= g_3' + (a_1 + a_2)g_2^2 + a_5g_2 + a_1g_1 + a_6,
\end{aligned}
$$

with the terminal conditions $g_1(T) = 0, g_2(T) = 0, g_3(T) = 0$.

Notice that g_1 satisfies a Riccati equation which admits an explicit solution. We can then solve explicitly the system of ODEs and write the solution with the model parameters

$$
g_1(t) = \frac{k_1k_2(1 - e^{(t-T)\sqrt{\triangle}})}{2(a_1 + a_2)(k_2 - k_1e^{(t-T)\sqrt{\triangle}})}, \tag{4.9}
$$

$$g_2(t) = e^{\int_t^T [a_3+2(a_1+a_2)g_1(\tau)]\,d\tau} \int_t^T [a_5 g_1(s) - 2a_4 r] e^{-\int_s^T [a_3+2(a_1+a_2)g_1(\tau)]\,d\tau}\, ds, \tag{4.10}$$

$$g_3(t) = \int_t^T [(a_1 + a_2)g_2^2(s) + a_5 g_2(s) + a_1 g_1(s) + a_6]\, ds. \tag{4.11}$$

With the above solution, we can write the value function explicitly as follows

$$V(t, w, \hat{\mu}) = \frac{w^{1-\alpha}}{1-\alpha} e^{\frac{1}{2}g_1(t)\hat{\mu}^2 + g_2(t)\hat{\mu} + g_3(t)}. \tag{4.12}$$

Taking derivatives, we obtain, in explicit form, the optimal robust portfolio choice and the worst-case scenario respectively

$$\begin{aligned} \pi^* &= \frac{\hat{\mu} - r + \frac{1-\alpha-\beta}{1-\alpha}(g_1\hat{\mu} + g_2)\gamma}{(\alpha+\beta)\sigma_S^2} \\ &= \frac{\hat{\mu} + \frac{1}{1-\alpha}(g_1\hat{\mu} + g_2)\gamma - r}{(\alpha+\beta)\sigma_S^2} - \frac{\frac{1}{1-\alpha}(g_1\hat{\mu} + g_2)\gamma}{\sigma_S^2}, \end{aligned} \tag{4.13}$$

$$\theta^* = -\beta \left[\frac{\hat{\mu} - r + \frac{1}{1-\alpha}(g_1\hat{\mu} + g_2)\gamma}{(\alpha+\beta)\sigma_S^2} \right], \tag{4.14}$$

where γ is defined in (2.16).

We summarize the previous results in the following theorem.

Theorem 4.1. *The value function $V(t, w, \hat{\mu})$ of the robust optimal investment problem (3.2) is given by (4.12), the robust optimal strategy π^* is given by (4.13) and the worst-case scenario theta* is defined by (4.14).*

A special case without the agent's attention is presented in the Section 5 to illustrate the difference between these two cases.

We provide some economic interpretations to the robust optimal strategy as well as the worst-case scenario. For the optimal portfolio choice in stock (4.13), there are two parts. The first part of π^* is the myopic investment strategy in stock, which is the estimated stock's excess return over the product of the stock variance and the adjusted risk aversion coefficient. In the presence of the model uncertainty, the agent's risk aversion coefficient is increased from α to $\alpha + \beta$, as shown in the denominator of the investment part. The second part of π^* is to hedge the estimation risk of the unobservable expected return rate. The hedging part does not depend on the agent's risk aversion. Since $\beta > 0$, the uncertainty about the true model drives the agent to invest less in the stock.

As for the worst case scenario θ^*, it is negatively related to the myopic investment in stock. If the agent holds a long investment position in stock, then θ^* is negative. The more is the agent's long investment position, the more negative is θ^*. Similarly, if the agent holds a short investment position in stock, then θ^* is positive. The more is the agent's short investment position, the more positive is θ^*. Intuitively, for the same value of ambiguity-aversion coefficient β, the agent is more concerned with the model uncertainty when her investment position in stock is more important. In other words, she does not care about the model uncertainty at all if she invests nothing in the stock. Furthermore, the larger is the agent's ambiguity-aversion coefficient β, with all other parameters remaining the same, the larger is the absolution value of θ^*, which means the worst-case scenario deviating further from the reference model.

In conclusion, the agent's limited attention to news a affects the robust optimal strategies and the worst-case scenario, which is presented in functions g_1, g_2. The agent's attention about the additional information also makes the variance of the estimation return lower. We will see from the numerical examples that the larger is a, the more is the agent's position in the stock.

5. Special case of no additional information

In this section, we present a brief description of the special case. We assume that the agent does not have any attention on the additional information or she even cannot have this very important additional information. Then the problem is reduced to a classical optimal investment problem with non-observable return. A letter with superscript A denotes the corresponding variable or function when the parameter $a = 0$ in the general case.

Similarly, we can obtain the following results by filter theory and dynamic programming

$$V^A(t, w, \hat{\mu}) = \frac{w^{1-\alpha}}{1-\alpha} e^{\frac{1}{2}g_1^A(t)\hat{\mu}^2 + g_2^A(t)\hat{\mu} + g_3^A(t)},$$

$$\begin{aligned} \pi^{A*} &= \frac{\hat{\mu} - r + \frac{1-\alpha-\beta}{1-\alpha}(g_1^A\hat{\mu} + g_2^A)\gamma^A}{(\alpha+\beta)\sigma_S^2} \\ &= \frac{\hat{\mu} + \frac{1}{1-\alpha}(g_1^A\hat{\mu} + g_2^A)\gamma^A - r}{(\alpha+\beta)\sigma_S^2} - \frac{\frac{1}{1-\alpha}(g_1^A\hat{\mu} + g_2^A)\gamma^A}{\sigma_S^2}, \end{aligned} \tag{5.1}$$

$$\theta^{A*} = -\beta\left[\frac{\hat{\mu} - r + \frac{1}{1-\alpha}(g_1^A\hat{\mu} + g_2^A)\gamma^A}{(\alpha+\beta)\sigma_S^2}\right], \tag{5.2}$$

where

$$\gamma^A = -\lambda_\mu\sigma_S^2 + \sqrt{\lambda_\mu^2\sigma_S^4 + \sigma_\mu^2\sigma_S^2}. \tag{5.3}$$

$$g_1^A(t) = \frac{k_1^A k_2^A(1 - e^{(t-T)\sqrt{\triangle}})}{2(a_1^A + a_2^A)(k_2^A - k_1^A e^{(t-T)\sqrt{\triangle}})}, \tag{5.4}$$

$$g_2^A(t) = e^{\int_t^T [a_3^A + 2(a_1^A + a_2^A)g_1^A(\tau)]\,d\tau} \int_t^T [a_5^A g_1^A(s) - 2a_4^A r] e^{-\int_s^T [a_3^A + 2(a_1^A + a_2^A)g_1^A(\tau)]\,d\tau}\,ds, \tag{5.5}$$

$$g_3^A(t) = \int_t^T [(a_1^A + a_2^A)(g_2^A)^2(s) + a_5^A g_2^A(s) + a_1^A g_1^A(s) + a_6^A]\,ds. \tag{5.6}$$

and

$$a_1^A = \frac{1}{2}\frac{(\gamma^A)^2}{\sigma_S^2}, \qquad a_2^A = -\frac{\beta(\gamma^A)^2}{2(1-\alpha)\sigma_S^2} + \frac{(1-\alpha-\beta)^2(\gamma^A)^2}{2\sigma_S^2(\alpha+\beta)(1-\alpha)}, \qquad a_3^A = \frac{(1-\alpha-\beta)\gamma^A}{(\alpha+\beta)\sigma_S^2} - \lambda_\mu,$$

$$a_4^A = \frac{1-\alpha}{2(\alpha+\beta)\sigma_S^2}, \qquad a_5^A = \lambda_\mu \bar{\mu} - \frac{(1-\alpha-\beta)\gamma^A}{\sigma_S^2(\alpha+\beta)} r, \qquad a_6^A = (1-\alpha)r + \frac{(1-\alpha)r^2}{2(\alpha+\beta)\sigma_S^2},$$

$$\triangle^A = 4(a_3^A)^2 - 16a_4^A(a_1^A + a_2^A), \qquad k_1^A = -a_3^A - \frac{\sqrt{\triangle^A}}{2}, \qquad k_2^A = -a_3^A + \frac{\sqrt{\triangle^A}}{2}.$$

Comparing to the general case with limited attention, there are several interesting observations listed as follows. From Eqs (4.13) and (5.1), we can not easily find the difference between the strategy of the general case and that of the special case. And the difference mainly depends on the functions g_1 and g_1^A, g_2 and g_2^A, whose relations are not obvious. In fact, the special case is the general case with the condition $a = 0$. The relation is clear in Figure 1 in the next section, the special case corresponds to the smallest proportion in the stock. That is, when the agent does not have the additional information or she does not pay attention on this key information, she takes the most conservative strategy. Checking Figures 6 and 7 in the next section, the agent without any attention on the additional information will adopt the smallest worst-case scenario and have largest variance of the estimation. Moreover, we can observe that $\gamma^A > \gamma$ for $a > 0$. This means that a positive attention on the additional signal can reduce the variance of the estimation for the stock return, thus improve the accuracy of the estimation.

6. Numerical examples

In this section, we analyze how the optimal strategy of the robust portfolio choice depends on the model parameters and how a change of the model parameter affects the agent's investment in the stock. First, we provide the basic values to the parameters see as in Table 1 and study the optimal strategy at time 0 for simplicity. The parameter $\hat{\mu}_t$ is assumed to be its long-run level m. The basic values of parameters are mainly referred to [10] and [27].

Table 1. Basic values of parameters.

σ_S	a	$\hat{\mu}$	λ_μ	$\bar{\mu}$	σ_μ	r	α	β	T
0.4	2	0.28	0.18	0.38	0.00747	0.04	2	1	6

Figure 1 shows the effects of the limited attention a and the risk aversion coefficient α on the robust optimal strategy in the stock, π^*. From Figure 1, we find that π^* increases along with the limited attention a and decreases with respect to the risk aversion coefficient α. When the agent pays more attention to the news, that is a increases, she obtains a more accurate estimation of the stock return, then she is confident enough to her investment action and increase the proportion of stock investment.

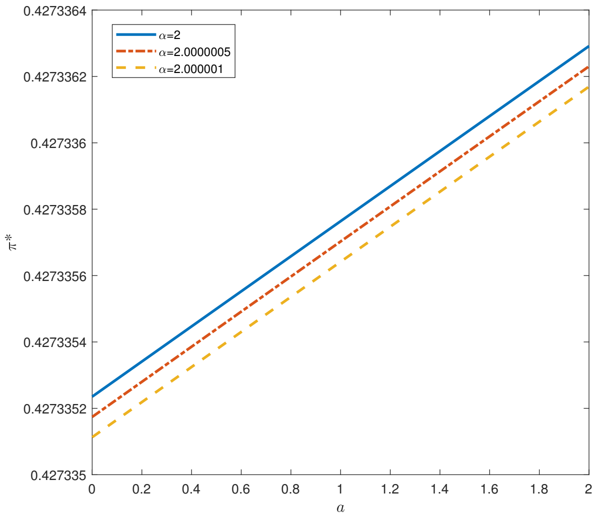

Figure 1. Effects of parameters a and α on π^*.

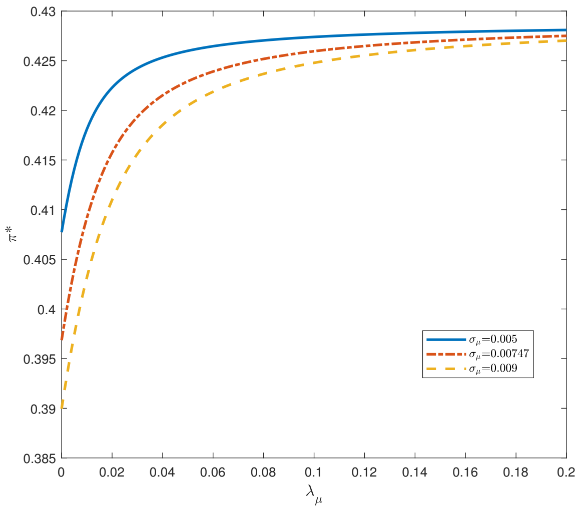

Figure 2. Effects of parameters λ_μ and σ_μ on π^*.

We now turn to the effects of the stock return's mean-reversion rate λ_μ and volatility σ_μ on the robust optimal investment strategy in the stock π^*, which is depicted in Figure 2. As λ_μ becomes larger, the stock return comes back to its long-term average faster, there is less uncertainty, and the agent increases her investment in the stock. Similarly, as σ_μ decreases, the stock return is less volatile and the agent increases her investment in the stock.

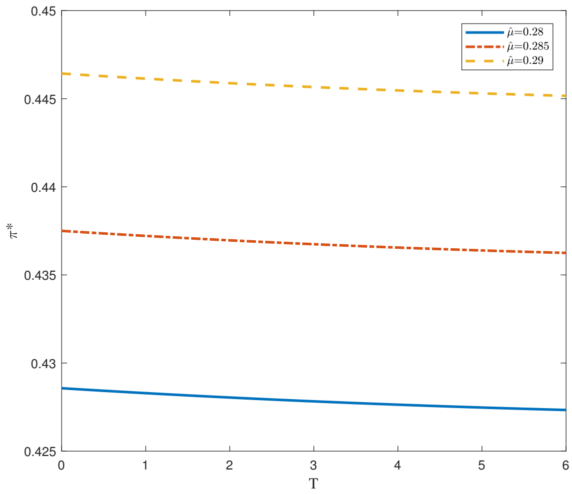

Figure 3. Effects of parameters T and $\hat{\mu}$ with the relative smaller value on π^*.

Figure 3 shows the effects of the investment horizon T and the estimated stock return $\hat{\mu}$ on the robust optimal investment strategy in the stock π^*. When the investment horizon T increases, the agent is more uncertain about her wealth at time T and decreased her investment in stock. As $\hat{\mu}$ becomes larger, the agent increases naturally her investment in the stock.

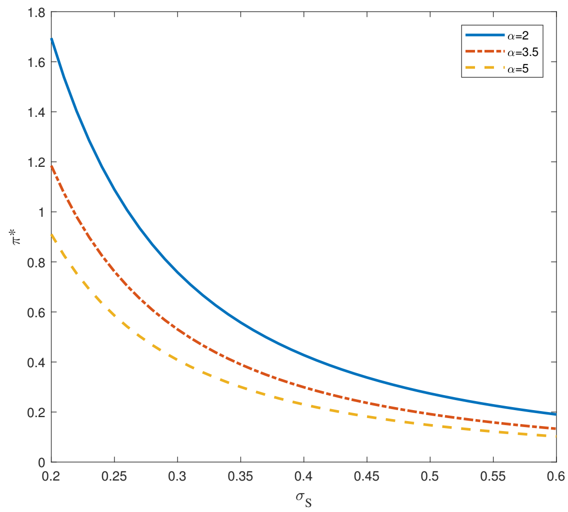

Figure 4. Effects of parameters σ_S and α on π^*.

Figure 4 reveals the effects of the stock volatility σ_S and the risk aversion coefficient α on the robust optimal strategy in the stock, π^*. From Figure 4, we find that π^* decreases along with the stock volatility σ_S and decreases with respect to α. When the stock is more volatile, that is when σ_S increases, the agent invests less in stock.

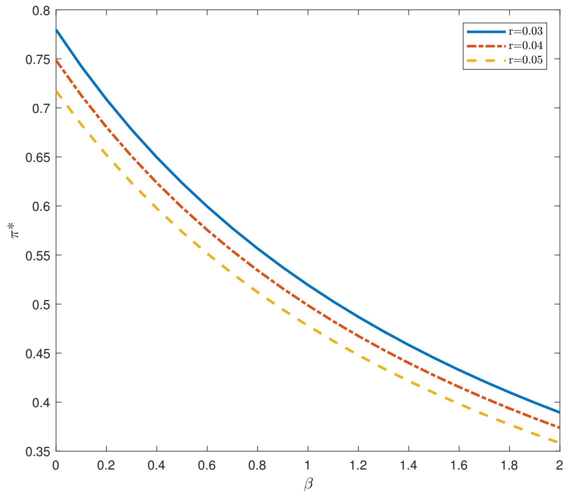

Figure 5. Effects of parameters β and r on π^*.

Figure 5 illustrates the effect of the agent's ambiguity aversion coefficient β and the risk-free rate r on the robust optimal investment strategy in the stock π^*. As parameter β becomes larger, the agent is more ambiguity averse, and hence more possible models around the reference model will be considered for the worst-case scenario candidates. In this way, the expected return under the worst-case scenario becomes lower, the agent scales down her strategy in the stock. When the agent is not that ambiguity-averse, she will be more confident on her understanding of the stock and her investment action and consequently will increase the proportion of stock investment. Naturally, when the risk-free rate r is higher, the agent increases her investment in the risk-free asset and decreases in the stock.

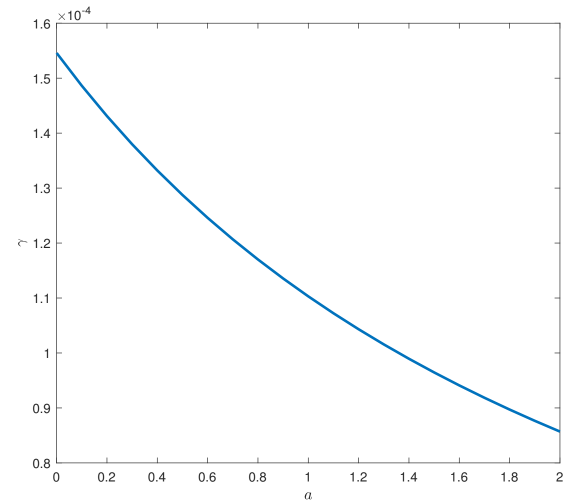

Figure 6. Effects of parameter a on γ.

Figure 6 shows that more attention makes γ, the variance of the estimation, smaller. This means that when the agent is more attentive to the information, she can process the signal in a more accurate way. We can see that when $a = 0$, i.e., in our special case, the investor considers the variance of the estimation to be larger. Since the investor cannot obtain additional information in this case, the estimate is not accurate enough.

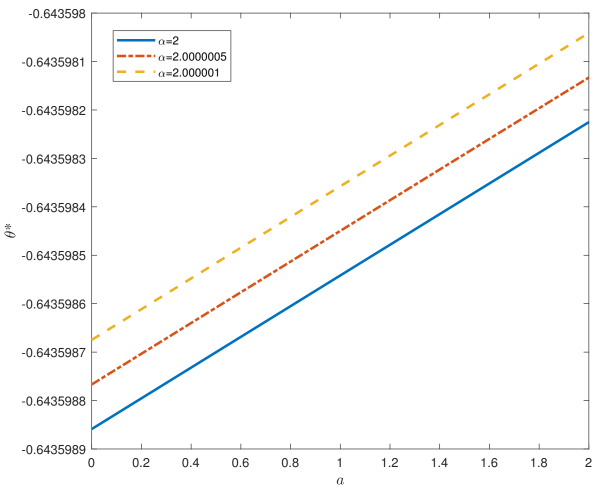

Figure 7. Effects of parameters a and α on θ^*.

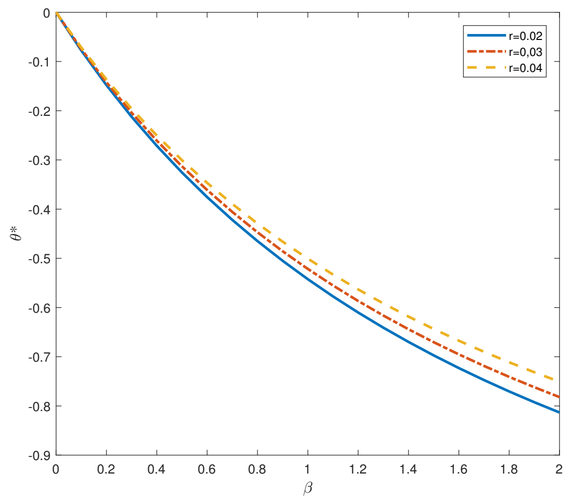

Figure 8. Effects of parameters β and r on θ^*.

Figure 7 illustrates the effect of the agent's attention a and the risk aversion coefficient α on the worst-case scenario θ^*. As shown in (2.20), the stock's expected return rate under the worst-case scenario is $\hat{\mu}_t + \sigma_S \theta_t^*$. With the parameters values in our numerical example, θ^* is negative while $\hat{\mu} + \sigma_S \theta^*$ is positive. $\hat{\mu}$ is actually the expected return rate under the reference model, a negative value θ^* means the worst-case expected return rate is lower than the reference model. The smaller is the absolute value of θ^*, the closer is the worst-case expected return rate to the reference model. Figure 7 shows that when the agent's attention a is higher, she is more confident on her estimation. When a equals 0, the value of θ^* is very small, which means that the investor is very ambiguity averse, and correspondingly, the worst-case model deviates further from the reference model. Thus the worst-case scenario deviates less from the reference model, and the worst-case expected return rate is better, which is consistent with the higher stock proportion for a higher attention in Figure 1. Regarding the relation to the risk aversion coefficient α, the higher is the α, the less the agent invests in the stock, the better is the worst-case expected return rate.

Figure 8 displays the effect of the agent's ambiguity aversion coefficient β and the risk-free rate r on the worst-case scenario θ^*. When $\beta = 0$, we can easily see from the Figure 8 that the worst-case scenario θ^* equals 0 which means the agent is ambiguity neural and she does not doubt her reference model. So the probability measure does not change at all. For higher levels of ambiguity aversion, the worst-case scenario θ^* is lower. The term $\hat{\mu}_t + \sigma_S \theta_t^*$ becomes smaller recalling (2.20) which means the lower stock's expected return rate under the worst-case scenario. In addition, the absolute value of θ^* increases in this case, the worst-case expected return rate is far away from the reference model. Therefore, she has to decrease her investment in the stock to follow a more conservative strategy corresponding to Figure 5. Obviously, the worst-case scenario θ^* is growing as the risk-free rate increases.

7. Conclusions and future directions

In this paper, we study the optimal strategies for an ambiguity-averse agent in the financial market with the agent's limited attention. Specifically, the agent can obtain an additional information related to the unobservable stock return to determine optimal decisions. Technically, this useful information will be presented as a signal form in the model. The optimization procedure can be decomposed into three stages. We firstly apply the standard Kalman filter to estimate optimal evolution of the stock return and obtain the approximating model. Then we regard this approximating model as the reference model for the robust optimal control problem. With the corresponding penalty function for alterative models, we finally find the worst-case scenario by first order condition and derive the closed-form solution of the robust optimization problem by stochastic dynamic programming. This study shows that both agent's attention and ambiguity aversion impact the levels and structures of the optimal strategy and the worst-case scenario, even the variance of the estimation return. We also presented a special case and discuss the difference between the general model with limited attention and the special case without considering the attention to highlight the importance of the agent's attention.

In the future, we consider more general settings. For example, instead of assuming a constant level of attention from the investor, we can introduce a cost function that is a deterministic function of the investor's attention, which means that a certain cost must be incurred to obtain higher levels of attention on the information. The concept of an investor's attention can also be viewed as a control that the agent can choose based on a certain cost. However, the optimal control problem will become significantly more complex as we need to search for not only the optimal strategy but also the optimal attention to information [35].

Empirical studies can also be conducted on the information processing cost and the investment strategies to validate our model and to understand the implications of technology developments on investment behaviour as well as on real economy [36–40].

Furthermore, we can also consider the topic of an investor's attention allocation across different financial assets. All these attempts allow us to explore how the ambiguity-aversion coefficient interacts with attention, which has significant economic implications. Moreover, by considering different utility functions or mean-variance criteria, we can analyze how attention affects the agent's decision-making process in various settings.

Another relevant area of study is the data-driven α-robust portfolio optimization method, as presented in [41]. By incorporating limited attention, we can further improve the performance of this approach.

In addition to portfolio optimization, attention can also play a crucial role in sustainable investment. Several studies have explored the intersection of limited attention and sustainable investment [42–45]. By considering both factors, we can devise more effective and ethical investment strategies that address both financial and social concerns.

However, incorporating limited attention into these models makes them more complex to solve. To address this challenges, we can adopt innovative mathematic methods, the martingale approach, or backward stochastic differential equations to efficiently solve these models. These methods allow us to take into account limited attention while maintaining the accuracy and validity of our results.

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Conflict of interest

The authors declare there is no conflicts of interest.

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