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Research article

Spectral analysis of discontinuous Sturm-Liouville operators with Herglotzs transmission

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Abstract: In this paper, we study the spectral properties of the Sturm-Liouville operator with eigenparameter-dependent boundary conditions and transmission conditions. In details, we introduce a Hilbert space formula, so that the problem we consider can be interpreted as an eigenvalue problem of an self-adjoint operator. Moreover, the Green's function and the resolvent of the related linear operator are obtained.

Keywords: spectral analysis; Sturm-Liouville operator; transmission conditions; eigenparameter dependent boundary condition; eigenvalue problem

1. Introduction

In the present work, we shall investigate the spectra of the Sturm-Liouville equation

$$Ly := -p(x)y'' + q(x)y = \lambda y,\ \ x \in J \equiv [-a, 0) \cup (0, b], \tag{1.1}$$

with the boundary condition

$$L_1 y := \lambda(\alpha_1' y(-a) - \alpha_2' y'(-a)) - (\alpha_1 y(-a) - \alpha_2 y'(-a)) = 0, \tag{1.2}$$

$$L_2 y : = \lambda(\beta_1' y(b) - \beta_2' y'(b)) + (\beta_1 y(b) - \beta_2 y'(b)) = 0. \tag{1.3}$$

The spectral parameters not only appear in boundary condition, but also depend on the Herglotzs function

$$\Delta' y := y'(0^+) - y'(0^-) = -y(0^+)\Big(\lambda\eta - \xi - \sum_{i=1}^{N} \frac{b_i^2}{\lambda - c_i}\Big), \tag{1.4}$$

$$\Delta y := y(0^+) - y(0^-) = y'(0^-)\Big(\lambda\kappa + \zeta - \sum_{j=1}^{M} \frac{a_j^2}{\lambda - d_j}\Big). \tag{1.5}$$

Here, $p(x) = \frac{1}{p_1^2}, x \in [-a, 0)$ and $p(x) = \frac{1}{p_2^2}, x \in (0, b]$; $q(x)$ is real valued continuous function in J; p_i, α_i, β_i, α_i' and β_i' $(i = 1, 2)$ are nonzero real numbers. In the Herglotzs function, all parameters satisfy the following conditions: $a_j, b_i > 0, c_i < c_{i+1}, d_j < d_{j+1}, i = \overline{1, N-1}, j = \overline{1, M-1}$; $\eta, \kappa > 0$, $\xi, \zeta \in \mathbb{R}$ and $N, M \in \mathbb{N}_0$. Let

$$\mu(\lambda) = \lambda\eta + \xi + \sum_{i=1}^{N} \frac{b_i^2}{\lambda - c_i}, \nu(\lambda) = \lambda\kappa + \zeta - \sum_{j=1}^{M} \frac{a_j^2}{\lambda - d_j}. \tag{1.6}$$

Then according to the properties of Herglotzs function (see [1]), we know that

$$\frac{1}{\mu(\lambda)} = \sigma - \sum_{i=1}^{N'} \frac{\varepsilon_i^2}{\lambda - \gamma_i}, \ \frac{1}{\nu(\lambda)} = \tau + \sum_{j=1}^{M'} \frac{\epsilon_j^2}{\lambda - \delta_j},$$

where $\sigma, \tau \in \mathbb{R}$ and $\varepsilon_i, \epsilon_j > 0$, for $i = \overline{1, N'-1}$, $j = \overline{1, M'-1}$; $\gamma_i < \gamma_{i+1}$, $\delta_j < \delta_{j+1}$. Therefore

$$y(0^+) = \frac{1}{\mu(\lambda)}\Delta' y, \ y'(0^-) = \frac{1}{\nu(\lambda)}\Delta y. \tag{1.7}$$

In many mathematical and physical models, it is necessary to study the eigenvalue of Sturm-Liouville problem and its corresponding eigenfunction. When the spectral parameters appear not only in differential equations, but also in boundary conditions, excellent results have been obtained (see [2–4]). Binding firstly studied the eigenvalue problem of Sturm-Liouville operator with boundary conditions dependent on spectral parameters

$$(a_k\lambda + b_k)y(0) = (c_k\lambda + d_k)(py)'(0), \ \ (-1)^k(a_kd_k - b_kc_k) \leq 0, k = 0, 1.$$

In addition, similar problems for differential equations with continuous coefficient ($p(x) \equiv 1$) and boundary conditions with spectral parameter were investigated in [5–7] and other works.

It is noteworthy that the boundary condition or the coefficient in the equation in the above results are all continuous. Now, the question is when the coefficient in the equation and the boundary condition are both discontinuous, could we still obtain the spectral properties of the linear eigenvalue problems? We know that discontinuous boundary value problems can also be found in many physical problems, such as, diffraction problem (see [8]), heat and mass transfer problem (see [9]) and vibrating problem (see [10]). To deal with the discontinuities, some conditions are necessary, such as, point interactions, impulsive conditions, transmission conditions, jump conditions or interface conditions (see [11–13]). For example, in [14], the author considered transmission conditions at one point and found asymptotic formulas of eigenvalues and corresponding eigenfunctions. Moreover, for similar problems, the work of literature [15] focused on Sturm-Liouville operators with a finite number of transmission conditions and established the self-adjointness of linear operator in a suitable Hilbert space.

Aspired by the above results, we consider the eigenvalue problems (1.1)–(1.5), where the coefficient $p(x)$ is discontinuous. In details, we consider the Sturm-Liouville equation in which the first coefficient may have the discontinuity at one point. Moreover, we allow boundary conditions and transmission conditions (Nevanlinna-Herglotz functions) to depend on spectral parameters. In Section 2, linear operator formulation is established, and the problems (1.1)–(1.5) can be interpreted as the eigenvalue problem of linear operator. The fundamental solutions and characteristic determinant are given in

Section 3. Based on the operator formulation in the Hilbert space, the resolvent operator and self-adjointness of linear operator are constructed in the last Section.

Finally, for the sake of reader's convenience, let us introduce the properties of the Nevanlinna-Herglotz function as follows.

(i) if $\lambda = c_i$ (c_i is the pole of $\mu(\lambda)$), then transmission condition (1.4) and (1.5) degenerates into $y(0^+) = 0$ and $y'(0^-)\nu(\lambda) = -y(0^-)$;

(ii) if $\mu(\lambda) = 0$ ($\lambda \neq c_i$ is the zero of $\mu(\lambda)$), then $\Delta' y = 0$;

(iii) if $\lambda = d_j$ (d_j is the pole of $\nu(\lambda)$), then transmission condition (1.4) and (1.5) degenerates into $y'(0^-) = 0$ and $y(0^+)\mu(\lambda) = y'(0^+)$;

(iv) if $\nu(\lambda) = 0$ ($\lambda \neq d_j$ is the zero of $\nu(\lambda)$), then $\Delta y = 0$.

2. The operator eigenvalue problem

In this section, we define a special inner product in the Hilbert space

$$\mathcal{H} = L^2(-a,b) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}^{N'} \oplus \mathbb{C}^{M'}$$

and a linear operator A defined on $\mathcal{H}$. Moreover, the Sturm-Liouville problems (1.1)–(1.5) can be considered as the operator eigenvalue problem.

Define

$$\rho_1 := \begin{vmatrix} \alpha_1' & \alpha_1 \\ \alpha_2' & \alpha_2 \end{vmatrix} > 0, \ \rho_2 := \begin{vmatrix} \beta_1' & \beta_1 \\ \beta_2' & \beta_2 \end{vmatrix} > 0.$$

For convenience's sake, we use the following notations:

$$\begin{gathered} \mathbf{f}^1 := (f_1^1, f_2^1, \cdots, f_{N'}^1)^T, \ \mathbf{f}^2 := (f_1^2, f_2^2, \cdots, f_{M'}^1)^T; \\ f_1 := \alpha_1' f(-a) - \alpha_2' f'(-a), \ f_2 := \beta_1' f(b) - \beta_2' f'(b), \\ f^1 := \alpha_1 f(-a) - \alpha_2 f'(-a), \ f^2 := \beta_1 f(b) - \beta_2 f'(b). \end{gathered}$$

For η, $\kappa > 0$, we define a new inner product in $\mathcal{H}$ by

$$\begin{aligned} \langle F, G\rangle :=& p_1^2 \int_{-a}^{0} f(x)\bar{g}(x)dx + p_2^2 \int_0^b f(x)\bar{g}(x)dx \\ &+ \frac{1}{\rho_1} f_1 \bar{g}_1 + \frac{1}{\rho_2} f_2 \bar{g}_2 + \langle \mathbf{f}^1, \mathbf{g}^1\rangle_1 + \langle \mathbf{f}^2, \mathbf{g}^2\rangle_1 \end{aligned} \tag{2.1}$$

for

$$F := (f, f_1, f_2, \mathbf{f}^1, \mathbf{f}^2)^T, \ G := (g, g_1, g_2, \mathbf{g}^1, \mathbf{g}^2)^T \in \mathcal{H},$$

where $\langle \cdot, \cdot \rangle_1$ denotes Euclidean inner product.

In the Hilbert space $\mathcal{H}$, we consider the operator A which is defined by

$$A \begin{pmatrix} f \\ f_1 \\ f_2 \\ \mathbf{f}^1 \\ \mathbf{f}^2 \end{pmatrix} = \begin{pmatrix} Lf \\ f^1 \\ -f^2 \\ \boldsymbol{\varepsilon}\Delta' f + [\gamma_i]\mathbf{f}^1 \\ \boldsymbol{\epsilon}\Delta f + [\delta_j]\mathbf{f}^2 \end{pmatrix} = \begin{pmatrix} Lf \\ \alpha_1 f(-a) - \alpha_2 f'(-a) \\ -(\beta_1 f(b) - \beta_2 f'(b)) \\ \boldsymbol{\varepsilon}\Delta' f + [\gamma_i]\mathbf{f}^1 \\ \boldsymbol{\epsilon}\Delta f + [\delta_j]\mathbf{f}^2 \end{pmatrix}$$

with the domain

$$D(A)=\{F=(f,f_1,f_2,\mathbf{f}^1,\mathbf{f}^2)^T: f\in AC[-a,b], f'\in AC[-a,0)\cup(0,b], Lf\in L^2(-a,b),$$
$$-f(0^+)+\sigma\Delta' f-\langle \mathbf{f}^1,\boldsymbol{\varepsilon}\rangle_1=0,\ f'(0^-)-\tau\Delta f-\langle \mathbf{f}^2,\boldsymbol{\epsilon}\rangle_1=0\},$$

where $[\gamma_i]:=\operatorname{diag}(\gamma_1,\cdots,\gamma_{N'})$, $[\delta_j]:=\operatorname{diag}(\delta_1,\cdots,\delta_{M'})$, $\boldsymbol{\varepsilon}:=(\varepsilon_i)$ and $\boldsymbol{\epsilon}:=(\epsilon_j)$.

Lemma 2.1. *The domain $D(A)$ is dense in $\mathcal{H}$.*

Proof. Let $W=(w,f_1,f_2,\mathbf{f}^1,\mathbf{f}^2)^T\in\mathcal{H}$, where $w\in C^\infty[-a,0)\cup(0,b]$ satisfying

$$w(-a)=w'(-a)=0, w(b)=w'(b)=0$$

and the condition

$$w(0^-)=\sigma\langle \mathbf{f}^1,\boldsymbol{\varepsilon}\rangle_1+(1-\sigma)\langle \mathbf{f}^2,\boldsymbol{\epsilon}\rangle_1,\ w(0^+)=(\sigma-1)\langle \mathbf{f}^1,\boldsymbol{\varepsilon}\rangle_1-\sigma\langle \mathbf{f}^2,\boldsymbol{\epsilon}\rangle_1,$$
$$w'(0^-)=-\tau\langle \mathbf{f}^1,\boldsymbol{\varepsilon}\rangle_1+(\tau+1)\langle \mathbf{f}^2,\boldsymbol{\epsilon}\rangle_1,\ w'(0^+)=(1-\tau)\langle \mathbf{f}^1,\boldsymbol{\varepsilon}\rangle_1+\tau\langle \mathbf{f}^2,\boldsymbol{\epsilon}\rangle_1.$$

Meanwhile,

$$\Delta w=\langle \mathbf{f}^2,\boldsymbol{\epsilon}\rangle_1-\langle \mathbf{f}^1,\boldsymbol{\varepsilon}\rangle_1,\ \Delta' w=\langle \mathbf{f}^1,\boldsymbol{\varepsilon}\rangle_{1-}\langle \mathbf{f}^2,\boldsymbol{\epsilon}\rangle_1.$$

Then, it is easy to verify that $W\in D(A)$. Next, as long as it is proved that the elements in $\mathcal{H}$ can be approximated by the elements in $D(A)$, the desired result can be obtained.

Since

$$(C_0^\infty(-a,0)\oplus C_0^\infty(0,b))\oplus\{0\}\oplus\{0\}\oplus\{\mathbf{0}\}\oplus\{\mathbf{0}\}\subseteq D(A)$$

and

$$\overline{(C_0^\infty(-a,0)\oplus C_0^\infty(0,b))}\supset L^2(-a,b),$$

there exists a sequence $\{m_n\}\in C_0^\infty(-a,0)\oplus C_0^\infty(0,b)$ with $m_n\to f-w$ as $n\to\infty$, where $M_n:=(m_n,0,0,\mathbf{0},\mathbf{0})^T\in D(A)$. Therefore, $W+M_n\to F$ as $n\to\infty$ giving that $\overline{D(A)}\supset\mathcal{H}$.

Theorem 2.1. *The operator eigenvalue problem $AF=\lambda F$ and the considered Sturm-Liouville problems (1.1)–(1.5) are equivalent and the eigenfunction is the first components of the corresponding eigenelements of the operator A. Moreover, for η, $\kappa>0$, we have following results:*

(i) if $\lambda\neq\gamma_i\ \forall i=\overline{1,N'}$, then $\mathbf{f}^1=(\lambda I-[\gamma_i])^{-1}\boldsymbol{\varepsilon}\Delta' f$; if $\lambda=\gamma_I\ \exists I\in\{\overline{1,N'}\}$, then $\mathbf{f}^1=\frac{-f(0^+)}{\varepsilon_I}e^I$;

(ii) if $\lambda\neq\delta_j\ \forall j=\overline{1,M'}$, then $\mathbf{f}^2=(\lambda I-[\delta_i])^{-1}\boldsymbol{\epsilon}\Delta f$; if $\lambda=\gamma_J\ \exists J\in\{\overline{1,M'}\}$, then $\mathbf{f}^2=\frac{f(0^-)}{\epsilon_J}e^J$, where $\mathbf{e}^n$ is the vector in $\mathbb{R}^n$ with and except the n-th element is 1, all other elements are 0.

Proof. We just need to show that the eigenelement f of the operator A obeys the boundary conditions (1.2) and (1.3) and transfer conditions (1.4) and (1.5). It is clear that f satisfies (1.2) and (1.3). The definition of A implies $\gamma_i f_i^1+\varepsilon_i\Delta' f=\lambda f_i^1$ for all i. Meanwhile, the domain of A gives $-f(0^+)+\sigma\Delta' f-\langle \mathbf{f}^1,\boldsymbol{\varepsilon}\rangle_1=0$. Thus, if $\lambda\neq\gamma_i$ for all i, then

$$f(0^+)=\Big(\sigma-\sum_{i=1}^{N'}\frac{\varepsilon_i^2}{\lambda-\gamma_i}\Big)\Delta' f.$$

If $\lambda=\gamma_I$ for some $I\in\{1,\cdots,N'\}$, then $-f(0^+)-\langle f_I^1,\varepsilon_I\rangle_1=0$. That is, $f_I^1=\frac{-f(0^+)}{\varepsilon_I}$. Hence, f satisfies (1.4).

Similarly, if $\lambda \neq \delta_j$ for all j, then $f_j^2 = \frac{\epsilon_j}{\lambda-\delta_j}\Delta f$ and

$$f'(0^-) = \Big(\tau + \sum_{j=1}^{M'} \frac{\epsilon_j^2}{\lambda - \delta_j}\Big)\Delta f.$$

While $\lambda = \delta_J$ for some $j \in \{1, \cdots, M'\}$, the domain condition forces $f_J^2 = \frac{f'(0^-)}{\epsilon_J}$ from which (1.5) follows.

Theorem 2.2. *The linear operator A is symmetric.*

Proof. Let $F, G \in D(A)$. Then it follows from the problem (1.1) and the relation (2.1) that

$$\begin{aligned}\langle AF, G\rangle - \langle F, AG\rangle =&(f\bar{g}')(0^-) - (f'\bar{g})(0^-) + (f'\bar{g})(0^+) - (f\bar{g}')(0^+)\\ &+ (f'\bar{g})(-a) - (f\bar{g}')(-a) + (f\bar{g}')(b) - (f'\bar{g})(b)\\ &+ \frac{1}{\rho_1} f^1\bar{g}_1 - \frac{1}{\rho_2} f^2\bar{g}_2 + \langle \boldsymbol{\varepsilon}\Delta' f + [\gamma_i]\mathbf{f}^1, \mathbf{g}^1\rangle_1 + \langle \boldsymbol{\epsilon}\Delta f + [\delta_j]\mathbf{f}^2, \mathbf{g}^2\rangle_1\\ &- \frac{1}{\rho_1} f_1\bar{g}^1 + \frac{1}{\rho_2} f_2\bar{g}^2 - \langle \mathbf{f}^1, \boldsymbol{\varepsilon}\Delta' g + [\gamma_i]\mathbf{g}^1\rangle_1 - \langle \mathbf{f}^2, \boldsymbol{\epsilon}\Delta g + [\delta_j]\mathbf{g}^2\rangle_1.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\rho_1^{-1}(f^1\bar{g}_1 - f_1\bar{g}^1) &= (f\bar{g}')(-a) - (\bar{g}f')(-a),\\ \rho_2^{-1}(f^2\bar{g}_2 - f_2\bar{g}^2) &= -((f\bar{g}')(b) - (\bar{g}f')(b)).\end{aligned}$$

Meanwhile, the vector components satisfy

$$\begin{aligned}\langle \boldsymbol{\varepsilon}\Delta' f + [\gamma_i]\mathbf{f}^1, \mathbf{g}^1\rangle_1 - \langle \mathbf{f}^1, \boldsymbol{\varepsilon}\Delta' g + [\gamma_i]\mathbf{g}^1\rangle_1 &= \langle \boldsymbol{\varepsilon}\Delta' f, \mathbf{g}^1\rangle_1 - \langle \mathbf{f}^1, \boldsymbol{\varepsilon}\Delta' g\rangle_1,\\ \langle \boldsymbol{\epsilon}\Delta f + [\delta_j]\mathbf{f}^2, \mathbf{g}^2\rangle_1 - \langle \mathbf{f}^2, \boldsymbol{\epsilon}\Delta g + [\delta_j]\mathbf{g}^2\rangle_1 &= \langle \boldsymbol{\epsilon}\Delta f, \mathbf{g}^2\rangle_1 - \langle \mathbf{f}^2, \boldsymbol{\epsilon}\Delta g\rangle_1,\end{aligned}$$

and the domain condition $D(A)$ implies

$$\begin{aligned}\langle \boldsymbol{\varepsilon}\Delta' f, \mathbf{g}^1\rangle_1 - \langle \mathbf{f}^1, \boldsymbol{\varepsilon}\Delta' g\rangle_1 &= \Delta' f[-\bar{g}(0^+) + \sigma\Delta'\bar{g}] - \Delta'\bar{g}[-f(0^+) + \sigma\Delta' f],\\ \langle \boldsymbol{\epsilon}\Delta f, \mathbf{g}^2\rangle_1 - \langle \mathbf{f}^2, \boldsymbol{\epsilon}\Delta g\rangle_1 &= \Delta f[\bar{g}'(0^-) - \tau\Delta\bar{g}] - \Delta\bar{g}[f'(0^-) - \tau\Delta f].\end{aligned}$$

Therefore,

$$\begin{aligned}\langle \boldsymbol{\varepsilon}\Delta' f + [\gamma_i]\mathbf{f}^1, \mathbf{g}^1\rangle_1 - \langle \mathbf{f}^1, \boldsymbol{\varepsilon}\Delta' g + [\gamma_i]\mathbf{g}^1\rangle_1 &= \Delta'\bar{g}f(0^+) - \Delta' f\bar{g}(0^+),\\ \langle \boldsymbol{\epsilon}\Delta f + [\delta_j]\mathbf{f}^2, \mathbf{g}^2\rangle_1 - \langle \mathbf{f}^2, \boldsymbol{\epsilon}\Delta g + [\delta_j]\mathbf{g}^2\rangle_1 &= \Delta f\bar{g}'(0^-) - \Delta\bar{g}f'(0^-).\end{aligned}$$

It can be obtained by simple calculation

$$\begin{aligned}&(f'\bar{g} - f\bar{g}')(0^+) - (f'\bar{g} - f\bar{g}')(0^-)\\ =&\bar{g}(0^+)\Delta' f - f(0^+)\Delta'\bar{g} - \bar{g}'(0^-)\Delta f + f'(0^-)\Delta\bar{g}.\end{aligned}$$

Thus, $\langle AF, G\rangle - \langle F, AG\rangle = 0$ and so A is symmetric.

Corollary 2.1. *All eigenvalues of the Sturm-Liouville problems (1.1)–(1.5) are real.*

Corollary 2.2. *Let λ_1 and λ_2 be two different eigenvalues of the Sturm-Liouville problems (1.1)–(1.5). Then the corresponding the eigenfunctions $u_1(x)$ and $u_2(x)$ are orthogonal, i.e.,*

$$\begin{aligned}&p_1^2\int_{-a}^{0} u_1(x)u_2(x)dx + p_2^2\int_0^b f_1(x)f_2(x)dx + \frac{1}{\rho_1} f_1(u_1)f_1(u_2)\\ &+\frac{1}{\rho_2} f_2(u_1)f_2(u_2) + \langle \mathbf{f}^1(u_1), \mathbf{g}^1(u_2)\rangle_1 + \langle \mathbf{f}^2(u_1), \mathbf{g}^2(u_2)\rangle_1 = 0.\end{aligned}$$

3. Fundamental solutions and characteristic determinant

Lemma 3.1. ([5]) *All eigenvalues of Sturm-Liouville problems (1.1)–(1.5), not at poles of $\mu(\lambda)$ or $\nu(\lambda)$, are geometrically simple. In the case, Herglotz condition (1.4) and (1.5) can be transformed into*

$$\begin{pmatrix} y(0^+) \\ y'(0^+) \end{pmatrix} = \begin{pmatrix} 1 & \nu(\lambda) \\ \mu(\lambda) & 1+\mu(\lambda)\nu(\lambda) \end{pmatrix} \begin{pmatrix} y(0^-) \\ y'(0^-) \end{pmatrix}.$$

Lemma 3.2. ([16]) *Let $q(x) \in C[-a,b]$, and $f(\lambda)$ and $g(\lambda)$ be given entire functions. Then for $\forall\lambda \in \mathbb{C}$, the equation*

$$-p(x)u'' + q(x)u = \lambda u, \quad x \in [-a,b]$$

has a unique solution $u = u(x,\lambda)$ satisfying the initial conditions

$$u(a) = f(\lambda),\ u'(a) = g(\lambda)\ (\text{or } u(b) = f(\lambda),\ u'(b) = g(\lambda)).$$

Moreover, for each fixed $x \in [-a,b]$, $u = u(x,\lambda)$ is an entire function of λ.

Lemma 3.3. *Let $u_-(x,\lambda), x \in [-a,0)$ be the solution of the Sturm-Liouville problem (1.1) satisfying conditions*

$$u_-(-a) = -\alpha_2 + \lambda\alpha_2',\ \ u_-'(-a) = -\alpha_1 + \lambda\alpha_1' \tag{3.1}$$

and $v(x,\lambda), x \in (0,b]$ denote the solution of the Sturm-Liouville problem (1.1) satisfying the conditions

$$v_+(b) = -\beta_2 + \lambda\beta_2',\ \ v_+'(b) = -\beta_1 + \lambda\beta_1'. \tag{3.2}$$

Then the Wronskian $W[u_-,v_+] = u_-v_+' - v_+u_-'$ is independent of x.

Proof. Direct computation, we have

$$\begin{aligned} \frac{\partial}{\partial x}W[u_-(x,\lambda),v_+(x,\lambda)] &= u_-(x,\lambda)\frac{\partial^2}{\partial x^2}v_+(x,\lambda) - v_+(x,\lambda)\frac{\partial^2}{\partial x^2}u_-(x,\lambda) \\ &= \frac{qv_+ - \lambda v_+}{p}u_- - \frac{qu_- - \lambda u_-}{p}v_+ = 0. \end{aligned}$$

It follows that the Wronskian $W[u_-(x,\lambda),v_+(x,\lambda)]$ is constant on $[-a,0)\cup(0,b]$ and by virtue of Lemma 3.2, it is a function of λ.

We know that the problem (1.1) exists two fundamental solutions on whole $[-a,0)\cup(0,b]$ satisfying the boundary conditions (1.2)–(1.5). First, we extend $u_-(x,\lambda)$ and $v_+(x,\lambda)$ by the zero function to $[-a,0)\cup(0,b]$, i.e., we define

$$u_-(x,\lambda) = \begin{cases} u_-(x,\lambda), & x \in [-a,0), \\ 0, & x \in (0,b] \end{cases}$$

and

$$v_+(x,\lambda) = \begin{cases} 0, & x \in [-a,0), \\ v_+(x,\lambda), & x \in (0,b]. \end{cases}$$

Now by virtue of Lemma 3.1, it's workable to extend $u_-(x,\lambda)$, $x\in[-a,0)$ and $v_+(x,\lambda)$, $x\in(0,b]$ by nontrivial solution $u_+(x,\lambda)$, $x\in(0,b]$ and $v_-(x,\lambda)$, $x\in[-a,0)$ satisfying the conditions

$$\begin{pmatrix}u_+(0^+)\\ u_+'(0^+)\end{pmatrix}=\begin{pmatrix}1 & \nu(\lambda)\\ \mu(\lambda) & 1+\mu(\lambda)\nu(\lambda)\end{pmatrix}\begin{pmatrix}u_-(0^-)\\ u_-'(0^-)\end{pmatrix}$$

and

$$\begin{pmatrix}v_-(0^-)\\ v_-'(0^-)\end{pmatrix}=\begin{pmatrix}1+\mu(\lambda)\nu(\lambda) & -\nu(\lambda)\\ -\mu(\lambda) & 1\end{pmatrix}\begin{pmatrix}v_+(0^+)\\ v_+'(0^+)\end{pmatrix}.$$

Moreover, we define two linearly independent solutions of the problem (1.1) on the whole $[-a,0)\cup(0,b]$ as

$$u(x,\lambda)=\begin{cases}u_-(x,\lambda), & x\in[-a,0),\\ u_+(x,\lambda), & x\in(0,b],\end{cases}\tag{3.3}$$

$$v(x,\lambda)=\begin{cases}v_-(x,\lambda), & x\in[-a,0),\\ v_+(x,\lambda), & x\in(0,b].\end{cases}\tag{3.4}$$

It note that u and v must satisfy boundary conditions (1.2)–(1.5). Let

$$y(x,\lambda)=\varphi(\lambda)u(x,\lambda)+\psi(\lambda)v(x,\lambda).\tag{3.5}$$

According to (1.4) and (1.5), we have

$$y(0^+)(\lambda\eta-\xi)+\Delta' y=y(0^+)\sum_{i=1}^{N}\frac{b_i^2}{\lambda-c_i},$$

$$-y'(0^-)\lambda(\kappa+\zeta)+\Delta y=-y'(0^-)\sum_{j=1}^{M}\frac{a_j^2}{\lambda-d_j}.$$

Moreover, we define

$$\begin{cases}U_1(y,\lambda):=-(y(0^+)(\lambda\eta-\xi)+\Delta' y)\prod_{i=1}^{N}(\lambda-c_i)+y(0^+)\sum_{i=1}^{N}b_i^2\prod_{k\neq i}(\lambda-c_k)=0,\\[2ex] U_2(y,\lambda):=(y'(0^-)(\lambda\kappa+\zeta)+\Delta y)\prod_{j=1}^{M}(\lambda-d_i)-y'(0^-)\sum_{j=1}^{M}a_j^2\prod_{k\neq j}(\lambda-d_k)=0\end{cases}\tag{3.6}$$

and any solution to problem (1.1) on $[-a,0)\cup(0,b]$ satisfying the boundary conditions (1.2) and (1.3) must be of the form (3.5). Let

$$\omega(\lambda)=\det\begin{pmatrix}U_1(u,\lambda) & U_1(v,\lambda)\\ U_2(u,\lambda) & U_2(v,\lambda)\end{pmatrix}.$$

Therefore, $\omega(\lambda)$ will be referred to as the characteristic determinant of (1.1)–(1.5). It is shown in Theorem 3.1 below that $\omega(\lambda)$ has the properties expected of the characteristic determinant.

Theorem 3.1. *The eigenvalue λ of the Sturm-Liouville problems (1.1)–(1.5) consists of the zero of the characteristic determinant.*

Proof. The relation (3.6) implies

$$U_k(y,\lambda) = \varphi(\lambda)U_k(u,\lambda) + \psi(\lambda)U_k(v,\lambda), \quad k = 1,2, \tag{3.7}$$

That is, λ is an eigenvalue of Sturm-Liouville problems (1.1)–(1.5) if and only if $U_1(y,\lambda) = 0$, $U_2(y,\lambda) = 0$. Moreover, these two equations exist nontrivial solution $\varphi(\lambda)$ and $\psi(\lambda)$ if and only if

$$\omega(\lambda) = \det\begin{pmatrix} U_1(u,\lambda) & U_1(v,\lambda) \\ U_2(u,\lambda) & U_2(v,\lambda)\end{pmatrix} = 0.$$

Therefore, we proved that the eigenvalue of Sturm-Liouville problems (1.1)–(1.5) coincides with the zero of $\omega(\lambda)$.

Similar to Lemma 3.3, let $W[u(x,\lambda), v(x,\lambda)] =: \varpi(\lambda)$. In view of Theorem 3.1, we define

$$g(x,\lambda) := \frac{v(x,\lambda)}{\varpi(\lambda)}\int_{-a}^{x} u(t,\lambda)h(t)dt + \frac{u(x,\lambda)}{\varpi(\lambda)}\int_{x}^{b} v(t,\lambda)h(t)dt, \; h \in L^2(-a,b).$$

and the Green's function of the Sturm-Liouville problems (1.1)–(1.5) is given by

$$G(x,t) = \begin{cases} \frac{u(t,\lambda)v(x,\lambda)}{\varpi(\lambda)}, & t < x, \; t \in [-a,0)\cup(0,b], \\ \frac{v(t,\lambda)u(x,\lambda)}{\varpi(\lambda)}, & x < t, \; t \in [-a,0)\cup(0,b]. \end{cases}$$

Theorem 3.2. *Let*

$$g(x,\lambda) = \int_{-a}^{b} G(x,t)h(t)dt := Th. \tag{3.8}$$

Then $g(x,\lambda)$ is the solution of operator equation $(\lambda - L)g = ph$ on J. Moreover, g satisfies the boundary conditions (1.2)–(1.5).

Proof. The relation (3.8) implies

$$g\varpi(\lambda) = v(x,\lambda)\int_{-a}^{x} u(t,\lambda)h(t)dt + u(x,\lambda)\int_{x}^{b} v(t,\lambda)h(t)dt. \tag{3.9}$$

Furthermore, we have

$$\frac{\partial}{\partial x}g\varpi(\lambda) = \frac{\partial}{\partial x}v(x,\lambda)\int_{-a}^{x} u(t,\lambda)h(t)dt + \frac{\partial}{\partial x}u(x,\lambda)\int_{x}^{b} v(t,\lambda)h(t)dt \tag{3.10}$$

and

$$\begin{aligned} p\frac{\partial^2}{\partial x^2}g\varpi(\lambda) =& \frac{\partial^2}{\partial x^2}v(x,\lambda)\int_{-a}^{x} u(t,\lambda)h(t)dt + \frac{\partial^2}{\partial x^2}u(x,\lambda)\int_{x}^{b} v(t,\lambda)h(t)dt + ph(x)\varpi \\ =&(q-\lambda)g\varpi(\lambda) + ph\varpi(\lambda). \end{aligned} \tag{3.11}$$

Therefore, $(\lambda - L)g = ph$ holds.

It remains only to show that g satisfies (1.2) and (1.3), (1.4) and (1.5). By (3.9) and (3.10), we obtain

$$g(-a) = \frac{u(-a,\lambda)}{\varpi(\lambda)}\int_{-a}^{b} v(t,\lambda)h(t)dt, \quad g'(-a) = \frac{u'(-a,\lambda)}{\varpi(\lambda)}\int_{-a}^{b} v(t,\lambda)h(t)dt.$$

Moreover, we know that u satisfies (1.2). Then, g satisfies (1.2). Similarly, g satisfies (1.3). Moreover,

$$\begin{pmatrix} g(0^{\pm}) \\ g'(0^{\pm})\end{pmatrix} = \frac{1}{\varpi(\lambda)}\begin{pmatrix} v(0^{\pm}) \\ v'(0^{\pm})\end{pmatrix}\int_{-a}^{0} u(t)h(t)dt + \frac{1}{\varpi(\lambda)}\begin{pmatrix} u(0^{\pm}) \\ u'(0^{\pm})\end{pmatrix}\int_{0}^{b} v(t)h(t)dt.$$

Obviously, (1.4) and (1.5) are obeyed.

4. The resolvent operator of A

In this section, we study the resolvent operator in the Hilbert space $\mathcal{H}$. We first consider nonhomogeneous conditions

$$-\boldsymbol{\varepsilon}\Delta' f + (\lambda I - [\gamma_i])\mathbf{f}^1 = p\mathbf{h}^1, \tag{4.1}$$

$$-\boldsymbol{\epsilon}\Delta f + (\lambda I - [\delta_j])\mathbf{f}^2 = p\mathbf{h}^2. \tag{4.2}$$

Meanwhile, the domain of the operator A implies

$$-f(0^+) + \sigma\Delta' f - \langle \mathbf{f}^1, \boldsymbol{\varepsilon}\rangle_1 = 0, \tag{4.3}$$

$$f'(0^-) - \tau\Delta f - \langle \mathbf{f}^2, \boldsymbol{\epsilon}\rangle_1 = 0. \tag{4.4}$$

If $\lambda \neq \gamma_i$ for all i, then from (4.1) we have

$$-f(0^+) + \sigma\Delta' f = \langle (\lambda I - [\gamma_i])^{-1}(p\mathbf{h}^1 + \boldsymbol{\varepsilon}\Delta' f), \boldsymbol{\varepsilon}\rangle_1.$$

By inner product calculation, we get

$$-f(0^+) + \sigma\Delta' f = \sum_{i=1}^{N'}\Big(\frac{ph_i^1\varepsilon_i}{\lambda - \gamma_i} + \frac{\varepsilon_i^2\Delta' f}{\lambda - \gamma_i}\Big).$$

Therefore, by (1.7), we have

$$-f(0^+) + \frac{1}{\mu(\lambda)}\Delta' f = \langle p\mathbf{h}^1, (\lambda I - [\gamma_i])^{-1}\boldsymbol{\varepsilon}\rangle_1.$$

If $\lambda = \gamma_I$ for some $I \in \{1, \cdots, N'\}$, then from (4.1) we have $\Delta' f = -\frac{ph_I^1}{\varepsilon_I}$. For $i \in \{1, \cdots, N'\} \setminus I$, $f_i^1 = \frac{h_i^1 + \varepsilon_i\Delta' f}{\gamma_I - \gamma_i}$. Thus, from (4.3) we get

$$-f(0^+) - \sigma\frac{ph_I^1}{\varepsilon_I} - \sum_{i\neq I}\frac{\varepsilon_i}{\varepsilon_I}\frac{\varepsilon_I ph_i^1 - \varepsilon_i ph_I^1}{\gamma_I - \gamma_i} = \varepsilon_I f_I^1.$$

Similarly, if $\lambda \neq \delta_j$ for all j, then

$$f'(0^-) - \frac{1}{\nu(\lambda)}\Delta f = \langle p\mathbf{h}^2, (\lambda I - [\delta_j])^{-1}\boldsymbol{\epsilon}\rangle_1.$$

If $\lambda = \delta_J$ for some $J \in \{1, \cdots, M'\}$, then

$$f'(0^-) + \tau\frac{ph_J^2}{\epsilon_J} - \sum_{j\neq J}\frac{\epsilon_j}{\epsilon_J}\frac{\epsilon_J ph_j^2 - \epsilon_j ph_J^2}{\delta_J - \delta_j} = \epsilon_I f_J^2.$$

Therefore, the operator equation

$$(\lambda I - A)Y = H, \;\; H = (ph, ph_1, ph_2, p\mathbf{h}^1, p\mathbf{h}^2)^T \in L^2(-a, b) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}^{N'} \oplus \mathbb{C}^{M'}$$

is equivalent to the discontinuous nonhomogeneous BVP

$$-p(x)y'' + q(x)y = \lambda y(x) - ph(x), \;\; x \in J,$$

together with inhomogeneous boundary condition

$$\lambda(\alpha_1' y(-a) - \alpha_2' y'(-a)) - (\alpha_1 y(-a) - \alpha_2 y'(-a)) = ph_1,$$
$$\lambda(\beta_1' y(b) - \beta_2' y'(b)) + (\beta_1 y(b) - \beta_2 y'(b)) = ph_2$$

and transmission conditions (the case of $\lambda \neq \gamma_i$ and $\lambda \neq \delta_j$)

$$y(0^+)\mu(\lambda) - \Delta' y = \langle p\mathbf{h}^1, (\lambda I - [\gamma_i])^{-1}\boldsymbol{\varepsilon}\rangle,$$
$$y'(0^+)\nu(\lambda) - \Delta y = \langle p\mathbf{h}^2, (\lambda I - [\delta_j])^{-1}\boldsymbol{\epsilon}\rangle.$$

We consider the resolvent set $\rho(A) = \{\lambda \in \mathbb{C} | (\lambda I - A)^{-1} \in D(A)\}$. Then, we need to show $(\lambda I - A)^{-1}$ is the resolvent operator, just prove $(\lambda I - A)^{-1} \in D(A)$.

Theorem 4.1. *Let λ not be an eigenvalue of operator A. Then*

$$(\lambda I - A)^{-1} H = \begin{pmatrix} T_\lambda h \\ (T_\lambda h)_1 \\ (T_\lambda h)_2 \\ (\lambda I - [\gamma_i])^{-1}\boldsymbol{\varepsilon}\Delta' T_\lambda h \\ (\lambda I - [\delta_j])^{-1}\boldsymbol{\epsilon}\Delta' T_\lambda h \end{pmatrix} =: \tilde{G}h.$$

Proof. Obviously, the resolvent operator $(\lambda I - A)^{-1}$ exists. It remains only to show $\tilde{G}h \in D(A)$. The definition of $T_\lambda h$ and Theorem 3.2 imply that $g \in AC[-a, b]$, $g' \in AC[-a, 0) \cup (0, b]$ and obeys the boundary conditions (1.2) and (1.3). Moreover, the equalities

$$g(0^+) = (T_\lambda h)(0^+) = \Big(\sigma - \sum_{i=1}^{N'} \frac{\varepsilon_i^2}{\lambda - \gamma_i}\Big)\Delta'(T_\lambda h) = \sigma\Delta' g - \langle \mathbf{g}^1, \boldsymbol{\varepsilon}\rangle_1, \quad \lambda \neq \gamma_i$$

and

$$g'(0^-) = (T_\lambda h)'(0^-) = \Big(\tau + \sum_{j=1}^{M'} \frac{\epsilon_j^2}{\lambda - \delta_j}\Big)\Delta(T_\lambda h) = \tau\Delta' g - \langle \mathbf{g}^2, \boldsymbol{\epsilon}\rangle_1 \quad \lambda \neq \delta_j$$

hold. Meanwhile, the properties of the Nevanlinna-Herglotz function imply that if $\lambda = \gamma_I$ for some $I \in \{1, \cdots, N'\}$, then $(\Delta' T_\lambda h) = 0$ and $\mathbf{g}^1 = \frac{-y(0^+)}{\varepsilon_I} e^I$. Therefore,

$$g(0^+) = -\langle \mathbf{g}^1, \boldsymbol{\varepsilon}\rangle_1 = \sigma\Delta' g - \langle \mathbf{g}^1, \boldsymbol{\varepsilon}\rangle_1.$$

Similarly, we have

$$g'(0^-) = \langle \mathbf{g}^2, \boldsymbol{\epsilon}\rangle_1 = \tau\Delta g + \langle \mathbf{g}^1, \boldsymbol{\epsilon}\rangle_1.$$

Thus, $\tilde{G}h \in D(A)$ and the desired result holds.

Theorem 4.2. *Let $R(\lambda, A) = (\lambda I - A)^{-1}$. Then*

$$\|R(\lambda, A)H\| \leq |\mathrm{Im}\lambda|^{-1}\|H\|, \quad H \in \mathcal{H}$$

holds, where $\forall \lambda \in \mathbb{C}$ satisfies $\mathrm{Im}\lambda \neq 0$.

Proof. For $\forall\ H = (ph, ph_1, ph_2, p\mathbf{h}_3, p\mathbf{h}_4)^T \in \mathcal{H}$ and $Y = R(\lambda, A)H$. Since $(\lambda I - A)Y = H$, we have

$$\langle AY, Y\rangle = \langle \lambda Y - H, Y\rangle = \lambda\langle Y, Y\rangle - \langle H, Y\rangle$$

and

$$\langle Y, AY\rangle = \langle Y, \lambda Y - H\rangle = \bar{\lambda}\langle Y, Y\rangle - \overline{\langle H, Y\rangle},$$

which imply $|\text{Im}\lambda|\|Y\|^2 = |\text{Im}(H, Y)|$. On the other side, in view of Cauchy-Schwartz inequality, we get

$$|\text{Im}(H, Y)| \leq |(H, Y)| \leq \|H\|\|Y\|.$$

Therefore, the inequality

$$\|R(\lambda, A)H\| = \|Y\| \leq |\text{Im}\lambda|^{-1}\|H\|, \quad H \in \mathcal{H}$$

holds.

Theorem 4.3. *In Hilbert space $\mathcal{H}$, the operator A is self-adjoint.*

Proof. Obviously, A is a dense symmetric operator. To show that A is self-adjoint, it remains only to verify that $D(A^*) = D(A)$. Let $H \in D(A^*)$. Then

$$\langle AH, G\rangle = \langle H, A^*G\rangle \text{ for all } G \in D(A). \tag{4.5}$$

It follows from (4.5) that

$$\langle (iI - A)G, H\rangle = \langle G, (-iI - A^*)H\rangle. \tag{4.6}$$

Note that $\lambda = -i$ is a regular point. Then, we have

$$(iI - A)Y = -iH - A^*H, \quad Y \in D(A). \tag{4.7}$$

Substituting (4.7) in (4.6), we have

$$\langle (iI - A)G, H\rangle = \langle (iI - A)G, Y\rangle. \tag{4.8}$$

Similarly, from $\lambda = i$ is a regular point, let $G = R(i, A)(H - Y)$. Then by (4.8), we have $H = Y$ and thus $H \in D(A)$.

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Conflict of interest

The authors declare there is no conflicts of interest.

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