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Research article

Ground states of Nehari-Pohožaev type for a quasilinear Schrödinger system with superlinear reaction

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Abstract: This article is devoted to study the following quasilinear Schrödinger system with super-quadratic condition:

$$\begin{cases} -\Delta u + V_1(x)u - \Delta(u^2)u = h(u,v), \ x \in \mathbb{R}^N, \\ -\Delta v + V_2(x)v - \Delta(v^2)v = g(u,v), \ x \in \mathbb{R}^N, \end{cases}$$

where $N \geq 3$, $V_1(x)$, $V_2(x)$ are variable potentials and h, g satisfy some conditions. By establishing a suitable Nehari-Pohožaev type constraint set and considering related minimization problem, we prove the existence of ground states.

Keywords: quasilinear Schrödinger systems; ground state solutions; Pohožaev manifold

1. Introduction

In this work, we consider the following quasilinear Schrödinger system:

$$\begin{cases} -\Delta u + V_1(x)u - \Delta(u^2)u = h(u,v), \ x \in \mathbb{R}^N, \\ -\Delta v + V_2(x)v - \Delta(v^2)v = g(u,v), \ x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $N \geq 3$, $V_1(x)$, $V_2(x)$ are potential functions and h, $g : \mathbb{R}^2 \to \mathbb{R}$ are nonlinear terms. Before stating our assumptions and main result, now introduce some related results that motivate the present work briefly.

1.1. Motivation and related results

The study of (1.1) was in part motivated by the nonlinear Schrödinger equation

$$i\partial_t z = -\Delta z + W(x)z - k(x,z) - \Delta l(|z|^2)l'(|z|^2)z, \tag{1.2}$$

where $z : \mathbb{R} \times \mathbb{R}^N \to C$, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential and $l : \mathbb{R} \to \mathbb{R}$ and $k : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are suitable functions. In recent years, this form of quasilinear equations have received much attention in mathematical physics. For various types of nonlinear term $l(s)$, (1.2) can be derived as models in different areas of physics. For instance, when $l(s) = s$, (1.2) models a superfluid equation in plasma physics by Kurihara [1] (also see [2]). Moreover, in the case $l(s) = \sqrt{1+s}$, (1.2) models self-channeling of a high-power ultrashort laser in matter, see [3–6]. It also appears in the theory of Heidelberg ferromagnets magnons, in dissipative quantum mechanics and in condensed matter theory, see for instance [7, 8]. For further detailed mathematical models and physics applications of (1.2), we infer also the readers to [9–16] and the references therein.

It is well known that (1.2) can be reduced to the following equation of elliptic type when setting $z(t, x) = \exp(-iEt)u(x)$, where $E \in \mathbb{R}$ and u is a real function (see [6]):

$$-\Delta u + V(x)u - \Delta l(u^2)l'(u^2)u = k(x, u),\ x \in \mathbb{R}^N. \tag{1.3}$$

Assuming that $l(s) = s$, then (1.3) turns out to be the superfluid film equation in plasma physics

$$-\Delta u + V(x)u - \Delta(u^2)u = k(x, u),\ x \in \mathbb{R}^N. \tag{1.4}$$

To our knowledge, (1.4) has been studied extensively. Especially, the following case that $k(x, u) = |u|^{p-2}u$, $4 \leq p < \frac{4N}{N-2}$ have attracted much attention:

$$-\Delta u + V(x)u - ku\Delta(u^2) = |u|^{p-2}u,\ x \in \mathbb{R}^N. \tag{1.5}$$

Poppenberg et al. [15] proved the existence of positive ground state of (1.5) by using a constrained minimization argument. Furthermore, with the help of changing of variables and the Mountain-pass Lemma, Liu and Wang [17] proved the existence of a positive solution for (1.5) in a new Orlicz space framework. In their work, the potential $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ satisfies

(A1) $\inf\limits_{x\in\mathbb{R}^N} V(x) \geq a > 0$, $\forall M > 0$, $\text{meas}\{x \in \mathbb{R}^N | V(x) \leq M\} < +\infty$.

Such kind of hypotheses was first introduced by Bartsch and Wang [18] to guarantee the compactness of

$$E := \{u \in H^1(\mathbb{R}^N) | \int_{\mathbb{R}^N} V(x)u^2 < \infty\} \hookrightarrow L^s(\mathbb{R}^N)$$

when $2 \leq s < \frac{2N}{N-2}$. Colin and Jeanjean [19] also proved the existence of solutions for (1.5) in the usual Sobolev space $H^1(\mathbb{R}^N)$ via changing variables. Recently, Ruiz and Siciliano [20] studied (1.5) under the case $2 < p < \frac{4N}{N-2}$. They used a technique of minimizing a functional under a Pohožaev constraint to show that (1.5) has ground state solution. Moreover, Chen et al. [21] proved the existence of ground states for (1.4) under super-quadratic condition, weaker monotonicity condition and a new decay condition on the potential $V(x)$.

From a mathematical point of view, quasilinear Schrödinger systems have been widely considered in the literature. Guo and Tang [22] studied the following system

$$\begin{cases} -\Delta u + (\lambda a(x) + 1)u - \frac{1}{2}\Delta(u^2)u = \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta},\ x \in \mathbb{R}^N, \\ -\Delta v + (\lambda b(x) + 1)v - \frac{1}{2}\Delta(v^2)v = \frac{2\alpha}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v,\ x \in \mathbb{R}^N, \end{cases}$$

where $\lambda > 0$ is a parameter, $\alpha > 2, \beta > 2, \alpha + \beta < \frac{4N}{N-2}$ for $N \geq 3$. They proved the existence of positive ground states via the Nehari manifold and the concentration compactness principle. In [23], under the assumptions of $(h, g) = \nabla F$ that

(G1) there exists $C > 0$ such that

$$|h(x,u,v)| + |g(x,u,v)| \leq C(1 + |(u,v)|^{p-1}),\ \forall (x,u,v) \in \mathbb{R}^N \times \mathbb{R}^2,$$

where $2 \leq p < \frac{4N}{N-2}$;

(G2) the following limits hold

$$\lim_{(u,v)\to(0,0)} \frac{h(x,u,v)}{|(u,v)|} = \lim_{(u,v)\to(0,0)} \frac{g(x,u,v)}{|(u,v)|} = 0 \ \text{ uniformly in } x \in \mathbb{R}^N;$$

(G3) there exists $\theta > 4$ such that

$$L_\theta(x,U) \leq 0,\ \forall (x,U) \in \mathbb{R}^N \times \mathbb{R}^2,$$

$$uh(x,u,v) \geq 0,\ vg(x,u,v) \geq 0 \text{ for all } (x,u,v) \in \mathbb{R}^N \times \mathbb{R}^2,$$

$$U = (u,v),\ L_\theta(x,U) = \theta F(x,U) - U \cdot \nabla F(x,U),$$

Severo and Silva considered the gradient system

$$\begin{cases} -\Delta u + V_1(x)u - \Delta(u^2)u = h(x,u,v),\ x \in \mathbb{R}^N, \\ -\Delta v + V_2(x)v - \Delta(v^2)v = g(x,u,v),\ x \in \mathbb{R}^N. \end{cases} \tag{1.6}$$

They obtained the existence of nontrivial solution under some other conditions on $V_1(x)$, $V_2(x)$ and (G1)–(G3) for the case of

$$F(x,u,v) = \frac{2}{\alpha+\beta}|u|^\alpha |v|^\beta \text{ with } \alpha > 2,\ \beta > 2 \text{ and } \alpha + \beta < \frac{4N}{N-2}.$$

Recently, Chen and Zhang [24] studied the following periodic quasilinear Schrödinger system with $2 < \alpha + \beta < \frac{4N}{N-2}$,

$$\begin{cases} -\Delta u + A(x)u - \frac{1}{2}\Delta(u^2)u = \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta,\ x \in \mathbb{R}^N, \\ -\Delta v + Bv - \frac{1}{2}\Delta(v^2)v = \frac{2\beta}{\alpha+\beta}|u|^\alpha |v|^{\beta-2}v,\ x \in \mathbb{R}^N, \end{cases}$$

and proved the existence of positive ground states with $2 < \alpha + \beta < \frac{4N}{N-2}$. Furthermore, they generalized the results by Guo and Tang [22] and obtained the similar results via a constrained minimization argument. There's also a lot of interesting work focusing on the existence or concentration of solutions to Choquard equations, Schrödinger equations and other elliptic equations, we can refer to [8, 23, 25–34] and the references therein.

Motivated by the mentioned papers [21, 23, 24], an interesting question is that, whether or not we can generalize the single quasilinear Schrödinger equation studied in Chen et al. [21] to a system, furthermore, whether we can find the existence of ground states for it with variable potential and super-quadratic condition. And thus, in the current paper, we shall establish the existence of ground states for quasilinear Schrödinger system (1.1) under some suitable assumptions via Pohožaev maniflod and the techniques in [21]. We point out that we consider two general classes of nonlinear terms $h(u,v)$ and $g(u,v)$, which include the particular nonlinearities treated in for example [22, 24, 35]. Besides, our potential functions do not depend on a real parameter.

1.2. Assumptions and main theorems

In order to establish a variational approach, we first suppose the following hypotheses:

(V1) $V_i(x) \in C(\mathbb{R}^N, [0,\infty))$, $i = 1,2$ satisfy

$$0 < V_{i,0} = \min_{x\in\mathbb{R}^N} V_i(x) < V_{i,\infty} = \lim_{|y|\to\infty} V_i(y).$$

(V2) $V_i(x)$, $i = 1,2$ satisfy the set $\{x \in \mathbb{R}^N : |\nabla V_i(x) \cdot x| \geq \varepsilon\}$ has finite Lebesgue measure for every $\varepsilon > 0$, and $t \mapsto [(N+2)V_i(tx) + \nabla V_i(tx) \cdot (tx)]/t^{p-2}$ is nonincreasing on $[0,\infty)$ for any $x \in \mathbb{R}^N$, where $p > 2$ is given by (F3).

(F1)

$$\lim_{(u,v)\to(0,0)} \frac{h(u,v)}{|(u,v)|} = \lim_{(u,v)\to(0,0)} \frac{g(u,v)}{|(u,v)|} = 0;$$
$$\lim_{|(u,v)|\to\infty} \frac{h(u,v)}{|(u,v)|^{2\cdot 2^*-1}} = \lim_{|(u,v)|\to\infty} \frac{g(u,v)}{|(u,v)|^{2\cdot 2^*-1}} = 0,$$

where $2^* = \frac{2N}{N-2}$ is the Sobolev critical exponent.

(F2) $\lim\limits_{|(u,v)|\to\infty} \frac{F(u,v)}{|(u,v)|^2} = +\infty$, where $\nabla F(u,v) = (h(u,v), g(u,v))$.

(F3) there exists a constant $p > 2$ such that $\frac{NF(t\tau_1,t\tau_2)+h(t\tau_1,t\tau_2)t\tau_1+g(t\tau_1,t\tau_2)t\tau_2}{|t\tau_1\tau_2|^p}$ is nondecreasing on both $t \in (-\infty,0)$ and $t \in (0,\infty)$, where $\tau_1, \tau_2 \in \mathbb{R}\backslash\{0\}$.

Remark 1.1. *We would like to point out that (F1) is much weaker than (G4) and (F2) is much weaker than (G6).*

Our result can be stated in the following form:

Theorem 1.1. *Suppose (V1), (V2) and (F1)–(F3) are satisfied. Then system (1.1) has a ground state solution.*

Now we introduce the following system

$$\begin{cases} -\Delta u + V_{1,\infty}u - \Delta(u^2)u = h(u,v), x \in \mathbb{R}^N, \\ -\Delta v + V_{2,\infty}v - \Delta(v^2)v = g(u,v), x \in \mathbb{R}^N, \end{cases} \tag{1.7}$$

which acts as a limit problem for (1.1) with $V_i(x) \equiv V_{i,\infty}$, $(i = 1,2)$. Applying Theorem 1.1 to (1.7), we have Corollary 1.2.

Corollary 1.2. *Suppose (F1)–(F3) are satisfied. Then system (1.7) has a ground state solution.*

1.3. Notations

Throughout the paper we make use of the following notations:

- we use C or C_i to denote various positive constants in context;
- $\int_{\mathbb{R}^N} \clubsuit$ denotes $\int_{\mathbb{R}^N} \clubsuit \, \mathrm{d}x$;
- $L^s(\mathbb{R}^N)$ denotes the Lebesgue space with norm $\|u\|_s = (\int_{\mathbb{R}^N} |u|^s \mathrm{d}x)^{\frac{1}{s}}$, where $1 \leq r < \infty$;
- $H^1(\mathbb{R}^N)$ denotes a Hilbert space with norm $\|u\|_{H^1(\mathbb{R}^N)} = (\int_{\mathbb{R}^N} |\nabla u|^2 + u^2)^{\frac{1}{2}}$;
- $D^{1,2}(\mathbb{R}^N)$ is a Hilbert space with norm $\|u\|_{D^{1,2}(\mathbb{R}^N)} = (\int_{\mathbb{R}^N} |\nabla u|^2)^{\frac{1}{2}}$;
- S is the best constant for the embedding of $D^{1,2}(\mathbb{R}^N)$ in $L^{2^*}(\mathbb{R}^N)$;
- the weak convergence in $H^1(\mathbb{R}^N)$ or H is denoted by $\rightharpoonup$, and the strong convergence is denoted by $\to$.

1.4. Outline

The rest of this paper is organized as follows: In Section 2, we use a change of variable and thus we are able to introduce a variational framework. In Section 3, we give some important primary results. In Section 4, we give the proof of Theorem 1.1.

2. The variational framework

Set $H := H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. It is no hard to verify that H is a complete space endowed with the norm

$$\|(u,v)\| = \Big(\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2 + u^2 + v^2) \Big)^{\frac{1}{2}}.$$

We note that, (1.1) is formally the Euler-Lagrange equation associated with the functional

$$\Phi(u,v) = \frac{1}{2}\int_{\mathbb{R}^N} [(1+2u^2)|\nabla u|^2 + (1+2v^2)|\nabla v|^2] + \frac{1}{2}\int_{\mathbb{R}^N} [V_1(x)u^2 + V_2(x)v^2] - \int_{\mathbb{R}^N} F(u,v).$$

However, Φ is not well defined in H because of the appearance of the nonlocal term $\int_{\mathbb{R}^N}(u^2|\nabla u|^2 + v^2|\nabla v|^2)$. We shall follow [17, 19, 36] and make the changing of variables $(u,v) = (f(z), f(w))$, where f is defined by

$$f'(t) = \frac{1}{[1+2f^2(t)]^{\frac{1}{2}}} \quad \text{on} \quad [0,\infty),$$
$$f(t) = -f(-t) \quad \text{on} \quad (-\infty, 0].$$

Then we obtain

$$\begin{aligned} I(z,w) :=& \Phi(f(z), f(w)) \\ =& \frac{1}{2}\int_{\mathbb{R}^N} [|\nabla z|^2 + |\nabla w|^2] + \frac{1}{2}\int_{\mathbb{R}^N} [V_1(x) f^2(z) + V_2(x) f^2(w)] \\ & - \int_{\mathbb{R}^N} F(f(z), f(w)). \end{aligned} \tag{2.1}$$

Under hypotheses (V1), (V2) and (F1)–(F3) it follows that $I \in C^1(H, \mathbb{R})$. In addition, for any φ_1, $\varphi_2 \in C_0^\infty(\mathbb{R}^N)$, $(z,w) \in H$ and $(z,w) + (\varphi_1, \varphi_2) \in H$, we compute the Gateaux derivative

$$\begin{aligned} \langle I'(z,w), (\varphi_1, \varphi_2)\rangle =& \int_{\mathbb{R}^N} (\nabla z \nabla \varphi_1 + \nabla w \nabla \varphi_2) + \int_{\mathbb{R}^N} [V_1(x) f(z) f'(z)\varphi_1 + V_2(x) f(w) f'(w) \varphi_2] \\ & - \int_{\mathbb{R}^N} [h(f(z), f(w)) f'(z)\varphi_1 + g(f(z), f(w)) f'(w)\varphi_2]. \end{aligned}$$

Then $(u,v) = (f(z), f(w))$ is a weak solution of (1.1) if and only if (z,w) is a critical point of I. Moreover, every critical point of I corresponds precisely to the weak solutions of the semilinear system

$$\begin{cases} -\Delta z + V_1(x) f(z) f'(z) = h(f(z), f(w)) f'(z) \ \text{in } \mathbb{R}^N, \\ -\Delta w + V_2(x) f(w) f'(w) = g(f(z), f(w)) f'(w) \ \text{in } \mathbb{R}^N. \end{cases} \tag{2.2}$$

Next, we define the Pohožaev type functional of (2.2):

$$
\begin{aligned}
P(z,w) :=& \frac{N-2}{2}\|\nabla z\|_2^2 + \frac{N-2}{2}\|\nabla w\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N}[NV_1(x) + \nabla V_1(x)\cdot x]f^2(z) \\
&+ \frac{1}{2}\int_{\mathbb{R}^N}[NV_2(x) + \nabla V_2(x)\cdot x]f^2(w) - N\int_{\mathbb{R}^N} F(f(z), f(w))
\end{aligned}
$$

for all $(z,w) \in H$. It is standard to prove that any solution (z,w) of (2.2) satisfies $P(z,w) = 0$ and $\langle I'(z,w), (\frac{f(z)}{f'(z)}, \frac{f(w)}{f'(w)})\rangle = 0$, where

$$
\begin{aligned}
&\left\langle I'(z,w), \left(\frac{f(z)}{f'(z)}, \frac{f(w)}{f'(w)}\right)\right\rangle \\
&= \int_{\mathbb{R}^N}\left[|\nabla z|^2\left(1 + \frac{2f^2(z)}{1+2f^2(z)}\right) + |\nabla w|^2\left(1 + \frac{2f^2(w)}{1+2f^2(w)}\right)\right] + \int_{\mathbb{R}^N}[V_1(x)f^2(z) + V_2(x)f^2(w)] \\
&- \int_{\mathbb{R}^N}[h(f(z), f(w))f(z) + g(f(z), f(w))f(w)].
\end{aligned}
\tag{2.3}
$$

Motivated by Chen et al. [21], we introduce the Nehari-Pohožaev manifold of I by

$$
\mathcal{M} := \{(z,w) \in H\backslash\{(0,0)\} : J(z,w) = 0\},
$$

where

$$
\begin{aligned}
J(z,w) :=& \left\langle I'(z,w), \left(\frac{f(z)}{f'(z)}, \frac{f(w)}{f'(w)}\right)\right\rangle + P(z,w) \\
=& \frac{N}{2}\|\nabla z\|_2^2 + \frac{N}{2}\|\nabla w\|_2^2 + \int_{\mathbb{R}^N}\frac{2f^2(z)}{1+2f^2(z)}|\nabla z|^2 + \int_{\mathbb{R}^N}\frac{2f^2(w)}{1+2f^2(w)}|\nabla w|^2 \\
&+ \frac{1}{2}\int_{\mathbb{R}^N}[(N+2)V_1(x) + \nabla V_1(x)\cdot x]f^2(z) + \frac{1}{2}\int_{\mathbb{R}^N}[(N+2)V_2(x) + \nabla V_2(x)\cdot x]f^2(w) \\
&- \int_{\mathbb{R}^N}[NF(f(z), f(w)) + h(f(z), f(w))f(z) + g(f(z), f(w))f(w)].
\end{aligned}
\tag{2.4}
$$

Then every non-trivial solution of (2.2) is contained in $\mathcal{M}$.

Similarly, for (1.7), we have

$$
\begin{aligned}
I^\infty(z,w) =& \frac{1}{2}\int_{\mathbb{R}^N}[|\nabla z|^2 + |\nabla w|^2] + \frac{1}{2}\int_{\mathbb{R}^N}[V_{1,\infty}f^2(z) + V_{2,\infty}f^2(w)] \\
&- \int_{\mathbb{R}^N} F(f(z), f(w)),
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
J^\infty(z,w) =& \frac{N}{2}\|\nabla z\|_2^2 + \frac{N}{2}\|\nabla w\|_2^2 + \int_{\mathbb{R}^N}\frac{2f^2(z)}{1+2f^2(z)}|\nabla z|^2 + \int_{\mathbb{R}^N}\frac{2f^2(w)}{1+2f^2(w)}|\nabla w|^2 \\
&+ \frac{N+2}{2}V_{1,\infty}\|f(z)\|_2^2 + \frac{N+2}{2}V_{2,\infty}\|f(w)\|_2^2 \\
&- \int_{\mathbb{R}^N}[NF(f(z), f(w)) + h(f(z), f(w))f(z) + g(f(z), f(w))f(w)]
\end{aligned}
\tag{2.6}
$$

and define

$$\mathcal{M}^\infty := \{(z, w) \in H\backslash\{(0,0)\} : J^\infty(z, w) = 0\}.$$

For any $z \in H^1(\mathbb{R}^N) \setminus \{0\}$ and $t > 0$, we make scaling refer to the method proposed by Chen et al. [21] that $z_t(x) = f^{-1}(tf(z(t^{-1}x)))$. Let $(z, w) \in H$, we have

$$\begin{aligned} I(z_t, w_t) =& \frac{t^N}{2} \int_{\mathbb{R}^N} \Big[\frac{1 + 2t^2 f^2(z)}{1 + 2f^2(z)} |\nabla z|^2 + \frac{1 + 2t^2 f^2(w)}{1 + 2f^2(w)} |\nabla w|^2 \Big] \\ &+ \frac{t^{N+2}}{2} \int_{\mathbb{R}^N} [V_1(tx) f^2(z) + V_2(tx) f^2(w)] - t^N \int_{\mathbb{R}^N} F(tf(z), tf(w)), \end{aligned} \tag{2.7}$$

$$\begin{aligned} J(z_t, w_t) =& \frac{N}{2} t^N \int_{\mathbb{R}^N} \Big[\frac{1 + 2t^2 f^2(z)}{1 + 2f^2(z)} |\nabla z|^2 + \frac{1 + 2t^2 f^2(w)}{1 + 2f^2(w)} |\nabla w|^2 \Big] \\ &+ 2t^{N+2} \int_{\mathbb{R}^N} \Big[\frac{f^2(z)}{1 + 2f^2(z)} |\nabla z|^2 + \frac{f^2(w)}{1 + 2f^2(w)} |\nabla w|^2 \Big] \\ &+ \frac{t^{N+2}}{2} \int_{\mathbb{R}^N} [(N+2) V_1(tx) + \nabla V_1(tx) \cdot (tx)] f^2(z) \\ &+ \frac{t^{N+2}}{2} \int_{\mathbb{R}^N} [(N+2) V_2(tx) + \nabla V_2(tx) \cdot (tx)] f^2(w) \\ &- t^N \int_{\mathbb{R}^N} [h(tf(z), tf(w)) tf(z) + g(tf(z), tf(w)) tf(w) + tNF(tf(z), tf(w))]. \end{aligned} \tag{2.8}$$

Define $m := \inf\limits_{\mathcal{M}} I(z, w)$ and $m^\infty := \inf\limits_{\mathcal{M}^\infty} I^\infty(z, w)$, then our aim is to prove that m is achieved.

3. Preliminaries

In this section we give some primary results used throughout the paper. First, for easy reference we present below some properties of the function f and its derivative, which are extensively used in the rest of the paper.

Lemma 3.1. [8, 19] *f(t) and its derivative have the following properties:*
(1) f is uniquely defined C^∞ function and invertible;
(2) $|f'(t)| \leq 1$ for all $t \in \mathbb{R}$;
(3) $|f(t)| \leq |t|$ for all $t \in \mathbb{R}$;
(4) $\frac{f(t)}{t} \to 1$ as $t \to 0$;
(5) $\frac{f(t)}{\sqrt{t}} \to 2^{\frac{1}{4}}$ as $t \to +\infty$;
(6) $\frac{f(t)}{2} \leq tf'(t) \leq f(t)$ for all $t \geq 0$ and $f(t)f'(t) \leq f^2(t)$ for all $t \in \mathbb{R}$;
(7) $|f(t)| \leq 2^{\frac{1}{4}} |t|^{\frac{1}{2}}$ for all $t \in \mathbb{R}$;
(8) the function $f^2(t)$ is strictly convex;
(9) there exists a positive constant C such that

$$|f(t)| \geq \begin{cases} C|t|, & |t| \leq 1, \\ C|t|^{\frac{1}{2}}, & |t| \geq 1; \end{cases}$$

(10) there exist positive contant C_1 and C_2 such that

$$|t| \leq C_1|f(t)| + C_2|f(t)|^2 \ for\ all\ t \in \mathbb{R};$$

(11) $|f(t)f'(t)| \leq \frac{1}{\sqrt{2}}$ for all $t \in \mathbb{R}$.

Inspired by [21, 37, 38], we establish the following important inequality.

Lemma 3.2. *Assume that (V1), (V2), (F1)–(F3) hold. Then*

$$I(z,w) \geq I(z_t,w_t) + \frac{1-t^{N+p}}{N+p}J(z,w) + \frac{1}{2}Q(t,z,w), \ \forall (z,w) \in H\backslash\{(0,0)\}, \ t>0,$$

where $Q(t,z,w)$ is defined as

$$\begin{aligned} Q(t,z,w) := &\int_{\mathbb{R}^N} \left\{\left[1 - \frac{t^N(1+2t^2f^2(z))}{1+2f^2(z)}\right] - \frac{1-t^{N+p}}{N+p}\left[N + \frac{4f^2(z)}{1+2f^2(z)}\right]\right\}|\nabla z|^2 \\ &+ \int_{\mathbb{R}^N} \left\{\left[1 - \frac{t^N(1+2t^2f^2(w))}{1+2f^2(w)}\right] - \frac{1-t^{N+p}}{N+p}\left[N + \frac{4f^2(w)}{1+2f^2(w)}\right]\right\}|\nabla w|^2. \end{aligned}$$

Proof. Set $Q_i(t,x) := V_i(x) - t^{N+2}V_i(tx) - \frac{1-t^{N+p}}{N+p}[(N+2)V_i(x) + \nabla V_i(x)\cdot x]$, $i = 1,2$, and

$$\begin{aligned} Q_3(t,\tau_1,\tau_2) :=& t^N F(t\tau_1,t\tau_2) - F(\tau_1,\tau_2) \\ &+ \frac{1-t^{N+p}}{N+p}[h(\tau_1,\tau_2)\tau_1 + g(\tau_1,\tau_2)\tau_2 + NF(\tau_1,\tau_2)]. \end{aligned}$$

It follows from (2.1), (2.4) and (2.5) that

$$\begin{aligned} &I(z,w) - I(z_t,w_t) \\ =&\frac{1}{2}\|\nabla z\|_2^2 + \frac{1}{2}\|\nabla w\|_2^2 - \frac{t^N}{2}\int_{\mathbb{R}^N}\left[\frac{1+2t^2f^2(z)}{1+2f^2(z)}|\nabla z|^2 + \frac{1+2t^2f^2(w)}{1+2f^2(w)}|\nabla w|^2\right] \\ &+ \frac{1}{2}\int_{\mathbb{R}^N}[V_1(x) - t^{N+2}V_1(tx)]f^2(z) + \frac{1}{2}\int_{\mathbb{R}^N}[V_2(x) - t^{N+2}V_2(tx)]f^2(w) \\ &+ \int_{\mathbb{R}^N}[t^N F(tf(z),tf(w)) - F(f(z),f(w))] \\ =&\frac{1-t^{N+p}}{N+p}\left\{\frac{N}{2}\|\nabla z\|_2^2 + \frac{N}{2}\|\nabla w\|_2^2 + \int_{\mathbb{R}^N}\frac{2f^2(z)}{1+2f^2(z)}|\nabla z|^2 + \int_{\mathbb{R}^N}\frac{2f^2(w)}{1+2f^2(w)}|\nabla w|^2\right. \\ &+ \frac{1}{2}\int_{\mathbb{R}^N}[(N+2)V_1(x) + \nabla V_1(x)\cdot x]f^2(z) + \frac{1}{2}\int_{\mathbb{R}^N}[(N+2)V_2(x) + \nabla V_2(x)\cdot x]f^2(w) \\ &\left.- \int_{\mathbb{R}^N}[NF(f(z),f(w)) + h(f(z),f(w))f(z) + g(f(z),f(w))f(w)]\right\} \\ &+ \frac{1}{2}\left\{\int_{\mathbb{R}^N}\left\{\left[1 - \frac{t^N(1+2t^2f^2(z))}{1+2f^2(z)}\right] - \frac{1-t^{N+p}}{N+p}\left[N + \frac{4f^2(z)}{1+2f^2(z)}\right]\right\}|\nabla z|^2\right. \\ &\left.+ \int_{\mathbb{R}^N}\left\{\left[1 - \frac{t^N(1+2t^2f^2(w))}{1+2f^2(w)}\right] - \frac{1-t^{N+p}}{N+p}\left[N + \frac{4f^2(w)}{1+2f^2(w)}\right]\right\}|\nabla w|^2\right\} \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_{\mathbb{R}^N}\left\{V_1(x)-t^{N+2}V_1(tx)-\frac{1-t^{N+p}}{N+p}[(N+2)V_1(x)+\nabla V_1(x)\cdot x]\right\}f^2(z)\\
&+\frac{1}{2}\int_{\mathbb{R}^N}\left\{V_2(x)-t^{N+2}V_2(tx)-\frac{1-t^{N+p}}{N+p}[(N+2)V_2(x)+\nabla V_2(x)\cdot x]\right\}f^2(w)\\
&+\int_{\mathbb{R}^N}\left\{t^N F(tf(z),tf(w))-F(tf(z),tf(w))+\frac{1-t^{N+p}}{N+p}[h(tf(z),tf(w))f(z)\right.\\
&\left.+g(tf(z),tf(w))f(w)+NF(tf(z),tf(w))]\right\}\\
=&\frac{1-t^{N+p}}{N+p}J(z,w)+\frac{1}{2}Q(t,z,w)+\frac{1}{2}\int_{\mathbb{R}^N}Q_1(t,x)f^2(z)+\frac{1}{2}\int_{\mathbb{R}^N}Q_2(t,x)f^2(w)\\
&+\int_{\mathbb{R}^N}Q_3(t,f(z),f(w)).
\end{aligned}
$$

It suffices to prove that

$$Q_i(t,x)\geq 0,\ \forall t\geq 0,\ x\in\mathbb{R}^N, i=1,2, \tag{3.1}$$

$$Q_3(t,\tau_1,\tau_2)\geq 0,\ \forall t\geq 0,\tau_1,\tau_2\in\mathbb{R}, \tag{3.2}$$

$$Q(t,z,w)\geq 0,\ \forall t\geq 0,(z,w)\in H. \tag{3.3}$$

In deed, through a simple calculation, for any $x\in\mathbb{R}^N$ and $t>0$, by (V2) we have

$$
\begin{aligned}
\frac{\mathrm{d}Q_i(t,x)}{\mathrm{d}t}&=t^{N+p-1}\left[(N+2)V_i(x)+\nabla V_i(x)\cdot x-\frac{(N+2)V_i(tx)+\nabla V_i(tx)\cdot(tx)}{t^{p-2}}\right]\\
&\begin{cases}\geq 0, t\geq 1,\\ \leq 0, 0<t<1,\end{cases}
\end{aligned}
$$

then, thanks to the continuity of $Q_i(t,x)$ on t, (3.1) holds. For $\tau_1,\tau_2\neq 0$, by (F3) we have

$$
\begin{aligned}
\frac{\mathrm{d}Q_3(t,\tau_1,\tau_2)}{\mathrm{d}t}=&t^{N+p-1}|\tau_1\tau_2|^p\left[\frac{NF(t\tau_1,t\tau_2)+h(t\tau_1,t\tau_2)t\tau_1+g(t\tau_1,t\tau_2)t\tau_2}{|t\tau_1\tau_2|^p}\right.\\
&\left.-\frac{h(\tau_1,\tau_2)\tau_1+g(\tau_1,\tau_2)\tau_2+NF(\tau_1,\tau_2)}{|\tau_1\tau_2|^p}\right]\begin{cases}\geq 0, t\geq 1,\\ \leq 0, 0<t<1,\end{cases}
\end{aligned}
$$

then, together with the continuity of $Q_3(t,\tau_1,\tau_2)$ on t imply that $Q_3(t,\tau_1,\tau_2)\geq Q_3(1,\tau_1,\tau_2)=0$ for all $t\geq 0$ and τ_1, $\tau_2\in\mathbb{R}\backslash\{0\}$, i.e., (3.2) holds. From the definition of f, we have

$$\|\nabla z\|_2\geq\|\nabla f(z)\|_2,\ \forall z\in\mathbb{R}^N. \tag{3.4}$$

Then, together with

$$
\begin{aligned}
\frac{\mathrm{d}Q(t,z,w)}{\mathrm{d}t}=&\ t^{N-1}\{t^2(t^{p-2}-1)[N\|\nabla z\|_2^2+2(\|\nabla z\|_2^2-\|\nabla f(z)\|_2^2)]+N(t^2-1)\|f(z)\|_2^2\}\\
&+t^{N-1}\{t^2(t^{p-2}-1)[N\|\nabla w\|_2^2+2(\|\nabla w\|_2^2-\|\nabla f(w)\|_2^2)]+N(t^2-1)\|f(w)\|_2^2\},
\end{aligned}
$$

imply that (3.3) holds. The proof is completed.

From Lemma 3.2, we have the following corollary.

Corollary 3.3. *Assume that (V1), (V2), (F1)–(F3) hold. Then for any* $(z, w) \in \mathcal{M}$,

$$I(z, w) = \max_{t>0} I(z_t, w_t). \tag{3.5}$$

Similar to the proof of Lemma 3.2 and Corollary 3.3, we can get the following corollary.

Corollary 3.4. *Assume that (F1)–(F3) hold. Then*

$$I^\infty(z, w) \geq I^\infty(z_t, w_t) + \frac{1 - t^{N+p}}{N + p} J^\infty(z, w) + \frac{1}{2} Q(t, z, w), \ \forall (z, w) \in H\backslash\{(0, 0)\}, \ t > 0.$$

Corollary 3.5. *Assume that (F1)–(F3) hold. Then for any* $(z, w) \in \mathcal{M}^\infty$,

$$I^\infty(z, w) = \max_{t>0} I^\infty(z_t, w_t). \tag{3.6}$$

Lemma 3.6. *Assume that (V1), (V2) hold. Then there exist two constants* d_1, $d_2 > 0$ *such that for all* $u \in H^1(\mathbb{R}^N)$,

$$\frac{N}{2}\|\nabla u\|_2^2 + \int_{\mathbb{R}^N} [(N + 2)V_i(x) + \nabla V_i(x) \cdot x]u^2 \geq d_i \|u\|^2_{H^1(\mathbb{R}^N)}, \ i = 1, 2.$$

Proof. Set $V_{i,\max} := \max_{x\in\mathbb{R}^N} V_i(x) \in (0, \infty)$, $i = 1, 2$, and $\hat{t}_i := (\frac{2V_{i,\max}}{V_{i,0}})^{\frac{1}{N+2}} > 1$. Then we have

$$\begin{aligned}
\frac{1}{N + p}[(N + 2)V_i(x) + \nabla V_i(x) \cdot x] &\geq \frac{1 - \hat{t}_i^{-(N+p)}}{N + p}[(N + 2)V_i(x) + \nabla V_i(x) \cdot x] \\
&\geq \hat{t}_i^{2-p}[V_i(\hat{t}_i x) - \hat{t}_i^{-(N+2)} V_i(x)] \\
&\geq \hat{t}_i^{2-p}[V_{i,0} - \hat{t}_i^{-(N+2)} V_{i,\max}] \\
&= \frac{V_{i,0}}{2}\Big(\frac{2V_{i,\max}}{V_{i,0}}\Big)^{\frac{2-p}{N+2}}, \ \forall x \in \mathbb{R}^N.
\end{aligned}$$

We conclude by taking $d_i = \min\Big\{\frac{N}{2}, (N + p)\frac{V_{i,0}}{2}\Big(\frac{2V_{i,\max}}{V_{i,0}}\Big)^{\frac{2-p}{N+2}}\Big\}$, $i = 1, 2$.

Lemma 3.7. *Assume that (V1), (V2) and (F1)–(F3) hold. Suppose that* $(z, w) \in H\backslash\{(0, 0)\}$, *then, there is a unique* $t_0 > 0$ *such that* $(z_{t_0}, w_{t_0}) \in \mathcal{M}$.

Proof. Let $(z, w) \in H\backslash\{(0, 0)\}$ be fixed. Define $Y(t) := I(z_t, w_t)$ on $(0, \infty)$. If

$$\begin{aligned}
Y'(t) =& \frac{N}{2} t^{N-1} \int_{\mathbb{R}^N} \Big[\frac{1 + 2t^2 f^2(z)}{1 + 2f^2(z)}|\nabla z|^2 + \frac{1 + 2t^2 f^2(w)}{1 + 2f^2(w)}|\nabla w|^2\Big] \\
&+ \frac{t^N}{2} \int_{\mathbb{R}^N} \Big[\frac{4t f^2(z)}{1 + 2f^2(z)}|\nabla z|^2 + \frac{4t f^2(w)}{1 + 2f^2(w)}|\nabla w|^2\Big] \\
&+ \frac{N + 2}{2} t^{N+1} \int_{\mathbb{R}^N} [V_1(tx) f^2(z) + V_2(tx) f^2(w)] \\
&+ \frac{t^{N+2}}{2} \int_{\mathbb{R}^N} [\nabla V_1(tx) \cdot x f^2(z) + \nabla V_2(tx) \cdot x f^2(w)]
\end{aligned}$$

$$- Nt^{N-1}\int_{\mathbb{R}^N} F(tf(z),tf(w)) - t^N\int_{\mathbb{R}^N}[h(tf(z),tf(w))f(z) + g(tf(z),tf(w))f(w)]$$
$$=0.$$

Obviously,

$$Y'(t) = 0 \ \Leftrightarrow\ \frac{J(z_t,w_t)}{t} = 0 \ \Leftrightarrow\ (z_t,w_t) \in \mathcal{M}.$$

From (V1), (V2) and (F2), it is easy to prove that $\lim_{t\to 0} Y(t) = 0$, $Y(t) > 0$ for $t > 0$ small enough and $Y(t) \to -\infty$ as $t \to +\infty$. This implies that $Y(t)$ attains its maximum. Let (z_{t_0}, w_{t_0}) be the unique point at which this maximum is achieved with $t_0 > 0$ such that $Y'(t_0) = 0$ and $(z_{t_0}, w_{t_0}) \in \mathcal{M}$.

Now we prove the uniqueness. Indeed, if there exists another positive constant $t_1 \neq t_0$ such that $(z_{t_1}, w_{t_1}) \in \mathcal{M}$, i.e., $J(z_{t_1}, w_{t_1}) = 0$, then (3.3) and Lemma 3.2 imply

$$\begin{aligned} I(z_{t_1},w_{t_1}) &> I(z_{t_0},w_{t_0}) + \frac{t_1^{N+p} - t_0^{N+p}}{(N+p)t_1^{N+p}} J(z_{t_1},w_{t_1}) \\ &= I(z_{t_0},w_{t_0}) \\ &> I(z_{t_1},w_{t_1}) + \frac{t_0^{N+p} - t_1^{N+p}}{(N+p)t_0^{N+p}} J(z_{t_0},w_{t_0}) \\ &= I(z_{t_1},w_{t_1}), \end{aligned}$$

which is a contradiction, this finishes the proof.

Corollary 3.8. *Assume that $V_i(x) = V_{i,\infty}$, $i = 1,2$ and (F1)–(F3) hold. Suppose that $(z,w) \in H\backslash\{(0,0)\}$, then, there is a unique $t_0 > 0$ such that $(z_{t_0}, w_{t_0}) \in \mathcal{M}^\infty$.*

By Corollary 3.3 and Lemma 3.7, we have $\mathcal{M} \neq \emptyset$ and the following lemma.

Lemma 3.9. *Assume that (V1), (V2) and (F1)–(F3) hold. Then*

$$\inf_{\mathcal{M}} I(z,w) = m = \inf_{(z,w)\in H\backslash\{(0,0)\}} \max_{t>0} I(z_t,w_t).$$

Proof. For any $(z,w) \in \mathcal{M}$, it follows from Corollary 3.3 and $\mathcal{M} \subset H \setminus \{(0,0)\}$ that

$$\inf_{\mathcal{M}} I(z,w) \geq \inf_{(z,w)\in H\backslash\{(0,0)\}} \max_{t>0} I(z_t,w_t). \tag{3.7}$$

On the other hand, by Lemma 3.7 there is a unique $t_0 > 0$ such that $(z_{t_0}, w_{t_0}) \in \mathcal{M}$, then $\max_{t>0} I(z_t,w_t) = I(z_{t_0},w_{t_0}) \geq \inf_{\mathcal{M}} I(z,w)$, which implies

$$\inf_{(z,w)\in H\backslash\{(0,0)\}} \max_{t>0} I(z_t,w_t) \geq \inf_{\mathcal{M}} I(z,w). \tag{3.8}$$

(3.7) and (3.8) complete the proof.

Lemma 3.10. *Assume that (V1), (V2) and (F1)–(F3) hold. Then*
(i) there exists $\rho^2 > 0$ such that $\|\nabla z\|_2^2 + \|\nabla w\|_2^2 \geqslant \rho^2$ for any $(z,w) \in \mathcal{M}$;
(ii) $m = \inf_{\mathcal{M}} I > 0$.

Proof. (i) For any $(z,w) \in \mathcal{M}$, we have $J(z,w) = 0$. By (F1), (2.4), (3.4), Lemma 3.1 (3), (7), Lemma 3.6 and the Sobolev embedding inequality, for any $\varepsilon > 0$, one has

$$\begin{aligned}
&\frac{N}{4}(\|\nabla z\|_2^2 + \|\nabla w\|_2^2) + \frac{\gamma_1}{2}\|f(z)\|_{H^1(\mathbb{R}^N)}^2 + \frac{\gamma_2}{2}\|f(w)\|_{H^1(\mathbb{R}^N)}^2 \\
\le&\frac{N}{4}(\|\nabla z\|_2^2 + \|\nabla w\|_2^2) + \frac{N}{4}(\|\nabla f(z)\|_2^2 + \|\nabla f(w)\|_2^2) \\
&+ \frac{1}{2}\int_{\mathbb{R}^N}[(N+2)V_1(x) + \nabla V_1(x)\cdot x]f^2(z) + \frac{1}{2}\int_{\mathbb{R}^N}[(N+2)V_2(x) + \nabla V_2(x)\cdot x]f^2(w) \\
\le&\frac{N}{2}(\|\nabla z\|_2^2 + \|\nabla w\|_2^2) + \int_{\mathbb{R}^N}\frac{2f^2(z)}{1+2f^2(z)}|\nabla z|^2 + \int_{\mathbb{R}^N}\frac{2f^2(w)}{1+2f^2(w)}|\nabla w|^2 \\
&+ \frac{1}{2}\int_{\mathbb{R}^N}[(N+2)V_1(x) + \nabla V_1(x)\cdot x]f^2(z) + \frac{1}{2}\int_{\mathbb{R}^N}[(N+2)V_2(x) + \nabla V_2(x)\cdot x]f^2(w) \\
=&\int_{\mathbb{R}^N}[NF(f(z),f(w)) + h(f(z),f(w))f(z) + g(f(z),f(w))f(w)] \\
\le&C_1\varepsilon\|f(z)\|_2^2 + C_1\varepsilon\|f(w)\|_2^2 + C_2C_\varepsilon\|z\|_{2^*}^{2^*} + C_2C_\varepsilon\|w\|_{2^*}^{2^*} \\
\le&C_1\varepsilon\|f(z)\|_2^2 + C_1\varepsilon\|f(w)\|_2^2 + C_2C_\varepsilon S^{-\frac{2^*}{2}}\|\nabla z\|_2^{2^*} + C_2C_\varepsilon S^{-\frac{2^*}{2}}\|\nabla w\|_2^{2^*}.
\end{aligned}$$

If we choose $\varepsilon = \frac{1}{2}\min\{\frac{\gamma_1}{2C_1\beta_1^2}, \frac{\gamma_2}{2C_1\beta_2^2}\}$ (β_1 and β_2 are embedding constants), then

$$\begin{aligned}
&\frac{N}{4}(\|\nabla z\|_2^2 + \|\nabla w\|_2^2) + (\frac{\gamma_1}{2} - C_1\varepsilon\beta_1^2)\|f(z)\|_{H^1(\mathbb{R}^N)}^2 + (\frac{\gamma_2}{2} - C_1\varepsilon\beta_2^2)\|f(w)\|_{H^1(\mathbb{R}^N)}^2 \\
\le&C_2C_\varepsilon S^{-\frac{2^*}{2}}(\|\nabla z\|_2^{2^*} + \|\nabla w\|_2^{2^*}).
\end{aligned} \tag{3.9}$$

By a simple calculation, we can deduce that there exists $\rho^2 > 0$ such that

$$\|\nabla z\|_2^2 + \|\nabla w\|_2^2 \ge \rho^2,\ \forall (z,w) \in \mathcal{M}.$$

(ii) By (3.1) and (3.2), we can deduce that

$$(p-2)V_i(x) - \nabla V_i(x)\cdot x \ge 0, \forall x \in \mathbb{R}^N, i = 1,2,$$

$$h(\tau_1,\tau_2)\tau_1 + g(\tau_1,\tau_2)\tau_2 - pF(\tau_1,\tau_2) \ge 0, \forall \tau_1,\tau_2 \in \mathbb{R}.$$

Note that

$$\begin{aligned}
&I(z,w) - \frac{1}{N+p}J(z,w) \\
=&\frac{p-2}{2(N+p)}(\|\nabla z\|_2^2 + \|\nabla w\|_2^2) + \frac{1}{N+p}(\|\nabla f(z)\|_2^2 + \|\nabla f(w)\|_2^2) \\
&+ \frac{1}{2(N+p)}\int_{\mathbb{R}^N}[(p-2)V_1(x) - \nabla V_1(x)\cdot x]f^2(z) \\
&+ \frac{1}{2(N+p)}\int_{\mathbb{R}^N}[(p-2)V_2(x) - \nabla V_2(x)\cdot x]f^2(w) \\
&+ \frac{1}{N+p}\int_{\mathbb{R}^N}[h(f(z),f(w))f(z) + g(f(z),f(w))f(w)
\end{aligned}$$

$$- pF(f(z), f(w))], \forall (z,w) \in \mathcal{M}.$$

Then for any $(z,w) \in \mathcal{M}$, by (i) we can get

$$\begin{aligned} I(z,w) &= I(z,w) - \frac{1}{N+p} J(z,w) \\ &\geq \frac{p-2}{2(N+p)} (\|\nabla z\|_2^2 + \|\nabla w\|_2^2) \\ &\geq \frac{p-2}{2(N+p)} \rho^2 \\ &> 0. \end{aligned}$$

This implies that $m = \inf_{\mathcal{M}} I > 0$.

Lemma 3.11. *Assume that (V1), (V2) and (F1)–(F3) hold. Then* $m^\infty \geq m$.

Proof. If the conclusion is false, that is, $m^\infty < m$, then set $\kappa := m - m^\infty > 0$ and there exists $(z_\kappa^\infty, w_\kappa^\infty)$ such that

$$(z_\kappa^\infty, w_\kappa^\infty) \in \mathcal{M}^\infty \text{ and } m^\infty + \frac{\kappa}{2} > I^\infty(z_\kappa^\infty, w_\kappa^\infty). \tag{3.10}$$

By Corollary 3.8, there is $t_\kappa > 0$ such that $((z_\kappa^\infty)_{t_\kappa}, (w_\kappa^\infty)_{t_\kappa}) \in \mathcal{M}^\infty$. It follows from (3.10) that

$$\begin{aligned} m^\infty + \frac{\kappa}{2} &> I^\infty(z_\kappa^\infty, w_\kappa^\infty) \\ &\geq I^\infty((z_\kappa^\infty)_{t_\kappa}, (w_\kappa^\infty)_{t_\kappa}) \\ &\geq I((z_\kappa^\infty)_{t_\kappa}, (w_\kappa^\infty)_{t_\kappa}) \\ &\geq m = m^\infty + \kappa, \end{aligned}$$

which is a contradiction. This completes the proof.

4. Proof of Theorem 1.1

Lemma 4.1. *Assume that (V1), (V2), (F1)–(F3) hold. If* $z_n \rightharpoonup \bar{z}$, $w_n \rightharpoonup \bar{w}$ *in* $H^1(\mathbb{R}^N)$, *then*

$$\begin{aligned} I(z_n, w_n) &= I(\bar{z}, \bar{w}) + I(z_n - \bar{z}, w_n - \bar{w}) + o(1); \\ J(z_n, w_n) &= J(\bar{z}, \bar{w}) + J(z_n - \bar{z}, w_n - \bar{w}) + o(1). \end{aligned}$$

Proof. Since

$$\begin{aligned} I(z,w) =& \frac{1}{2} \int_{\mathbb{R}^N} [|\nabla z|^2 + |\nabla w|^2] + \frac{1}{2} \int_{\mathbb{R}^N} [V_1(x) f^2(z) + V_2(x) f^2(w)] \\ &- \int_{\mathbb{R}^N} F(f(z), f(w)), \end{aligned}$$

it suffices to prove

$$\int_{\mathbb{R}^N} (|\nabla z_n|^2 + |\nabla w_n|^2) = \int_{\mathbb{R}^N} (|\nabla \bar{z}|^2 + |\nabla \bar{w}|^2) - \int_{\mathbb{R}^N} (|\nabla (z_n - \bar{z})|^2 + |\nabla (w_n - \bar{w})|^2) + o(1), \tag{4.1}$$

$$\int_{\mathbb{R}^N} V_1(x)f^2(z_n) = \int_{\mathbb{R}^N} V_1(x)f^2(\bar{z}) + \int_{\mathbb{R}^N} V_1(x)f^2(z_n - \bar{z}) + o(1), \tag{4.2}$$

$$\int_{\mathbb{R}^N} V_2(x)f^2(w_n) = \int_{\mathbb{R}^N} V_2(x)f^2(\bar{w}) + \int_{\mathbb{R}^N} V_2(x)f^2(w_n - \bar{w}) + o(1), \tag{4.3}$$

$$\int_{\mathbb{R}^N} F(f(z_n), f(w_n)) = \int_{\mathbb{R}^N} F(f(\bar{z}), f(\bar{w})) + \int_{\mathbb{R}^N} F(f(z_n - \bar{z}), f(w_n - \bar{w})) + o(1). \tag{4.4}$$

By the Brezis-Lieb Lemma,

$$\int_{\mathbb{R}^N} |\nabla(z_n - \bar{z})|^2 = \int_{\mathbb{R}^N} |\nabla z_n|^2 - \int_{\mathbb{R}^N} |\nabla \bar{z}|^2 + o(1),$$
$$\int_{\mathbb{R}^N} |\nabla(w_n - \bar{w})|^2 = \int_{\mathbb{R}^N} |\nabla w_n|^2 - \int_{\mathbb{R}^N} |\nabla \bar{w}|^2 + o(1).$$

Then we have (4.1). Note that by Lemma 3.1 (2), (6), (7) and the Young inequality, one has

$$\begin{aligned}
|f^2(z_n - \bar{z} + \bar{z}) - f^2(z_n - \bar{z})| &= \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f^2(z_n - \bar{z} + t\bar{z})\mathrm{d}t \right| \\
&= \left| 2\int_0^1 f(z_n - \bar{z} + t\bar{z}) f'(z_n - \bar{z} + t\bar{z})\bar{z}\mathrm{d}t \right| \\
&\le 2 \cdot 2^{\frac{1}{2}}(|z_n - \bar{z}||\bar{z}| + \bar{z}^2) \\
&\le C_1(\varepsilon|z_n - \bar{z}|^2 + C_\varepsilon \bar{z}^2).
\end{aligned}$$

Then by Lemma 3.1 (3), we can get

$$|f^2(z_n - \bar{z} + \bar{z}) - f^2(z_n - \bar{z}) - f^2(\bar{z})| \le C_1\varepsilon|z_n - \bar{z}|^2 + (C_2 + C_1C_\varepsilon)\bar{z}^2.$$

Define $G_n(x) := \max\{|f^2(z_n - \bar{z} + \bar{z}) - f^2(z_n - \bar{z}) - f^2(\bar{z})| - C_1\varepsilon|z_n - \bar{z}|^2, 0\}$, which satisfies that

$$G_n(x) \to 0 \text{ a.e. in } \mathbb{R}^N,$$

$$0 \le G_n(x) \le (C_2 + C_1C_\varepsilon)\bar{z}^2.$$

Hence, by the Lebesgue dominated covergence theorem, we have $\int_{\mathbb{R}^N} G_n(x) \to 0$ as $n \to \infty$. By the definition of G_n, we obtain

$$\int_{\mathbb{R}^N} |f^2(z_n - \bar{z} + \bar{z}) - f^2(z_n - \bar{z}) - f^2(\bar{z})| \le \int_{\mathbb{R}^N} 2 \cdot 2^{\frac{1}{2}}\varepsilon|z_n - \bar{z}|^2 + \int_{\mathbb{R}^N} G_n,$$

which implies that

$$\int_{\mathbb{R}^N} |f^2(z_n) - f^2(z_n - \bar{z}) - f^2(\bar{z})| = o(1),$$

thus gives (4.2). Analogously, we get (4.3).

To prove (4.4), we claim that

$$\int_{\mathbb{R}^N} |F(f(z_n), f(w_n)) - F(f(z_n - \bar{z}), f(w_n - \bar{w})) + F(f(\bar{z}), f(\bar{w}))| = o(1). \tag{4.5}$$

Define

$$Z(z,w) := F(f(z), f(w)), \tag{4.6}$$

then by (F1) and Lemma 3.1 (3), (7), we have

$$\lim_{(z,w)\to(0,0)} \frac{Z(z,w)}{|(z,w)|^2} = \lim_{(z,w)\to(0,0)} \frac{F(f(z), f(w))}{|(f(z), f(w))|^2} \cdot \frac{|(f(z), f(w))|^2}{|(z,w)|^2} = 0$$

and

$$\lim_{|(z,w)|\to\infty} \frac{Z(z,w)}{|(z,w)|^{2^*}} = \lim_{|(z,w)|\to\infty} \frac{F(f(z), f(w))}{|(f(z), f(w))|^{2\cdot 2^*}} \cdot \frac{|(f(z), f(w))|^{2\cdot 2^*}}{|(z,w)|^{2^*}} = 0,$$

from which we can deduce that, given any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for all $(z,w) \in \mathbb{R}^2$,

$$|Z(z,w)| \le \varepsilon(|(z,w)|^2 + |(z,w)|^{2^*}) + C_\varepsilon |(z,w)|^r, \tag{4.7}$$

$$|\tilde{h}(z,w)| \le \varepsilon(|(z,w)| + |(z,w)|^{2^*-1}) + C_\varepsilon |(z,w)|^{r-1}, \tag{4.8}$$

$$|\tilde{g}(z,w)| \le \varepsilon(|(z,w)| + |(z,w)|^{2^*-1}) + C_\varepsilon |(z,w)|^{r-1}, \tag{4.9}$$

where $\nabla Z(z,w) = (\tilde{h}, \tilde{g})$ and $r \in (2, 2^*)$. By (4.6)–(4.9), the Mean value theorem and the Young inequality, one has

$$\begin{aligned}
&\left|Z(z_n - \bar{z} + \bar{z}, w_n - \bar{w} + \bar{w}) - Z(z_n - \bar{z}, w_n - \bar{w})\right| \\
\le&\left|Z(z_n - \bar{z} + \bar{z}, w_n - \bar{w} + \bar{w}) - Z(z_n - \bar{z}, w_n - \bar{w} + \bar{w})\right| \\
&+ \left|Z(z_n - \bar{z}, w_n - \bar{w} + \bar{w}) - Z(z_n - \bar{z}, w_n - \bar{w})\right| \\
\le&\left|\int_0^1 \tilde{h}(z_n - \bar{z} + t\bar{z}, w_n - \bar{w} + \bar{w})\bar{z}\mathrm{d}t\right| + \left|\int_0^1 \tilde{g}(z_n - \bar{z}, w_n - \bar{w} + t\bar{w})\bar{w}\mathrm{d}t\right| \\
\le& \int_0^1 \Big[\varepsilon|(z_n - \bar{z} + t\bar{z}, w_n)||\bar{z}| + \varepsilon|(z_n - \bar{z} + t\bar{z}, w_n)|^{2^*-1}|\bar{z}| + C_\varepsilon|(z_n - \bar{z} + t\bar{z}, w_n)|^{r-1}|\bar{z}|\Big]\mathrm{d}t \\
&+ \int_0^1 \Big[\varepsilon|(z_n - \bar{z}, w_n - \bar{w} + t\bar{w})||\bar{w}| + \varepsilon|(z_n - \bar{z}, w_n - \bar{w} + t\bar{w})|^{2^*-1}|\bar{w}| \\
&+ C_\varepsilon|(z_n - \bar{z}, w_n - \bar{w} + t\bar{w})|^{r-1}|\bar{w}|\Big]\mathrm{d}t \\
\le& C_3\varepsilon(|z_n - \bar{z}|^2 + |w_n - \bar{w}|^2 + |z_n - \bar{z}|^{2^*} + |w_n - \bar{w}|^{2^*} + \bar{z}^2 + w_n^2 + |\bar{z}|^{2^*} + |w_n|^{2^*} \\
&+ \bar{w}^2 + |\bar{w}|^{2^*}) + C_4 C_\varepsilon(|z_n - \bar{z}|^r + |w_n - \bar{w}|^r + |\bar{z}|^r + |w_n|^r + |\bar{w}|^r)
\end{aligned}$$

and

$$\begin{aligned}
&|Z(z_n, w_n) - Z(z_n - \bar{z}, w_n - \bar{w}) - Z(\bar{z}, \bar{w})| \\
\le&|Z(x, z_n, w_n) - Z(x, z_n - \bar{z}, w_n - \bar{w})| + |Z(x, \bar{z}, \bar{w})| \\
\le& C_5\varepsilon(|z_n - \bar{z}|^2 + |w_n - \bar{w}|^2 + |z_n - \bar{z}|^{2^*} + |w_n - \bar{w}|^{2^*} + \bar{z}^2 + w_n^2 + |\bar{z}|^{2^*} + |w_n|^{2^*} \\
&+ \bar{w}^2 + |\bar{w}|^{2^*}) + C_6 C_\varepsilon(|z_n - \bar{z}|^r + |w_n - \bar{w}|^r + |\bar{z}|^r + |w_n|^r + |\bar{w}|^r).
\end{aligned}$$

Define the function

$$H_n(x) = \max\{|Z(z_n, w_n) - Z(z_n - \bar{z}, w_n - \bar{w}) - Z(\bar{z}, \bar{w})| - C_5\varepsilon(|z_n - \bar{z}|^2 + |w_n - \bar{w}|^2$$

$$+|z_n-\bar{z}|^{2^*}+|w_n-\bar{w}|^{2^*})-C_6C_\varepsilon(|z_n-\bar{z}|^r+|w_n-\bar{w}|^r),0\},$$

which satisfies that

$$H_n\to 0 \text{ a.e. in } \mathbb{R}^N,$$

$$0\le H_n(x)\le C_5\varepsilon(\bar{z}^2+w_n^2+|\bar{z}|^{2^*}+|w_n|^{2^*}+\bar{w}^2+|\bar{w}|^{2^*})+C_6C_\varepsilon(|\bar{z}|^r+|w_n|^r+|\bar{w}|^r).$$

Then similar to the proof of (4.2), one can get (4.5).

The first half of the proof of the lemma is completed. Similarly, we can prove the second half of the lemma.

Lemma 4.2. *Assume that (V1), (V2) and (F1)–(F3) hold. Then m is achieved.*

Proof. Let $\{(z_n,w_n)\}\subset\mathcal{M}$ so that $I(z_n,w_n)\to m$. Then, as the proof in Lemma 3.10, we have

$$m+o(1)=I(z_n,w_n)=I(z_n,w_n)-\frac{1}{N+p}J(z_n,w_n)\ge\frac{p-2}{2(N+p)}(\|\nabla z_n\|_2^2+\|\nabla w_n\|_2^2),$$

which shows that $\{\|\nabla z_n\|_2\}$ and $\{\|\nabla w_n\|_2\}$ are bounded. In view of (3.9), we have

$$\begin{aligned}&\frac{\gamma_1}{4}\|f(z_n)\|^2_{H^1(\mathbb{R}^N)}+\frac{\gamma_2}{4}\|f(w_n)\|^2_{H^1(\mathbb{R}^N)}\\ \le&C_1C_\varepsilon S^{-\frac{2^*}{2}}(\|\nabla z_n\|_2^{2^*}+\nabla w_n\|_2^{2^*}),\end{aligned}$$

which implies that $\{\|f(z_n)\|_{H^1(\mathbb{R}^N)}\}$ and $\{\|f(w_n)\|_{H^1(\mathbb{R}^N)}\}$ are bounded. Then it follows from Lemma 3.1 (9) and the Sobolev embedding inequality that

$$\begin{aligned}\int_{\mathbb{R}^N}z_n^2&=\int_{|z_n|\le1}z_n^2+\int_{|z_n|>1}z_n^2\\&\le C_2\int_{|z_n|\le1}|f(z_n)|^2+\int_{\mathbb{R}^N}|z_n|^{2^*}\\&\le C_3\|f(z_n)\|_2^2+S^{-\frac{2^*}{2}}\|\nabla z_n\|_2^{2^*},\end{aligned}$$

which shows that $\{z_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Similarly, $\{w_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Up to a subsequence, there exist $\bar{z},\bar{w}\in H^1(\mathbb{R}^N)$ such that

$$\begin{aligned}&z_n\rightharpoonup\bar{z},\ w_n\rightharpoonup\bar{w}\text{ in }H^1(\mathbb{R}^N);\\&z_n\to\bar{z},\ w_n\to\bar{w}\ \text{ in }L^s_{loc}(\mathbb{R}^N),\ 2\le s<2^*;\\&z_n\to\bar{z},\ w_n\to\bar{w}\ \text{ a.e. in }\mathbb{R}^N.\end{aligned}$$

If $(\bar{z},\bar{w})=O$, then

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}[V_{i,\infty}-V_i(x)]f^2(z_n)=0,i=1,2. \tag{4.10}$$

In fact, it follows from (V1) that for any $\varepsilon>0$, there exists $R>0$ such that

$$|V_i(x)-V_{i,\infty}|<\varepsilon,\ \forall|x|\ge R.$$

On the one hand, we have

$$\left|\int_{|x|\le R}[V_{i,\infty}-V_i(x)]f^2(z_n)\right|\le 2V_{i,\infty}\int_{|x|\le R}z_n^2.$$

On the other hand, it is easy to check that

$$\lim_{n\to\infty}\int_{|x|>R}[V_{i,\infty}-V_i(x)]z_n^2=\lim_{n\to\infty}\varepsilon\int_{|x|>R}z_n^2\to 0 \text{ as } n\to\infty.$$

It is clear that (4.10) holds. Now we prove

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\nabla V_i(x)\cdot xf^2(z_n)=0, i=1,2. \tag{4.11}$$

Let $A_\varepsilon=\{x\in\mathbb{R}^3:|\nabla V_i(x)\cdot x|\ge\varepsilon\}$, by the Openset construction theorem, $A_\varepsilon=\bigcup_{n=1}^{+\infty}(A_\varepsilon\cap B_n)$, then we have $\lim\limits_{n\to\infty}\text{meas}(A_\varepsilon\cap B_n)=\text{meas}(A_\varepsilon)<+\infty$, which implies

$$\lim_{n\to\infty}\text{meas}(A_\varepsilon\cap B_n^c)=0. \tag{4.12}$$

From (3.1) we have

$$\lim_{t\to 0}Q_i(t,x)\ge 0,\ \forall x\in\mathbb{R}^N, i=1,2,$$
$$\lim_{t\to\infty}\frac{Q_i(t,x)}{t^{N+p}}\ge 0,\ \forall x\in\mathbb{R}^N, i=1,2,$$

then we can obtain $-(N+2)V_i(x)\le\nabla V_i(x)\cdot x\le(p-2)V_i(x)$, this implies

$$|\nabla V_i(x)\cdot x|\le C_4V_i(x)\le C_4V_{i,\infty}:=M. \tag{4.13}$$

By Lemma 3.1 (3), (4.12), (4.13) and the Hölder inequality, for large R we have

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla V_i(x)\cdot x|f^2(z_n)\\
&\le\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla V_i(x)\cdot x|z_n^2\\
&=\lim_{n\to\infty}\left[\int_{B_R}|\nabla V_i(x)\cdot x|z_n^2+\int_{B_R^c\cap A_\varepsilon}|\nabla V_i(x)\cdot x|z_n^2+\int_{B_R^c\cap A_\varepsilon^c}|\nabla V_i(x)\cdot x|z_n^2\right]\\
&\le\lim_{n\to\infty}\left[M\int_{B_R}z_n^2+M\int_{B_R^c\cap A_\varepsilon}z_n^2+\int_{B_R^c\cap A_\varepsilon^c}\varepsilon z_n^2\right]\\
&\le\lim_{n\to\infty}\left[M\int_{B_R}z_n^2+M\int_{B_R^c\cap A_\varepsilon}z_n^2+\varepsilon\int_{\mathbb{R}^N}z_n^2\right]\\
&\le\lim_{n\to\infty}\left[M\int_{B_R}z_n^2++M\Big(\int_{B_R^c\cap A_\varepsilon}1\Big)^{\frac{2}{N}}\Big(\int_{B_R^c\cap A_\varepsilon}z_n^{\frac{2N}{N-2}}\Big)^{\frac{N-2}{N}}+\varepsilon\int_{\mathbb{R}^N}z_n^2\right]\\
&\le\lim_{n\to\infty}\left[M\int_{B_R}z_n^2+M(\text{meas}(B_R^c\cap A_\varepsilon))^{\frac{2}{N}}\|z_n\|_{2^*}^2+\varepsilon\int_{\mathbb{R}^N}z_n^2\right]
\end{aligned}$$

$= 0.$

Next, we claim that there exist $\delta_1, \delta_2 > 0$ and $\{y_{n_1}\}, \{y_{n_2}\} \subset \mathbb{R}^N$ such that $\int_{B_1(y_{n_1})} z_n^2 > \delta_1$, $\int_{B_2(y_{n_2})} w_n^2 > \delta_2$. Otherwise, by Lions' concentration compactness principle lemma, we know

$$z_n, w_n \to 0 \text{ in } L^r(\mathbb{R}^N),\ \forall r \in (2, 2^*). \tag{4.14}$$

Under (F1), (F2) and Lemma 3.1 (3), for any $\varepsilon > 0$ and some $p \in (2, 2\cdot 2^*)$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}&\int_{\mathbb{R}^N}[h(f(z), f(w))f(z) + g(f(z), f(w))f(w) + F(f(z), f(w))]\\ \leq& C_5\varepsilon[\|f(z)\|_2^2 + \|f(w)\|_2^2] + C_5C_\varepsilon[\|z\|_{2^*}^{2^*} + \|w\|_{2^*}^{2^*}\\ &+ \|z\|_{2\cdot 2^*}^{2\cdot 2^*} + \|w\|_{2\cdot 2^*}^{2\cdot 2^*} + \|z\|_p^p + \|w\|_p^p].\end{aligned} \tag{4.15}$$

By (2.4), (3.4), (4.15) and Lemma 3.6, one can get

$$\begin{aligned}&\frac{N}{4}\rho^2 + \frac{d_1}{2}\|f(z_n)\|^2 + \frac{d_2}{2}\|f(w_n)\|^2\\ \leq&\frac{N}{4}\|\nabla z_n\|_2^2 + \frac{N}{4}\|\nabla w_n\|_2^2 + \frac{N}{4}\|\nabla f(z_n)\|_2^2 + \frac{N}{4}\|\nabla f(w_n)\|_2^2\\ &+ \frac{1}{2}\int_{\mathbb{R}^N}[(N+2)V_1(x) + \nabla V_1(x)\cdot x]f^2(z_n) + \frac{1}{2}\int_{\mathbb{R}^N}[(N+2)V_2(x) + \nabla V_2(x)\cdot x]f^2(w_n)\\ \leq&\frac{N}{2}\|\nabla z_n\|_2^2 + \frac{N}{2}\|\nabla w_n\|_2^2 + \int_{\mathbb{R}^N}\frac{2f^2(z_n)}{1+2f^2(z_n)}|\nabla z_n|^2 + \int_{\mathbb{R}^N}\frac{2f^2(w_n)}{1+2f^2(w_n)}|\nabla w_n|^2\\ &+ \frac{1}{2}\int_{\mathbb{R}^N}[(N+2)V_1(x) + \nabla V_1(x)\cdot x]f^2(z_n) + \frac{1}{2}\int_{\mathbb{R}^N}[(N+2)V_2(x) + \nabla V_2(x)\cdot x]f^2(w_n)\\ =&\int_{\mathbb{R}^N}[NF(f(z_n), f(w_n)) + h(f(z_n), f(w_n))f(z_n) + g(f(z_n), f(w_n))f(w_n)]\\ \leq&(1+N)C_5\varepsilon[\|f(z_n)\|_2^2 + \|f(w_n)\|_2^2] + (1+N)C_5C_\varepsilon[\|z_n\|_{2^*}^{2^*} + \|w_n\|_{2^*}^{2^*}\\ &+ \|z_n\|_{2\cdot 2^*}^{2\cdot 2^*} + \|w_n\|_{2\cdot 2^*}^{2\cdot 2^*} + \|z_n\|_p^p + \|w_n\|_p^p],\end{aligned}$$

which contradicts to (4.14).

If we set $\hat{z_n}(x) = z_n(x+y_{n_1})$, $\hat{w}_n(x) = w_n(x+y_{n_2})$, then we have $\|\hat{z_n}\|_{H^1(\mathbb{R}^N)} = \|z_n\|_{H^1(\mathbb{R}^N)}$ and $\|\hat{w}_n\|_{H^1(\mathbb{R}^N)} = \|w_n\|_{H^1(\mathbb{R}^N)}$. By (2.1), (2.4)–(2.6), (4.10) and (4.11), we have

$$\begin{aligned}I^\infty(z_n, w_n) &\to I(z_n, w_n) \to m,\\ J^\infty(z_n, w_n) &\to J(z_n, w_n) = 0.\end{aligned}$$

Then

$$I^\infty(\hat{z_n}, \hat{w}_n) \to m,\ J^\infty(\hat{z_n}, \hat{w}_n) = o(1), \tag{4.16}$$

$$\int_{B_1(0)}|\hat{z_n}|^2 > \delta_1,\ \int_{B_2(0)}|\hat{w}_n|^2 > \delta_2. \tag{4.17}$$

Moreover, there exist $\hat{z}, \hat{w} \in H^1(\mathbb{R}^N)\setminus\{0\}$ such that, passing to a subsequence,

$$\hat{z_n} \rightharpoonup \hat{z},\ \hat{w}_n \rightharpoonup \hat{w} \text{ in } H^1(\mathbb{R}^N);$$

$$\hat{z_n} \to \hat{z},\ \hat{w_n} \to \hat{w} \text{ in } L^s_{\text{loc}}(\mathbb{R}^N),\ 1 \le s < 2^*;$$
$$\hat{z_n} \to \hat{z},\ \hat{w_n} \to \hat{w} \text{ a.e. on } \mathbb{R}^N.$$

Let $(r_n, s_n) = (\hat{z_n} - \hat{z}, \hat{w_n} - \hat{w})$. Then from (4.16) and Lemma 4.1 we have

$$I^\infty(\hat{z_n}, \hat{w_n}) = I^\infty(\hat{z}, \hat{w}) + I^\infty(r_n, s_n) + o(1), \tag{4.18}$$

$$J^\infty(\hat{z_n}, \hat{w_n}) = J^\infty(\hat{z}, \hat{w}) + J^\infty(r_n, s_n) + o(1). \tag{4.19}$$

Define

$$\Psi^\infty(z, w) := I^\infty(z, w) - \frac{1}{N+p} J^\infty(z, w),\ \forall (z, w) \in H. \tag{4.20}$$

From (4.18)–(4.20), one can get

$$\Psi^\infty(r_n, s_n) = m - \Psi^\infty(\hat{z}, \hat{w}) + o(1), \tag{4.21}$$

$$J^\infty(r_n, s_n) = -J^\infty(\hat{z}, \hat{w}) + o(1). \tag{4.22}$$

If there exists a subsequence $\{(r_{n_i}, s_{n_i})\}$ of $\{(r_n, s_n)\}$ such that $(r_{n_i}, s_{n_i}) = 0$, i.e., $r_{n_i} = s_{n_i} = 0$, then we have

$$I^\infty(\hat{z}, \hat{w}) = m,\ J^\infty(\hat{z}, \hat{w}) = 0. \tag{4.23}$$

Next, we assume that $(r_n, s_n) \neq O$. We claim that $J^\infty(\hat{z}, \hat{w}) \le 0$. If not, that is, $J^\infty(\hat{z}, \hat{w}) > 0$, then it follows from (4.22) that $J^\infty(r_n, s_n) < 0$ for n large enough. From Corollary 3.8, there exists $t_n > 0$ such that $((r_n)_{t_n}, (s_n)_{t_n}) \in \mathcal{M}^\infty$ for large n. By (4.20)–(4.22), Corollary 3.4 and Lemma 3.11, one has

$$\begin{aligned} m - \Psi^\infty(\hat{z}, \hat{w}) + o(1) = \Psi^\infty(r_n, s_n) &= I^\infty(r_n, s_n) - \frac{1}{N+p} J^\infty(r_n, s_n) \\ &> I^\infty((r_n)_{t_n}, (s_n)_{t_n}) - \frac{t_n^{N+p}}{N+p} J^\infty(r_n, s_n) \\ &\ge m^\infty \ge m, \end{aligned}$$

which is a contradiction with $\Psi^\infty(\hat{z}, \hat{w}) > 0$. Thus, $J^\infty(\hat{z}, \hat{w}) \le 0$. By Corollary 3.8, there exists $t_\infty > 0$ such that $(\hat{z}_{t_\infty}, \hat{w}_{t_\infty}) \in \mathcal{M}^\infty$. Moreover, by using (4.15), (4.21), the Fatou's lemma and Lemma 3.11 we have

$$\begin{aligned} m &= \lim_{n\to\infty} I^\infty(\hat{z_n}, \hat{w_n}) \\ &= \lim_{n\to\infty} \left[I^\infty(\hat{z_n}, \hat{w_n}) - \frac{1}{N+p} J^\infty(\hat{z_n}, \hat{w_n}) \right] \\ &\ge I^\infty(\hat{z}, \hat{w}) - \frac{1}{N+p} J^\infty(\hat{z}, \hat{w}) \\ &\ge I^\infty(\hat{z}_{t_\infty}, \hat{w}_{t_\infty}) - \frac{t_\infty^{N+p}}{N+p} J^\infty(\hat{z}, \hat{w}) \\ &\ge m^\infty \ge m, \end{aligned}$$

which shows that (4.23) holds also. By Lemma 3.7, there exists $\hat{t} > 0$ such that $(\hat{z}_{\hat{t}}, \hat{w}_{\hat{t}}) \in \mathcal{M}$. By (V1), Corollary 3.5, (4.26), we have

$$m \le I(\hat{z}_{\hat{t}}, \hat{w}_{\hat{t}}) \le I^\infty(\hat{z}_{\hat{t}}, \hat{w}_{\hat{t}}) \le I^\infty(\hat{z}, \hat{w}) = m,$$

which implies that m is achieved at $(\hat{z}_{\hat{t}}, \hat{w}_{\hat{t}}) \in \mathcal{M}$.

If $(\bar{z}, \bar{w}) \ne O$. In this case, similar to the proof of (4.23), by using I and J instead of I^∞ and J^∞, we can prove that $I(\bar{z}, \bar{w}) = m,\ J(\bar{z}, \bar{w}) = 0$.

Proof of Theorem 1.1. Let $(\bar{z}, \bar{w}) \in \mathcal{M}$ be a minimizer of the functional $I|_{\mathcal{M}}$. Then, from Lemma 3.9, one has

$$I(\bar{z}, \bar{w}) = m = \inf_{(z,w)\in H\setminus\{(0,0)\}} \max_{t>0} I(z_t, w_t).$$

Suppose by contradiction that $(\bar{z}, \bar{w})$ is not a critical point of I, then one can find $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R}^N)$ such that

$$\begin{aligned}
&\langle I'(\bar{z}, \bar{w}), (\varphi_1, \varphi_2)\rangle \\
&= \int_{\mathbb{R}^N} (\nabla \bar{z} \nabla \varphi_1 + \nabla \bar{w} \nabla \varphi_2) + \int_{\mathbb{R}^N} [V_1(x) f(\bar{z}) f'(\bar{z})\varphi_1 + V_2(x) f(\bar{w}) f'(\bar{w})\varphi_2] \\
&\quad - \int_{\mathbb{R}^N} [h(f(\bar{z}), f(\bar{w})) f'(\bar{z})\varphi_1 + g(f(\bar{z}), f(\bar{w})) f'(\bar{w})\varphi_2] \\
&< -1.
\end{aligned}$$

We choose small $\varepsilon > 0$ such that

$$\langle I'(\bar{z}_t + \sigma\varphi_1, \bar{w}_t + \sigma\varphi_2), (\varphi_1, \varphi_2)\rangle \le -\frac{1}{2},\ |t-1|,\ |\sigma| \le \varepsilon,$$

and introduce a cut-off function $0 \le \xi \le 1$ satisfying $\xi(t) = 1$ for $|t-1| \le \frac{\varepsilon}{2}$ and $\xi(t) = 0$ for $|t-1| \ge \varepsilon$. For $t \ge 0$, we define

$$\eta_1(t) := \begin{cases} \bar{z}_t, & |t-1| \ge \varepsilon, \\ \bar{z}_t + \varepsilon\xi(t)\varphi_1, & |t-1| < \varepsilon, \end{cases}$$

$$\eta_2(t) := \begin{cases} \bar{w}_t, & |t-1| \ge \varepsilon, \\ \bar{w}_t + \varepsilon\xi(t)\varphi_2, & |t-1| < \varepsilon. \end{cases}$$

Then $\eta_i(t)$ is a continuous curve in the metric $(H^1(\mathbb{R}^N), d)$, and eventually choosing a smaller ε, we get $d_{H^1(\mathbb{R}^N)}(\eta_i(t), 0) > 0$ for $|t-1| < \varepsilon$, where $d_{H^1(\mathbb{R}^N)}(u, v) = \|u - v\|_{H^1(\mathbb{R}^N)}$.

Next, we claim that $\sup_{t\ge0} I(\eta_1(t), \eta_2(t)) < m$. Indeed, if $|t-1| \ge \varepsilon$, then by Lemma 3.2, for all $t > 0$ we have

$$I(\eta_1(t), \eta_2(t)) = I(\bar{z}_t, \bar{w}_t) \le I(\bar{z}, \bar{w}) - \frac{1}{2}Q(t, \bar{z}, \bar{w}) = m - \frac{1}{2}Q(t, \bar{z}, \bar{w}) < m.$$

If $|t-1| < \varepsilon$, $t \ge 0$, by using the mean value theorem to the C^1-map $[0, \varepsilon] \ni \sigma \mapsto I'(\bar{z}_t + \sigma\xi(t)\varphi_1, \bar{w}_t + \sigma\xi(t)\varphi_2) \in \mathbb{R}$, we find, for a suitable $\bar{\sigma} \in (0, \varepsilon)$,

$$\begin{aligned}
I(\eta_1(t), \eta_2(t)) &= I(\bar{z}_t + \sigma\xi(t)\varphi_1, \bar{w}_t + \sigma\xi(t)\varphi_2) \\
&= I(\bar{z}_t, \bar{w}_t) + \langle I'(\bar{z}_t + \bar{\sigma}\xi(t)\varphi_1, \bar{w}_t + \bar{\sigma}\xi(t)\varphi_2), (\xi(t)\varphi_1, \xi(t)\varphi_2)\rangle \\
&\le I(\bar{z}_t, \bar{w}_t) - \frac{1}{2}\xi(t) \\
&< m.
\end{aligned} \tag{4.24}$$

Define $\Psi_0(t) := J(\eta_1(t), \eta_2(t))$. It follows from (F1), (F2) that there exist $T_1 \in (0,1)$ and $T_2 \in (1,\infty)$ such that $J(\bar{z}_{T_1}, \bar{w}_{T_1}) > 0$ and $J(\bar{z}_{T_2}, \bar{w}_{T_2}) < 0$. Then we observe

$$\Psi_0(T_1) = J(\eta_1(T_1), \eta_2(T_1)) = J(\bar{z}_{T_1}, \bar{w}_{T_1}) > 0$$

and

$$\Psi_0(T_2) = J(\eta_1(T_2), \eta_2(T_2)) = J(\bar{z}_{T_2}, \bar{w}_{T_2}) > 0,$$

which, together with the continuity of $\Psi_0(t)$ on $t \in [0,\infty)$, imply that there exists $t_0 \in [T_1, T_2]$ such that $J(\eta_1(t_0), \eta_2(t_0)) = 0$. Thus, we have

$$(\eta_1(t_0), \eta_2(t_0)) \cap \mathcal{M} \neq \emptyset$$

and

$$I(\eta_1(t_0), \eta_2(t_0)) < m.$$

This contradicts to the definition of m.

To conclude, there exists $(\bar{z}, \bar{w}) \in \mathcal{M}$ such that

$$I(\bar{z}, \bar{w}) = m = \inf_{(z,w)\in H\setminus\{(0,0)\}} \max_{t>0} I(z_t, w_t), \ I'(\bar{z}, \bar{w}) = 0.$$

This shows $(\bar{z}, \bar{w})$ is a ground state solution of (1.1) such that $I(\bar{z}, \bar{w}) = \inf_{\mathcal{M}} I > 0$.

The proof is completed.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. S. Kurihara, Large-amplitude quasi-solitons in superfluids films, *J. Phys. Soc. Jpn.*, **50** (1981), 3262–3267. https://doi.org/10.1143/JPSJ.50.3262
2. E. W. Laedke, K. H. Spatschek, L. Stenflo, Evolution theorem for a class of perturbed envelope soliton solutions, *J. Math. Phys.*, **24** (1963), 2764–2769. https://doi.org/10.1063/1.525675
3. A. V. Borovskii, A. L. Galkin, D. Book, Dynamical modulation of an ultrashort high-intensity laser pulse in matter, *J. Exp. Theor. Phys.*, **77** (1993), 562–573.
4. H. S. Brandi, C. Manus, G. Mainfray, Relativistic and ponderomotive self-focusing of a laser beam in a radially inhomogeneous plasma, *Physics Fluids B Plasma Phys.*, **5** (1993), 3539–3550. https://doi.org/10.1063/1.860828

5. X. L. Chen, R. N. Sudan, Necessary and sufficient conditions for self-focusing of short ultraintense laser pulse, *Phys. Rev. Lett.*, **70** (1993), 2082–2085. https://doi.org/10.1103/PhysRevLett.70.2082

6. S. Cuccagna, On instability of excited states of the nonlinear Schrödinger equation, *Phys. D. Nonlinear Phenom.*, **238** (2009), 38–54. https://doi.org/10.1016/j.physd.2008.08.010

7. H. Lange, B. Toomire, P. F. Zweifel, Time-dependent dissipation in nonlinear Schrödinger systems, *J. Math. Phys.*, **36** (1995), 1274–1283. https://doi.org/10.1063/1.531120

8. X. H. Tang, S. T. Chen, Singularly perturbed choquard equations with nonlinearity satisfying Berestycki-Lions assumptions, *Adv. Nonlinear Anal.*, **9** (2019), 413–437. https://doi.org/10.1515/anona-2020-0007

9. F. G. Bass, N. N. Nasanov, Nonlinear electromagnetic-spin waves, *Phys. Rep.*, **189** (1990), 165–223. https://doi.org/10.1016/0370-1573(90)90093-H

10. L. Briill, H. Lange, Solitary waves for quasilinear Schrödinger equations, *Expo. Math.*, **4** (1986), 279–288.

11. R. W. Hasse, A general method for the solution of nonlinear soliton and kink Schrödinger equations, *Z. für Phys. B Condens. Matter*, **37** (1980), 83–87. https://doi.org/10.1007/BF01325508

12. A. M. Kosevich, B. A. Ivanov, A. S. Kovalev, Magnetic solitons, *Phys. Rep.*, **194** (1990), 117–238. https://doi.org/10.1016/0370-1573(90)90130-T

13. H. Lange, M. Poppenberg, H. Teismann, Nash-Moser methods for the solution of quasilinear Schrödinger equations, *Commun. Partial Differ. Equations*, **24** (1999), 1399–1418. https://doi.org/10.1080/03605309908821469

14. V. G. Makhankov, V. K. Fedyanin, Nonlinear effects in quasi-one-dimensional models and condensed matter theory, *Phy. Rep.*, **104** (1984), 1–86.

15. M. Poppenberg, K. Schmitt, Z. Q. Wang, On the existence of soliton solutions to quasilinear Schrödinger equations, *Calc. Var. Partial Differ. Equations*, **14** (2002), 329–344. https://doi.org/10.1007/s005260100105

16. G. R. W. Quispel, H. W. Capel, Equation of motion for the Heisenberg spin chain, *Phys. A Stat. Mech. Appl.*, **110** (1982), 41–80. https://doi.org/10.1016/0378-4371(82)90104-2

17. J. Q. Liu, Y. Q. Wang, Z. Q. Wang, Soliton solutions for quasilinear Schrödinger equations, II, *J. Differ. Equations*, **187** (2003), 473–493. https://doi.org/10.1016/S0022-0396(02)00064-5

18. T. Bartsch, Z. Q. Wang, Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^N$, *Commun. Partial Differ. Equations*, **20** (1995), 1725–1741. https://doi.org/10.1080/03605309508821149

19. M. Colin, L. Jeanjean, Solutions for a quasilinear Schrödinger equation: a dual approach, *Nonlinear Anal.*, **56** (2004), 213–226. https://doi.org/10.1016/j.na.2003.09.008

20. D. Ruiz, G. Siciliano, Existence of ground states for a modified nonlinear Schrödinger equations, *Nonlinearity*, **23** (2010), 1221–1233. 10.1088/0951-7715/23/5/011

21. S. T. Chen, V. D. Rădulescu, X. H. Tang, B. L. Zhang, Ground state solutions for quasilinear Schrödinger equations with variable potential and superlinear reaction, *Rev. Mat. Iberoam.*, **36** (2020), 1549–1570. https://doi.org/10.4171/RMI/1175

22. Y. X. Guo, Z. W. Tang, Ground state solutions for quasilinear Schrödinger systems, *J. Math. Anal. Appl.*, **389** (2012), 322–339. https://doi.org/10.1016/j.jmaa.2011.11.064

23. U. Severo, E. da Silva, On the existence of standing wave solutions for a class of quasilinear Schrödinger systems, *J. Math. Anal. Appl.*, **412** (2014), 763–775. https://doi.org/10.1016/j.jmaa.2013.11.012

24. J. Q. Chen, Q. Zhang, Ground state solution of Nehari-Pohožaev type for periodic quasilinear Schrödinger system, *J. Math. Phys.*, **61** (2020), 101510. https://doi.org/10.1063/5.0014321

25. J. H. Chen, B. T. Cheng, X. J. Huang, Ground state solutions for a class of quasilinear Schrödinger equations with Choquard type nonlinearity, *Appl. Math. Lett.*, **102** (2020), 106141. https://doi.org/10.1016/j.aml.2019.106141

26. J. H. Chen, X. J. Huang, B. T. Cheng, Positive solutions for a class of quasilinear Schrödinger equations with superlinear condition, *Appl. Math. Lett.*, **87** (2019), 165–171. https://doi.org/10.1016/j.aml.2018.07.035

27. J. H. Chen, X. J. Huang, B. T. Cheng, C. X. Zhu, Some results on standing wave solutions for a class of quasilinear Schrödinger equations, *J. Math. Phys.*, **60** (2019), 091506. https://doi.org/10.1063/1.5093720

28. J. H. Chen, X. J. Huang, D. D. Qin, B. T. Cheng, Existence and asymptotic behavior of standing wave solutions for a class of generalized quasilinear Schrödinger equations with critical Sobolev exponents, *Asymptotic Anal.*, **120** (2020), 199–248. https://doi.org/10.3233/ASY-191586

29. J. H. Chen, Q. F. Wu, X. J. Huang, C. X. Zhu, Positive solutions for a class of quasilinear Schrödinger equations equations with two parameters, *Bull. Malays. Math. Sci. Soc.*, **43** (2020), 2321–2341. https://doi.org/10.1007/s40840-019-00803-y

30. Y. T. Shen, Y. J. Wang, Soliton solutions for generalized quasilinear Schrödinger equations, *Nonlinear Anal. Theory Methods Appl.*, **80** (2013), 194–201. https://doi.org/10.1016/j.na.2012.10.005

31. J. Chen, Q. Zhang, Existence of positive ground state solutions for quasilinear Schrödinger system with positive parameter, *Appl. Anal.*, (2022). https://doi.org/10.1080/00036811.2022.2033232

32. W. Zhang, J. Zhang, V. D. Rădulescu, Concentrating solutions for singularly perturbed double phase problems with nonlocal reaction, *J. Differ. Equations*, **347** (2023), 56–103. https://doi.org/10.1016/j.jde.2022.11.033

33. J. Zhang, W. Zhang, Semiclassical states for coupled nonlinear Schrödinger system with competing potentials, *J. Geom. Anal.*, **32** (2022), 1–36.

34. G. Li, On the existence of nontrivial solutions for quasilinear Schrödinger systems, *Boundary Value Probl.*, **1** (2022), 1–17.

35. J. Q. Chen, Q. Zhang, Existence of ground state solution of Nehari-Pohožaev type for a quasilinear Schrödinger system, *Differential Integral Equations*, **34** (2021), 1-20. https://doi.org/10.57262/die/1610420451

36. J. M. do *Ó*, U. B. Severo, Quasilinear Schrödinger equations involving concave and convex nonlinearities, *Commun. Pure Appl. Anal.*, **8** (2009), 621–644. http://doi.org/10.3934/cpaa.2009.8.621

37. S. T. Chen, X. H. Tang, F. F. Liao, Existence and asymptotic behavior of sign-changing solutions for fractional Kirchhoff-type problems in low dimensions, *Nonlinear Differ. Equations Appl. NoDEA*, **25** (2018), 1–23. https://doi.org/10.1007/s00030-018-0531-9

38. S. T. Chen, B. T. Zhang, X. H. Tang, Existence and non-existence results for Kirchhoff-type problems with convolution nonlinearity, *Adv. Nonlinear Anal.*, **9** (2018), 148–167. https://doi.org/10.1515/anona-2018-0147