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Research article

Positive solutions for fractional iterative functional differential equation with a convection term

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Abstract: In this paper, we deal with the fractional iterative functional differential equation nonlocal boundary value problem with a convection term. By using the fixed point theorems, some results about existence, uniqueness, continuous dependence and multiplicity of positive solutions are derived.

Keywords: fractional iterative functional differential equation; convection term; integral boundary condition; positive solution; fixed point theorems

1. Introduction

In this paper, we study the fractional iterative functional differential equation with a convection term and nonlocal boundary condition

$$\begin{cases} -^C D_{0^+}^{\alpha} u(t) + \lambda u'(t) = f(u^{[0]}(t), u^{[1]}(t), \cdots, u^{[N]}(t)), \quad 0 < t < 1, \\ u'(0) = 0, \quad u(1) = \varphi(u), \end{cases} \tag{1.1}$$

where ${}^C D_{0^+}^{\alpha}$ denotes the Caputo derivative of order α, $1 < \alpha < 2$, $\lambda \in \mathbb{R}$, $u^{[0]}(t) = t$, $u^{[1]}(t) = u(t), \cdots, u^{[N]}(t) = u^{[N-1]}(u(t))$. $\varphi(u) = \int_0^1 u(s)dA(s)$ is a Stieltjes integral with a signed measure, that is, A is a function of bounded variation.

During the recent few decades, a vast literature on fractional differential equations has emerged, see [1–6] and the references therein. On the excellent survey of these related documents it is pointed out that the applicability of the theoretical results to fractional differential equations arising in various fields, for instance, chaotic synchronization [3], signal propagation [4], viscoelasticity [5], dynamical networks with multiple weights [6], and so on. Recently, we notice that the study of Caputo fractional differential equations with a convection term has become a heat topic (see [7–10]).

In [7], Meng and Stynes considered the Green function and maximum principle for the following

Caputo fractional boundary value problem (BVP)

$$\begin{cases} -^C D_{0^+}^{\alpha} u(t) + bu'(t) = f(t), \quad 0 < t < 1, \\ u(0) - \beta_0 u'(0) = \gamma_0, \quad u(1) + \beta_1 u'(1) = \gamma_1, \end{cases}$$

where $1 < \alpha < 2$, b, β_0, β_1, γ_0, $\gamma_1 \in \mathbb{R}$ and $f \in C[0,1]$ are given. Bai et al. [8] studied the Green function of the above problem, and the results obtained improve some conclusions of [7] to some degree.

Wang et al. [9] used operator theory to establish the Green function for the following problem

$$\begin{cases} -^C D_{a^+}^{\alpha} u(t) + \lambda u'(t) = h(t), \quad a < t < b, \\ u(a) - \beta_0 u'(a) = \gamma_0, \quad u(b) + \beta_1 u'(b) = \gamma_1, \end{cases}$$

where $1 < \alpha < 2$, the constants $\lambda, \beta_0, \beta_1, \gamma_0, \gamma_1$ and the function $h \in C[a,b]$ are given. The methods are entirely different from those used in [7, 8], and the results generalize corresponding ones in [7, 8].

In [10], Wei and Bai investigated the following fractional order BVP

$$\begin{cases} -^C D_{0^+}^{\alpha} u(t) + bu'(t) = f(t, u(t)), \quad x \in (0,1), \\ u(0) - \beta_0 u'(0) = 0, \quad u(1) + \beta_1 u'(1) = 0, \end{cases}$$

where $1 < \alpha \leq 2$ and b, β_0, $\beta_1 \in \mathbb{R}$ are constants. By employing the Guo-Krasnoselskii fixed point theorem and Leggett-Williams fixed point theorem, the existence and multiplicity results of positive solutions are presented.

Now, not only fractional differential equations have been studied constantly (see [11–17]), but also iterative functional differential equations have been discussed extensively as valuable tools in the modeling of many phenomena in various fields of scientific and engineering disciplines, for example, see [18–26] and the references cited therein. In [22], Zhao and Liu used the Krasnoselskii fixed point theorem to discuss the existence of periodic solutions of an iterative functional differential equation

$$u'(t) = c_1(t)u(t) + c_1(t)u^{[2]}(t) + \ldots + c_n(t)u^{[N]}(t) + F(t).$$

For the general iterative functional differential equation

$$u'(t) = f(u^{[0]}(t), u^{[1]}(t), \cdots, u^{[N]}(t)),$$

the existence, uniqueness, boundedness and continuous dependence on initial data of positive solutions was considered in [23].

In [24], the authors studied the following BVP

$$\begin{cases} u''(t) + h(u^{[0]}(t), u^{[1]}(t), \cdots, u^{[N]}(t)) = 0, \quad -b \leq t \leq b, \\ u(-b) = \eta_1, \quad u(b) = \eta_2, \quad \eta_1, \eta_2 \in [-b, b], \end{cases}$$

where $h : [-b, b] \times \mathbb{R}^N \to \mathbb{R}$ is continuous. By using the fixed point theorems, the authors established the existence, uniqueness and continuous dependence of a bounded solution.

To the best of our knowledge, there are few researches on fractional iterative functional differential equations integral boundary value problem with a convection term. Motivated by the above works and for the purpose to contribute to filling these gaps in the literature, this paper mainly focuses on handing with the existence, uniqueness, continuous dependence and multiplicity of positive solutions for the fractional iterative functional differential equation nonlocal BVP (1.1).

By a positive solution u of (1.1), we mean $u(t) > 0$ for $t \in [0, 1]$ and satisfies (1.1).

2. Preliminaries and lemmas

Definition 2.1.([1], [2]) *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by*

$$I^{\alpha}_{0^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided the right-hand side is pointwise defined on $(0, +\infty)$, where $\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt$ $(x > 0)$ is the gamma function.

Definition 2.2.([1], [2]) *The Caputo fractional derivative of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by*

$$^{C}D^{\alpha}_{0^+} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds,$$

provided the right-hand side is pointwise defined on $(0, +\infty)$, where $m = [\alpha] + 1$.

Definition 2.3. *[1] The two-parameter Mittag-Leffler function is defined by*

$$E_{\alpha,\gamma}(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \gamma)}, \quad \text{for } \alpha > 0,\ \gamma > 0 \text{ and } x \in \mathbb{R}.$$

Lemma 2.1. *[7] Let $F_\beta(x) = x^{\beta-1} E_{\alpha-1,\beta}(\lambda x^{\alpha-1})$. Then F_β has the following properties:*
$(P_1) : [F_{\beta+1}(x)]' = F_\beta(x)$ *for* $\beta \geq 0$ *and* $x \geq 0$*;*
$(P_2) : F_1(0) = 1,\ F_\beta(0) = 0$ *for* $\beta > 1$*;*
$(P_3) : F_1(x) > 0$ *for* $x > 0,\ F_2(x)$ *is increasing for* $x \geq 0$*;*
$(P_4) : F_{\alpha-1}(x) > 0$ *for* $x > 0,\ F_\alpha(x)$ *is increasing for* $x > 0$*;*
$(P_5) : F_1(x) = \lambda F_\alpha(x) + 1$ *for* $0 \leq x \leq 1$.

For $\beta > 0$ and $\nu > 0$ one has by [1]

$$(I^{\beta}_{0^+} F_\nu)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} s^{\nu-1} E_{\alpha-1,\nu}(\lambda s^{\alpha-1}) ds = t^{\beta+\nu-1} E_{\alpha-1,\beta+\nu}(\lambda t^{\alpha-1}).$$

That is to say,

$$(I^{\beta}_{0^+} F_\nu)(t) = F_{\beta+\nu}(t), \quad 0 < t \leq 1. \tag{2.1}$$

Lemma 2.2. *Suppose that $h \in AC[0, 1]$ and $\varphi(\mathbf{1}) \neq 1$. Then $u \in AC^2[0, 1]$ is the solution of*

$$\begin{cases} -^{C} D^{\alpha}_{0^+} u(t) + \lambda u'(t) = h(t), \quad 0 < t < 1,\ 1 < \alpha < 2, \\ u'(0) = 0,\ u(1) = \varphi(u), \end{cases} \tag{2.2}$$

if and only if the function u satisfies $u(t) = \int_0^1 H(t, s) h(s) ds$, where

$$H(t, s) = \frac{1}{1 - \varphi(\mathbf{1})} \mathcal{G}_A(s) + G(t, s),$$

$$\mathcal{G}_A(s) = \int_0^1 G(t, s) dA(t), \quad G(t, s) = \begin{cases} F_\alpha(1-s) - F_\alpha(t-s), & 0 \leq s \leq t \leq 1, \\ F_\alpha(1-s), & 0 \leq t \leq s \leq 1. \end{cases}$$

Proof. Applying I^{α}_{0+} to the both sides of the Eq (2.2), we know by simple calculation that the general solution of (2.2) is given by

$$u(t) = C_0 + C_1 F_2(t) - \int_0^t F_\alpha(t-s)h(s)ds, \quad t \in [0,1]. \tag{2.3}$$

Then

$$u'(t) = C_1 F_1(t) - \int_0^t F_{\alpha-1}(t-s)h(s)ds.$$

In view of $u'(0) = 0$ and $u(1) = \varphi(u)$, we deduce by (2.3) that

$$C_1 = 0, \quad C_0 = \varphi(u) + \int_0^1 F_\alpha(1-s)h(s)ds.$$

Therefore,

$$\begin{aligned} u(t) &= \varphi(u) + \int_0^1 F_\alpha(1-s)h(s)ds - \int_0^t F_\alpha(t-s)h(s)ds \\ &= \varphi(u) + \int_0^1 G(t,s)h(s)ds, \quad t \in [0,1]. \end{aligned} \tag{2.4}$$

Direct computations yield

$$\begin{aligned} \varphi(u) &= \int_0^1 \varphi(u)dA(t) + \int_0^1 \int_0^1 G(t,s)h(s)dsdA(t) \\ &= \varphi(u)\varphi(\mathbf{1}) + \int_0^1 \int_0^1 G(t,s)dA(t)h(s)ds \\ &= \varphi(u)\varphi(\mathbf{1}) + \int_0^1 \mathcal{G}_A(s)h(s)ds. \end{aligned}$$

It follows that

$$\varphi(u) = \frac{1}{1-\varphi(\mathbf{1})} \int_0^1 \mathcal{G}_A(s)h(s)ds.$$

Substituting it to (2.4), we have $u(t) = \int_0^1 H(t,s)h(s)ds, \ t \in [0,1]$.

Conversely, due to

$$u(t) = \int_0^1 H(t,s)h(s)ds = \int_0^1 \left(\frac{1}{1-\varphi(\mathbf{1})} \mathcal{G}_A(s) + G(t,s) \right) h(s)ds,$$

we obtain

$$\varphi(u) = \frac{\varphi(\mathbf{1})}{1-\varphi(\mathbf{1})} \int_0^1 \mathcal{G}_A(s)h(s)ds + \int_0^1 \mathcal{G}_A(s)h(s)ds = \frac{1}{1-\varphi(\mathbf{1})} \int_0^1 \mathcal{G}_A(s)h(s)ds,$$

and

$$u(t) = \varphi(u) + \int_0^1 G(t,s)h(s)ds = \varphi(u) + \int_0^1 F_\alpha(1-s)h(s)ds - \int_0^t F_\alpha(t-s)h(s)ds.$$

Then, $u(1) = \varphi(u)$ and $u'(0) = 0$.

Let

$$H(t) = \int_0^t F_\alpha(t-s)h(s)ds = \int_0^t F_\alpha(s)h(t-s)ds, \quad 0 \le t \le 1.$$

Then, for almost all $t \in [0, 1]$,

$$H'(t) = h(0)F_\alpha(t) + \int_0^t F_\alpha(s)h'(t-s)ds = h(0)F_\alpha(t) + \int_0^t F_\alpha(t-s)h'(s)ds,$$

and

$$H''(t) = h(0)F_{\alpha-1}(t) + \int_0^t F_{\alpha-1}(t-s)h'(s)ds.$$

Then using (2.1), we calculate

$$\begin{aligned}
{}^C D_{0^+}^\alpha H(t) &= (I_{0^+}^{2-\alpha} H'')(t) \\
&= h(0)F_1(t) + \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-r)^{1-\alpha} \int_0^r F_{\alpha-1}(r-s)h'(s)dsdr \\
&= h(0)F_1(t) + \int_0^t h'(s)[\frac{1}{\Gamma(2-\alpha)} \int_0^{t-s} (t-s-\tau)^{1-\alpha} F_{\alpha-1}(\tau)d\tau]ds \\
&= h(0)F_1(t) + \int_0^t h'(s)(I^{2-\alpha}F_{\alpha-1})(t-s)ds \\
&= h(0)F_1(t) + \int_0^t h'(s)F_1(t-s)ds,
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
\lambda H'(t) &= \lambda h(0)F_\alpha(t) + \lambda \int_0^t F_\alpha(t-s)h'(s)ds \\
&= \lambda h(0)F_\alpha(t) + \int_0^t [F_1(t-s) - 1]h'(s)ds.
\end{aligned} \tag{2.6}$$

Combining (2.5) with (2.6), we obtain

$$\begin{aligned}
&{}^C D_{0^+}^\alpha H(t) - \lambda H'(t) \\
=&h(0)F_1(t) - \lambda h(0)F_\alpha(t) + \int_0^t h'(s)ds \\
=&h(t) + h(0)[F_1(t) - \lambda F_\alpha(t) - 1] \\
=&h(t), \quad 0 < t < 1,
\end{aligned}$$

where we utilize $h \in AC[0, 1]$ and (P_5). Consequently, we obtain $-{}^C D_{0^+}^\alpha u(t) + \lambda u'(t) = h(t)$. The proof is finished.

As a direct consequence of the previous results, we deduce the following properties that, as we will see, will be fundamental for subsequent studies.

Lemma 2.3. *Assume that* $0 \le \varphi(\mathbf{1}) < 1$ *and* $\mathcal{G}_A(s) \ge 0$ *for* $s \in [0, 1]$. *Then for* $t, s \in [0, 1]$,

1). $G(t, s)$ *and* $H(t, s)$ *are continuous;*

2). $0 \le G(t,s) \le F_\alpha(1-s)$, and $G(t,s)$ is decreasing with respect to t;

3). $0 \le H(t,s) \le \omega(s)$, where $\omega(s) = \frac{1}{1-\varphi(\mathbf{1})}\mathcal{G}_A(s) + F_\alpha(1-s)$, and $H(t,s)$ is decreasing with respect to t.

Let $E = C[0,1]$. Then E is a Banach space with the usual maximum norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. For $0 \le P \le 1$ and $L > 0$, define

$$\Omega(P,L) = \{u \in E : 0 \le u(t) \le P,\ |u(t_2) - u(t_1)| \le L|t_2 - t_1|,\ \forall t, t_1, t_2 \in [0,1]\}.$$

It is easy to show that $\Omega(P,L)$ is a convex and compact set.

Define an operator $T_\lambda : E \to E$ as follows:

$$(Tu)(t) = \int_0^1 H(t,s) f(u^{[0]}(s), u^{[1]}(s), \cdots, u^{[N]}(s))ds, \quad t \in [0,1].$$

By Lemma 2.2, we can easily know that u is a solution of BVP (1.1) iff u is the fixed point of the operator T.

Suppose that

(H_1) for the function $f : [0,1]^{N+1} \to [0,+\infty)$, there exist constants $0 < k_0, k_1, \cdots, k_N < +\infty$ such that

$$|f(t, u_1, u_2, \cdots, u_N) - f(s, v_1, \cdots, v_N)| \le k_0|t-s| + \sum_{j=1}^{N} k_j|u_j - v_j|.$$

(H_2) $0 \le \varphi(\mathbf{1}) < 1$, and A is a function of bounded variation such that $\mathcal{G}_A(s) \ge 0$ for $s \in [0,1]$.

Clearly, using (H_1), we obtain

$$|f(t, u_1, u_2, \cdots, u_N)| \le k_0|t| + \beta + \sum_{j=1}^{N} k_j|u_j|, \tag{2.7}$$

where $\beta = |f(0,0,\cdots,0)|$.

Lemma 2.4. *[22] For any $u, v \in \Omega(P,L)$,*

$$\|u^{[n]} - v^{[n]}\| \le \sum_{i=0}^{n-1} L^i \|u - v\|, \quad n = 1, 2, \cdots, N.$$

Lemma 2.5. *[27] Let P be a cone in a real Banach space E, $P_c = \{u \in P : \|u\| \le c\}$, $P(\theta, a, b) = \{u \in P : a \le \theta(u),\ \|u\| \le b\}$. Suppose $A : P_c \to P_c$ is completely continuous, and suppose there exists a concave positive functional θ with $\theta(u) \le \|u\|$ ($u \in P$) and numbers a, b and d with $0 < d < a < b \le c$, satisfying the following conditions:*

$(C1)$ $\{u \in P(\theta,a,b) : \theta(u) > a\} \ne \emptyset$ and $\theta(Au) > a$ if $u \in P(\theta,a,b)$;

$(C2)$ $\|Au\| < d$ if $u \in P_d$;

$(C3)$ $\theta(Au) > a$ for all $u \in P(\theta,a,c)$ with $\|Au\| > b$.

Then A has at least three fixed points $u_1, u_2, u_3 \in P_c$ with

$$\|u_1\| < d,\ \theta(u_2) > a,\ \|u_3\| > d,\ \theta(u_3) < a.$$

Remark. *If $b = c$, then $(C1)$ implies $(C3)$.*

3. Existence and uniqueness of positive solution

Theorem 3.1. *Suppose that* (H_1) *and* (H_2) *hold. If*

$$(k_0 + \beta + \sum_{j=1}^{N} \sum_{i=0}^{j-1} k_j L^i P) \int_0^1 \omega(s) ds \le P, \tag{3.1}$$

and

$$(k_0 + \beta + \sum_{j=1}^{N} \sum_{i=0}^{j-1} k_j L^i P) F_\alpha(1) \le L, \tag{3.2}$$

then problem (1.1) *has a unique nonnegative solution. If in addition A is an increasing function, and there exists* $t_0 \in [0, 1]$ *such that* $f(t_0, 0, \cdots, 0) > 0$*, then problem* (1.1) *has a unique positive solution.*

Proof. For any $u \in \Omega(P, L)$, in view of (2.7) and Lemma 2.4, we deduce that

$$\begin{aligned} &|f(u^{[0]}(s), u^{[1]}(s) \cdots, u^{[N]}(s))| \\ \le & k_0|s| + \beta + \sum_{j=1}^{N} \sum_{i=0}^{j-1} k_j L^i \|u\| \\ \le & k_0 + \beta + \sum_{j=1}^{N} \sum_{i=0}^{j-1} k_j L^i P, \quad 0 \le s \le 1, \end{aligned}$$

and hence

$$\begin{aligned} |(Tu)(t)| &\le \int_0^1 \omega(s) |f(u^{[0]}(s), u^{[1]}(s), \cdots, u^{[N]}(s))| ds \\ &\le (k_0 + \beta + \sum_{j=1}^{N} \sum_{i=0}^{j-1} k_j L^i P) \int_0^1 \omega(s) ds \le P, \quad t \in [0, 1]. \end{aligned}$$

Therefore, $0 \le (Tu)(t) \le P$ for $t \in [0, 1]$.

On the other hand, for any $t_1, t_2 \in [0, 1]$ and $t_1 < t_2$, by means of Lagrange mean value theorem, there exists $\xi \in (t_1, t_2) \subset (0, 1)$ such that

$$\begin{aligned} \int_0^1 |H(t_2, s) - H(t_1, s)| \, ds &= \int_0^1 (G(t_1, s) - G(t_2, s)) ds \\ &= \int_0^{t_2} F_\alpha(t_2 - s) \, ds - \int_0^{t_1} F_\alpha(t_1 - s) \, ds \\ &= F_{\alpha+1}(t_2) - F_{\alpha+1}(t_1) \\ &= F_\alpha(\xi)(t_2 - t_1) \\ &\le F_\alpha(1)(t_2 - t_1). \end{aligned}$$

It follows from (3.2) that

$$\begin{aligned}|(Tu)(t_2)-(Tu)(t_1)| &\le (k_0+\beta+\sum_{j=1}^{N}\sum_{i=0}^{j-1}k_jL^iP)\int_0^1|H(t_2,s)-H(t_1,s)|\,ds\\ &\le (k_0+\beta+\sum_{j=1}^{N}\sum_{i=0}^{j-1}k_jL^iP)F_\alpha(1)(t_2-t_1)\\ &\le L(t_2-t_1).\end{aligned}$$

Therefore, $T(\Omega(P,L))\subset\Omega(P,L)$.

Next, we show that T is a contraction mapping on $\Omega(P,L)$. Indeed, let $u,v\in\Omega(P,L)$. Then

$$\begin{aligned}&|f(u^{[0]}(s),u^{[1]}(s),\cdots,u^{[N]}(s))-f(v^{[0]}(s),v^{[1]}(s),\cdots,v^{[N]}(s))|\\ &\le\sum_{j=1}^{N}k_j\|u^{[j]}-v^{[j]}\|\le\sum_{j=1}^{N}\sum_{i=0}^{j-1}k_jL^i\|u-v\|,\end{aligned}$$

and

$$\begin{aligned}&\|Tu-Tv\|\\ &\le\int_0^1\omega(s)|f(u^{[0]}(s),u^{[1]}(s),\cdots,u^{[N]}(s))-f(v^{[0]}(s),v^{[1]}(s),\cdots,v^{[N]}(s))|ds\\ &\le\sum_{j=1}^{N}\sum_{i=0}^{j-1}k_jL^i\|u-v\|\int_0^1\omega(s)ds.\end{aligned}$$

It follows from (3.1) that

$$\sum_{j=1}^{N}\sum_{i=0}^{j-1}k_jL^i\int_0^1\omega(s)ds<1.$$

This shows that T is a contraction mapping on $\Omega(P,L)$. It follows from the contraction mapping theorem that T has a unique fixed point u in $\Omega(P,L)$. In other words, problem (1.1) has a unique nonnegative solution.

Suppose u is the nonnegative solution to problem (1.1). Then

$$u(t)=\int_0^1H(t,s)f(u^{[0]}(s),\cdots,u^{[N]}(s))ds,\quad t\in[0,1].$$

By the monotonicity of $H(t,s)$, we have $u(t)\ge u(1)\ge0$ for $t\in[0,1]$. If A is an increasing function, and there exists $t_0\in[0,1]$ such that $f(t_0,0,\cdots,0)>0$, we must have $u(1)>0$. Otherwise, $u(1)=0$ and we have $\int_0^1u(s)dA(s)=\varphi(u)=u(1)=0$. Then $u(t)\equiv0$ for $t\in[0,1]$. By the equation of (1.1), we conclude $f(t,0,\cdots,0)\equiv0$ for $t\in[0,1]$, which is a contradiction. We have thus proved $u(1)>0$ and $u(t)\ge u(1)>0$ for $t\in[0,1]$.

4. Continuous dependence

Theorem 4.1. *Assume that the conditions of Theorem 3.1 are satisfied. Then the unique positive solution of problem* (1.1) *continuously depends on function* f.

Proof. For two continuous functions $f_1, f_2 : [0,1]^{N+1} \to [0,+\infty)$, they correspond respectively to unique solutions u_1 and u_2 in $\Omega(P,L)$ such that

$$u_i(t) = \int_0^1 H(t,s) f_i(u_i^{[0]}(s), u_i^{[1]}(s), \cdots, u_i^{[N]}(s))ds, \quad t \in [0,1],\ i = 1,2.$$

By (H_1), we find that

$$\begin{aligned}
&|f_2(u_2^{[0]}(s), u_2^{[1]}(s), \cdots, u_2^{[N]}(s)) - f_1(u_1^{[0]}(s), u_1^{[1]}(s), \cdots, u_1^{[N]}(s))| \\
\le& |f_2(u_2^{[0]}(s), u_2^{[1]}(s), \cdots, u_2^{[N]}(s)) - f_2(u_1^{[0]}(s), u_1^{[1]}(s), \cdots, u_1^{[N]}(s))| \\
&+ |f_2(u_1^{[0]}(s), u_1^{[1]}(s), \cdots, u_1^{[N]}(s)) - f_1(u_1^{[0]}(s), u_1^{[1]}(s), \cdots, u_1^{[N]}(s))| \\
\le& \|f_2 - f_1\| + \sum_{j=1}^{N} \sum_{i=0}^{j-1} k_j L^i \|u_2 - u_1\|.
\end{aligned}$$

It follows from Lemma 2.3 that

$$|u_2(t) - u_1(t)| \le (\|f_2 - f_1\| + \sum_{j=1}^{N} \sum_{i=0}^{j-1} k_j L^i \|u_2 - u_1\|) \int_0^1 \omega(s)ds, \quad t \in [0,1].$$

Then

$$\|u_2 - u_1\| \le \frac{\int_0^1 \omega(s)ds}{1 - \sum_{j=1}^{N} \sum_{i=0}^{j-1} k_j L^i \int_0^1 \omega(s)ds} \|f_2 - f_1\|.$$

The proof is complete.

5. Multiplicity of positive solutions

Define

$$\Omega = \{u \in E : u(t) \ge 0,\ t \in [0,1]\}, \quad \Omega_c = \{u \in \Omega : \|u\| < c\},$$

$$M = \left(\int_0^1 \omega(s)ds \right)^{-1},\ m = \left(\int_{\frac{1}{6}}^{\frac{5}{6}} \min_{t \in [\frac{1}{6}, \frac{5}{6}]} H(t,s)ds \right)^{-1},$$

$$\theta(u) = \min_{1/6 \le t \le 5/6} |u(t)|, \quad \Omega(\theta, b, d) = \{u \in \Omega : b \le \theta(u),\ \|u\| \le d\}.$$

Obviously, θ is a continuous concave functional and $\theta(u) \le \|u\|$ for $u \in \Omega$.

Theorem 5.1. *Assume that* $f \in AC([0,1]^{N+1}, [0,+\infty))$ *and* (H_2) *hold. If there exist constants* $0 < a < \frac{1}{6} \le b < c \le \frac{5}{6}$ *such that*

$(D1)$ $f(t, u_1, u_2, \cdots, u_N) < Ma, \quad (t, u_1, u_2, \cdots, u_N) \in [0,1] \times [0,a]^N$;

$(D2)$ $f(t, u_1, u_2, \cdots, u_N) > mb, \quad (t, u_1, u_2, \cdots, u_N) \in [\frac{1}{6}, \frac{5}{6}] \times [b, c]^N;$
$(D3)$ $f(t, u_1, u_2, \cdots, u_N) \leq Mc, \quad (t, u_1, u_2, \cdots, u_N) \in [0, 1] \times [0, c]^N,$
then problem (1.1) *has three non-negative solutions* $u_1, u_2, u_3 \in \overline{\Omega_c}$ *with*

$$\|u_1\| < a,\ \theta(u_2) > b,\ \|u_3\| > a,\ \theta(u_3) < b.$$

Proof. We first prove $T : \overline{\Omega_c} \to \overline{\Omega_c}$ is completely continuous. For $u \in \overline{\Omega_c}$, we have $\|u\| \leq c < 1$. Then $0 \leq u^{[j]}(s) \leq c$ for $j = 1, 2, \cdots, N$ and $0 \leq s \leq 1$. It follows from $(D3)$ that

$$\|Tu\| \leq \int_0^1 \omega(s) f(u^{[0]}(s), u^{[1]}(s), \cdots, u^{[N]}(s))ds \leq Mc \int_0^1 \omega(s)ds = c.$$

Therefore, $T(\overline{\Omega_c}) \subset \overline{\Omega_c}$ and TD is uniformly bounded for any bounded set $D \subset \overline{\Omega_c}$. We denote $\overline{M}$ as the maximum of f on $[0, 1]^{N+1}$. Since $H(t, s)$ is uniformly continuous on $[0, 1] \times [0, 1]$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $u \in D$, $t_1, t_2 \in [0, 1]$ and $|t_2 - t_1| < \delta$, we have $|H(t_2, s) - H(t_1, s)| < \frac{\varepsilon}{\overline{M}}$. Then,

$$|Tu(t_2) - Tu(t_1)| \leq \overline{M} \int_0^1 |H(t_2, s) - H(t_1, s)|ds < \varepsilon,$$

which implies that TD is equicontinuous. Clearly, the fact that f is continuous implies that T is continuous. Hence, $T : \overline{\Omega_c} \to \overline{\Omega_c}$ is completely continuous.

For any $u \in \overline{\Omega_a}$, due to $(D1)$, we conclude

$$\|Tu\| \leq \int_0^1 \omega(s) f(u^{[0]}(s), u^{[1]}(s), \cdots, u^{[N]}(s))ds \leq \int_0^1 \omega(s)Mads = a.$$

Then, $T(\overline{\Omega_a}) \subset \overline{\Omega_a}$, which implies that (C_2) in Lemma 2.5 holds.

Choose $v(t) = \frac{c+b}{2}$, $0 \leq t \leq 1$. Obviously, $v \in \{u \in \Omega(\theta, b, c) : \theta(u) > b\}$. Then $\{u \in \Omega(\theta, b, c) : \theta(u) > b\} \neq \emptyset$. For $u \in \Omega(\theta, b, c)$, we have $\frac{1}{6} \leq b \leq u(t) \leq c \leq \frac{5}{6}$ for $\frac{1}{6} \leq t \leq \frac{5}{6}$. Then $b \leq u^{[j]}(s) \leq c$ for $j = 1, 2, \cdots, N$ and $\frac{1}{6} \leq s \leq \frac{5}{6}$. It follows from $(D2)$ that

$$\theta(Tu) = \min_{1/6 \leq t \leq 5/6} |(Tu)(t)| > \int_{\frac{1}{6}}^{\frac{5}{6}} \min_{1/6 \leq t \leq 5/6} H(t, s)mbds = b,$$

which implies that (C_1) in Lemma 2.5 holds. By remark under Lemma 2.5, we know that (C_3) in Lemma 2.5 holds. Then according to Lemma 2.5, problem (1.1) has three nonnegative solutions $u_1, u_2, u_3 \in \overline{\Omega_c}$ with

$$\|u_1\| < a,\ \theta(u_2) > b,\ \|u_3\| > a,\ \theta(u_3) < b.$$

6. An example

We consider the following BVP

$$\begin{cases} -{}^C D_{0^+}^{\frac{3}{2}} u(t) + u'(t) = f(t, u(t), u^{[2]}(t), u^{[3]}(t)), \quad 0 < t < 1, \\ u'(0) = 0,\ u(1) = \int_0^1 u(s)dA(s), \end{cases} \tag{6.1}$$

where $f(t,u(t),u^{[2]}(t),u^{[3]}(t)) = \frac{1}{200}t + \frac{1}{100}sin(u(t)) + \frac{1}{100}sin(u^{[2]}(t)) + \frac{1}{100}sin(u^{[3]}(t))$, $A(s) = \frac{1}{2}s$. It follows that $\varphi(\mathbf{1}) = \int_0^1 d(\frac{1}{2}s) = \frac{1}{2}$ and $\mathcal{G}_A(s) = \int_0^1 G(t,s)d(\frac{1}{2}t) \geq 0$, which implies (H_2) holds. Since

$$
\begin{aligned}
&|f(t,u(t),u^{[2]}(t),u^{[3]}(t)) - f(s,u(s),u^{[2]}(s),u^{[3]}(s))| \\
\leq &\frac{1}{200}|t-s| + \frac{1}{100}|u(t)-u(s)| + \frac{1}{100}|u^{[2]}(t)-u^{[2]}(s)| + \frac{1}{100}|u^{[3]}(t)-u^{[3]}(s)|,
\end{aligned}
$$

we obtain $k_0 = \frac{1}{200}$, $k_1 = \frac{1}{100}$, $k_2 = \frac{1}{100}$, $k_3 = \frac{1}{100}$ and $\beta = 0$, which implies (H_1) holds.

Direct computation shows that $\int_0^1 \omega(s)ds = \int_0^1 [\frac{1}{1-\varphi(\mathbf{1})}\mathcal{G}_A(s) + F_{\frac{3}{2}}(1-s)]ds < \frac{159}{50}$. Choose $P = \frac{3}{4}$ and $L = 1$, we have

$$
(k_0 + \beta + \sum_{j=1}^{3}\sum_{i=0}^{j-1} k_j L^i P)\int_0^1 \omega(s)ds < \frac{159}{1000},
$$

and

$$
(k_0 + \beta + \sum_{j=1}^{3}\sum_{i=0}^{j-1} k_j L^i P)F_{\frac{3}{2}}(1) < \frac{3}{10}.
$$

Taking $t_0 = \frac{1}{2} \in [0,1]$, we find that $f(t_0,0,0,0) = \frac{1}{400} > 0$. By Theorem 3.1 we conclude that BVP (6.1) has a unique positive solution.

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Conflict of interest

The authors declare there is no conflicts of interest.

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