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ERA, 31(11): 6881–6897.
DOI: 10.3934/era.20233348
Received: 18 August 2023
Revised: 04 October 2023
Accepted: 08 October 2023
Published: 26 October 2023

Research article

Dynamical behavior of Benjamin-Bona-Mahony system with finite distributed delay in 3D

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Abstract: We study the Benjamin-Bona-Mahony model with finite distributed delay in 3D, which depicts the dispersive impact of long waves. Based on the well-posedness of model, the family of pullback attractors for the evolutionary processes generated by a global weak solution has been obtained, which is unique and minimal, via verifying asymptotic compactness in functional space with delay C_V and topological space $V \times C_V$, where the energy equation method and a retarded Gronwall inequality are utilized.

Keywords: Benjamin-Bona-Mahony system; distributed delay; pullback attractor

1. Introduction

In studying the dispersive impact of long waves in shallow water, Benjamin, Bona and Mahony discovered the following physical model (called the Benjamin-Bona-Mahony equations)

$$u_t + u_x + uu_x - u_{xxt} = 0,$$

also called BBM equations for short (see [1]). In addition, this model covers many kinds of waves, such as the surface wave, acoustic-gravity wave, hydromagnetic wave, acoustic waves and so on.

In previous decades, there have been many interesting results on the BBM equations subject to different conditions. In 1985, the existence results of solutions were extended to all dimensions in [2], and it was shown that the supremum solution norm decayed to zero like the expression $s^{-2/3}$ as $s \to \infty$ in considering the generalized BBM equations in 2D with small initial data (see [3]). Moreover, the relating existence of solutions in non-cylindrical domains can be found in [4], some conclusions on well-posedness on the energy space and numerical analyses can be seen in [5].

For existence, dimension estimate, regularity, smoothness of global attractor and determining nodes, many meaningful results can be found in [6–9]. Literatures [10, 11] have shown that the global weak attractors to the BBM equations exist in H^2_{per} and H^1 respectively, which are actually the global strong attractors via the energy equation method.

About the asymptotic behavior of BBM model, via the Littlewood-Paley projection operator, a sufficient condition was given in [12, 13], and an attractor was obtained by showing that the BBM system had the point dissipative property and asymptotic compactness, and the regularity of the system attractor was finally given. On an unbounded domain, in 2009 B. Wang studied the stochastic BBM system, obtained a random attractor in [14], showed that under the forward flow the attractor was invariant and had the property of pullback attraction to any random set, and by the tail-estimate method derived the asymptotic compactness of corresponding dynamical systems. Other results, such as the multiple-order breathers for the BBM system, can be seen in [15] and literatures therein.

In the industrial and economic fields, the delay/memory effect arises naturally, which leads to the idea that some motion depends on the present state together with the past state, for which some related interesting works can be seen in [16–21] for the dynamical behaviour of Navier-Stokes equations with delay, [22, 23] for long-time behaviour of solutions to the BBM system the delay/memory, [24] for the Brinkman-Forchheimer equation with delay and [25] for a viscoelastic system with memory and delay. However, results involving dynamics of the BBM model with finite distributed delay are few, and we aim to consider the dynamical behavior of the following BBM equations in 3D with finite distributed delay on a bounded domain $\Theta \subset \mathbb{R}^3$

$$\begin{cases} u_t - \triangle u_t - \nu\triangle u + \nabla\cdot\overrightarrow{F}(u) = \int_{-h}^{0}\mathcal{G}(t,s,u^t)ds, \ (t,x)\in\Theta_\tau, \\ u(t,x_i+Le_i) = u(t,x_i), \ (t,x)\in\partial\Theta_\tau, \\ u(\sigma,x) = u_0(x), \ x\in\Theta, \\ u(t+\sigma,x) = \eta(t,x), \ (t,x)\in[-h,0]\times\Theta, \end{cases} \tag{1.1}$$

where the boundary $\partial\Theta$ is smooth, $\Theta_\sigma = (\sigma,+\infty)\times\Theta$, $\partial\Theta_\sigma = (\sigma,+\infty)\times\partial\Theta$, and $\sigma\in\mathbb{R}$ is the initial time. $u(t,x)$ denotes the velocity vector field unknown, ν the kinematic viscosity of fluid, and $\int_{-h}^{0}\mathcal{G}(t,s,u^t)ds$ is the finite distributed delay, where

$$u^t(s) = u(t+s), \ s\in[-h,0], \ h>0.$$

Also, u_0 and the delay term η in $[-h,0]$ satisfy that $u_0 = \eta(0)$. $\overrightarrow{F}(t) = (F_1(t), F_2(t), F_3(t))$ is a nonlinear vector function on $\mathbb{R}$, where $F_k(t)$ $(k = 1,2,3)$ are smooth functions satisfying

$$F_k(0) = 0, \ |F_k(t)| \le C(|t| + |t|^2).$$

To the system (1.1) in 2D, if $F(u) = u + \frac{1}{2}u^2$, then it can be reduced into the generalized BBM equations

$$u_t + u_x + uu_x - \nu u_{xx} - u_{xxt} = g$$

which reflect the dispersive impact together with the dissipative effect. The main characteristics and difficulty encountered in this paper can be summed up in the following two points.

(i) For the system (1.1), we give some Banach spaces, some hypotheses on $\int_{-h}^{0}\mathcal{G}(t,s,u^t)ds$ and $\overrightarrow{F}$, and the definition of a weak solution together with the theory on dynamics in Section 2. Then, we derive the global well-posedness of system (1.1) via Fadeo-Galerkin approximation method in Section 3.

(ii) In Section 4, the novelty in this paper is to use the retarded Gronwall inequality, construct a tempered universe $\mathcal{D}$, and show that the $\mathcal{D}$-pullback absorbing set exists. Via the energy equation method, we show the process $U(\cdot,\cdot)$ to (1.1) has the property of $\mathcal{D}$-pullback asymptotic compactness, and get the pullback attractor in C_V and $V \times C_V$.

2. Preliminaries

2.1. Some spaces

Let H be $\overline{(C_0^\infty(\Theta))^3}$ in $(L^2(\Theta))^3$ topology with inner product $(\cdot,\cdot)$ and norm $|\cdot|$, V denotes $\overline{(C_0^\infty(\Theta))^3}$ in $(H^1(\Theta))^3$ topology with inner product $((\cdot,\cdot))$ and norm $\|\cdot\|$, and W is a homogeneous space of all functions in $(H^2(\Theta))^3$. Let $V^\star$ denote the dual space of V with norm $\|\cdot\|_\star$, $\langle\cdot,\cdot\rangle$ is the dual product between V and $V^\star$, and there holds the embedding relation that $V \hookrightarrow H \hookrightarrow V^\star$.

Under the periodic boundary condition, the elliptic operator $A = -\Delta$ is positively self-adjoint in H, and in space H the inverse operator A^{-1} is also compact. The properties of A lead to the fact that the eigenvalues $\{\lambda_k\}_{k=1}^\infty$ of A exist together with eigenfunctions $\{\omega_k\}_{k=1}^\infty$, which are orthonormal and satisfy

$$0 < \lambda_1 \le \lambda_2 \le \cdots, \ \lim_{k\to+\infty} \lambda_k = +\infty.$$

Define the retarded Banach spaces as

$$C_Y = C([-h,0];Y), \ L_Y^2 = L^2(-h,0;Y), \ Y = H, V$$

with the norm

$$\|u\|_{C_Y} = \sup_{s\in[-h,0]} \|u(\cdot+s)\|_Y, \ \|u\|_{L_Y^2} = \int_{-h}^0 \|u(\cdot+s)\|_Y ds.$$

2.2. The retarded Gronwall inequality

Lemma 2.1. *([26]) $u(t) \in Y$, Y is a Banach space, and there holds for any $t \ge \sigma \ge 0$ that*

$$\|u(t)\|_Y \le E(t,\sigma)\|u^\sigma\|_{C_Y} + \int_\sigma^t K_1(t,s)\|u^s\|_{C_Y} ds + \int_t^\infty K_2(t,s)\|u^s\|_{C_Y} ds + C_0, \tag{2.1}$$

where the functions $E(\cdot,\cdot)$, $K_1(\cdot,\cdot)$, $K_2(\cdot,\cdot) \ge 0$ are measurable in $\mathbb{R}^2$, and $C_0 \ge 0$. Assume that

$$\kappa(K_1,K_2) = \kappa_0 = \sup_{t\ge\sigma}\Big(\int_\sigma^t K_1(t,s)ds + \int_t^\infty K_2(t,s)ds\Big) < +\infty,$$

and

$$\lim_{t\to+\infty} E(t+l,l) = 0, \ \forall\, l \in \mathbb{R}^+.$$

Let $\vartheta = \sup_{t\ge s\ge\sigma} E(t,s)$, then it holds that

(R1) When $\kappa_0 < 1$, then for $\forall\, \varepsilon > 0$ and $R > 0$, there is a positive constant $T = T(\varepsilon,R)$ such that for any $t > T$

$$\|u^t\|_{C_Y} < \mu C_0 + \varepsilon,$$

where $u(t) \in C([-h,\infty);Y)$ *satisfying* (2.1) *with* $\|u^\sigma\|_{C_Y} \le R$, *and* $\mu = \frac{1}{1-\kappa_0}$.

(R2) When $\kappa_0 < \frac{1}{1+\vartheta}$, *there are positive constants* M_0 *and* ι *such that for any* $t \ge \sigma$

$$\|u^t\|_{C_Y} \le M_0\|u^\sigma\|_{C_Y}e^{-\iota t} + \gamma C_0,$$

where $u(t) \in C([-h,\infty);Y)$ *satisfying* (2.1), $\gamma = \frac{\mu+1}{1-\kappa_0 c}$, *and* $c = \max\{\frac{\vartheta}{1-\kappa_0}, 1\}$.

(R3) When $\kappa_0 < \frac{1}{1+\vartheta}$ *and* $\kappa_0 c < 1$, $u(t)$ *reduces to the trivial case.*

3. Global well-posedness

3.1. Some hypotheses

To prove the existence of a solution to (1.1), we let $g(t,u^t) = \int_{-h}^0 \mathcal{G}(t,s,u^t)ds$ and give the following conditions.

(C1) the measurable function $g : \mathbb{R}\times C_H \to (L^2(\Omega))^3$ satisfies for any $t \in \mathbb{R}$ that $g(t,0) = 0$, and there is a constant $L_g > 0$ satisfying for any $u^t, v^t \in C_H$ that

$$\|g(t,u^t) - g(t,v^t)\|_{L^2} \le L_g\|u^t - v^t\|_{C_H}.$$

(C2) $\exists\, C_g > 0$ satisfying

$$\int_\sigma^t \|g(s,u^s) - g(s,v^s)\|_{L^2}^2 ds \le C_g \int_{\sigma-h}^t \|u(s) - v(s)\|_H^2 ds.$$

(C3) denote

$$f_i(t) = F_i'(t),\ \mathcal{F}_i(t) = \int_0^t F_i(r)dr,$$

which satisfy

$$|f_i(t)| \le C(1 + |t|),\ |\mathcal{F}_i(t)| \le C(|t|^2 + |t|^3),$$

where

$$\overrightarrow{f}(t) = (f_1(t), f_2(t), f_3(t)),\ \overrightarrow{\mathcal{F}}(t) = (\mathcal{F}_1(t), \mathcal{F}_2(t), \mathcal{F}_3(t)).$$

3.2. Well-posedness

Let $G(u) = \nabla\cdot\overrightarrow{F}(u)$, then the system (1.1) is reduced to the following form

$$\begin{cases} \frac{\partial u}{\partial t} + A\frac{\partial u}{\partial t} + \nu Au + G(u) = g(t,u^t), \\ u(\sigma) = u_0,\ u(s+\sigma) = \eta(s,x),\ s \in [-h,0]. \end{cases} \tag{3.1}$$

Definition 3.1. *Assume* $u_0 \in W$, $\eta \in C_V$, *a function* $u(\cdot,x) :\ [\sigma,\infty) \to V$, *satisfying* $u(\sigma+s,x) = \eta(s)$ *in* $[-h,0]$, *is said to be a weak solution to* (3.1) *if it holds for any* $T > \sigma$ *that*

(i) $u \in C([\sigma,T];V)$, $\frac{\partial u}{\partial t} \in L^2(\sigma,T;H)$.

(ii) for any $w \in V$ *it holds that*

$$< u(t) + Au(t), w > +\nu\int_s^t < Au(l), w > dl + \int_s^t (G(u_m), w)dl$$

$$=< u(s) + Au(s), w > + \int_s^t (g(l, u_m^l), w)dl, \ \forall\ s, t \in [\sigma, T).$$

(iii) the energy equality holds

$$\frac{1}{2}\frac{d}{dt}|u(t,x)|^2 + \frac{1}{2}\frac{d}{dt}\|u(t,x)\|^2 + \nu\|u(t,x)\|^2 = (g(t,u^t), u(t,x)). \tag{3.2}$$

Moreover, the Eq (3.2) *could be expressed as*

$$-\frac{1}{2}\int_\sigma^T (|u(l)|^2 + \|u(l)\|^2)\zeta'(l)dl + \nu\int_\sigma^T \|u(l)\|^2\zeta(l)dl + \int_\sigma^T (G(u(l)), u(l))\zeta(l)dl$$
$$= \int_\sigma^T (g(l,u^l), u(l))\zeta(l)dl, \ \forall\ \zeta \in C_0^\infty[\sigma, T].$$

To sum up, the main results on well-posedness of solution to (3.1) are stated as follows.

Theorem 3.2. *Suppose $u_0 \in W, \eta \in C_V$, and assumptions (C1)–(C3) hold. Then the existence of solution $u(t,\sigma)$ to* (3.1) *holds, it is unique and depends on η continuously, and the system process $U(\cdot,\cdot)$ is generated by $u(t,\sigma)$.*

Proof. The Faedo-Galerkin method will be used to obtain the conclusion.

Procedure I. Existence of solution to the Galerkin equation
Considering the orthogonal eigenfunctions $\{\omega_1,\ \omega_2, \cdots,\ \omega_k, \cdots\}$ in V and letting

$$V_k = span\{\omega_1, \omega_2, \cdots, \omega_k\},$$

we can denote an approximate solution as

$$u_k(t) = \sum_{j=1}^{k} \chi_{jk}(t)\omega_k \ (j = 1, 2, \cdots, k)$$

for system (3.1) in V_k, which satisfies the corresponding differential equation of (3.1)

$$\frac{d}{dt}(u_k, \omega_j) + \frac{d}{dt} < Au_k, \omega_j > +\nu < Au_k(t), \omega_j > +(G(u_k), \omega_j) = (g(u_k^t), \omega_j), \tag{3.3}$$
$$u_k(\sigma + s) = P_k\eta(s) = \eta_k(s), \ s \in [-h, 0]. \tag{3.4}$$

where $\chi_{jk}(t)$ is undetermined and $P_k : H \to V_k$ is the orthogonal projection operator, and $\eta_k \to \eta$ in C_V as $k \to \infty$.

From the conclusion on ordinary differential equations, the local solution to systems (3.3) and (3.4), which has finite dimension, can be derived uniquely.

Procedure II. Conclusions on a priori estimate
Multiplying (3.3) with χ_{jk} and summing from $j = 1$ to k, from $\mathcal{F}(0) = 0$ and the divergence theorem we have

$$\int_\Theta (\nabla \cdot \overrightarrow{F}(u_k))u_k dx = -\int_\Theta \overrightarrow{F}(u_k) \cdot \nabla u_k dx$$

$$= -\int_{\partial\Theta} \overrightarrow{\mathcal{F}}(u_k) \cdot \overrightarrow{n}\, dx = -\int_{\partial\Theta} \overrightarrow{\mathcal{F}}(0) \cdot \overrightarrow{n}\, dx = 0, \tag{3.5}$$

it follows that

$$\frac{1}{2}\frac{d}{dt}|u_k|^2 + \frac{1}{2}\frac{d}{dt}\|u_k\|^2 + \nu\|u_k\|^2 = (g(u_k^t), u_k) \leq \frac{\nu}{2}\|u_k\|^2 + \frac{1}{2\nu\lambda_1}|g(u_k^t)|^2. \tag{3.6}$$

Integrating (3.6) over $[\sigma, t]$ and using conditions (C1) and (C2), we deduce

$$\begin{aligned} & |u_k(t)|^2 + \|u_k(t)\|^2 + \nu \int_\sigma^t \|u_k(l)\|^2 dl \\ \leq\ & |u_0|^2 + \|u_0\|^2 + \frac{1}{\nu\lambda_1}\int_\sigma^t |g(u_k^l)|^2 dl \\ \leq\ & |u_0|^2 + \|u_0\|^2 + \frac{C_g}{\nu\lambda_1}\int_{\sigma-h}^t |u_k(l)|^2 dl \\ \leq\ & |u_0|^2 + \|u_0\|^2 + \frac{C_g}{\nu\lambda_1}\|\eta\|^2_{L^2_H} + \frac{C_g}{\nu\lambda_1}\int_\sigma^t |u_k(l)|^2 dl, \end{aligned} \tag{3.7}$$

and assumptions on the initial conditions together with the Gronwall Lemma lead to

$$\{u_k(t)\} \subset L^\infty(\sigma, T; V) \cap L^2(\sigma, T; V). \tag{3.8}$$

Multiplying (3.3) with $A\chi_{jk}$ and summing from $j = 1$ to k, we show

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|u_k(t)\|^2 + \frac{1}{2}\frac{d}{dt}|Au_k(t)|^2 + \nu|Au_k(t)|^2 \\ \leq\ & |(g(u_k^t), Au_k(t))| + |(\nabla \cdot \overrightarrow{F}(u_k), Au_k(t))| \\ \leq\ & \frac{\nu}{6}|Au_k(t)|^2 + \frac{3}{2\nu}|g(u_k^t)|^2 + |(\nabla \cdot \overrightarrow{F}(u_k), Au_k(t))|^2. \end{aligned} \tag{3.9}$$

By using the conditions (C1) and (C2) and the interpolation inequalities such as

$$\|u_k(t,x)\|_{L^4} \leq C|\nabla u_k(t,x)|^{3/4}|u_k(t,x)|^{1/4}, \ \|\nabla u_k(t,x)\|_{L^4} \leq C|Au_k(t,x)|^{3/4}|\nabla u_k(t,x)|^{1/4},$$

we have

$$\begin{aligned} |(\nabla \cdot \overrightarrow{F}(u_k), Au_k(t))| &\leq \int |F'(u_k)||\nabla u_k||Au_k| dx \\ &\leq C\int_\Theta (1 + |u_k|)|\nabla u_k||Au_k| dx \\ &\leq C\int_\Theta |\nabla u_k||Au_k| dx + C\int_\Theta |u_k||\nabla u_k||Au_k| dx, \end{aligned} \tag{3.10}$$

where

$$C\int_\Theta |\nabla u_k||Au_k| dx \leq \frac{C}{\nu}|\nabla u_k|^2 + \frac{\nu}{6}|Au_k|^2, \tag{3.11}$$

and

$$
\begin{aligned}
C\int_{\Theta}|u_k(x)||\nabla u_k(x)||Au_k(x)|dx &\leq \|u_k(x)\|_{L^4}|Au_k(x)|\|\nabla u_k(x)\|_{L^4}\\
&\leq C|u_k|^{1/4}|\nabla u_k|^{3/4}|Au_k||Au_k|^{3/4}|\nabla u_k|^{1/4}\\
&\leq C|u_k|^2|\nabla u_k|^8+\frac{\nu}{6}|Au_k|^2. \qquad (3.12)
\end{aligned}
$$

Integrating (3.9) over $[\sigma, t]$, we show that

$$
\begin{aligned}
&|\nabla u_k(t)|^2+|Au_k(t)|^2+\nu\int_\sigma^t|Au_k(l)|^2dl\\
\leq\ &\|u_0\|^2+\|u_0\|_W^2+\frac{3}{\nu}\int_\sigma^t|g(u_k^l)|^2dl\\
&+C\int_\sigma^t|u_k(l)|^2\|u_k(l)\|^8dl+\frac{C}{\nu}\int_\sigma^t\|u_k(l)\|^2dl \qquad (3.13)\\
\leq\ &\|u_0\|^2+\|u_0\|_W^2+\frac{3C_g}{\nu}\|\eta\|_{L_H^2}^2+\frac{3C_g}{\nu}\int_\sigma^t|u_k(l)|^2dl\\
&+C\int_\sigma^t|u_k(l)|^2\|u_k(l)\|^8dl+\frac{C}{\nu}\int_\sigma^t\|u_k(l)\|^2dl, \qquad (3.14)
\end{aligned}
$$

the fact that $u_k\in L^\infty(\sigma,T;V)\cap L^2(\sigma,T;V)$ together with the Gronwall Lemma lead to

$$
\{u_k(t)\}\subset L^\infty(\sigma,T;W)\cap L^2(\sigma,T;W). \qquad (3.15)
$$

Procedure III. Compact argument

From (3.1) we see that

$$
(I+A)\frac{\partial}{\partial t}u_k=-\nu Au_k-G(u_k)+g(t,u_k^t), \qquad (3.16)
$$

and the above results make us know $Au_k, g(u_k^t)\in L^2(\sigma,T;H)$. Moreover, from condition (C3), we derive

$$
\begin{aligned}
\|G(u_k)\|_{L^2(\sigma,T;H)}^2 &= \leq C\int_\sigma^T\int_\Theta|(|u_k|+1)\nabla u_k|^2dxds\\
&\leq C\int_\sigma^T\|u_k\|^2ds+C\int_\sigma^T\int_\Theta|u_k|^2|\nabla u_k|^2dxds\\
&\leq C\int_\sigma^T\|u_k\|^2ds+C\int_\sigma^T\|u_k\|_{L^4}^{1/2}\|\nabla u_k\|_{L^4}^{1/2}ds\\
&\leq C\int_\sigma^T\|u_k\|^2ds+C\int_\sigma^T(\|u_k\|_{L^4}+\|\nabla u_k\|_{L^4})ds\\
&\leq C\int_\sigma^T(\|u_k\|^2+1+|Au_k|)ds, \qquad (3.17)
\end{aligned}
$$

and from the result in Procedure II we show $G(u_k)\in L^2(\sigma,T;H)$. Thus, it follows that

$$
(I+A)\frac{du_k}{dt}\in L^2(\sigma,T;H).
$$

For the operator $A: D(A) \to H$, the property of positive self-adjoint operator makes us know there is a unique determined resolution

$$\{E_\lambda\}_{\lambda \geq 0}$$

which is a family of projection operators, called the resolution of the identity I, and some properties are presented in [27]. Therefore, we can consider the following resolvent

$$(I + A)^{-1} = \int_0^\infty (1 + \lambda)^{-1} dE(\lambda),$$

with the operator norm

$$\|(I + A)^{-1}\|_{\mathcal{L}}^2 = \int_0^\infty (1 + \lambda)^{-2} d\|E_\lambda\|^2 \leq 1,$$

and it holds that

$$\frac{\partial u_k}{\partial t} \in L^2(\sigma, T; H).$$

The Aubin-Lions Lemma together with the above results leads to

$$\begin{cases} u_k \rightharpoonup^* u \text{ weakly in } L^\infty(\sigma, T; W), \\ u_k \rightharpoonup u \text{ weakly in } L^2(\sigma, T; W), \\ \frac{\partial}{\partial t} u_k \rightharpoonup \frac{\partial}{\partial t} u \text{ weakly in } L^2(\sigma, T; H), \\ u_k \to u \text{ strongly in } L^2(\sigma, T; V), \\ u_k \to u \text{ strongly in } V, \ a.e. t \in (\sigma, T), \end{cases} \tag{3.18}$$

and from the Lions-Aubin-Simon Lemma with (3.18) we get $u \in C([\sigma, T]; V)$.

Procedure IV. Limit process

From (3.18) we can obtain

$$|u_k(t, x)|^2 + \|u_k(t, x)\|^2 \to |u(t, x)|^2 + \|u(t, x)\|^2, \ k \to \infty$$

and

$$2\nu \int_s^t < Au_k(l), w > dl \to 2\nu \int_s^t < Au_k(l), w > dl, \ \forall \ w \in V.$$

Since $G(u_k) \in L^2(\sigma, T; H)$, the property of sequential compactness in L^2 ensures the existence of subsequence satisfying in $L^2(\sigma, T; H)$ that

$$G(u_k) \rightharpoonup G(u),$$

and

$$2 \int_s^t (G(u_k(l)), w) dl \to 2 \int_s^t (G(u_k(l)), w) dl.$$

From the conditions (C1) and (C2), the fact that $\eta \in L_H^2$ leads to $g(u_k^t) \in L^2(\sigma, T; H)$, and it also holds that

$$2 \int_s^t (g(u_k^l), w) dl \to 2 \int_s^t (g(u^l), w) dl.$$

Making the limit procedure on (3.3), we get that u is a solution to (3.2), and from (3.18) we can also obtain the following weak convergence in V

$$u_k(\sigma) \rightharpoonup u(\sigma).$$

Procedure V. Uniqueness

Assume that $u(t), v(t)$ are two solutions to (3.1) with initial conditions η_1 and η_2 respectively, then $\hat{u}(t) = u(t) - v(t)$ satisfies the equation

$$\hat{u}_t + A\hat{u}_t + \nu A\hat{u} + G(u) - G(v) = g(t, u^t) - g(t, v^t), \tag{3.19}$$

where $\eta_w = \eta_1 - \eta_2$. Multiplying (3.19) by $\hat{u}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\hat{u}(t)|^2 + \frac{1}{2}\frac{d}{dt}\|\hat{u}(t)\|^2 + \nu\|\hat{u}(t)\|^2 \\
&\leq |(G(u) - G(v), \hat{u})| + |(g(u^t) - g(v^t), \hat{u})| \\
&\leq \int_\Theta |F(u) - F(v)||\nabla\hat{u}|dx + \int_\Theta |g(u^t) - g(v^t)||\hat{u}|dx \\
&\leq \int_\Theta |\hat{u}||\nabla\hat{u}|dx + \int_\Theta |g(u^t) - g(v^t)||\hat{u}|dx \\
&\leq \frac{\nu}{4}\|\hat{u}(t)\|^2 + C|\hat{u}(t)|^2 + \frac{1}{\nu\lambda_1}|g(u^t) - g(v^t)|^2 + \frac{\nu}{4}\|\hat{u}(t)\|^2.
\end{aligned} \tag{3.20}$$

Integrating (3.20) with respect to t, we show

$$\begin{aligned}
&|\hat{u}(t)|^2 + \|\hat{u}(t)\|^2 + \nu\int_\sigma^t \|\hat{u}(s)\|^2 ds \\
\leq\ & |\hat{u}_0|^2 + \|\hat{u}_0\|^2 + C\int_\sigma^t |\hat{u}(s)|^2 ds + \frac{1}{\nu\lambda_1}\int_\sigma^t |g(u^s) - g(v^s)|^2 ds \\
\leq\ & |\hat{u}_0|^2 + \|\hat{u}_0\|^2 + \frac{C_g}{\nu\lambda_1}\|\eta_w\|^2_{L^2_H} + C\int_\sigma^t |\hat{u}(s)|^2 ds,
\end{aligned} \tag{3.21}$$

it follows from Gronwall's inequality that

$$|\hat{u}(t)|^2 \leq \Big(|\hat{u}_0|^2 + \|\hat{u}_0\|^2 + \frac{C_g}{\nu\lambda_1}\|\eta_w\|^2_{L^2_H}\Big)e^{C(T-\sigma)}. \tag{3.22}$$

Therefore, the uniqueness of the solution holds naturally together with the dependence on initial conditions, it follows that the continuous process $U(\cdot,\cdot)$ in the space C_V is finally generated.

4. Tempered pullback dynamics for BBM equation with finite distributed delay

4.1. Theory on tempered pullback dynamics

We will offer in this part some conclusions relating to tempered pullback dynamic theory (see [20]), and we first denote $\mathcal{P}(Y)$ as the family consisting of all subsets nonempty in Banach space Y. Let $\mathcal{D}$ be a nonempty class, whose element is the family $\widehat{D} = \{D(t)\}_{t\in\mathbb{R}}$ in $\mathcal{P}(Y)$, and $\mathcal{D}$ is said to be a universe in $\mathcal{P}(Y)$.

Definition 4.1. *For any $t \in \mathbb{R}$, a subset family $\widehat{D_0} = \{D_0(t)\}$ in $\mathcal{P}(Y)$ is said to be $\mathcal{D}-$pullback absorbing with respect to $U(\cdot,\cdot)$ on Y if, for any $\widehat{D} \in \mathcal{D}$, there is always a positive constant $T(t,\widehat{D}) \le t$ satisfying that*

$$U(t,\sigma)D(\sigma) \subset D_0(t),\ \forall\ \sigma \le t.$$

Definition 4.2. *For any $t \in \mathbb{R}$, $\widehat{D} \in \mathcal{D}$, $\{\sigma_n\} \subset (-\infty, t]$ satisfying $\sigma_n \to -\infty$ when $n \to \infty$, and any sequence $y_n \in D(\sigma_n)$, we say that the process $U(\cdot,\cdot)$ is $\mathcal{D}-$pullback asymptotically compact on Y if it always holds that the sequence $\{U(t,\sigma_n)y_n\}$ has the property of relative compactness in space Y.*

Definition 4.3. *For any $t \in \mathbb{R}$, for the family $\mathcal{A} = \{A(t)\}$ in Y if the following hold*

1) Property of pullback invariance: $U(t,\sigma)\mathcal{A}_{\mathcal{D}}(\sigma) = \mathcal{A}_{\mathcal{D}}(t),\ \forall\ \sigma \le t$,

2) Property of pullback attraction:

$$\lim_{\sigma\to-\infty} dist_Y(U(t,\sigma)B, \mathcal{A}) = 0,\ \forall\ B \in \mathcal{D},$$

then $\mathcal{A}$ is called a $\mathcal{D}$-pullback attractor to $U(t,\sigma)$.

Definition 4.4. *Assume that $\widehat{M} = \{M(t)\}$ is a family consisting of closed sets in $\mathcal{P}(Y)$ satisfying for any $\widehat{D} = \{D(t)\} \in \mathcal{D}$ that*

$$\lim_{\sigma\to-\infty} dist_Y(U(t,\sigma)D(\sigma), M(t)) = 0.$$

If $\mathcal{A}_{\mathcal{D}}(t) \subset M(t)$, then we say that $\mathcal{A}_{\mathcal{D}}$ is minimal.

Theorem 4.5. *Let be $U(\cdot,\cdot) : \mathbb{R}^2_d \times Y \to Y$ a closed process, which has the $\mathcal{D}-$pullback absorbing set $\widehat{D}_0 = \{D_0(t)\}$ in $\mathcal{P}(Y)$, and has the property of $\mathcal{D}-$pullback asymptotical compactness. Then, the $\mathcal{D}$-pullback attractor $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t)\}$ exists and is shown as for any $t \in \mathbb{R}$*

$$\mathcal{A}_{\mathcal{D}}(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}} \Gamma(\widehat{D}, t)}^{Y},$$

where

$$\Gamma(\widehat{D}, t) = \bigcap_{s\le t} \overline{\bigcup_{\sigma<s} U(t,\sigma)D(\sigma)}^{Y}.$$

Moreover, the family $\mathcal{A}_{\mathcal{D}}$ is minimal.

4.2. $\mathcal{D}$-pullback absorbing set

For any $t \in \mathbb{R}$, we first construct a universe $\mathcal{D} = \{D(t)\}$ in $\mathcal{P}(C_V)$ satisfying that

$$\lim_{\sigma\to-\infty} e^{\tilde{r}\sigma} \sup_{\eta\in D(\sigma)} \|\eta\|^2_{C_V} = 0,\ \tilde{r} = \frac{\lambda_1}{1+\lambda_1}\nu.$$

Lemma 4.6. *Let assumptions (C1)–(C3) hold, and $\eta \in C_V$. Then, the process $\{U(\cdot,\cdot)\}$ to* (3.1) *has the $\mathcal{D}$-pullback absorbing set $\mathcal{D}_0 = \{D_0(t)\}$ in C_V in which*

$$D_0(t) = \bar{B}_{C_V}(0, \tilde{\rho}(t))$$

with radius

$$\tilde{\rho}(t) = M_0(\|\eta\|^2_{C_H} + \|\eta\|^2_{C_V})e^{-\iota t} + (\gamma + 1)C_0, \tag{4.1}$$

where $M_0, \iota, C_0 > 0$ are positive constants and

$$\gamma = \frac{2 - \kappa_0}{(1 - \kappa_0)(1 - \kappa_0 c)},\ c = max\{\frac{1}{1 - \kappa_0},\ 1\}.$$

Proof. Multiplying (3.1) by u leads to

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu\|u(t)\|^2 = (g(u^t), u(t)) \le \frac{\nu}{2}\|u(t)\|^2 + \frac{1}{2\nu\lambda_1}|g(u^t)|^2, \tag{4.2}$$

i.e.,

$$\frac{d}{dt}|u(t)|^2 + \frac{d}{dt}\|u(t)\|^2 + \nu\|u\|^2 \le \frac{1}{\nu\lambda_1}|g(u^t)|^2, \tag{4.3}$$

and

$$\begin{aligned} &\frac{d}{dt}(e^{\tilde{r}(t-\sigma)}(|u(t)|^2 + \|u(t)\|^2)) \\ &\le e^{\tilde{r}(t-\sigma)}(\tilde{r}|u(t)|^2 + \tilde{r}\|u(t)\|^2 - \nu\|u(t)\|^2) + \frac{1}{\nu\lambda_1}e^{\tilde{r}(t-\sigma)}|g(u^t)|^2 \\ &\le \frac{1}{\nu\lambda_1}e^{\tilde{r}(t-\sigma)}|g(u^t)|^2, \end{aligned} \tag{4.4}$$

where $\tilde{r} = \frac{\lambda_1}{1+\lambda_1}\nu$. Integrating (4.4) with respect to t leads to

$$\begin{aligned} &|u(t, x)|^2 + \|u(t, x)\|^2 \\ &\le e^{\tilde{r}(\sigma-t)}(|u_0|^2 + \|u_0\|^2) + \frac{1}{\nu\lambda_1}\int_\sigma^t e^{\tilde{r}(s-t)}|g(u^s)|^2 ds \\ &\le e^{\tilde{r}(\sigma-t)}(|u_0|^2 + \|u_0\|^2) + \frac{C_f}{\nu\lambda_1}\int_{\sigma-h}^t e^{\tilde{r}(s-t)}|u(s)|^2 ds \\ &\le e^{\tilde{r}(\sigma-t)}(|u_0|^2 + \|u_0\|^2) + \frac{C_f}{\nu\lambda_1}e^{\tilde{r}h}\int_\sigma^t e^{\tilde{r}(s-t)}|u^s|^2 ds \\ &\quad + \frac{C_f}{\nu\lambda_1}\int_{t-h}^t e^{\tilde{r}(s-t)}|u(s)|^2 ds + \frac{C_f}{\nu\lambda_1}\int_{\sigma-h}^\sigma e^{\tilde{r}(s-t)}|u(s)|^2 ds, \end{aligned} \tag{4.5}$$

i.e.,

$$\begin{aligned} &|u(t, x)|^2 + \|u(t, x)\|^2 \\ &\le e^{\tilde{r}(\tau-t)}(|u_0|^2 + \|u_0\|^2) + \frac{C_f}{\nu\lambda_1}e^{\tilde{r}h}\int_\sigma^t e^{\tilde{r}(s-t)}|u^s|^2 ds + \frac{C_f}{\nu\lambda_1}\|\eta\|^2_{L^2_H} ds + \frac{C_f h}{\nu\lambda_1}\|u\|_{L^\infty}. \end{aligned} \tag{4.6}$$

From the retarded integral inequality, we can set

$$E(t, s) = e^{\tilde{r}(s-t)},\ K_1(t, s) = \frac{C_f}{\nu\lambda_1}e^{\tilde{r}h}e^{\tilde{r}(s-t)},\ C_0 = \frac{C_f}{\nu\lambda_1}\|\eta\|^2_{L^2_H} ds + \frac{C_f h}{\nu\lambda_1}\|u\|_{L^\infty},$$

where

$$\lim_{t\to+\infty} E(t+\cdot,\cdot)=0,\ \vartheta=\sup_{t\geq s\geq\sigma} E(t,s)=1,\ \kappa_0=\kappa(K_1,0)=\sup_{t\geq\sigma}\int_\sigma^t K_1(t,s)ds,$$

and choosing a suitable t could lead to

$$\frac{C_f}{\nu\lambda_1}e^{\tilde{r}h}\int_\sigma^t e^{\tilde{r}(s-t)}ds\leq 1/2,\ \kappa_0=\kappa(K_1,0)<\frac{1}{1+\vartheta}.$$

It follows from Lemma 2.1 that there exist positive constants M_0 and ι satisfying

$$\|u\|_{C_H}^2+\|u\|_{C_V}^2\leq M_0(\|\eta\|_{C_H}^2+\|\eta\|_{C_V}^2)e^{-\iota t}+\gamma C_0, \tag{4.7}$$

and by using (4.7) in (4.6) we can obtain

$$\begin{aligned}&|u(t,x)|^2+\|u(t,x)\|^2\\ \leq\ & e^{\tilde{r}(\sigma-t)}(|u_0|^2+\|u_0\|^2)+\frac{1}{2}(M_0(\|\eta\|_{C_H}^2+\|\eta\|_{C_V}^2)e^{-\iota t}+\gamma C_0)+C_0.\end{aligned} \tag{4.8}$$

It follows that the following pullback absorbing set exists

$$\hat{D}_0=\{D_0(t)\}_{t\in\mathbb{R}},$$

where $D_0(t)=\{u|\|u\|_{C_V}\leq M_0(\|\eta\|_{C_H}^2+\|\eta\|_{C_V}^2)e^{-\iota t}+(\gamma+1)C_0\}$.

Remark 4.1. *Let $\|(u(t),u_t)\|_{V\times C_V}$ be the norm of topology of $V\times C_V$, conditions (C1)–(C3) hold, and $\eta\in C_V$. Then, the tempered pullback absorbing set $\mathcal{D}^\circ=\{D^\circ(t)\}$ in $V\times C_V$ exists for the system* (3.1)*, and*

$$D^\circ(t)=\bar{B}_{V\times C_V}(0,\tilde{\rho}^\circ(t)),$$

where

$$\tilde{\rho}^\circ(t)=2M_0(\|\eta\|_{C_H}^2+\|\eta\|_{C_V}^2)e^{-\iota t}+(\gamma+3)C_0. \tag{4.9}$$

In fact, combining (4.7) and (4.8), we can obtain the above conclusion directly.

4.3. $\mathcal{D}$-pullback asymptotic compactness

The following aims to use the energy equation method(see [28]) to show the process $U(\cdot,\cdot)$ to (3.1) has the property of tempered pullback asymptotic compactness.

Lemma 4.7. *Assume that the conditions (C1)–(C3) hold, and $\eta\in C_V$. Then, for the system* (3.1)*, the $\mathcal{D}$-pullback asymptotical compactness of $U(\cdot,\cdot)$ in C_V holds.*

Proof. We will achieve the goal via two procedures.

Procedure I. Convergence of $\{u_k\}$ in $[t_0-h,t_0]$ and V

For $t_0 \in \mathbb{R}$, we assume that $\{u_k\} \subset D(t, \sigma_k; \phi_k)$, $\{\phi_k\} \subset D(\sigma_k)$ is bounded in C_V, and $\{\sigma_k\} \subset (-\infty, t_0 - 2h]$, where $\sigma_k \to -\infty$ as $k \to +\infty$. From Theorem 3.2 with the Aubin-Lions Lemma, a subsequence $\{u_k\}$ exists and satisfies

$$\begin{cases} u_k \rightharpoonup^* u \text{ in } L^\infty(t_0 - 2h, t_0; W), \\ u_k \rightharpoonup u \text{ in } L^2(t_0 - 2h, t_0; W), \\ \frac{\partial}{\partial t} u_k \rightharpoonup \frac{\partial}{\partial t} u \text{ in } L^2(t_0 - h, t_0; H), \\ u_k \to u \text{ in } L^2(t_0 - h, t_0; V), \\ u_k \to u \text{ in } V, \ a.e.\ t \in (t_0 - h, t_0). \end{cases} \tag{4.10}$$

From (4.10) we can use the Lions-Aubin-Simon Lemma to derive that there exists a subsequence $\{u_k\}$ satisfying

$$u_k \to u \text{ in } C([t_0 - h, t_0]; V),$$

and there holds in V

$$u_k(s_k) \rightharpoonup u(s), \tag{4.11}$$

where $\{s_k\} \subset [t_0 - h, t_0]$ and $s_k \to s \in [t_0 - h, t_0]$ as $k \to \infty$.

Also, the hypotheses on $G(\cdot)$ and $g(t, u_t)$ lead to $G(u_k) \rightharpoonup G(u)$ weakly in $L^2(t_0 - 2h, t_0; H)$ and $g(t, u_{kt}) \rightharpoonup g(t, u_t)$ weakly in $L^2(t_0 - t, t_0; H)$, and we can conclude that u satisfies the system (3.1) in $[t_0 - h, t_0]$.

Procedure II. Strong convergence of $\{u_k\}$

In this part, the energy equation method will be used to show the tempered pullback asymptotical compactness for $U(\cdot, \cdot)$, that is,

$$\|u_k(s_k) - u(s)\| \to 0 \text{ as } k \to +\infty. \tag{4.12}$$

Claim 1.

$$\liminf_{k\to\infty} \|u_k(s_k)\| \geq \|u(s)\|. \tag{4.13}$$

The weak convergence (4.11) together with the Banach-Steinhaus Theorem leads to the fact that (4.13) holds, which means the possible energy loss.

Claim 2.

$$\limsup_{k\to\infty} \|u_k(s_k)\| \leq \|u(s)\|. \tag{4.14}$$

Multiplying Eq (3.1) by u, we get that

$$\frac{d}{dt}|u(t, x)|^2 + \frac{d}{dt}\|u(t, x)\|^2 = -2\nu\|u(t, x)\|^2 + 2(g(u^t), u(t, x)), \tag{4.15}$$

and integrating yields that

$$|u(s)|^2 + \|u(s)\|^2 = |u(l)|^2 + \|u(l)\|^2 + 2\int_l^s ((g(u^r), u(r)) - \nu\|u(r)\|^2)dr.$$

In the interval $[t_0 - h, t_0]$ we define the following functionals

$$Q(t) = |u(s)|^2 + \|u(s)\|^2 - 2\int_{t_0-h}^{s} (g(u^r), u(r))dr \tag{4.16}$$

and

$$Q_k(s) = |u_k(s)|^2 + \|u_k(s)\|^2 - 2\int_{t_0-h}^{s} (g(u_k^r), u_k(r))dr, \tag{4.17}$$

where $Q(s)$ and $Q_k(s)$ are continuous and decreasing in $[t_0 - h, t_0]$, and the above conclusion that the subsequence $\{u_k\}$ is convergent leads to that, as $k \to \infty$,

$$Q_k(s) \to Q(s)\ a.e.\ s \in (t_0 - h, t_0). \tag{4.18}$$

Therefore, for $\forall\ \varepsilon > 0$, $\exists\ \tilde{k} \in \mathbb{N}$, and when $k \geq \tilde{k}$ and $\{s_k\} \subset [t_0 - h, t_0]$, it always holds that

$$|Q_k(s_k) - Q(s_k)| \leq \frac{\varepsilon}{2}. \tag{4.19}$$

The continuity of $Q(s)$ in $[t_0 - h, t_0]$ leads to uniform continuity, and thus it follows that, for $\forall\ \varepsilon > 0$ and any sequence $\{s_k\} \subset [t_0 - h, t_0]$ with $s_k \to s^*$ as $k \to \infty$, that there is $\tilde{\tilde{k}} \in \mathbb{N}$ satisfying

$$|Q(s_k) - Q(s^*)| \leq \frac{\varepsilon}{2},\ k > \tilde{\tilde{k}}. \tag{4.20}$$

Let $\bar{k} = \max\{\tilde{k}, \tilde{\tilde{k}}\}$, then

$$|Q_k(s_k) - Q(s^*)| \leq |Q_k(s_k) - Q(s_k)| + |Q(s_k) - Q(s^*)| < \varepsilon,\ \forall\ k > \bar{k}, \tag{4.21}$$

and from the arbitrariness of ε we know that, for any $\{s_k\} \subset [t_0 - h, t_0]$, it holds that

$$\limsup_{k\to\infty} Q_k(s_k) \leq Q(s^*), \tag{4.22}$$

and with

$$\limsup_{k\to\infty} \left(|u_k(s_k)|^2 + \|u_k(s_k)\|^2\right) \leq |u(s^*)|^2 + \|u(s^*)\|^2, \tag{4.23}$$

it follows that

$$\|u_k(s_k)\| \to \|u(s^*)\|. \tag{4.24}$$

The convergence relations (4.24) and (4.11) together with the Radon Theorem lead to (4.12), and we finish the proof on the asymptotic compactness in C_V naturally.

The process $U(t, \sigma)$ to (3.1) in $V \times C_V$ also has the property of tempered pullback asymptotical compactness.

4.4. The tempered pullback dynamics in C_V

The main results of pullback dynamics to system (3.1) is stated as the following.

Theorem 4.8. *Let assumptions (C1)–(C3) hold, $u_0 \in W$, and $\eta \in C_V$. Then, the $\mathcal{D}$-pullback attractor $\mathcal{A}_{C_V}(t)$ in C_V to the process $U(t,\sigma)$ of* (3.1)*, and it is minimal in addition.*

Proof. Theorem 3.2 shows that the continuity of process $U(t,\sigma)$ is satisfied in C_V, the existence of $\mathcal{D}$-pullback absorbing set in C_V is derived in Lemma 4.6, and in Lemma 4.7 the tempered pullback asymptotic compactness is established. From Theorem 4.5 we can obtain the minimal pullback attractor in C_V to system (3.1). These complete the proof.

Remark 4.2. *Let assumptions (C1)–(C3) hold, $u_0 \in W$, and $\eta \in C_V$. To the process $U(t,\sigma)$ of* (3.1)*, the $\mathcal{D}$-pullback attractor $\mathcal{A}_{V\times C_V}(t)$ exists in $V \times C_V$, and has the property of minimality.*

Proof. The proof is similar as in the above theorem by using the same technique in [29], and here we skip the details.

5. Further research

Based on the result of well-posedness of the BBM model with finite distributed delay, we finish the proof on existence and minimality of pullback attractors $\mathcal{A}_{C_V}(t)$. If the delay approaches infinity, the problem relating to pullback dynamics and continuity of attractors is still open, which is our next objective.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare there is no conflict of interest.

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