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Research article

On the construction of recurrent fractal interpolation functions using Geraghty contractions

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Abstract: The recurrent iterated function systems (RIFS) were first introduced by Barnsley and Demko and generalized the usual iterated function systems (IFS). This new method allowed the construction of more general sets, which do not have to exhibit the strict self similarity of the IFS case and, in particular, the construction of recurrent fractal interpolation functions (RFIF). Given a data set $\{(x_n, y_n) \in I \times \mathbb{R}, n = 0, 1, \ldots, N\}$ where $I = [x_0, x_N]$, we ensured that attractors of RIFS constructed using Geraghty contractions were graphs of some continuous functions which interpolated the given data. Our approach goes beyond the classical framework and provided a wide variety of systems for different approximations problems and, thus, gives more flexibility and applicability of the fractal interpolation method. As an application, we studied the error rates of time series related to the vaccination of COVID-19 using RFIF, and we compared them with the obtained results on the FIF.

Keywords: recurrent iterated function system (RIFS); Geraghty contraction; recurrent fractal interpolation function

1. Introduction

In 1986, Barnsley [1] introduced the concept of the fractal interpolation functions (FIFs) as continuous functions interpolating a given set of data points. Following this, an iterative process for interpolating data that sat on a discrete closed subset of $\mathbb{R}^n$ was introduced by Dubuc [2, 3] and a general construction was given by Massopust [4]. This construction is based on the Read-Bajraktarevic operator for the FIFs. More precisely, FIFs are generated by an iterated function system (IFS) and arise as fixed points of the Read-Bajraktarevic operator defined on suitable function space. Several authors were interested in the study of many significant properties of FIFs including calculus, dimension, smoothness, stability, perturbation error, and more ([5–7]). The problem of the existence of FIF returns to the study of the existence and uniqueness of some fixed points on the fractal space, and the most widely studied FIFs are based on the Banach fixed point theorem. Recently, many researchers have studied the existence of FIFs by using different well-known results obtained in the fixed point theory, like Rakotch and Matkowski's theorems [8–11], Geraghty's theorem [12] and so on.

Fractal interpolation has found applications in the field of image segmentation, image analysis, image compression and signal processing ([13,14]). In particular, uniformly sampled signal is required for the discrete wavelet transform computation when we consider the multifractal analysis to study the heart rate variability [15]. In order to gain more flexibility in natural-shape generation or in image compression, a new class of IFSs were introduced and were called RIFS [16, 17]. The RIFS use elements of stochastic processes, which can provide more flexible methods since they involve local IFS [18, 19]. It induces recurrent fractal interpolation functions (RFIFs) which main advantage is that they are not self-similar and they reflect only local similarities in contrast to the FIFs induced by the standard IFSs. Some properties of RFIFs, including statistical properties [20, 21] and box dimension [22], have been studied (see also [17,23]). However, only RFIF, whose existence is ensured by the Banach fixed point theorem, has been considered.

In this paper, we aim to construct non-affine RFIFs using Geraghty contractions. Therefore, we present in the next section some results on RIFS and we give new conditions on the existence of the attractor of a new class of RIFS. These results will be explored in the third paragraph to define a new class of RFIF using Gheraghty contractions. Section four will be devoted to the study of the normalized root mean square errors of time series related to the vaccination of COVID-19 using RFIF, and we compare them with the obtained results on the FIF.

2. RIFS

2.1. Preliminary results on RIFS

Let $(\mathcal{X}, d)$ be a complete, locally compact metric space and $w_j : \mathcal{X} \to \mathcal{X}$, for $j = 1, \ldots, n$ are self-mappings on $\mathcal{X}$. We define $P = (p_{ij})$ to be an irreducible $n \times n$ row stochastic matrix associated to these maps, i.e.,

$$\sum_{j=1}^{n} p_{ij} = 1, \text{ for all } i = 1, \ldots, n \text{ and } p_{ij} \geq 0, \text{ for all } 1 \leq i, j \leq n, \tag{2.1}$$

and for any $1 \leq i, j \leq n$, there exists a sequence $i_1, i_2, \ldots, i_k$ with $i_1 = i$ and $i_k = j$ such that

$$p_{i_1 i_2} p_{i_2 i_3} \ldots p_{i_{k-1} i_k} > 0. \tag{2.2}$$

A RIFS consists of an IFS $(\mathcal{X}, w_i, i = 1, \ldots, n)$ together with a matrix P as defined above. They have their origins in the work of Barnsley and Demko [24] and have been developed by Barnsley et al. [17]. In fact, they are a natural generalization of the IFS that permits them to construct fractals as fixed points of some operators on complete metric spaces [16, 25].

A RIFS is, in essence, a Markov chain with n states. Given an initial set that is transformed through the mapping w_i, we should say that the chain is in state i. The value p_{ij} is the transition probability from state i to state j. That is, assuming that the mapping w_i has been employed at the previous iteration, the probability of selecting the map w_j for the next iteration is given by p_{ij}. The motivation underlying the necessary properties, that matrix (p_{ij}) should satisfy becomes apparent as a consequence of the Markov chain rationale. The first condition given by (2.1) ensures that the chain must be stochastic, that is, at every state the transition probabilities to any other state should sum up to one. The second one (2.2) means that the chain is irreducible. Namely, that we may transfer from any state of the chain to any other state. Therefore, we can be sure that, given sufficient time, all the mappings of the RIFS will be used. Otherwise, if P is reducible, it is possible that some of the mappings of the RIFS will be rendered useless.

Remark 1. *An IFS with probabilities $\{\mathcal{X}, w_1, \ldots, w_n, p_1, \ldots, p_n\}$ [16] provides a simple example of an RIFS $\{\mathcal{X}, w_1, \ldots, w_n, (p_{ij})_{1\le i,j\le n}\}$ when $p_{ij} = p_j$, for all $i, j \in \{1, \ldots, n\}$. In addition, if $p_1 = p_2 = \cdots = p_n := p$, then the RIFS reduces to the usual IFS denoted simply by $(\mathcal{X}, w, p)$. Therefore, RIFSs provide more flexibility so that their attractors need not exhibit the self-similarity or self-tiling properties characteristic of IFS attractors.*

Let $\mathcal{H}$ be the set of all nonempty compact subsets of $\mathcal{X}$. We define the Hausdorff metric h by

$$h(A, B) = \max\{d(A, B), d(B, A)\}, \quad A, B \in \mathcal{H},$$

with

$$d(A, B) = \sup_{x\in A} \inf_{y\in B} d(x, y) \text{ and } d(B, A) = \sup_{x\in B} \inf_{y\in A} d(x, y).$$

Then [16], the space $(\mathcal{H}, h)$ is complete, and compact whenever $\mathcal{X}$ is compact. Let $\{\mathcal{X}, w_1, \ldots, w_n, (p_{ij})_{1\le i,j\le n}\}$ be a RIFS and let $W_{ij} : \mathcal{H} \to \mathcal{H}$ be defined as

$$W_{ij}(A) = \begin{cases} w_i(A) & \text{if } p_{ji} > 0 \\ \emptyset & . \end{cases}$$

Now, consider the space $\mathcal{H}^n = \mathcal{H} \times \cdots \times \mathcal{H}$ endowed with the metric $\tilde{h}$ defined as

$$\tilde{h}((A_1, \ldots, A_n), (B_1, \ldots, B_n)) = \max_{1\le i\le n} h(A_i, B_i)$$

and let $W : \mathcal{H}^n \to \mathcal{H}^n$ be given by

$$W\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & \ldots & W_{1n} \\ W_{21} & W_{22} & \ldots & W_{2n} \\ \vdots & \vdots & & \vdots \\ W_{n1} & W_{n2} & \ldots & W_{nn} \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} = \begin{pmatrix} W_{11}(A_1) \cup W_{12}(A_2) \cup \ldots \cup W_{1n}(A_n) \\ W_{21}(A_1) \cup W_{22}(A_2) \cup \ldots \cup W_{2n}(A_n) \\ \vdots \\ W_{n1}(A_1) \cup W_{n2}(A_2) \cup \ldots \cup W_{nn}(A_n) \end{pmatrix}.$$

Thus,

$$W\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} = \begin{pmatrix} \bigcup_{j\in I(1)} w_1(A_j) \\ \bigcup_{j\in I(2)} w_2(A_j) \\ \vdots \\ \bigcup_{j\in I(n)} w_n(A_j) \end{pmatrix}, \tag{2.3}$$

where $I(i) = \{1 \le j \le n\ ;\ p_{ji} > 0\}$, for $i = 1, \ldots, n$.

Example 1. *We consider* $\mathbb{I} = \left\{[0,1]^2, w_i, 1 = 1, \ldots, 4\right\}$ *the RIFS equipped with the mappings*

$$w_1\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1/2 & 0\\0 & 1/2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix}, \qquad w_2\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1/2 & 0\\0 & 1/2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}0\\50\end{pmatrix}$$

$$w_3\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1/2 & 0\\0 & 1/2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}50\\0\end{pmatrix}, \qquad w_4\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1/2 & 0\\0 & 1/2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}50\\50\end{pmatrix}. \tag{2.4}$$

If we consider the transition matrix

$$P = \begin{pmatrix}1/2 & 0 & 1/2 & 0\\1/3 & 1/3 & 0 & 1/3\\1/2 & 1/2 & 0 & 0\\0 & 1/2 & 0 & 1/2\end{pmatrix},$$

then, the function W is defined by

$$W = \begin{pmatrix}w_1 & w_1 & w_1 & \emptyset\\\emptyset & w_2 & w_2 & w_2\\w_3 & \emptyset & \emptyset & \emptyset\\\emptyset & w_4 & \emptyset & w_4\end{pmatrix}.$$

Now, let $A = (A_1, A_2, A_3, A_4)^T \in \mathcal{H}^4$ *be an initial point and assume that* A_i *was generated by applying the mapping* w_i *to an unknown set for each* $i = 1, 2, 3, 4$*, then* $W(A)$ *is computed as*

$$W\begin{pmatrix}A_1\\A_2\\A_3\\A_4\end{pmatrix} = \begin{pmatrix}w_1(A_1) \cup w_1(A_2) \cup w_1(A_3)\\w_2(A_2) \cup w_2(A_3) \cup w_2(A_4)\\w_3(A_1)\\w_4(A_2) \cup w_4(A_4)\end{pmatrix}.$$

Therefore, only the maps w_j *with positive transition probabilities, i.e.,* $p_{ij} > 0$ *can be applied to the set* A_i*. In particular, only the map* w_3 *is applied to* A_1*.*

2.2. Existence of attractor for a new class of RIFS

Although the Banach fixed point theorem is the most useful tool for the existence (and uniqueness) of attractors of IFS, there are numerous results in the literature related to some generalizations of the Banach principle that give sufficient conditions for the existence of a contractive fixed point (see for example [26–28]).

Definition 1.

1) *Let* (X, ρ) *be a metric space. If for some function* $\varphi : (0, \infty) \to (0, \infty)$ *and a self-map* f*, we have :*

$$\forall\ x, y \in X, \quad \rho(f(x), f(y)) \leq \varphi(\rho(x, y)),$$

then we say that f *is a* φ*-contraction.*

2) *If* f *is a* φ*-contraction for some function* $\varphi : (0, \infty) \to (0, \infty)$ *such that for any* $t > 0$, $\alpha(t) = \dfrac{\varphi(t)}{t} < 1$ *and the function* $t \mapsto \dfrac{\varphi(t)}{t}$ *is nonincreasing (so-called Geraghty I) (or nondecreasing (Geraghty II), or continuous (Geraghty III)), then we call such a function a Geraghty contraction.*

It is easy to see that each Banach contraction is a Geraghty contraction. It suffices to consider the function $\varphi(t) = \alpha t$, for some $0 \leq \alpha < 1$. Furthermore, the notions of Geraghty I and III contractions coincide when the metric space (X, ρ) is compact [28].

Theorem 1. *[26] Let (X, ρ) be a complete metric space and $f : X \to X$ be a Geraghty contraction then there is a unique fixed point $x_\infty \in X$ of f, and for each $x \in X$, $\lim_{n\to+\infty} f^n(x) = x_\infty$.*

Theorem 2. *Assume that $w_i : \mathcal{X} \to \mathcal{X}$, for $i = 1, \dots, n$ are Geraghty contractions of a RIFS, then there exists a unique set $A = (A_1, \dots, A_n)$, called the attractor of the RIFS, such that $W(A) = A$ and*

$$A_i = \bigcup_{j\in I(i)} w_i(A_j), \quad \text{for } i = 1, \dots, n.$$

We identify A to the union of all A_i, that is, $A = \bigcup_{i=1}^{n} A_i$.

Proof. Let $\varphi : (0, \infty) \to (0, \infty)$ be a nondecreasing function satisfying $\frac{\varphi(t)}{t} < 1$ for all $t > 0$, and $t \mapsto \frac{\varphi(t)}{t}$ is nonincreasing (or nondecreasing, or continuous), such that

$$d(w_i(x), w_i(y)) \leq \varphi(d(x, y)), \quad \forall x, y \in \mathcal{X}.$$

Then, we have

$$\begin{aligned} h(W(A), W(B)) &= h\Big(\bigcup_{i=1}^{n} w_i(A), \bigcup_{i=1}^{n} w_i(B)\Big) \leq \max_{1\leq i\leq n} h(w_i(A), w_i(B)) \\ &= \max_{1\leq i\leq n}\Big(\max\left(d(w_i(A), w_i(B)), d(w_i(B), w_i(A))\right)\Big) \\ &\leq \max\left(\varphi(d(A, B)), \varphi(d(B, A))\right) \leq \varphi(h(A, B)). \end{aligned}$$

Hence, the Hutchinson operator W is a Geraghty contraction and has a unique fixed compact set (Theorem 1). Using (2.3), this means that there exists n compact subsets A_i, for $i = 1, \dots, n$, such that $A = (A_1, \dots, A_n)$ and

$$A_i = \bigcup_{j\in I(i)} w_i(A_j).$$

□

An RIFS $\{\mathcal{X}, w_1, \dots, w_n, (p_{ij})_{1\leq i,j\leq n}\}$ is said to satisfy the open set condition (OSC) if there exists bounded nonempty open sets $U_1, U_2, \dots, U_n$ with the property that

$$\bigcup_{j, p_{ji}>0} w_i(U_j) \subset U_i, \quad w_i(U_j) \cap w_i(U_k) = \emptyset, \text{for all } \; ki \neq lj \; \text{ with } \; p_{ji}p_{kj} > 0. \tag{2.5}$$

When the RIFS satisies the OSC, we can compute the Hausdorff dimension of the attractor generated by the system. In fact, the attractor's Hausdorff dimension reflects how space-filling or complex the attractor is. Calculating the Hausdorff dimension of the attractor is a common method of quantifying its complexity. The condition (2.5) is satisfied in the case of RFIF in section three. It prevents the interpolated curve or fractal from collapsing into itself or having degenerate, overlapping segments. It ensures that the points used for interpolation are well separated and distinct, allowing for the generation of a meaningful and nondegenerate fractal or curve.

Example 2. *This example involves four transformations that take the square into itself. For $1 \le i \le 4$, the range of w_i is the quadrant of the square labeled i. If the transition probability p_{ij} is zero, no points on the attractor lie in the sub-quadrant labeled ij. While the structure of the attractor of the RIFS depends only on whether or not p_{ij} is nonzero and not on the values of the p_{ij}s, the coloring and texture of the image can be controlled through the probability values (see [16] for more details). To illustrate that, different choices for the transition probabilities are considered. Let us consider again the mappings $w_i, i = 1,2,3,4$ defined in Example 1. To construct the attractor, we apply one map of the IFS to a random selected point. Then, at each step, we apply a new map according to the probabilities of the transition of the matrix. In Figure 1 (a1, a2 and a3), we illustrate the attractors of RIFS equipped respectively with the following transition matrices :*

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad P_2 = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

Now, we consider $\widetilde{\mathbb{I}} = \left\{[0,1]^2, \widetilde{w}_i, i = 1,\dots,4\right\}$, the RIFS equipped with the mappings

$$\widetilde{w}_i(x,y) = \left(a_i x + b_i, c_i x + \frac{y}{1+\beta y} + f_i\right), \text{ for } i = 1,\dots,4, \tag{2.6}$$

where $\beta = 0.01$. These mappings are Geraghty contractions since they satisfy

$$\left|c_i x_1 + \frac{y_1}{1+\beta y_1} - c_i x_2 - \frac{y_2}{1+\beta y_2}\right| \le |c_i||x_1 - x_2| + \varphi(|y_1 - y_2|), \text{ for all } y_1, y_2 \in [0,1],$$

with $\varphi(t) = \dfrac{t}{1+\beta t}$ (this fact can be proved using Lemma 1). The attractors of $\widetilde{\mathbb{I}}$, equipped with the transition matrices P_1, P_2 and P_3, are illustrated in Figure 1 (b1, b2 and b3), respectively.

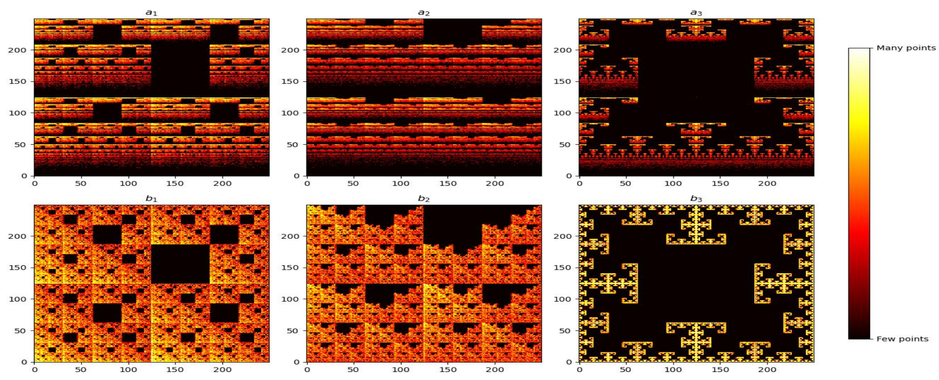

Figure 1. The RIFSs equipped with the transition matrices P_1, P_2 and P_3.

3. RFIFs

For $n \ge 2$, let $\{(x_i, y_i) \in \mathbb{R}^2,\ i = 0,\dots,n\}$ be a given data set, where $x_0 < x_1 < \dots < x_n$ and $y_i \in [a,b]$ for all $i = 0,\dots,n$. For $i = 1,\dots,n$, we set $I = [x_0, x_n]$ and $I_i = [x_{i-1}, x_i]$. For each interval I_i, we define $D_i = [x_{l(i)}, x_{r(i)}]$, with $l(i), r(i) \in \{0, \dots, n\}$ such that $l(i) < r(i)$ and $x_{r(i)} - x_{l(i)} > x_i - x_{i-1}$. For each $i = 1, \dots, n$, we consider $L_i : [x_{l(i)}, x_{r(i)}] \to [x_{i-1}, x_i]$ as the homeomorphism given by $L_i(x) = a_i x + b_i$ and $F_i : D_i \times \mathbb{R} \to \mathbb{R}$ as continuous mappings satisfying

$$F_i(x_{l(i)}, y_{l(i)}) = y_{i-1} \quad \text{and} \quad F_i(x_{r(i)}, y_{r(i)}) = y_i. \tag{3.1}$$

Assume that F_i is k-Lipschitz with respect to the first variable and a Geraghty contraction with respect to the second variable; that is

$$|F_i(x,y) - F_i(x',y')| \leq k|x-x'| + \varphi(|y-y'|), \text{ for all } x, x' \in D_i, \text{ and } y, y' \in \mathbb{R}, \tag{3.2}$$

where $k \geq 0$ and $\varphi : (0,\infty) \to (0,\infty)$ is a nondecreasing function such that, for any $t > 0$, $\alpha(t) := \dfrac{\varphi(t)}{t} < 1$ and the function α is nonincreasing (or nondecreasing, or continuous). We define $w_i : D_i \times \mathbb{R} \to I_i \times \mathbb{R}$ by $w_i(x,y) = (L_i(x), F_i(x,y))$. Set $I(i) = \{j \in \{1, \dots, n\}\ ;\ I_j \subset D_i\}$.

The proof of the following lemma mimics that of [9, Theorem 3.7] and [12, Theorem 3.3].

Lemma 1. *There exists a metric d_θ in $I \times [a,b]$ equivalent to the Euclidean metric, such that for all $i = 1, \dots, n$, the mappings w_i defined above, are Geraghty contractions with respect to d_θ.*

Proof. We define a metric d_θ on $I \times [a,b]$ by

$$d_\theta\left((x,y),(x',y')\right) := |x-x'| + \theta|y-y'|,$$

where θ is a real number that is specified in the sequel. Then, for all $(x,y),(x',y') \in I \times [a,b]$, we have:

$$\begin{aligned}
d_\theta\left(w_i(x,y), w_i(x',y')\right) &= d_\theta\left((L_i(x), F_i(x,y)), (L_i(x'), F_i(x',y'))\right) \\
&= |L_i(x) - L_i(x')| + \theta|F_i(x,y) - F_i(x',y')| \\
&\leq |a_i||x-x'| + \theta k|x-x'| + \theta\varphi(|y-y'|) \\
&= (|a_i| + \theta k)|x-x'| + \theta\varphi(|y-y'|) \\
&= \left(|a_i| + \theta k + \theta\frac{\varphi(|y-y'|)}{|x-x'|+|y-y'|}\right)|x-x'| + \theta\frac{\varphi(|y-y'|)}{|x-x'|+|y-y'|}|y-y'| \\
&\leq \left(|a_i| + \theta k + \theta\frac{\varphi(|x-x'|+|y-y'|)}{|x-x'|+|y-y'|}\right)|x-x'| + \theta\frac{\varphi(|x-x'|+|y-y'|)}{|x-x'|+|y-y'|}|y-y'| \\
&\leq (|a_i| + \theta k + \theta)|x-x'| + \theta\frac{\varphi(|x-x'|+|y-y'|)}{|x-x'|+|y-y'|}|y-y'| \\
&\leq \max\left(a + \theta k + \theta, \frac{\varphi(|x-x'|+|y-y'|)}{|x-x'|+|y-y'|}\right) d_\theta((x,y),(x',y')),
\end{aligned}$$

with $a = \max\limits_{1\leq i\leq n}|a_i|$. Set $\theta := \dfrac{1-a}{2(k+1)}$, then $0 < \theta < 1$ and $0 < a + \theta k + \theta < 1$. Moreover, for $t > 0$, let $\beta(t) := \max(a + \theta k + \theta, \frac{\varphi(t)}{t})$. Then, $\beta : (0,\infty) \to [0,1)$ is a nonincreasing (or nondecreasing, or continuous), since $t \mapsto \frac{\varphi(t)}{t}$ is a nonincreasing (or nondecreasing, or continuous). Furthermore, we obtain:

$$d_\theta(w_i(x,y), w_i(x',y')) \leq \beta\left(d((x,y),(x',y'))\right) d_\theta((x,y),(x',y')),$$

where $d((x,y),(x',y')) := |x-x'| + |y-y'|$. Since $0<\theta<1$, then, for all $(x,y)\neq(x',y')$, we have:

$$\theta|x-x'| + \theta|y-y'| \leq |x-x'| + \theta|y-y'| \leq |x-x'| + |y-y'|.$$

Thus,

$$d_\theta((x,y),(x',y')) \leq d((x,y),(x',y')) \leq \theta^{-1} d_\theta((x,y),(x',y')).$$

If $\beta : (0,\infty) \to [0,1)$ is nonincreasing, then

$$\begin{aligned} d_\theta(w_i(x,y), w_i(x',y')) &\leq \beta\left(d((x,y),(x',y'))\right) d_\theta((x,y),(x',y')) \\ &\leq \beta\left(d_\theta((x,y),(x',y'))\right) d_\theta((x,y),(x',y')). \end{aligned}$$

If $\beta : (0,\infty) \to [0,1)$ is nondecreasing, then

$$\begin{aligned} d_\theta(w_i(x,y), w_i(x',y')) &\leq \beta\left(d((x,y),(x',y'))\right) d_\theta((x,y),(x',y')) \\ &\leq \beta\left(\theta^{-1} d_\theta((x,y),(x',y'))\right) d_\theta((x,y),(x',y')) \\ &\leq \beta'\left(d_\theta((x,y),(x',y'))\right) d_\theta((x,y),(x',y')), \end{aligned}$$

where $\beta' : (0,\infty) \to [0,1)$ is a nondecreasing function defined as $\beta'(t) := \beta(\theta^{-1}t)$.
If $\beta : (0,\infty) \to [0,1)$ is continuous, then

$$\begin{aligned} d_\theta(w_i(x,y), w_i(x',y')) &\leq \beta\left(d((x,y),(x',y'))\right) d_\theta((x,y),(x',y')) \\ &\leq \beta''\left(d_\theta((x,y),(x',y'))\right) d_\theta((x,y),(x',y')), \end{aligned}$$

where $\beta'' : (0,\infty) \to [0,1)$ is a continuous function defined as $\beta''(t) := \max_{x\in[t,\theta^{-1}t]} \beta(x)$. Hence, w_i are Geraghty contractions in $(I\times[a,b], d_\theta)$. Moreover, we have:

$$\begin{aligned} \min(1,\theta)\sqrt{|x-x'|^2 + |y-y'|^2} &\leq \min(1,\theta)(|x-x'| + |y-y'|) \\ &\leq d_\theta((x,y),(x',y')) \\ &\leq (1+\theta)\sqrt{|x-x'|^2 + |y-y'|^2}. \end{aligned}$$

That is, d_θ is equivalent to the Euclidean metric on $I\times[a,b]$. In particular, $(I\times[a,b], d_\theta)$ is a complete metric space. □

Denote by $C(I)$ the set of continuous functions $f : I = [x_0, x_n] \to [a,b]$. Let $C^*(I)$ be the set of continuous functions on I that pass through the interpolation points; that is

$$C^*(I) = \{f \in C(I) \mid f(x_i) = y_i, \ \forall\, i = 0,\ldots,n\}.$$

Then, $C(I)$ and $C^*(I)$ are complete metric spaces with respect to the uniform metric ρ defined by

$$\rho(f_1, f_2) = \max_{x\in I} |f_1(x) - f_2(x)|.$$

Define the Read-Bajraktarevic operator T on $C^*(I)$ by

$$Tf(x) = F_i(L_i^{-1}(x), f(L_i^{-1}(x))), \text{ for } x \in I_i \text{ and } i = 1,\ldots,n.$$

It is easy to verify that Tf is continuous on I and that $Tf(x_i) = y_i$, for all $i = 0,\ldots,n$.

Our main result in this section is given by the following theorem. This extends [17, Corollaries 3.5] and [17, Corollaries 3.7] and the results given in [9, 12] regarding the existence of FIF defined with Rakotch and Geraghty contractions.

Theorem 3. *For $n \geq 2$, let $\{(x_i, y_i);\ i = 0, \ldots, n\}$ be a given data set and let $\{D_i \times \mathbb{R}, w_i;\ i = 1, \ldots, n\}$ be the RIFS constructed above. Then, there exists a unique function $\tilde{f} \in C(I)$ such that $\tilde{f}(x_i) = y_i$, for all $i = 0, \ldots, n$. Moreover, let G_i be the graph of the restriction of $\tilde{f}$ to I_i, then $G = (G_1, \ldots, G_n)$ is the attractor of the RIFS $\{D_i \times \mathbb{R}, w_i;\ i = 1, \ldots, n\}$, i.e.,*

$$G_i = \bigcup_{j \in I(i)} w_i(G_j), \text{ for } i = 1, \ldots, n.$$

Furthermore, the RFIF $\tilde{f}$ satisfies the functional equation

$$\tilde{f}(x) = F_i(L_i^{-1}(x), \tilde{f} \circ L_i^{-1}(x)), \text{ for all } x \in I_i \text{ and } i = 1, \ldots, n.$$

Proof. Let's first notice that the RIFS $\{D_i \times \mathbb{R}, w_i;\ i = 1, \ldots, n\}$ has a unique attractor by Lemma 1 and Theorem 1. Moreover, for $f, g \in C^*(I)$, we have:

$$\begin{aligned}
\rho(Tf, Tg) &= \max_{x \in I} \left| Tf(x) - Tg(x) \right| = \max_{\substack{1 \leq i \leq n \\ x \in I_i}} \left| Tf(x) - Tg(x) \right| \\
&= \max_{\substack{1 \leq i \leq n \\ x \in I_i}} \left| F_i(L_i^{-1}(x), f(L_i^{-1}(x))) - F_i(L_i^{-1}(x), g(L_i^{-1}(x))) \right| \\
&\leq \max_{1 \leq i \leq n} \sup_{x \in I_i} \varphi\Big(\left| f(L_i^{-1}(x)) - g(L_i^{-1}(x)) \right| \Big).
\end{aligned}$$

Since, for each $i \in \{1, \ldots, n\}$ and every $x \in I_i$, we have:

$$\begin{aligned}
\varphi\Big(\left| f(L_i^{-1}(x)) - g(L_i^{-1}(x)) \right| \Big) &\leq \varphi\Big(\max_{x \in I_i} \left| f(L_i^{-1}(x)) - g(L_i^{-1}(x)) \right| \Big) \\
&\leq \varphi\Big(\max_{x \in I} \left| f(x) - g(x) \right| \Big) = \varphi(\rho(f, g)),
\end{aligned}$$

then, we obtain :

$$\rho(Tf, Tg) \leq \varphi(\rho(f, g)).$$

Thus, T is a Geraghty contraction and has a unique fixed point $\tilde{f} \in C^*(I)$, i.e. there exists a unique continuous function $\tilde{f}$ that interpolates the points (x_i, y_i), $i = 0, \ldots, n$ and satisfies

$$\tilde{f}(x) = F_i(L_i^{-1}(x), \tilde{f} \circ L_i^{-1}(x)),\ x \in I_i,\ i = 1, \ldots, n.$$

Moreover, let G be the graph of f and let G_i be the graph of the restriction of f to I_i, for $i = 1, \ldots, n$, then,

$$G = \bigcup_{i=1}^{n} G_i.$$

Furthermore, we have:

$$\begin{aligned}
G_i &= \left\{ (x, \tilde{f}(x));\ x \in I_i \right\} \\
&= \left\{ \Big(x, F_i(L_i^{-1}(x), \tilde{f}(L_i^{-1}(x))) \Big) ; x \in I_i \right\} \\
&= \left\{ \Big(L_i(x), F_i(x, \tilde{f}(x)) \Big) ; x \in D_i \right\}
\end{aligned}$$

$$= \left\{ w_i(x, \tilde{f}(x)); x \in D_i \right\}.$$

Hence, $(G_1, \ldots, G_n)$ is the invariant set of W, i.e., the attractor of the RIFS $\{D_i \times \mathbb{R}, w_i;\ i = 1, \ldots, n\}$, and we have:

$$G_i = \bigcup_{j \in I(i)} w_i(G_j), \text{ for } i = 1, \ldots, n.$$

□

Example 3. *Considering the interpolation data*

$$\{(0, 9); (0.25, 11); (0.5, 10); (0.75, 11); (1, 8)\},$$

we illustrate the RFIFs corresponding to $\widetilde{\mathbb{I}}$ *equipped with the transition matrices* P_1*,* P_2 *and* P_3 *given in Example 2 in Figure 2 (l1–l3), respectively. The subsequent figure (f1–f3) illustrates the RFIF equipped with affine transformations of Example 1 with the same respective transition matrices so that,*

	D_1	D_2	D_3	D_4
P_1	$[\frac{1}{2}, 1]$	$[0, \frac{1}{2}]$	$[\frac{1}{4}, \frac{3}{4}]$	$[\frac{1}{2}, 1]$
P_2	$[0, \frac{1}{2}]$	$[0, \frac{3}{4}]$	$[0, 1]$	$[0, 1]$
P_3	$[\frac{1}{2}, 1]$	$[0, \frac{3}{4}]$	$[\frac{1}{2}, 1]$	$[0, \frac{3}{4}]$.

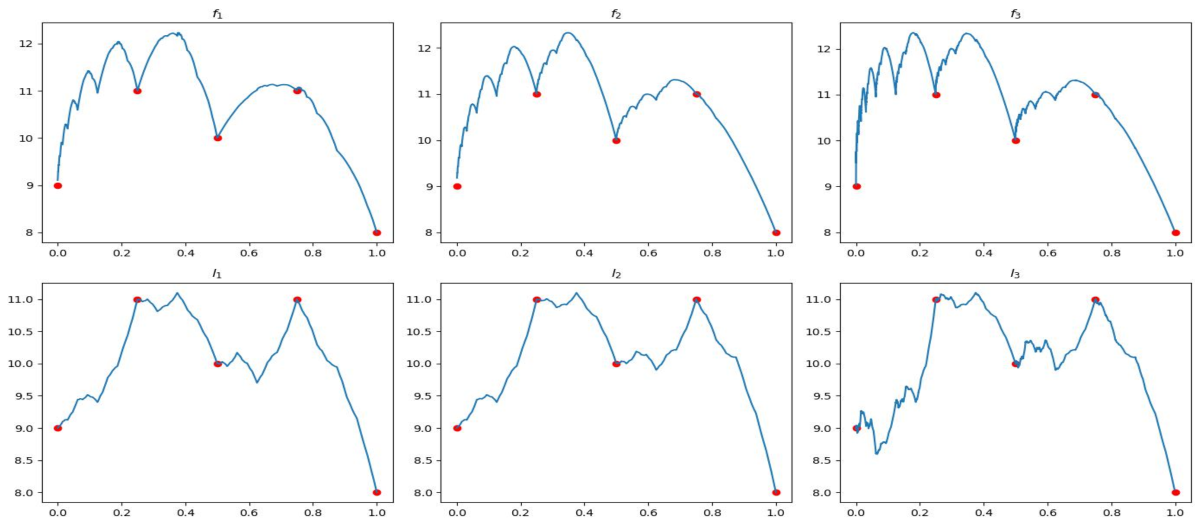

Figure 2. The RFIFs equipped with the transition matrices P_1, P_2 and P_3.

4. Application

In this section, we will compare RFIF with the classical FIF technique in terms of error rates of the time-series representing the daily vaccinations data. We can show that the fractal interpolation method gives lower error rates than the cubic splines, which is often used to avoid the problem of Runge's phenomenon [29]. For this, we collect (from the website ourworldindata.org) daily vaccination data from the following four countries: Germany, Italy, Tunisia and France (see Table 1). As seen in Figure

3, the irregularities of this data is very visible and the reconstruction of the curve from the point of view of fractal interpolation is significantly relevant. This can be an effective way to recover missing information due to insufficient testing, and to analyze the complexity of the evolution of COVID-19 given the importance of vaccination data for each day. In addition, the variations of the FIF incorporate the behavior of the data (see Figure 4). Two methods of fractal interpolation namely linear interpolation, where

$$F_i(x, y) = \alpha y + c_i x + d_i,$$

as well as fractal interpolation defined by Geraghty contractions, with

$$F_i(x, y) = \frac{y}{1 + \beta y} + c_i' x + d_i',$$

are considered.

Table 1. To effectively test the goodness of fit of the interpolation techniques that we will use, the interpolation data for each country that we used is the sliced original data.

	Tun	Fra	Ita	Ger
total data obtained	273	669	671	670
interpolation data	69	168	168	168

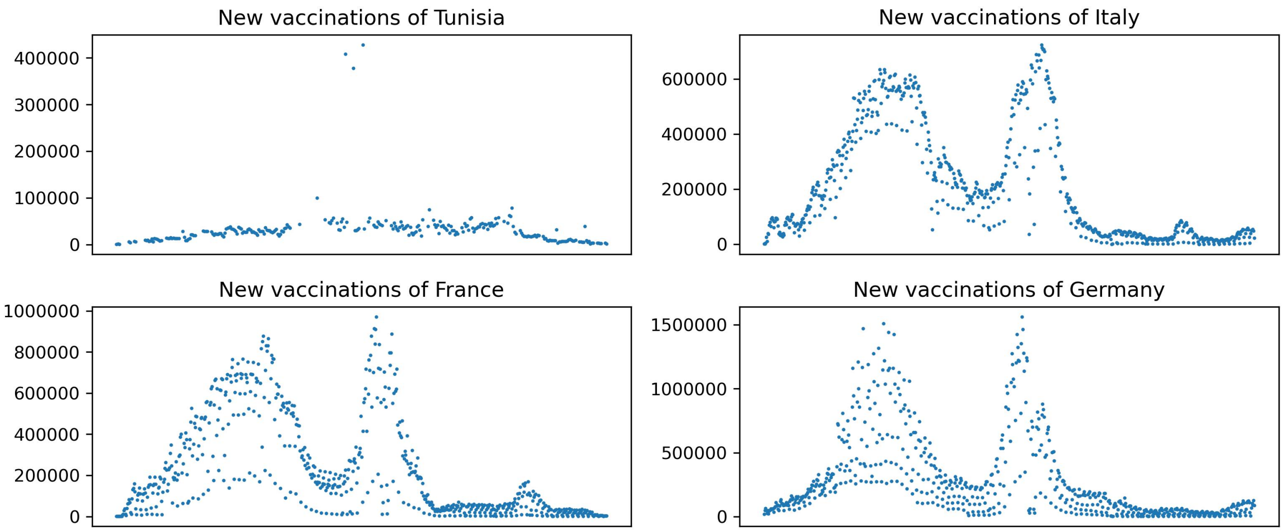

Figure 3. The data of new vaccinations for Tunisia, Italy, France and Germany.

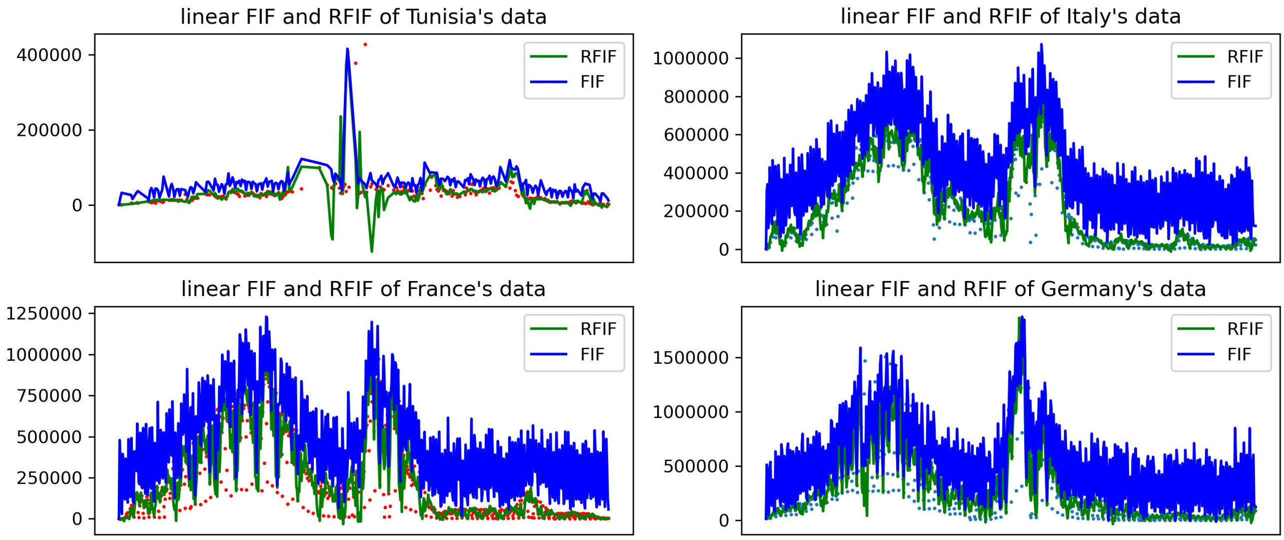

Figure 4. The Graphs of linear FIFs and RFIFs induced from the interpolation data of Tunisia, Italy, France and Germany.

A common way of measuring the quality of the fit of the interpolation is the normalized root mean square error (NRMSE), which is a statistical error indicator that facilitates the comparison between datasets with different scales and ensures a more universal measure of comparison. There exists various different methods of RMSE normalizations, and we will use

$$NRMSE = \frac{\sqrt{\sum_{i=0}^{n}(y_i - \hat{y}_i)^2}}{\frac{1}{n}\sum_{i=0}^{n} y_i},$$

where y_i is the ith observation of y, $\hat{y}_i$ is the predicted y value given by the interpolation technique and n is the number of total data considered.

In Table 2, we present the results obtained for different parameters α and β concerning the daily vaccination data of France. The parameters α and β have been chosen arbitrarily. We can see that the RFIF is less sensitive to parameter change compared to the FIF which has big differences in the NMRSE for the various parameter values. This may be explained by the fact that other parameters are involved in the RFIF, namely, the coefficients of the stochastic matrix. Even though we did not get relevant information on the evolution of the disease, we can see the fractal structure of the epidemic curve. Moreover, the presence of parameters α and β help to get a wide variety of mappings for approximating problems. The obtained values show a slight improvement when the recurrent fractal interpolation is deployed. Nevertheless, there are still questions about optimal values of the entries of the stochastic matrix that can be managed to determine an optimal choice of the intervals D_i (as defined in Section 3), as well as optimal choice of β (as given in [13, 30] for the choice of the vertical scaling factor) so that the obtained curves fit as closely as possible to the real values.

Table 2. The NRMSE of fractal interpolation and recurrent fractal interpolation on France's data.

		Fractal interpolation	Recurrent fractal interpolation
$F_i(x, y) = \alpha y + c_i x + d_i$	$\alpha = 0.1$	273.819	264.536
	$\alpha = 0.2$	310.896	270.743
	$\alpha = 0.3$	379.131	281.771
	$\alpha = 0.5$	636.935	320.554
$F_i(x, y) = \dfrac{y}{1 + \beta y} + c'_i x + d'_i$	$\beta = 0.015$	263.342	263.252
	$\beta = 0.025$	263.283	263.283
	$\beta = 0.045$	263.354	263.299
	$\beta = 0.1$	263.278	263.345

5. Conclusions

RIFS can be seen as improvements of the notion of IFS and were introduced in order to gain more flexibility in natural-shape generation and image compression. In this paper, we gave new conditions on the existence of the attractor of a new class of RIFS based on the condition of Geraghty contractions. Consequently, we constructed a new class of RFIF.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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