Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 31(11): 6833–6843.
DOI: 10.3934/era.2023345
Received: 06 June 2023
Revised: 31 August 2023
Accepted: 09 October 2023
Published: 23 October 2023

*Research article*

# Fractional integral associated with the Schrödinger operators on variable exponent space

**Huali Wang**[1] **and Ping Li**[2,*]

[1] School of Mathematics and Statistics, Hubei University of Arts and Science, Xiangyang 441000, China

[2] School of Informations and Mathematics, Yangtze University, Jingzhou 434000, China

* **Correspondence:** Email: liping@whu.edu.cn.

**Abstract:** Let $\mathcal{L} = -\Delta + V$ be the Schrödinger operators on $\mathbb{R}^n$ with nonnegative potential $V$ belonging to the reverse Hölder class $RH_q$ for some $q \geq \frac{n}{2}$. We prove the boundedness of fractional integral operator $\mathcal{I}_\alpha$ related to the Schrödinger operators $\mathcal{L}$ from strong and weak variable exponent Lebesgue spaces into suitable variable exponent Lipschitz type spaces.

**Keywords:** Lipschitz spaces; fractional integral operators; Schrödinger operators; variable exponent spaces

## 1. Introduction

In this paper, we consider the Schrödinger operators

$$\mathcal{L} = -\triangle + V(x),\ x \in \mathbb{R}^n,\ n \geq 3,$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial_{x_i}^2}$ and $V(x)$ is a nonnegative potential belonging to the reverse Hölder class $RH_q$ for some $q \geq \frac{n}{2}$. Assume that $f$ is a nonnegative locally $L^q(\mathbb{R}^n)$ integrable function on $\mathbb{R}^n$, then we say that $f$ belongs to $RH_q$ $(1 < q \leq \infty)$ if there exists a positive constant $C$ such that the reverse Hölder's inequality

$$\Big(\frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|^q dy\Big)^{\frac{1}{q}} \leq \frac{C}{|B(x,r)|}\int_{B(x,r)} |f(y)| dy$$

holds for $x$ in $\mathbb{R}^n$, where $B(x,r)$ denotes the ball centered at $x$ with radius $r < \infty$ [1]. For example, the nonnegative polynomial $V \in RH_\infty$, in particular, $|x|^2 \in RH_\infty$.

Let the potential $V \in RH_q$ with $q \geq \frac{n}{2}$, and the critical radius function $\rho(x)$ is defined as

$$\rho(x) = \sup_{r>0}\Big\{r : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)dy \leq 1\Big\}, x \in \mathbb{R}^n. \tag{1.1}$$

We also write $\rho(x) = \frac{1}{m_V(x)}, x \in \mathbb{R}^n$. Clearly, $0 < m_V(x) < \infty$ when $V \neq 0$, and $m_V(x) = 1$ when $V = 1$. For the harmonic oscillator operator ( Hermite operator ) $H = -\Delta + |x|^2$, we have $m_V(x) \sim (1 + |x|)$.

Thanks to the heat diffusion semigroup $e^{-t\mathcal{L}}$ for enough good function $f$, the negative powers $\mathcal{L}^{-\frac{\alpha}{2}}(\alpha > 0)$ related to the Schrödinger operators $\mathcal{L}$ can be written as

$$\mathcal{I}_\alpha f(x) = \mathcal{L}^{-\frac{\alpha}{2}} f(x) = \int_0^\infty e^{-t\mathcal{L}} f(x) t^{\frac{\alpha}{2}-1} dt, \ 0 < \alpha < n. \tag{1.2}$$

Applying Lemma 3.3 in [2] for enough good function $f$ holds that

$$\mathcal{I}_\alpha f(x) = \int_{\mathbb{R}^n} K_\alpha(x, y) f(y) dy, \ 0 < \alpha < n,$$

and the kernel $K_\alpha(x, y)$ satisfies the following inequality

$$K_\alpha(x, y) \leq \frac{C_k}{(1 + |x - y|(m_V(x) + m_V(y)))^k} \frac{1}{|x - y|^{n-\alpha}}. \tag{1.3}$$

Moreover, we have $K_\alpha(x, y) \leq \frac{C}{|x-y|^{n-\alpha}}, 0 < \alpha < n$.

Shen [1] obtained $L^p$ estimates of the Schrödinger type operators when the potential $V \in RH_q$ with $q \geq \frac{n}{2}$. For Schrödinger operators $\mathcal{L} = -\Delta + V$ with $V \in RH_q$ for some $q \geq \frac{n}{2}$, Harboure et al. [3] established the necessary and sufficient conditions to ensure that the operators $\mathcal{L}^{-\frac{\alpha}{2}}(\alpha > 0)$ are bounded from weighted strong and weak $L^p$ spaces into suitable weighted $BMO_{\mathcal{L}}(w)$ space and Lipschitz spaces when $p \geq \frac{n}{\alpha}$. Bongioanni Harboure and Salinas proved that the fractional integral operator $\mathcal{L}^{-\alpha/2}$ is bounded form $L^{p,\infty}(w)$ into $BMO^\beta_{\mathcal{L}}(w)$ under suitable conditions for weighted $w$ [4]. For more backgrounds and recent progress, we refer to [5–7] and references therein.

Ramseyer, Salinas and Viviani in [8] studied the fractional integral operator and obtained the boundedness from strong and weak $L^{p(\cdot)}$ spaces into the suitable Lipschitz spaces under some conditions on $p(\cdot)$. In this article, our main interest lies in considering the properties of fractional integrals operator $\mathcal{L}^{-\frac{\alpha}{2}}(\alpha > 0)$, related to $\mathcal{L} = -\Delta + V$ with $V \in RH_q$ for some $q \geq \frac{n}{2}$ in variable exponential spaces.

We now introduce some basic properties of variable exponent Lebsegue spaces, which are used frequently later on.

Let $p(\cdot) : \Omega \to [1, \infty)$ be a measurable function. For a measurable function $f$ on $\mathbb{R}^n$, the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \Big\{f : \int_\Omega \Big|\frac{f(x)}{s}\Big|^{p(x)} dx < \infty\Big\},$$

where $s$ is a positive constant. Then $L^{p(\cdot)}(\Omega)$ is a Banach space equipped with the follow norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf\Big\{s > 0 : \int_\Omega \Big|\frac{f(x)}{s}\Big|^{p(x)} dx \leq 1\Big\}.$$

We denote

$$p^- := \operatorname*{ess\,inf}_{x\in\Omega} p(x) \ \text{ and } \ p^+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x).$$

Let $\mathcal{P}(\mathbb{R}^n)$ denote the set of all measurable functions $p$ on $\mathbb{R}^n$ that take value in $[1,\infty)$, such that $1 < p_-(\mathbb{R}^n) \le p(\cdot) \le p_+(\mathbb{R}^n) < \infty$.

Assume that $p$ is a real value measurable function $p$ on $\mathbb{R}^n$. We say that $p$ is locally log-Hölder continuous if there exists a constant $C$ such that

$$|p(x) - p(y)| \le \frac{C}{\log(e + 1/|x-y|)},\ x, y \in \mathbb{R}^n,$$

and we say $p$ is log-Hölder continuous at infinity if there exists a positive constant $C$ such that

$$|p(x) - p(\infty)| \le \frac{C}{\log(e + |x|)},\, x \in \mathbb{R}^n,$$

where $p(\infty) := \lim_{|x|\to\infty} p(x) \in \mathbb{R}$.

The notation $\mathcal{P}^{\log}(\mathbb{R}^n)$ denotes all measurable functions $p$ in $\mathcal{P}(\mathbb{R}^n)$, which states $p$ is locally log-Hölder continuous and log-Hölder continuous at infinity. Moreover, we have that $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, which implies that $p'(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$.

**Definition 1.1.** [8] Assume that $p(\cdot)$ is an exponent function on $\mathbb{R}^n$. We say that a measurable function $f$ belongs to $L^{p(\cdot),\infty}(\mathbb{R}^n)$, if there exists a constant $C$ such that for $t > 0$,

$$\int_{\mathbb{R}^n} t^{p(x)} \chi_{\{|f|>t\}}(x)dx \le C.$$

It is easy to check that $L^{p(\cdot),\infty}(\mathbb{R}^n)$ is a quasi-norm space equipped with the following quasi-norm

$$\|f\|_{p(\cdot),\infty} = \inf\Big\{s > 0 : \sup_{t>0} \int_{\mathbb{R}^n} \Big(\frac{t}{s}\Big)^{p(x)} \chi_{\{|f|>t\}}(x)dx \le 1\Big\}.$$

Next, we define $Lip^{\mathcal{L}}_{\alpha,p(\cdot)}$ spaces related to the nonnegative potential $V$.

**Definition 1.2.** Let $p(\cdot)$ be an exponent function with $1 < p^- \le p^+ < \infty$ and $0 < \alpha < n$. We say that a locally integrable function $f \in Lip^{\mathcal{L}}_{\alpha,p(\cdot)}(\mathbb{R}^n)$ if there exist constants $C_1, C_2$ such that for every ball $B \subset \mathbb{R}^n$,

$$\frac{1}{|B|^{\frac{\alpha}{n}} \|\chi_B\|_{p'(\cdot)}} \int_B |f(x) - m_B f| dx \le C_1, \tag{1.4}$$

and for $R \ge \rho(x)$,

$$\frac{1}{|B|^{\frac{\alpha}{n}} \|\chi_B\|_{p'(\cdot)}} \int_B |f(x)| dx \le C_2, \tag{1.5}$$

where $m_B f = \frac{1}{|B|}\int_B f$. The norm of space $Lip^{\mathcal{L}}_{\alpha,p(\cdot)}(\mathbb{R}^n)$ is defined as the maximum value of two infimum of constants $C_1$ and $C_2$ in (1.4) and (1.5).

**Remark 1.1.** $Lip^{\mathcal{L}}_{\alpha,p(\cdot)}(\mathbb{R}^n) \subset \mathfrak{L}_{\alpha,p(\cdot)}(\mathbb{R}^n)$ is introduced in [8]. In particular, when $p(\cdot) = C$ for some constant, then $Lip^{\mathcal{L}}_{\alpha,p(\cdot)}(\mathbb{R}^n)$ is the usual weighted $BMO$ space $BMO^{\beta}_{\mathcal{L}}(w)$, with $w = 1$ and $\beta = \alpha - \frac{n}{p}$ [4].

**Remark 1.2.** It is easy to see that for some ball $B$, the inequality (1.5) leads to inequality (1.4) holding, and the average $m_B f$ in (1.4) can be replaced by a constant $c$ in following sense

$$\frac{1}{2}\|f\|_{Lip^{\mathcal{L}}_{\alpha,p(\cdot)}} \le \sup_{B\in\mathbb{R}^n} \inf_{c\in\mathbb{R}} \frac{1}{|B|^{\frac{\alpha}{n}} \|\chi_B\|_{p'(\cdot)}} \int_B |f(x) - c| dx \le \|f\|_{Lip^{\mathcal{L}}_{\alpha,p(\cdot)}}.$$

In 2013, Ramseyer et al. in [8] studied the Lipschitz-type smoothness of fractional integral operators $\mathcal{I}_\alpha$ on variable exponent spaces when $p^+ > \frac{\alpha}{n}$. Hence, when $p^+ > \frac{\alpha}{n}$, it will be an interesting problem to see whether or not we can establish the boundedness of fractional integral operators $\mathcal{L}^{-\frac{\alpha}{2}}(\alpha > 0)$ related to Schrödinger operators from Lebesgue spaces $L^{p(\cdot)}$ into Lipschitz-type spaces with variable exponents. The main aim of this article is to answer the problem above.

We now state our results as the following two theorems.

**Theorem 1.3.** *Let potential $V \in RH_q$ for some $q \geq n/2$ and $p(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$. Assume that* $1 < p^- \leq p^+ < \frac{n}{(\alpha-\delta_0)^+}$ *where* $\delta_0 = \min\{1, 2 - n/q\}$, *then the fractional integral operator $\mathcal{I}_\alpha$ defined in* (1.2) *is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ into $Lip^{\mathcal{L}}_{\alpha,p(\cdot)}(\mathbb{R}^n)$.*

**Theorem 1.4.** *Let the potential $V \in RH_q$ with $q \geq n/2$ and $p(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$. Assume that* $1 < p^- \leq p^+ < \frac{n}{(\alpha-\delta_0)^+}$ *where* $\delta_0 = \min\{1, 2 - n/q\}$. *If there exists a positive number $r_0$ such that $p(x) \leq p^\infty$ when $|x| > r_0$, then the fractional integral operator $\mathcal{I}_\alpha$ defined in* (1.2) *is bounded from $L^{p(\cdot),\infty}(\mathbb{R}^n)$ into $Lip^{\mathcal{L}}_{\alpha,p(\cdot)}(\mathbb{R}^n)$.*

To prove Theorem 1.3, we first need to decompose $\mathbb{R}^n$ into the union of some disjoint ball $B(x_k, \rho(x_k))(k \geq 1)$ according to the critical radius function $\rho(x)$ defined in (1.1). According to Lemma 2.6, we establish the necessary and sufficient conditions to ensure $f \in Lip^{\mathcal{L}}_{\alpha,p(\cdot)}(\mathbb{R}^n)$. In order to prove Theorem 1.3, by applying Corollary 1 and Remark 1.2, we only need to prove that the following two conditions hold:

(i) For every ball $B = B(x_0, r)$ with $r < \rho(x_0)$, then

$$\int_B |\mathcal{I}_\alpha f(x) - c| dx \leq C|B|^{\frac{\alpha}{n}} \|\chi_B\|_{p'(\cdot)} \|f\|_{p(\cdot)};$$

(ii) For any $x_0 \in \mathbb{R}^n$, then

$$\int_{B(x_0,\rho(x_0))} \mathcal{I}_\alpha(|f|)(x) dx \leq C|B(x_0, \rho(x_0))|^{\frac{\alpha}{n}} \|\chi_{B(x_0,\rho(x_0))}\|_{p'(\cdot)} \|f\|_{p(\cdot)}.$$

In order to check the conditions (i) and (ii) above, we need to find the accurate estimate of kernel $K_\alpha(x, y)$ of fractional integral operator $\mathcal{I}_\alpha$ (see Lemmas 2.8 and 2.9, then use them to obtain the proof of this theorem; the proof of the Theorem 1.4 proceeds identically).

The paper is organized as follows. In Section 2, we give some important lemmas. In Section 3, we are devoted to proving Theorems 1.3 and 1.4.

Throughout this article, $C$ always means a positive constant independent of the main parameters, which may not be the same in each occurrence. $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$, $B_k = B(x_0, 2^k R)$ and $\chi_{B_k}$ are the characteristic functions of the set $B_k$ for $k \in \mathbb{Z}$. $|S|$ denotes the Lebesgue measure of $S$. $f \sim g$ means $C^{-1}g \leq f \leq Cg$.

## 2. Some useful lemmas

In this section, we give several useful lemmas that are used frequently later on.

**Lemma 2.1.** *[9] Assume that the exponent function $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

*where $r_p = 1 + 1/p^- - 1/p^+$.*

**Lemma 2.2.** *[8] Assume that $p(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$ and $1 < p^- \leq p^+ < \infty$, and $p(x) \leq p(\infty)$ when $|x| > r_0 > 1$. For every ball $B$ and $f \in L^{p(\cdot),\infty}$ we have*

$$\int_B |f(x)|dx \leq C\|f\|_{L^{p(\cdot),\infty}} \|\chi_B\|_{L^{p'(\cdot)}},$$

*where the constant $C$ only depends on $r_0$.*

Fo the following lemma see Corollary 4.5.9 in [10].

**Lemma 2.3.** *Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, then for every ball $B \subset \mathbb{R}^n$ we have*

$$\|\chi_B\|_{p(\cdot)} \sim |B|^{\frac{1}{p(x)}}, \quad if \quad |B| \leq 2^n, \ \ x \in B,$$

*and*

$$\|\chi_B\|_{p(\cdot)} \sim |B|^{\frac{1}{p(\infty)}}, \quad if \quad |B| \geq 1.$$

**Lemma 2.4.** *Assume that $p(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$, then for all balls $B$ and all measurable subsets $S := B(x_0, r_0) \subset B := B(x_1, r_1)$ we have*

$$\frac{\|\chi_S\|_{p'(\cdot)}}{\|\chi_B\|_{p'(\cdot)}} \leq C\Big(\frac{|S|}{|B|}\Big)^{1-\frac{1}{p^-}}, \quad \frac{\|\chi_B\|_{p'(\cdot)}}{\|\chi_S\|_{p'(\cdot)}} \leq C\Big(\frac{|B|}{|S|}\Big)^{1-\frac{1}{p^+}}. \tag{2.1}$$

*Proof.* We only prove the first inequality in (2.1), and the second inequality in (2.1) proceeds identically. We consider three cases below by applying Lemma 2.3, and it holds that

1) if $|S| < 1 < |B|$, then $\dfrac{\|\chi_S\|_{p'(\cdot)}}{\|\chi_B\|_{p'(\cdot)}} \sim \dfrac{|S|^{\frac{1}{p'(x_S)}}}{|B|^{\frac{1}{p'(\infty)}}} \leq \Big(\dfrac{|S|}{|B|}\Big)^{\frac{1}{(p')^+}} = \Big(\dfrac{|S|}{|B|}\Big)^{1-\frac{1}{p^-}}$;

2) if $1 \leq |S| < |B|$, then $\dfrac{\|\chi_S\|_{p'(\cdot)}}{\|\chi_B\|_{p'(\cdot)}} \sim \dfrac{|S|^{\frac{1}{p'(\infty)}}}{|B|^{\frac{1}{p'(\infty)}}} \leq \Big(\dfrac{|S|}{|B|}\Big)^{\frac{1}{(p')^+}} = \Big(\dfrac{|S|}{|B|}\Big)^{1-\frac{1}{p^-}}$;

3) if $|S| < |B| < 1$, then $\dfrac{\|\chi_S\|_{p'(\cdot)}}{\|\chi_B\|_{p'(\cdot)}} \sim \dfrac{|S|^{\frac{1}{p'(x_S)}}}{|B|^{\frac{1}{p'(x_S)}}} |B|^{\frac{1}{p'(x_S)} - \frac{1}{p'(x_B)}} \leq C\Big(\dfrac{|S|}{|B|}\Big)^{\frac{1}{(p')^+}} = C\Big(\dfrac{|S|}{|B|}\Big)^{1-\frac{1}{p^-}}$, where $x_S \in S$ and $x_B \in B$.

Indeed, since $|x_B - x_S| \leq 2r_1$, by using the local-Hölder continuity of $p'(x)$ we have

$$\Big|\frac{1}{p'(x_S)} - \frac{1}{p'(x_B)}\Big| \log\frac{1}{r_1} \leq \frac{\log\frac{1}{r_1}}{\log(e + \frac{1}{|x_S - x_B|})} \leq \frac{\log\frac{1}{r_1}}{\log(e + \frac{1}{2r_1})} \leq C.$$

We end the proof of this lemma.

**Remark 2.1.** Thanks to the second inequality in (2.1), it is easy to prove that

$$\|\chi_{2B}\|_{p'(\cdot)} \leq C\|\chi_B\|_{p'(\cdot)}.$$

**Lemma 2.5.** *[1] Suppose that the potential $V \in B_q$ with $q \geq n/2$, then there exists positive constants $C$ and $k_0$ such that*

1) $\rho(x) \sim \rho(y)$ *when* $|x-y| \leq C\rho(x)$;

2) $C^{-1}\rho(x)\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{-k_0} \leq \rho(y) \leq C\rho(x)\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{k_0/(k_0+1)}$.

**Lemma 2.6.** *[11] There exists a sequence of points $\{x_k\}_{k=1}^{\infty}$ in $\mathbb{R}^n$ such that $B_k := B(x_k, \rho(x_k))$ satisfies*

1) $\mathbb{R}^n = \bigcup_k B_k$,

2) *For every* $k \geq 1$, *then there exists* $N \geq 1$ *such that* card $\{j : 4B_j \cap 4B_k \neq \emptyset\} \leq N$.

**Lemma 2.7.** *Assume that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $0 < \alpha < n$. Let sequence $\{x_k\}_{k=1}^{\infty}$ satisfy the propositions of Lemma* 2.6. *Then a function $f \in Lip^{\mathcal{L}}_{\alpha,p(\cdot)}(\mathbb{R}^n)$ if and only if $f$ satisfies* (1.4) *for every ball, and*

$$\frac{1}{|B(x_k,\rho(x_k))|^{\frac{\alpha}{n}}\|\chi_{B(x_k,\rho(x_k))}\|_{p'(\cdot)}}\int_{B(x_k,\rho(x_k))}|f(x)|dx \leq C, \ for\ all\ k \geq 1. \tag{2.2}$$

*Proof.* Let $B := B(x,R)$ denote a ball with center $x$ and radius $R > \rho(x)$. Noting that $f$ satisfies (1.4), and thanks to Lemma 2.6 we obtain that the set $G = \{k : B \cap B_k \neq \emptyset\}$ is finite.

Applying Lemma 2.5, if $z \in B_k \cap B$, we get

$$\begin{aligned}\rho(x_k) &\leq C\rho(z)\Big(1+\frac{|x_k-z|}{\rho(x_k)}\Big)^{k_0} \leq C2^{k_0}\rho(z)\\ &\leq C2^{k_0}\rho(x)\Big(1+\frac{|x-z|}{\rho(x)}\Big)^{\frac{k_0}{k_0+1}}\\ &\leq C2^{k_0}\rho(x)\Big(1+\frac{R}{\rho(x)}\Big) \leq C2^{k_0}R.\end{aligned}$$

Thus, for every $k \in G$, we have $B_k \subset CB$.

Thanks to Lemmas 2.4 and 2.6, it holds that

$$\begin{aligned}\int_B |f(x)|dx &= \int_{B\bigcap\bigcup_k B_k}|f(x)|dx = \int_{\bigcup_{k\in G}(B\bigcap B_k)}|f(x)|dx\\ &\leq \sum_{k\in G}\int_{B\cap B_k}|f(x)|dx \leq \sum_{k\in G}\int_{B_k}|f(x)|dx\\ &\leq C\sum_{k\in G}|B_k|^{\frac{\alpha}{n}}\|\chi_{B_k}\|_{p'(\cdot)}\\ &\leq C|B|^{\frac{\alpha}{n}}\|\chi_B\|_{p'(\cdot)}.\end{aligned}$$

The proof of this lemma is completed.

**Corollary 1.** *Assume that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $0 < \alpha < n$, then a measurable function $f \in Lip^{\mathcal{L}}_{\alpha,p(\cdot)}$ if and only if $f$ satisfies* (1.4) *for every ball $B(x,R)$ with radius $R < \rho(x)$ and*

$$\frac{1}{|B(x,\rho(x))|^{\frac{\alpha}{n}}\|\chi_{B(x,\rho(x))}\|_{p'(\cdot)}}\int_{B(x,\rho(x))}|f(x)|dx \leq C. \tag{2.3}$$

Let $k_t(x,y)$ denote the kernel of heat semigroup $e^{-t\mathcal{L}}$ associated to $\mathcal{L}$, and $K_\alpha(x,y)$ be the kernel of fractional integral operator $\mathcal{I}_\alpha$, then it holds that

$$K_\alpha(x,y) = \int_0^\infty k_t(x,y)t^{\frac{\alpha}{2}}dt. \tag{2.4}$$

Some estimates of $k_t$ are presented below.

**Lemma 2.8.** *[12] There exists a constant $C$ such that for $N > 0$,*

$$k_t(x,y) \le Ct^{-n/2}e^{-\frac{|x-y|^2}{Ct}}\Big(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\Big)^{-N},\ x,y \in \mathbb{R}^n.$$

**Lemma 2.9.** *[13] Let $0 < \delta < \min(1, 2 - \frac{n}{q})$. If $|x - x_0| < \sqrt{t}$, then for $N > 0$ the kernel $k_t(x,y)$ defined in (2.4) satisfies*

$$|k_t(x,y) - k_t(x_0,y)| \le C\Big(\frac{|x-x_0|}{\sqrt{t}}\Big)^\delta t^{-n/2}e^{-\frac{|x-y|^2}{Ct}}\Big(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\Big)^{-N},$$

*for all $x, y$ and $x_0$ in $\mathbb{R}^n$.*

## 3. Proof of theorems

In this section, we are devoted to the proof of Theorems 1.3 and 1.4. To prove Theorem 1.3, thanks to Corollary 1 and Remark 1.2, we only need to prove that the following two conditions hold:

(i) For every ball $B = B(x_0, r)$ with $r < \rho(x_0)$, then

$$\int_B |\mathcal{I}_\alpha f(x) - c|dx \le C|B|^{\frac{\alpha}{n}}\|\chi_B\|_{p'(\cdot)}\|f\|_{p(\cdot)};$$

(ii) For any $x_0 \in \mathbb{R}^n$, then

$$\int_{B(x_0,\rho(x_0))} \mathcal{I}_\alpha(|f|)(x)dx \le C|B(x_0,\rho(x_0))|^{\frac{\alpha}{n}}\|\chi_{B(x_0,\rho(x_0))}\|_{p'(\cdot)}\|f\|_{p(\cdot)}.$$

We now begin to check that these conditions hold. First, we prove (ii).

Assume that $B = B(x_0, R)$ and $R = \rho(x_0)$. We write $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\mathbb{R}^n\setminus 2B}$. Hence, by the inequality (1.3), we have

$$\int_B \mathcal{I}_\alpha(|f_1|)(x)dx = \int_B \mathcal{I}_\alpha(|f\chi_{2B}|)(x)dx \le C\int_B\int_{2B}\frac{|f(y)|}{|x-y|^{n-\alpha}}dydx.$$

Applying Tonelli theorem, Lemma 2.1 and Remark 1.2, we get the following estimate

$$\begin{aligned}\int_B \mathcal{I}_\alpha(|f_1|)(x)dx &\le C\int_{2B}|f(y)|\int_B\frac{dx}{|x-y|^{n-\alpha}}dy\\ &\le CR^\alpha\int_{2B}|f(y)|dy\\ &\le C|B|^{\frac{\alpha}{n}}\|\chi_B\|_{p'(\cdot)}\|f\|_{p(\cdot)}.\end{aligned} \tag{3.1}$$

To deal with $f_2$, let $x \in B$ and we split $\mathcal{I}_\alpha f_2$ as follows:

$$\mathcal{I}_\alpha f_2(x) = \int_0^{R^2} e^{-t\mathcal{L}} f_2(x) t^{\frac{\alpha}{2}-1} dt + \int_{R^2}^{\infty} e^{-t\mathcal{L}} f_2(x) t^{\frac{\alpha}{2}-1} dt := I_1 + I_2.$$

For $I_1$, if $x \in B$ and $y \in \mathbb{R}^n \setminus 2B$, we note that $|x_0 - y| < |x_0 - x| + |x - y| < C|x - y|$. By Lemma 2.8, it holds that

$$\begin{aligned}
I_1 &= \Big| \int_0^{R^2} \int_{\mathbb{R}^n \setminus 2B} k_t(x,y) f(y) dy \, t^{\frac{\alpha}{2}-1} dt \Big| \\
&\le C \int_0^{R^2} \int_{\mathbb{R}^n \setminus 2B} t^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{t}} |f(y)| dy \, t^{\frac{\alpha}{2}-1} dt \\
&\le C \int_0^{R^2} t^{-\frac{n+\alpha}{2}-1} \int_{\mathbb{R}^n \setminus 2B} \Big(\frac{t}{|x-y|^2}\Big)^{M/2} |f(y)| dy dt \\
&\le C \int_0^{R^2} t^{\frac{M-n+\alpha}{2}-1} dt \int_{\mathbb{R}^n \setminus 2B} \frac{|f(y)|}{|x_0 - y|^M} dy,
\end{aligned}$$

where the constant $C$ only depends the constant $M$.

Applying Lemma 2.1 to the last integral, we get

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus 2B} \frac{|f(y)|}{|x_0 - y|^M} dy &= \sum_{i=1}^{\infty} \int_{2^{i+1}B \setminus 2^i B} \frac{|f(y)|}{|x_0 - y|^M} dy \\
&\le \sum_{i=1}^{\infty} (2^i R)^{-M} \int_{2^{i+1}B} |f(y)| dy \\
&\le C \sum_{i=1}^{\infty} (2^i R)^{-M} \|\chi_{2^{i+1}B}\|_{p'(\cdot)} \|f\|_{p(\cdot)}.
\end{aligned}$$

By using Lemma 2.4, we arrive at the inequality

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus 2B} \frac{|f(y)|}{|x_0 - y|^M} dy &\le C \sum_{i=1}^{\infty} (R)^{-M} (2^i)^{n - \frac{n}{p^+} - M} \|\chi_B\|_{p'(\cdot)} \|f\|_{p(\cdot)} \\
&\le C R^{-M} \|f\|_{p(\cdot)} \|\chi_B\|_{p'(\cdot)}. \qquad (3.2)
\end{aligned}$$

Here, the series above converges when $M > n - \frac{n}{p^+}$. Hence, for such $M$,

$$\begin{aligned}
\Big| \int_0^{R^2} e^{-t\mathcal{L}} f_2(x) t^{\frac{\alpha}{2}-1} dt \Big| &\le C R^{-M} \|f\|_{p(\cdot)} \|\chi_B\|_{p'(\cdot)} \int_0^{R^2} t^{\frac{M-n+\alpha}{2}-1} dt \\
&\le C |B|^{\frac{\alpha}{n}-1} \|f\|_{p(\cdot)} \|\chi_B\|_{p'(\cdot)}.
\end{aligned}$$

For $I_2$, thanks to Lemma 2.8, we may choose $M$ as above and $N \ge M$, then it holds that

$$\Big| \int_{R^2}^{\infty} e^{-t\mathcal{L}} f_2(x) t^{\frac{\alpha-2}{2}} dt \Big| = \Big| \int_{R^2}^{\infty} \int_{\mathbb{R}^n \setminus 2B} k_t(x,y) f(y) dy t^{\frac{\alpha-2}{2}} dt \Big|$$

$$\leq C\int_{R^2}^{\infty}\int_{\mathbb{R}^n\setminus 2B} t^{\frac{\alpha-n-N-2}{2}}\rho(x)^N e^{-\frac{|x-y|^2}{t}}|f(y)|dydt$$
$$\leq C\rho(x)^N\int_{R^2}^{\infty} t^{\frac{\alpha-n-N-2}{2}}\int_{\mathbb{R}^n\setminus 2B}\Big(\frac{t}{|x-y|^2}\Big)^{M/2}|f(y)|dydt.$$

As $x\in B$, thanks to Lemma 2.5, $\rho(x)\sim\rho(x_0)=R$. Hence we have

$$\Big|\int_{R^2}^{\infty} e^{-t\mathcal{L}}f_2(x)t^{\frac{\alpha}{2}-1}dt\Big|\leq CR^N\int_{R^2}^{\infty} t^{\frac{M+\alpha-n-N}{2}-1}dt\int_{\mathbb{R}^n\setminus 2B}\frac{|f(y)|}{|x_0-y|^M}dy.$$

Since $M+\alpha-n-N<0$, the integral above for variable $t$ converges, and by applying inequality (3.2) we have

$$\Big|\int_{R^2}^{\infty} e^{-t\mathcal{L}}f_2(x)t^{\frac{\alpha}{2}-1}dt\Big|\leq C|B|^{\frac{\alpha}{n}-1}\|f\|_{p(\cdot)}\|\chi_B\|_{p'(\cdot)},$$

thus we have proved (ii).

We now begin to prove that the condition (i) holds. Let $B=B(x_0,r)$ and $r<\rho(x_0)$. We set $f=f_1+f_2$ with $f_1=f\chi_{2B}$ and $f_2=f\chi_{\mathbb{R}^n\setminus 2B}$. We write

$$c_r=\int_{r^2}^{\infty} e^{-t\mathcal{L}}f_2(x_0)t^{\frac{\alpha}{2}-1}dt. \tag{3.3}$$

Thanks to (3.1), it holds that

$$\int_B|\mathcal{I}_\alpha(f(x))-c_r|\leq\int_B\mathcal{I}_\alpha(|f_1|)(x)dx+\int_B|\mathcal{I}_\alpha(f_2)(x)-c_r|dx$$
$$\leq C|B|^{\frac{\alpha}{n}-1}\|\chi_B\|_{p'(\cdot)}\|f\|_{p(\cdot)}+\int_B|\mathcal{I}_\alpha(f_2)(x)-c_r|dx.$$

Let $x\in B$ and we split $\mathcal{I}_\alpha f_2(x)$ as follows:

$$\mathcal{I}_\alpha f_2(x)=\int_0^{r^2} e^{-t\mathcal{L}}f_2(x)t^{\frac{\alpha}{2}-1}dt+\int_{r^2}^{\infty} e^{-t\mathcal{L}}f_2(x)t^{\frac{\alpha}{2}-1}dt:=I_3+I_4.$$

For $I_3$, by the same argument it holds that

$$I_3=\Big|\int_0^{r^2} e^{-t\mathcal{L}}f_2(x)t^{\frac{\alpha}{2}-1}dt\Big|\leq C|B|^{\frac{\alpha}{n}-1}\|f\|_{p(\cdot)}\|\chi_B\|_{p'(\cdot)}.$$

For $I_4$, by Lemma 2.9 and (3.3), it follows that

$$\Big|\int_{r^2}^{\infty} e^{-t\mathcal{L}}f_2(x)t^{\frac{\alpha}{2}-1}dt-c_r\Big|\leq\int_{r^2}^{\infty}\int_{\mathbb{R}^n\setminus 2B}|k_t(x,y)-k_t(x_0,y)||f(y)|dy\;t^{\frac{\alpha}{2}-1}dt$$
$$\leq C_\delta\int_{r^2}^{\infty}\int_{\mathbb{R}^n\setminus 2B}\Big(\frac{|x-x_0|}{\sqrt{t}}\Big)^{\delta}t^{-n/2}e^{-\frac{|x-y|^2}{Ct}}|f(y)|dy\;t^{\frac{\alpha}{2}-1}dt$$
$$\leq C_\delta r^\delta\int_{\mathbb{R}^n\setminus 2B}|f(y)|\int_{r^2}^{\infty} t^{-(n-\alpha+\delta)/2}e^{-\frac{|x-y|^2}{Ct}}\frac{dt}{t}dy.$$

Let $s = \frac{|x-y|^2}{t}$, then we obtain the following estimate

$$\Big|\int_{r^2}^{\infty} e^{-t\mathcal{L}} f_2(x) t^{\frac{\alpha}{2}-1} dt - c_r\Big| \le C_\delta r^\delta \int_{\mathbb{R}^n \setminus 2B} \frac{|f(y)|}{|x-y|^{n-\alpha+\delta}} dy \int_0^\infty s^{\frac{n-\alpha+\delta}{2}} e^{-\frac{s}{C}} \frac{ds}{s}.$$

Notice that the integral above for variable $s$ is finite, thus we only need to compute the integral above for variable $y$. Thanks to inequality (3.2), it follows that

$$\begin{aligned}\Big|\int_{r^2}^{\infty} e^{-t\mathcal{L}} f_2(x) t^{\frac{\alpha}{2}-1} dt - c_r\Big| &\le C_\delta r^\delta \int_{\mathbb{R}^n \setminus 2B} \frac{|f(y)|}{|x-y|^{n-\alpha+\delta}} dy \\ &\le C \sum_{i=1}^{\infty} R^{\alpha-n} (2^i)^{\alpha - \frac{n}{p^+} - \delta} \|\chi_B\|_{p'(\cdot)} \|f\|_{p(\cdot)} \\ &\le C|B|^{\frac{\alpha-n}{n}} \|f\|_{p(\cdot)} \|\chi_B\|_{p'(\cdot)},\end{aligned}$$

so (i) is proved.

**Remark 3.1.** By the same argument as the proof of Theorem 1.3, thanks to Lemma 2.2 we immediately obtained that the conclusions of Theorem 1.4 hold.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Acknowledgments

Ping Li is partially supported by NSFC ( No. 12371136). The authors would like to thank the anonymous referees for carefully reading the manuscript and providing valuable suggestions, which substantially helped in improving the quality of this paper. We also thank Professor Meng Qu for his useful discussions.

## Conflict of interest

The authors declare there are no conflicts of interest.

## References

1. Z. Shen, $L^p$ estimates for Schrödinger operators with certain potentials, in *Annales de l'institut Fourier*, **45** (1995), 513–546. https://doi.org/10.5802/aif.1463
2. L. Tang, Weighted norm inequalities for Schrödinger type operators, *Forum Math.*, **27** (2015), 2491–2532.
3. E. Harboure, O. Salinas, B. Viviani, Boundedness of the fractional integral on weighted Lebesgue and Lipschitz spaces, *Trans. Am. Math. Soc.*, **349** (1997), 235–255. https://doi.org/10.1090/S0002-9947-97-01644-9

4. B. Bongioanni, E. Harboure, O. Salinas, Weighted inequalities for negative powers of Schrödinger operators, *J. Math. Anal. Appl.*, **348** (2008), 12–27. https://doi.org/10.1016/j.jmaa.2008.06.045
5. W. Wu, J. Zhou, Two-weight norm inequalities for some fractional type operators related to Schrödinger operator on weighted Morrey spaces, *Turk. J. Math.*, **45** (2021), 2646–2663. https://doi.org/10.3906/mat-2105-64
6. Q. Zhang, L. Tang, Variation operators on weighted Hardy and BMO spaces in the Schrödinger setting, *Bull. Malays. Math. Sci. Soc.*, **45** (2022), 2285–2312. https://doi.org/10.1007/s40840-022-01339-4
7. L. Yang, P. Li, Boundedness and compactness of commutators related with Schrödinger operators on Heisenberg groups, *J. Pseudo-Differ. Oper. Appl.*, **14** (2023), 8. https://doi.org/10.1007/s11868-022-00504-4
8. M. Ramseyer, O. Salinas, B. Viviani, Lipschitz type smoothness of the fractional integral on variable exponent spaces, *J. Math. Anal. Appl.*, **403** (2013), 95–106. https://doi.org/10.1016/j.jmaa.2012.12.074
9. O. Kováčik, J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czech. Math. J.*, **41** (1991), 592–618. https://doi.org/10.21136/CMJ.1991.102493
10. L. Diening, P. Harjulehto, P. Hästö, M. Ružička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer, 2011. https://doi.org/10.1007/978-3-642-18363-8
11. J. Dziubański, J. Zienkiewicz, Hardy spaces $H^1$ associated to Schrödinger operators with potential satisfying reverse Hölder inequality, *Rev. Mat. Iberoam.*, **15** (1999), 277–295.
12. K. Kurata, An estimate on the heat kernel of magnetic Schrödinger operators and uniformly elliptic operators with non-negative potentials, *J. London Math. Soc.*, **62** (2000), 885–903. https://doi.org/10.1112/S002461070000137X
13. J. Dziubański, J. Zienkiewicz, $H_p$ spaces associated with Schrödinger operator with potential from reverse Hölder classes, *Colloq. Math.*, **98** (2003), 5–38. https://doi.org/10.4064/cm98-1-2