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Research article

Existence of traveling waves in a delayed convecting shallow water fluid model

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Abstract: This paper investigates a delayed shallow water fluid model that has not been studied in previous literature. Applying geometric singular perturbation theory, we prove the existence of traveling wave solutions for the model with a nonlocal weak delay kernel and local strong delay convolution kernel, respectively. When the convection term contains a nonlocal weak generic delay kernel, the desired heteroclinic orbit is obtained by using Fredholm theory and linear chain trick to prove the existence of two kink wave solutions under certain parametric conditions. When the model contains local strong delay convolution kernel and weak backward diffusion, under the same parametric conditions to the previous case, the corresponding traveling wave system can be reduced to a near-Hamiltonian system. The existence of a unique periodic wave solution is established by proving the uniqueness of zero of the Melnikov function. Uniqueness is proved by utilizing the monotonicity of the ratio of two Abelian integrals.

Keywords: shallow water fluid model; geometric singular perturbation theory; Abelian integral; Fredholm theory; traveling waves

1. Introduction

Traveling waves in nonlinear wave equations can explain nonlinear complex phenomena in many subjects, such as chemistry, physics, biology, optics and mechanics. The well-known KdV equation is extremely important in modeling the motion of shallow water, which is given by

$$u_t + \alpha u u_x + \beta u_{xxx} = 0. \tag{1.1}$$

It was first proposed by Korteweg and de Vries in 1895 and is usually used as a model to govern the one-dimensional propagation of small-amplitude, weakly dispersive waves [1]. In (1.1), the first two terms cause the classic overtaking phenomenon, while the last term prevents the formation of discontinuities. It is worth mentioning that the balance between the nonlinear convection term uu_x and the dispersion effect term u_{xxx} in Eq (1.1) gives rise to solitons [2, 3]. Some unusual nonlinear interactions among solitary wave pulses propagating in nonlinear dispersive media were observed in the numerical solutions. According to the important role in nonlinear models, there are a lot of investigations on finding the traveling wave solutions for KdV (1.1) and its generalized forms. In 1993, Derks and Gils [4] discussed the uniqueness of traveling waves in a perturbed KdV equation

$$u_t + uu_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0, \tag{1.2}$$

where ε is a positive parameter. Ogawa [5] studied the existence of solitary waves and periodic waves of (1.2) and gave the relationship between the amplitude and the wavelength. With a higher degree in convection term and by using the geometric singular perturbation theory, Yan et al. [6] proved the existence of solitary wave solutions and periodic wave solutions for a perturbed modified KdV equation

$$u_t + u^n u_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0.$$

Moreover, the KdV-mKdV equation

$$u_t + uu_x \pm u^2 u_x + u_{xxx} = 0$$

describes the internal solitary waves in shallow seas [7], which have been studied by the various methods [8–17]. Some new exact explicit solutions for a combined KdV-mKdV equation were obtained by means of the Bäcklund transformation [18] and the exact solutions for a new fractal unsteady KdV model with the non-smooth boundary by means of the sub-equation method were studied [19]. More precisely, Song [20] considered the diffusive single species model with Allee effect and distributed delay time, proving the existence of traveling wavefront solutions for the model with local strong and nonlocal weak generic delay kernels. Sun [21, 22] studied a dispersive-dissipative solid model with weakly external dissipation and provided a rigorous proof for the existence of a unique periodic wave as well as investigated the following KdV equation with three perturbed terms

$$u_t + \lambda_1 u^q u_x + \lambda_3 u_{xxx} + \varepsilon(\lambda_2 u_{xx} + \lambda_4 u_{xxxx} + \lambda_5 (uu_x)_x) = 0$$

with $q = 1, 2$. They proved the model possesses periodic waves with a range of wave speed and gave the explicit amplitude. Du [23] studied the existence of solitary wave solutions for the following generalized KdV-mKdV equation with local weak generic kernel delay

$$u_t + \alpha u_x + \beta(f * u)u^{p-1}u_x + u_{xxx} + \gamma u_{xx} = 0 \tag{1.3}$$

by applying the geometric singular perturbation theory. Here, $f * u$ represents a convolution as a spatial-temporal variable. When $\tau \to 0$, (1.3) reduces to a non-delayed model

$$u_t + \alpha u_x + \beta u^p u_x + u_{xxx} + \gamma u_{xx} = 0. \tag{1.4}$$

Xu [24] established the existence of traveling wave solutions for $(1.3)_{p=1}$. The parametric condition on the traveling wave fronts persisted was given. Now, we are interested in the wave motion model containing a special generic delay kernel in convection term. Consequently, in this paper, we investigate the following delay convecting shallow water fluid model

$$u_t + \alpha u_x + \beta((f * u)u)_x + u^2 u_x + ((1-q)\tau - q)u_{xx} + u_{xxx} = 0, \tag{1.5}$$

where $(\alpha,\beta) \in R^2$, $q = \{0,1\}$, $\tau > 0$ is a small parameter and $f * u$ represents a convolution in the spatial-temporal variable in (1.5); that is, there is a time delay in the lower order convection term. u_{xx} is backward diffusion effect. When $q = 0$, $\tau \to 0$, (1.5) reduces to $(1.4)_{p=1,\gamma=1}$. To our knowledge, no literature has considered the traveling wave solutions for when $q = 1$. Therefore, the existence of traveling waves for the (1.5) is unknown. When the model contains different delay convolution kernels, are the traveling wave solutions persisted or vanished? If traveling wave solutions persisted, what is the type? What is the number? To solve these questions, we discuss the corresponding ordinary differential equation for (1.5) with a nonlocal weak and local strong delay convolution kernel, respectively. Geometric singular perturbation theory is utilized to reduce the singular perturbed system to regular perturbed system. The existence of traveling wave solutions is proved by different techniques in two cases: $q = 1$ and $q = 0$.

The rest of this paper is organized as follows.In section two, we introduce the geometric singular perturbation theory, which is a key to deal with the delayed equations. In section three, the delay convecting shallow water fluid model (1.5) in the case $q = 1$ without delay is analyzed by qualitative theory. We prove that there are two heteroclinic orbits between the unstable node and saddles. For (1.5) in the case $q = 1$ with a nonlocal weak generic delay kernel, the existence of locally invariant manifold in a small neighborhood of critical manifold is obtained, which reducing the singular perturbed system into a regular perturbed system. The existence of kink wave solutions for (1.5) in the case of $q = 1$ is proved by the Fredholm theory and the linear chain trick. In Section 4, (1.5) in the case $q = 0$ with local strong delay and the weak backward diffusion effect is considered. The singular perturbed system is reduced into regular perturbed system, which is a near-Hamiltonian system. We discuss the existence of periodic waves on certain parametric conditions by analyzing the monotonicity of ratio of two Abelian integrals in the Melnikov function. Section five is a simplified conclusion.

2. Preliminaries

We first introduce the following results on invariant manifolds according to [25, 26]. The basic equations considered are of the form

$$\begin{cases} \dfrac{dx}{dt} = f(x,y,\varepsilon), \\ \dfrac{dy}{dt} = \varepsilon g(x,y,\varepsilon), \end{cases} \tag{2.1}$$

where $x = (x_1, x_2, \ldots, x_k)^T \in R^k$, $y = (y_1, y_2, \ldots, y_l)^T \in R^l$ and $0 < \varepsilon \ll 1$ is a real parameter. Functions f, g are C^∞ on the set $U \times V$, where $U \subset R^{k+l}$ and V is an open interval containing zero.

With a change of time scaling $z = \varepsilon t$, (2.1) can be written as

$$\begin{cases} \varepsilon\dfrac{dx}{dz} = f(x,y,\varepsilon), \\ \dfrac{dy}{dz} = g(x,y,\varepsilon), \end{cases} \tag{2.2}$$

then z is called the slow time scale and t is the fast time scale. Clearly, when $\varepsilon \neq 0$, (2.1) and (2.2) are equivalent. System (2.1) is called *the fast system*, while (2.2) is called *the slow system*. At the limit $\varepsilon \to 0$, system (2.1) reduces to a layer system

$$\begin{cases} x'(t) = f(x, y, 0), \\ y'(t) = 0, \end{cases} \tag{2.3}$$

and x is called the fast variable, whereas y is called the slow variable. When $\varepsilon \to 0$, the limit system of (2.2) is given by

$$\begin{cases} f(x, y, 0) = 0, \\ \dot{y} = g(x, y, 0), \end{cases} \tag{2.4}$$

which is called a reduced system. Assume that for $\varepsilon = 0$, the system has a compact, normally hyperbolic manifold of critical manifold M_0, which is contained in the set $\{f(x, y, 0) = 0\}$.

Definition 2.1. *The manifold M_0 is normally hyperbolic if the linearization of (2.1) at each point in M_0 has exactly l eigenvalues with zero real part, where l is the dimension of the slow variable y.*

Definition 2.2. *A set M is locally invariant under the flow from (2.1) if it has neighborhood V so that no trajectory can leave M without also leaving V. In other words, it is locally invariant if for all $x \in M$, $x \cdot [0, t] \subset V$ implies that $x \cdot [0, t] \subset M$, where the notation $x \cdot t$ is used to denote the application of a flow after time t to the initial condition x. Similarly with $[0, t]$ replaced by $[t, 0]$, when $t < 0$.*

Under the previous hypotheses, the statement holds.

Lemma 2.1. *If M_0 is compact and normally hyperbolic, then, for any $0 < r < +\infty$, if $\varepsilon > 0$ is sufficiently small, there exists a manifold M_ε, satisfying*
(i) *which is locally invariant under the flow of (2.1);*
(ii) *which is C^r in x, y and ε;*
(iii) $M_\varepsilon = \{(x, y) : x = h^\varepsilon(y)\}$ *for some C^r function $h^\varepsilon(y)$ and y in some compact K;*
(iv) *there exists locally invariant stable and unstable manifolds $W^s(M_\varepsilon)$ and $W^u(M_\varepsilon)$ that lie within $O(\varepsilon)$, and are diffeomorphic to $W^s(M_0)$ and $W^u(M_0)$, respectively.*

Geometric singular perturbation theory is a powerful tool for analyzing high-dimensional systems and exploiting a differential equation's geometric structures, such as its slow (center) manifolds and their fast stable and unstable fibers [27–31].

3. Kink wave solution for (1.5) in the case $q = 1$

In this section, the delayed convecting shallow water fluid model (1.5) in the case $q = 1$ is analyzed. We discuss the existence of heteroclinic orbits connecting an unstable node to a saddle when (1.5) is without delay. For (1.5) in the case $q = 1$ with a nonlocal weak generic delay kernel, a locally invariant manifold in a small neighborhood of a normal hyperbolic critical manifold is established, then a singular perturbed system is reduced to a regular perturbed system. The existence of kink wave solutions is proved by Fredholm theory and the linear chain trick.

3.1. The model without delay

According to the property of f when $\tau \to 0$, (1.5) is the non-delay. In the case $q = 1$, (1.5) reduces to

$$u_t + \alpha u_x + \beta u_x^2 + u^2 u_x - u_{xx} + u_{xxx} = 0. \tag{3.1}$$

For a given wave speed $c > 0$, substituting $u(x,t) = u(x+ct) = \phi(\xi)$ into (3.1), the following traveling wave equation is obtained

$$(c+\alpha)\phi' + \beta(\phi^2)' + \phi^2\phi' - \phi'' + \phi''' = 0, \tag{3.2}$$

where $\prime = \frac{d}{d\xi}$. Integrating (3.2) once and neglecting the integration constant, it can be simplified to

$$(c+\alpha)\phi + \beta\phi^2 + \frac{\phi^3}{3} - \phi' + \phi'' = 0, \tag{3.3}$$

which is equivalent to a two-dimensional first-order system

$$\begin{cases} \phi' = y, \\ y' = -(c+\alpha)\phi - \beta\phi^2 - \dfrac{\phi^3}{3} + y. \end{cases} \tag{3.4}$$

Clearly, (3.4) is a non-Hamiltonian system. Assume that $0 < \beta < \frac{\sqrt{3}}{3}$ and $0 < c + \alpha < \frac{3\beta^2}{4}$. Denote $\Delta := \beta^2 - \frac{4(c+\alpha)}{3}$. When $\Delta > 0$, it is easy to find that (3.4) has three equilibria, $E_0(0,0)$, $E_1(\frac{3}{2}(-\beta + \sqrt{\Delta}), 0)$ and $E_2(\frac{3}{2}(-\beta - \sqrt{\Delta}), 0)$. E_0 is an unstable node and E_1 and E_2 are saddles. Now the existence of heteroclinic orbits between E_0 and E_1 is discussed. For a suitable value $\delta > 0$, there is a negative invariant triangular set

$$D := \{(\phi, y) : 0 \le \phi \le \frac{3(-\beta + \sqrt{\Delta})}{2}, 0 \le y \le \delta\phi\}.$$

Let $\overrightarrow{m}$ be the vector defined by the righthand side of (3.4) and $\overrightarrow{n} = (-\delta, 1)$ be the outward normal vector on the boundary of D. On the side of $y = \delta\phi$, we have

$$\begin{aligned} \overrightarrow{m} \cdot \overrightarrow{n} &= \left(y, -(c+\alpha)\phi - \beta\phi^2 - \frac{\phi^3}{3} + y\right) \cdot (-\delta, 1) \mid_{(\phi,\delta\phi)} \\ &= -\delta^2\phi - (c+\alpha)\phi - \beta\phi^2 - \frac{\phi^3}{3} + \delta\phi \\ &\le \phi(-\delta^2 + \delta - (c+\alpha)). \end{aligned} \tag{3.5}$$

It is clear that $-\delta^2 + \delta - (c+\alpha) = 0$ has two positive roots, $\delta_1 = \frac{1-\sqrt{1-4(c+\alpha)}}{2}$ and $\delta_2 = \frac{1+\sqrt{1-4(c+\alpha)}}{2}$. Therefore, when choosing $\delta \le \delta_1$ or $\delta \ge \delta_2$, it has $\overrightarrow{m} \cdot \overrightarrow{n} \le 0$. Thus, one branch of the unstable manifold at $E_0(0,0)$ always stays in the region D and joins the saddle $E_1(\frac{3(-\beta+\sqrt{\Delta})}{2}, 0)$, which deduces the desired heteroclinic orbit that exists.

Similarly, the existence of a heteroclinic orbit between E_0 and E_2 can be proved. Therefore, from the relation between heteroclinic orbit and kink wave solution, the following statement holds.

Theorem 3.1. *When* $0 < \beta < \frac{\sqrt{3}}{3}$ *and* $0 < c + \alpha < \frac{3\beta^2}{4}$*, there is a heteroclinic orbit connecting the unstable node* $E_0(0, 0)$ *to the saddle* $E_1(\frac{3(-\beta+\sqrt{\Delta})}{2}, 0)$ *for* (3.4)*. There is another heteroclinic orbit connecting the unstable node* $E_0(0, 0)$ *to the saddle* $E_2(\frac{3(-\beta-\sqrt{\Delta})}{2}, 0)$*. Further, there are two kink wave solutions,* $u_1(x + ct) = \phi_1(\xi)$ *and* $u_2(x + ct) = \phi_2(\xi)$*, which satisfy that* $\phi_1(-\infty) = 0, \phi_1(+\infty) = \frac{3(-\beta+\sqrt{\Delta})}{2}$ *and* $\phi_2(-\infty) = 0, \phi_2(+\infty) = \frac{3(-\beta-\sqrt{\Delta})}{2}$ *with c as the wave speed.*

3.2. *The model* (1.5) *in the case* $q = 1$ *with nonlocal delay*

From Section 3.1, when $0 < \beta < \frac{\sqrt{3}}{3}$ and $0 < c+\alpha < \frac{3\beta^2}{4}$, there are two heteroclinic orbits connecting the unstable node $E_0(0, 0)$ to the saddle $E_1(\frac{3(-\beta+\sqrt{\Delta})}{2}, 0)$ and connecting $E_0(0, 0)$ to $E_2(\frac{3(-\beta-\sqrt{\Delta})}{2}, 0)$, respectively, so we shall verify the heteroclinic orbit persists when the model contains the nonlocal delay. Due to the diffusion, the delay needs to be incorporated in a way that allows for associated spatial averaging. Based on the idea first introduced by Britton [32], the system is changed into a slow system. By geometric singular perturbation theory, the existence of locally invariant manifold in a small neighborhood of critical manifold is obtained, which reduces the singular perturbed system to a regular perturbed system. The existence of kink wave solutions for (1.5) is proved by the Fredholm theory and linear chain trick. The convolution $f * u$ is denoted by

$$(f * u)(x, t) = \int_{-\infty}^{t} \int_{-\infty}^{\infty} f(x - y, t - s)u(y, s)dyds.$$

The kernel function $f(x, t)$ satisfies the normalization condition

$$f : [0, +\infty) \times [0, +\infty) \to [0, +\infty) \text{ and } \int_{0}^{\infty} \int_{-\infty}^{\infty} f(x, t)dxdt = 1,$$

so that the kernel does not affect the spatially uniform steady-state. Particularly, the nonlocal weak generic delay kernel is defined as follows

$$f(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \frac{1}{\tau} e^{-\frac{t}{\tau}},$$

where the parameter $\tau > 0$ measures the average time delay. Denote that

$$\eta(x, t) = (f * u)(x, t) = \int_{-\infty}^{t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t - s)}} e^{-\frac{(x-y)^2}{4(t-s)}} \frac{1}{\tau} e^{-\frac{t-s}{\tau}} u(y, s)dyds.$$

By direct computation, we obtain

$$\eta_t = \eta_{xx} + \frac{1}{\tau}(u - \eta).$$

Thus, (1.5) in the case $q = 1$ is equivalent to a two-dimensional system as the form

$$\begin{cases} u_t + \alpha u_x + \beta(\eta u)_x + u^2 u_x - u_{xx} + u_{xxx} = 0, \\ \eta_t = \eta_{xx} + \frac{1}{\tau}(u - \eta). \end{cases} \tag{3.6}$$

To find the traveling wave solution of (3.6), the transformations $u(x,t) = \phi(\xi), \eta(x,t) = \varphi(\xi), \xi = x + ct$ are taken and we obtain a traveling wave system satisfying the boundary conditions $\phi(-\infty) = 0$, $\phi(+\infty) = \frac{3(-\beta+\sqrt{\Delta})}{2}$ and $\phi'(\pm\infty) = 0$, which is given by

$$\begin{cases}(c+\alpha)\phi' + \beta(\varphi\phi)' + \phi^2\phi' - \phi'' + \phi''' = 0,\\ c\varphi' - \varphi'' - \dfrac{1}{\tau}(\phi - \varphi) = 0,\end{cases} \tag{3.7}$$

where $' = \frac{d}{d\xi}$. Integrating the first equation of (3.7) once, it obtains

$$(c+\alpha)\phi + \beta\varphi\phi + \frac{\phi^3}{3} - \phi' + \phi'' = 0,$$

then (3.7) changes to the following second order ordinary differential equation

$$\begin{cases}(c+\alpha)\phi + \beta\varphi\phi + \dfrac{\phi^3}{3} - \phi' + \phi'' = 0,\\ c\varphi' - \varphi'' - \dfrac{1}{\tau}(\phi - \varphi) = 0.\end{cases} \tag{3.8}$$

The small parameter $\tau > 0$ represents the delay in the original system, which is regarded as the perturbed parameter. By defining new variables $\phi' = y, \varphi' = \omega$, (3.8) is reformulated as a four-dimensional system

$$\begin{cases}\phi' = y,\\ y' = -(c+\alpha)\phi - \beta\varphi\phi - \dfrac{\phi^3}{3} + y,\\ \varphi' = \omega,\\ \omega' = c\omega - \dfrac{1}{\tau}(\phi - \varphi).\end{cases} \tag{3.9}$$

Setting that $\tau = \varepsilon^2$ and defining a new variable $\mu = \varepsilon\varphi'$, (3.9) is rewritten as a four-dimensional singular perturbed system

$$\begin{cases}\phi' = y,\\ y' = -(c+\alpha)\phi - \beta\varphi\phi - \dfrac{\phi^3}{3} + y,\\ \varepsilon\varphi' = \mu,\\ \varepsilon\mu' = c\varepsilon\mu - \phi + \varphi.\end{cases} \tag{3.10}$$

Undoubtedly, (3.10) is a slow system. When $\varepsilon \to 0$, (3.9) reduces to (3.4). From Theorem 3.1, we know that (3.10) possesses a heteroclinic orbit connecting E_0 to E_1. Notice that when $\varepsilon \neq 0$, it does not define a dynamic in R^4. Therefore, by the transformation $\xi = \varepsilon z$, we change (3.10) into the form

$$\begin{cases}\dot{\phi} = \varepsilon y,\\ \dot{y} = \varepsilon\left(-(c+\alpha)\phi - \beta\varphi\phi - \dfrac{\phi^3}{3} + y\right),\\ \dot{\varphi} = \mu,\\ \dot{\mu} = \varepsilon c\mu - \phi + \varphi,\end{cases} \tag{3.11}$$

where $\dot{}$ is the derivative respect to z. System (3.11) is the fast system. Systems (3.10) and (3.11) are equivalent when $\varepsilon > 0$. When $\varepsilon = 0$, the slow system defines a set

$$M_0 = \left\{(\phi, y, \varphi, \mu) \in R^4 : \mu = 0, \varphi = \phi\right\},$$

which is an invariant manifold of (3.10) with $\varepsilon = 0$. Since the linearized matrix of (3.11) restricted to M_0 is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix},$$

it is easy to obtain that the eigenvalues are 0, 0, 1, 1, the number of the eigenvalues with a zero real part are equal to dimM_0 and the other eigenvalues are hyperbolic. Thus, the slow manifold M_0 is normally hyperbolic. From geometric singular perturbation theory presented in section two, for sufficiently small $\varepsilon > 0$, there exists a locally invariant manifold M_ε in a small neighborhood of M_0 of the perturbed system (3.10), which is expressed as

$$M_\varepsilon = \left\{(\phi, y, \varphi, \mu) \in R^4 : \mu = g(\phi, y, \varepsilon), \varphi = \phi + h(\phi, y, \varepsilon)\right\},$$

where $g(\phi, y, \varepsilon), h(\phi, y, \varepsilon)$ are smooth functions and satisfy $g(\phi, y, 0) = 0$, $h(\phi, y, 0) = 0$. Thus the functions $g(\phi, y, \varepsilon)$ and $h(\phi, y, \varepsilon)$ can be expanded into a Taylor series as follows

$$\begin{aligned} g(\phi, y, \varepsilon) &= \varepsilon g_1(\phi, y) + \varepsilon^2 g_2(\phi, y) + O(\varepsilon^3), \\ h(\phi, y, \varepsilon) &= \varepsilon h_1(\phi, y) + \varepsilon^2 h_2(\phi, y) + O(\varepsilon^3). \end{aligned}$$

Substituting $\varphi = \phi + h(\phi, y, \varepsilon)$, $\mu = g(\phi, y, \varepsilon)$ into the slow system (3.10), we have

$$\begin{aligned} &c\varepsilon\left\{\frac{\partial g_1}{\partial \phi} y + \frac{\partial g_2}{\partial y}\left(-(c+\alpha)\phi - \beta\varphi\phi - \frac{\phi^3}{3} + y\right)\right\} + O(\varepsilon^3) = c\varepsilon^2 g_1 + \varepsilon h_1 + \varepsilon^2 h_2 + O(\varepsilon^3), \\ &c\varepsilon\left\{y + \varepsilon\left(\frac{\partial h_1}{\partial \phi} y + \frac{\partial h_1}{\partial y}\left(-(c+\alpha)\phi - \beta\varphi\phi - \frac{\phi^3}{3} + y\right)\right)\right\} + O(\varepsilon^3) = \varepsilon g_1 + \varepsilon^2 g_2 + +O(\varepsilon^3). \end{aligned}$$

By comparing coefficients of ε and ε^2, we obtain

$$g_1(\phi, y) = y,\ g_2(\phi, y) = 0,\ h_1(\phi, y) = 0,\ h_2(\phi, y) = -(c+\alpha)\phi - \beta\phi^2 - \frac{\phi^3}{3} - (c-1)y.$$

Thus, the dynamics of (3.10) on M_ε is determined by the following regular perturbed system

$$\begin{cases} \phi' = y, \\ y' = -(c+\alpha)\phi - \beta\phi^2 - \dfrac{\phi^3}{3} + y + \varepsilon^2 K(\phi, y) + O(\varepsilon^3), \end{cases} \tag{3.12}$$

where $K(\phi, y) = \beta(c+\alpha)\phi^2 + \beta^2\phi^3 + \frac{\beta\phi^4}{3} + (c-1)\beta\phi y$. Clearly, when $\varepsilon = 0$, (3.12) reduces to (3.4). Denote the equilibria of (3.12) are $E_{\varepsilon 0}$, $E_{\varepsilon 1}$ and $E_{\varepsilon 2}$, which lying in a small neighborhood of E_0, E_1 and E_2, respectively. In order to prove the existence of kink wave solutions of (1.5), we aim to establish the two heteroclinic orbits connecting $E_{\varepsilon 0}$ to $E_{\varepsilon 1}$, and another connecting $E_{\varepsilon 0}$ to $E_{\varepsilon 2}$, respectively. From Lemma 2.1, we know that such two heteroclinic orbits exist when $\varepsilon = 0$.

Let (ϕ, y) and (u_0, v_0) be the solutions of (3.12) and (3.4), respectively. For $\varepsilon > 0$, note that

$$\phi = u_0 + \varepsilon^2 u_1 + O(\varepsilon^3), \quad y = v_0 + \varepsilon^2 v_1 + O(\varepsilon^3). \tag{3.13}$$

Substitute ϕ and y in (3.13) into (3.12) and compare the coefficients of ε and ε^2, then u_1 and v_1 satisfy the following differential equation system

$$\frac{d}{d\xi}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ c+\alpha+2\beta u_0 + u_0^2 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \beta(c+\alpha)u_0^2 + \beta^2 u_0^3 + \frac{\beta u_0^4}{3} \end{pmatrix}. \tag{3.14}$$

Notice that our goal is finding the traveling wave solution satisfying (3.14) and $u_1(\pm\infty) = 0$, $v_1(\pm\infty) = 0$. Denote L^2 as the space of square integrable functions with inner production, that is

$$\langle u_1(\xi), v_1(\xi)\rangle = \int_{-\infty}^{+\infty} (u_1(\xi), v_1(\xi)) d\xi,$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product on R^2. From the Fredholm theory, 3.14 has a solution if and only if the following integral equation is satisfied

$$\int_{-\infty}^{+\infty}\left(u_1(\xi), \begin{pmatrix} 0 \\ \beta(c+\alpha)u_0^2 + \beta^2 u_0^3 + \frac{\beta u_0^4}{3} \end{pmatrix}\right) d\xi = 0,$$

for all functions $u_1(\xi)$ in the kernel of the adjoint of operator L defined by the left-hand side of (3.14). Denote L^* as the adjoint of operator L, then

$$L^* = -\frac{d}{d\xi} + \begin{pmatrix} 0 & c+\alpha+2\beta u_0 + u_0^2 \\ -1 & -1 \end{pmatrix}.$$

Implying that for all $u_1(\xi) \in KerL^*$, it has

$$\frac{du_1(\xi)}{d\xi} = \begin{pmatrix} 0 & c+\alpha+2\beta u_0 + u_0^2 \\ -1 & -1 \end{pmatrix} u_1(\xi). \tag{3.15}$$

Since the matrix in (3.15) is a variable coefficient matrix, the general solution is difficult to derive. Therefore, we aim to prove that only the zero solution satisfies $u_0(\pm\infty) = 0$ and we deduce the existence of homoclinic orbit. Even if the exact expression can not be found, $u_0(\xi)$ is a solution for the unperturbed system and satisfies the boundary condition $u_0(-\infty) = 0$. Thus on the limit status $\xi \to -\infty$, the matrix in (3.15) approaches to a constant coefficient matrix

$$\begin{pmatrix} 0 & c+\alpha \\ -1 & -1 \end{pmatrix}.$$

Clearly, the corresponding eigenvalues are determined by $\lambda^2 + \lambda + c + \alpha = 0$. Since $0 < c + \alpha < \frac{1}{4}$, there are two real negative eigenvalues $\lambda_{1,2} = \frac{-1\pm\sqrt{1-4(c+\alpha)}}{2} < 0$. Hence, when $\xi \to -\infty$, the solution of (3.15) must be decreasing exponentially with respect to ξ, except for the zero solution. Therefore, the solution satisfying $u_1(\pm\infty) = 0$ must be a zero solution, then the Fredholm orthogonality condition holds trivially, implying that such solutions of (3.15) exist and satisfy $\phi(-\infty) = 0$ and $y(\pm\infty) = 0$. Consequently, we conclude that for sufficiently small $\varepsilon > 0$, there exists two heteroclinic orbits of (3.15): One connects $E_{\varepsilon 0}$ to $E_{\varepsilon 1}$, and the other connects $E_{\varepsilon 0}$ to $E_{\varepsilon 2}$.

Theorem 3.2. *In the case* $q = 1$*, when* $0 < \beta < \frac{\sqrt{3}}{3}$ *and* $0 < c + \alpha < \frac{3\beta^2}{4}$*, for* $\tau > 0$ *is sufficiently small, the delayed convecting shallow water fluid model* (1.5) *with the nonlocal weak generic kernel*

$$(f * u)(x,t) = \int_{-\infty}^{t}\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{(x-y)^2}{4(t-s)}} \frac{1}{\tau} e^{-\frac{t-s}{\tau}} u(y,s)dyds$$

possesses two kink wave solutions $u_1(x,t) = \phi_1(\xi)$ *and* $u_2(x,t) = \phi_2(\xi)$*, where* $\phi_{1,2}(\xi)$ *satisfy* $\phi_1(-\infty) = 0$*,* $\phi_1(+\infty) = \frac{3(-\beta+\sqrt{\Delta})}{2}$ *and* $\phi_2(-\infty) = 0$*,* $\phi_2(+\infty) = \frac{3(-\beta-\sqrt{\Delta})}{2}$*. Here,* c *is the wave speed.*

Remark 3.1. *In the previous references [20, 24], only one heteroclinic orbit was obtained. In our results, two heteroclinic orbits are proved under certain parametric conditions since there are three equilibria for the system.*

4. The model (1.5) in the case $q = 0$ with local delay

In this section, we consider the traveling wave solution for Eq (1.5) in the case $q = 0$ with local delay, that is, $f(t) = \frac{t}{\tau^2} e^{-\frac{t}{\tau}}$, $t \in [0, +\infty)$. Similar to the case $q = 1$, making a traveling wave transformation $\xi = x + ct$ to (1.5) in the case $q = 0$ and integrating once, we obtain the traveling wave system

$$(c+\alpha)\phi + \beta\phi\omega + \frac{\phi^3}{3} + \phi'' + \tau\phi' = 0, \tag{4.1}$$

where

$$\omega = \int_0^{+\infty} \frac{s}{\tau^2} e^{-\frac{s}{\tau}} \phi(\xi - cs)ds.$$

By direct calculation, we obtain that

$$\frac{d\omega}{d\xi} = \frac{1}{c\tau}(\zeta - \omega), \quad \frac{d\zeta}{d\xi} = \frac{1}{c\tau}(\phi - \zeta), \tag{4.2}$$

where

$$\zeta = \int_0^{+\infty} \frac{1}{\tau} e^{-\frac{s}{\tau}} \phi(\xi - cs)ds.$$

Introducing new variable $\phi' = y$ and combining with (4.2), (4.1) is changed to the following four-dimensional system

$$\begin{cases} \phi' = y, \\ y' = -\left((c+\alpha)\phi + \beta\phi\omega + \dfrac{\phi^3}{3} + \tau y\right), \\ c\tau\omega' = \zeta - \omega, \\ c\tau\zeta' = \phi - \zeta, \end{cases} \tag{4.3}$$

where $'$ is derivative respect to ξ. System (4.3) is the slow system. When $\tau \neq 0$, a time scale transformation $\xi = \tau s$ is considered to change the slow system (4.3) into a fast system

$$\begin{cases} \dot{\phi} = \tau y, \\ \dot{y} = -\tau\left((c+\alpha)\phi + \beta\phi\omega + \dfrac{\phi^3}{3} + \tau y\right), \\ c\dot{\omega} = \zeta - \omega, \\ c\dot{\zeta} = \phi - \zeta, \end{cases} \tag{4.4}$$

where $\dot{}$ is derivative respect to s. Systems (4.3) and (4.4) are equivalent when $\tau > 0$. The two different time scales correspond to two different limiting systems. When $\tau \to 0$, (4.4) tends to the layer system

$$\begin{cases} \dot{\phi} = 0, \\ \dot{y} = 0, \\ c\dot{\omega} = \zeta - \omega, \\ c\dot{\zeta} = \phi - \zeta, \end{cases} \tag{4.5}$$

and (4.3) tends to the reduced system

$$\begin{cases} \phi' = y, \\ y' = -\left((c+\alpha)\phi + \beta\phi\omega + \dfrac{\phi^3}{3} + \tau y\right), \\ 0 = \zeta - \omega, \\ 0 = \phi - \zeta. \end{cases} \tag{4.6}$$

Similarly, the critical manifold is given by

$$M_0 = \{(\phi, y, \omega, \varphi) \in \mathbb{R}^4 : \omega = \phi, \zeta = \phi\},$$

which is a slow invariant manifold. The linearized matrix of (4.5) is given as the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{c} & \frac{1}{c} \\ \frac{1}{c} & 0 & 0 & -\frac{1}{c} \end{pmatrix}.$$

It is not difficult to verify that the number of the eigenvalues with zero real part equals to $dimM_0$ and the other eigenvalues are hyperbolic, then M_0 is normally hyperbolic. Similarly, there exists a manifold M_τ for (4.3) with sufficiently small $\tau > 0$, which is locally invariant and diffeomorphic to M_0 under the flow of (4.3). Then, M_τ can be expressed by

$$M_\tau = \{(\phi, y, \omega, \zeta) \in \mathbb{R}^4 : \omega = \phi + k(\phi, y, \tau), \zeta = \phi + l(\phi, y, \tau)\},$$

where $k(\phi, y, \tau), l(\phi, y, \tau)$ are smooth functions and satisfy $k(\phi, y, 0) = 0$, $l(\phi, y, 0) = 0$. Thus $k(\phi, y, \tau), l(\phi, y, \tau)$ can be expanded into Taylor series

$$k(\phi, y, \tau) = \tau k_1(\phi, y) + O(\tau^2),\ l(\phi, y, \tau) = \tau l_1(\phi, y) + O(\tau^2).$$

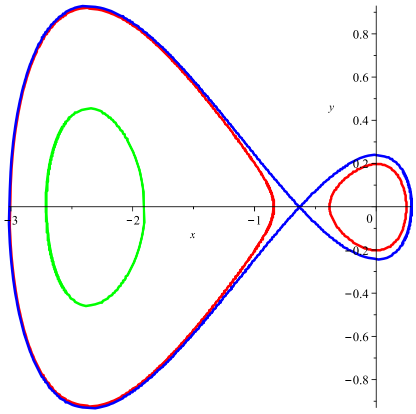

Figure 1. Phase portrait of (4.3) with $c+\alpha>0, \beta^2>\frac{4}{3}(c+\alpha)$.

Substituting $\omega=\phi+k(\phi,y,\tau)$, $\zeta=\phi+l(\phi,y,\tau)$ into the last equation of slow system (4.3), we have

$$c\tau(y+O(\tau))=\tau(l_1-k_1)+O(\tau^2),\ \ c\tau(y+O(\tau))=-\tau l_1+O(\tau^2).$$

By comparing the coefficients of τ, we get $k_1(\phi,y)=-2cy,\ l_1(\phi,y)=-cy$. Thus, the slow system (4.3) restricted on M_τ reduces into a regular perturbed system

$$\begin{cases}\phi'=y,\\ y'=-(c+\alpha)\phi-\beta\phi^2-\dfrac{\phi^3}{3}+\tau(2\beta c\phi y-y)+O(\tau^2),\end{cases}\tag{4.7}$$

which is a near-Hamiltonian system. When $\tau\to 0$, (4.7) reduces to a Hamiltonian system

$$\begin{cases}\phi'=y,\\ y'=-(c+\alpha)\phi-\beta\phi^2-\dfrac{\phi^3}{3},\end{cases}\tag{4.8}$$

with the Hamiltonian function is given by

$$H(\phi,y)=\frac{y^2}{2}+\frac{c+\alpha}{2}\phi^2+\frac{\beta}{3}\phi^3+\frac{1}{12}\phi^4.\tag{4.9}$$

Notice that $\Delta=\beta^2-\frac{4(c+\alpha)}{3}$. When $\Delta>0$, there are three equilibria $E_0(0,0)$, $E_1(\frac{3}{2}(-\beta+\sqrt{\Delta}),0)$ and $E_2(\frac{3}{2}(-\beta-\sqrt{\Delta}),0)$. When $c+\alpha>0$, E_0 is a center, E_1 and E_2 are saddles. The corresponding energy function values are $H(0,0)=0$, $h_1:=H(\frac{3}{2}(-\beta+\sqrt{\Delta}),0)$ and $h_2:=H(\frac{3}{2}(-\beta-\sqrt{\Delta}),0)$, respectively. Since we discuss the traveling wave for two models under the same parametric condition from the analysis in Section 3.1, we do not consider the case $\Delta=0$. With the help of the energy function $H(\phi,y)$ on parametric conditions $c+\alpha>0, \beta^2>\frac{4}{3}(c+\alpha)$, we give the phase portrait of (4.8) in Figure 1.

Suppose that there exists a closed orbit Γ_h of (4.8) surrounding E_0. $A(h)\in\Gamma_h$ is the rightmost point on the positive ϕ-axis. For $0<|h_\tau-h|\ll 1$, let Γ_{h_τ} de a piece of the orbit of the perturbed (4.7) starting from $A(h)$ to the next intersection point $B(h_\tau)$ with the positive ϕ-axis. Then, the displacement function [33] is given by

$$d(h,\tau)=\int_{\widehat{AB}}dH=\tau(M(h)+O(\tau)),$$

where

$$\begin{aligned} M(h) &= \oint_{\Gamma_h} (2\beta c\phi y - y)d\phi \\ &= 2\beta c J_1(h) - J_0(h) \\ &= 2\beta c J_0(h)\left(\frac{J_1(h)}{J_0(h)} - \frac{1}{2\beta c}\right), \end{aligned}$$

which is called the Melnikov function with $J_1(h) = \oint_{\Gamma_h} \phi y d\phi$ and $J_0(h) = \oint_{\Gamma_h} y d\phi = \iint_{int\Gamma_h} d\phi dy > 0$. We shall show that the Abelian integral ratio $P(h) := \frac{J_1(h)}{J_0(h)}$ is strictly monotonic with respect to h, and further prove there exists a unique periodic wave solution for (1.5) in the case $q = 0$. The following lemma provides a simple criterion to verify the monotonic of $P(h)$.

Lemma 4.1. *([34]) Assume that the Hamiltonian function $H(\phi, y)$ can be written as $\frac{y^2}{2} + \Phi(\phi)$, satisfying*

$$\Phi'(\phi)(\phi - a) > 0, \ for\ \phi \in (\gamma, A),$$

then $U'(h) > 0$ (or $U'(h) < 0$) in (h_1, h_2) implies $P'(h) > 0$ (or $P'(h) < 0$) in (h_1, h_2). Here,

$$U(h) := \mu(h) + \nu(h), P(h) := \frac{\oint_{\Gamma_h} \phi y d\phi}{\oint_{\Gamma_h} y d\phi},$$

$\mu(h)$ and $\nu(h)$ are the inverse functions of the corresponding maps Φ: $(\gamma, a) \mapsto (h_1, h_2)$ and $(a, A) \mapsto (h_1, h_2)$, then it has $\gamma < \mu(h) < a < \nu(h) < A$ and

$$\Phi(\mu(h)) \equiv \Phi(\nu(h)) \equiv h, \ h_1 < h < h_2.$$

For (1.5) in the case $q = 0$, we have the following results.

Theorem 4.1. *In the case $q = 0$, for any sufficient small $\tau > 0$, there exist some suitable c, α, β that satisfy $0 < \beta < \frac{\sqrt{3}}{3}$ and $0 < c + \alpha < \frac{3\beta^2}{4}$, such that (1.5) has two isolated periodic wave solutions locatd at two sides of $\phi = \frac{3}{2}(-\beta + \sqrt{\beta^2 - \frac{4(c+\alpha)}{3}})$ with $c > 0$ as the wave speed.*

Proof. According to the previous analysis, we discuss existence of periodic orbit near the family of closed orbits surrounding E_0. Existence of periodic orbit near the family of closed orbits surrounding E_2 can be proved similarly. Let $\Gamma_h := \{(\phi, y) : H(\phi, y) = h\}$, which corresponds closed orbits of (4.8) for each $h \in (0, h_1)$ and bounded in a homoclinic loop connecting the saddle point E_1. Then, $\Phi(\phi) := \frac{c+\alpha}{2}\phi^2 + \frac{\beta}{3}\phi^3 + \frac{1}{12}\phi^4$ is analytic in the interval $(0, A)$ and satisfying that $\Phi(0) = \Phi(A)$, where A is the rightmost intersection point between the homoclinic loop and positive ϕ-axis. For $c + \alpha > 0$ and $\frac{3}{2}(-\beta + \sqrt{\Delta}) < \phi < A$, it has

$$\Phi'(\phi)\phi = \frac{\phi^2}{3}\left(3(c + \alpha) + 3\beta\phi + \phi^2\right) > 0$$

implying that $\Phi(\phi)$ has a minimum at $\phi = 0$ and is strictly monotonic on $(\frac{3}{2}(-\beta + \sqrt{\Delta}), 0)$ and $(0, A)$, respectively. Let $\mu(h)$ and $\nu(h)$ be inverse functions of $\Phi(\phi)$ on these two intervals, respectively, and $\frac{3}{2}(-\beta + \sqrt{\Delta}) < \mu(h) < 0 < \nu(h) < A$. Define two functions

$$w(h) := \frac{\mu(h) + \nu(h)}{2}, \ z(h) := \frac{\nu(h) - \mu(h)}{2}.$$

Then, the criterion function in $s \in [0, z(h)]$ is given by

$$G(s) := \Phi(w(h) + s) - \Phi(w(h) - s) = \frac{2s}{3}((\beta + w)s^2 + 3\beta w^2 + w^3 + 3(c + \alpha)w).$$

Since 0, $z(h)$ and $-z(h)$ are the real roots of $G(s)$, we can rewrite $G(s)$ as

$$G(s) = \frac{2s(\beta + w)}{3}(s^2 - z(h)^2) < 0$$

for $s \in (0, z(h))$.

On the following, we prove $U(h)$ is monotonic for $h \in (0, h_1)$ by contradiction argument. Assume that there exists $\tilde{h}$ and $\bar{h}$ in $(0, h_1)$, $\tilde{h} < \bar{h}$, such that $U(\tilde{h}) = U(\bar{h})$, then it has $w(\tilde{h}) = w(\bar{h})$ and $z(\tilde{h}) < z(\bar{h})$. Setting that $h = \bar{h}$, it yields

$$G(s) = \Phi(w(\bar{h}) + s) - \Phi(w(\bar{h}) - s) < 0, \;\; s \in (0, z(h)).$$

Letting $s = z(\tilde{h})$ and $h = \bar{h}$ in $G(s)$, we have

$$\begin{aligned} G(s) &= \Phi(w(\bar{h}) + z(\tilde{h})) - \Phi(w(\bar{h}) - z(\tilde{h})) \\ &= \Phi(w(\tilde{h}) + z(\tilde{h})) - \Phi(w(\tilde{h}) - z(\tilde{h})) \\ &= \Phi(\mu(\tilde{h})) - \Phi(\nu(\tilde{h})) = 0, \end{aligned}$$

which contradicts to $G(s) < 0$ for all $s \in (0, z(h))$. Therefore, $U(h)$ is strictly monotonic for $h \in (0, h_1)$. From Lemma 4.1, it has $P(h)$ as strictly monotonic, which means there exists at most one $h^* \in (0, h_1)$ such that $M(h^*) = 0$ and $M'(h^*) \neq 0$. By the implicit function theorem for sufficiently small $\tau > 0$, there exists at most one $h = h^* + O(\tau)$ such that $d(h, \tau) = 0$, then there exists at most one periodic wave for (1.5) in the case $q = 0$.

Similarly, the existence of a unique periodic waves near the family of closed orbits surrounding E_2 can be proved. The proof of Theorem 4.1 is completed.

5. Conclusions

This paper mainly discussed a convecting a shallow water fluid model in two cases with different generic delay kernels under ceratin parametric conditions. The existence of traveling waves for the model were given by different techniques. By applying the geometric singular perturbation theory, the existence of locally invariant manifold in a small neighborhood of critical manifold was obtained and the desired orbit was established. According to the relationship between traveling wave solution and orbit on a phase plane of the associated ordinary differential equation, the existence of traveling wave solution was proved. For the model in the case $q = 1$ with nonlocal weak delay kernel, the heteroclinic orbit was established by the Fredholm theory and linear chain trick, which was an effective method to deal with physical models of delay. If the nonlocal weak delay kernel in presented paper was replaced by another delay kernel, the Fredholm theory and linear chain trick was also valid to establish the desired orbits for the corresponding traveling wave system. For the case $q = 0$, (1.5) contained a local strong delay convolution kernel and a weak backward diffusion effect. It can be reduced to a near-Hamiltonian system, then to the existence of periodic wave solutions by investigating the monotonicity of ratio of two Abelian integrals in the Melnikov function. It is worth pointing out that no literature has considered both near-Hamiltonian and non-near-Hamiltonian cases of a delayed model. Consequently, it is an interesting work to be further researched in the future.

Use of AI tools declaration

The author declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material, and further inquiries can be directed to the corresponding author.

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