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Research article

The Riccati-Bernoulli subsidiary ordinary differential equation method to the coupled Higgs field equation

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Abstract: By using the Riccati-Bernoulli (RB) subsidiary ordinary differential equation method, we proposed to solve kink-type envelope solitary solutions, periodical wave solutions and exact traveling wave solutions for the coupled Higgs field (CHF) equation. We get many solutions by applying the Bäcklund transformations of the CHF equation. The proposed method is simple and efficient. In fact, we can deal with some other classes of nonlinear partial differential equations (NLPDEs) in this manner.

Keywords: RB method; CHF equation; Bäcklund transformation; traveling wave solution; NLPDEs

1. Introduction

As we all know, nonlinear partial differential equations (NLPDEs) can describe various phenomena in physics [1–3], biology [4], chemistry [5,6] and finance [7], as well as several other fields [8–10]. The study of exact solutions for NLPDEs plays a significant role in the research of nonlinear physical phenomena. In the recent decades, a good many of valuable approaches was used to obtain exact wave solutions of NLPDEs, such as the inverse scattering method [11,12], iterative technique [13–16], test function method [17,18], Bäcklund transformation method [2,19,20], the sub-equation method [21,22], extended F-expansion method [22,23], Darboux transformation method [24], Hirota's bilinear method [25–28], the homogeneous balance method [29–31], the (G'/G)-expansion method [32,33], first integral method [34,35], tanh-sech method [36,37], extended homoclinic test method [38,39], Jacobi elliptic function method [40,41] and the Riccati-Bernoulli (RB) subsidiary ordinary differential equation method [42–45]. On the other hand, to obtain further information about natural phenomena, some analytical techniques and methods have also been developed for solving diverse differential equations, such as the fixed point theorem [9,46], upper and lower method [3,13,47] and dual approach [48]. In addition, due to the ability to better explain natural phenomena, many researchers have recently become interested in fractional order partial differential equations [1,9,13]. In this paper, we study the following the coupled Higgs field (CHF) equation [49]

$$u_{tt} - u_{xx} - \alpha u + \beta |u|^2 u - 2uv = 0, \tag{1a}$$

$$v_{tt} + v_{xx} - \beta\left(|u|^2\right)_{xx} = 0, \tag{1b}$$

where $u_{tt} = \frac{\partial^2 u}{\partial t^2}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, $v_{tt} = \frac{\partial^2 v}{\partial t^2}$, $v_{xx} = \frac{\partial^2 v}{\partial x^2}$, $\alpha > 0$ and $\beta > 0$ are known constants and $|u|$ denotes the modulus of the u.

Equation (1) is a coupled NLPDE, which describe the interactions between conserved scalar nucleons and neutral scalar mesons. Here, $v(x,t)$ stands for a complex scalar nucleon field and $u(x,t)$ stands for a real scalar meson field. For $\alpha < 0, \beta < 0$, Eq (1) is called the coupled nonlinear Klein-Gordon equation. Hu et al. constructed analytic expressions of homoclinic orbits for Eq (1) by the Hirota's bilinear method [50]. Based on the first integral method [51], Taghizadeh et al. obtained exact solutions for Eq (1), and by applying an algebraic method [52], Hon et al. obtained exact solitary wave solutions for Eq (1).

Yang et al. first proposed the RB method to obtain the exact solutions of complex NLPDEs [42]. This method provided efficient and simple math tools for solving some NLPDEs in mathematical physics. We choose the RB method to solve Eq (1). This paper is organized as follows: Section two briefly describes the RB method. In section three, we apply the RB method to Eq (1). In section four, we give the Bäcklund transformations of Eq (1). Finally, some available conclusions are obtained and summarized in section five.

2. Description of the RB method

Next, we consider the following NLPDE

$$F(\Omega, \Omega_x, \Omega_t, \Omega_{xx}, \Omega_{xt}, \cdots) = 0, \tag{2}$$

where F is usually a polynomial function, $\Omega(x,t)$ is assumed to be a solution of Eq (2) and the subscripts denote the partial derivatives. Below, the main steps for the RB method are provided.

Step 1. Introduce a new variable η as follows:

$$\eta = \mu(x + \lambda t), \tag{3}$$

where μ is a constant and λ stands for speed of localized wave. Then, $\Omega(x,t)$ is transformed into univariate functions

$$\Omega(x,t) = \Omega(\eta). \tag{4}$$

We can convert Eq (2) to an ordinary differential equation by using Eqs (3) and (4)

$$F(\Omega, \Omega', \Omega'', \Omega''', \cdots) = 0, \tag{5}$$

where Ω' denotes $\frac{d\Omega}{d\eta}$.

Step 2. Supposing Eq (5) is a solution of the following equation

$$\Omega' = a\Omega^{2-n} + b\Omega + c\Omega^n, \tag{6}$$

where a, b, c and n are constants that can be determined subsequently, taking the derivative of η on both sides of the Eq (6) , we get

$$\Omega'' = \left(a(2-n)\Omega^{1-n} + cn\Omega^{n-1} + b\right)\Omega', \tag{7}$$

$$\Omega''' = \begin{pmatrix} a^2(n-2)(2n-3)\Omega^{2-2n} + ab(n-3)(n-2)\Omega^{1-n} + \\ n(2n-1)c^2\Omega^{2n-2} + bcn(n+1)\Omega^{n-1} + \left(2ac+b^2\right) \end{pmatrix}\Omega'. \tag{8}$$

Remark 1. To avoid introducing new terminology, Eq (6) is called the RB equation. Clearly, if $n = 0$ and $ac \neq 0$, Eq (6) reduces to the Riccati equation. If $n \neq 1$, $a \neq 0$ and $c = 0$, Eq (6) reduces to the Bernoulli equation. Thus, Eq (6) includes the Riccati equation and the Bernoulli equation.

We present solutions of Eq (6) as follows:

Case 1. If $n = 1$, Eq (6) has the following solution

$$\Omega(\eta) = C_1 e^{\eta}, \tag{9}$$

where $C_1 = Ce^{(a+b+c)}$ and C is an arbitrary constant.

Case 2. If $n \neq 1$ and $a = b = 0$, Eq (6) has the following solution

$$\Omega(\eta) = (c(1-n)(\eta + C))^{\frac{1}{1-n}}. \tag{10}$$

Case 3. If $n \neq 1$, $a = 0$ and $b \neq 0$, Eq (6) has the following solution

$$\Omega(\eta) = \left(-\frac{c}{b} + Ce^{-b(n-1)\eta}\right)^{\frac{1}{1-n}}. \tag{11}$$

Case 4. If $n \neq 1$, $a \neq 0$ and $\Delta < 0$, Eq (6) has the following solution

$$\Omega(\eta) = \left(\frac{-b + \sqrt{-\Delta}\tan\left(\frac{(1-n)\sqrt{-\Delta}}{2}(\eta + C)\right)}{2a}\right)^{\frac{1}{1-n}}, \tag{12}$$

where

$$\Delta = b^2 - 4ac. \tag{13}$$

Case 5. If $n \neq 1$, $a \neq 0$ and $\Delta > 0$, Eq (6) has the following solution

$$\Omega(\eta) = \left(\frac{\sqrt{\Delta}}{a\left(1 - Ce^{\eta(1-n)\sqrt{\Delta}}\right)} - \frac{b + \sqrt{\Delta}}{2a}\right)^{\frac{1}{1-n}}. \tag{14}$$

Case 6. If $n \neq 1$, $a \neq 0$ and $\Delta = 0$, Eq (6) has the following solution

$$\Omega(\eta) = \left(\frac{1}{a(n-1)\eta + C} - \frac{b}{2a}\right)^{\frac{1}{1-n}}. \tag{15}$$

Step 3. First, we substitute the derivatives of Ω into Eq (5) and compare each coefficient of Ω^i, and then we yield a set of algebraic equations for n, a, b, c and λ. Second we solve the algebraic equations and substitute n, a, b, c, λ and Eq (3) into Eqs (9)–(15), then we can get exact traveling wave solutions of Eq (2).

3. Solutions of the CHF equation

To verify the effectiveness of the RB method, we use it to solve Eq (1) in this section. We applied the traveling wave transformation

$$u(x,t) = h(\eta)\, e^{i(\gamma x+\delta t)}. \tag{16}$$

Equation (1) became the following equations

$$\mu^2\left(1-\lambda^2\right)h'' + 2i\mu(\gamma-\lambda\delta)h' + \left(2v-\gamma^2+\delta^2+\alpha\right)h - \beta h^3 = 0, \tag{17}$$

$$\mu^2\left(\lambda^2 v'' + v'' - \beta\left(h^2\right)''\right) = 0, \tag{18}$$

where μ, λ, γ, δ are constants that can be determined subsequently, and h, h', h'', v denote $h(\eta)$, $\frac{dh(\eta)}{d\eta}$, $\frac{d^2h(\eta)}{d\eta^2}$, $v(\eta)$, respectively.

If we take

$$\gamma = \delta\lambda, \tag{19}$$

Eq (17) becomes

$$\mu^2\left(1-\lambda^2\right)h'' + \left(-\delta^2\lambda^2+\delta^2+2v+\alpha\right)h - \beta h^3 = 0. \tag{20}$$

To avoid generating trivial solution, let $\mu \neq 0$. Integrating Eq (18) twice and setting the first integration to zero, we have

$$v = \frac{\beta h^2 + A}{1+\lambda^2}, \tag{21}$$

where A is the second integration constant. We substitute Eq (21) into (20), then we have

$$\mu^2\left(1-\lambda^4\right)h'' + \left(-\delta^2\lambda^4+\alpha\lambda^2+\delta^2+2A+\alpha\right)h + \beta\left(1-\lambda^2\right)h^3 = 0. \tag{22}$$

Case I. When $\lambda = \pm 1$ from Eq (22), setting the coefficient of h to zero, we get

$$A = -\alpha. \tag{23}$$

From Eqs (22) and (23), an arbitrary function $h = h(\eta)$ is the solution of Eq (22). According to Eqs (15), (16), (19), (21) and (22), we have exact solutions

$$u(x,t) = h(\eta)\, e^{i(\pm\delta x+\delta t)}, \tag{24a}$$

$$v(x,t) = \frac{\beta(h(\eta))^2 - \alpha}{2}, \tag{24b}$$

$$\eta = \mu(x \pm t), \tag{24c}$$

where μ, δ are arbitrary constants and $h = h(\eta)$ is an arbitrary function.

Case II. When $\lambda \neq \pm 1$, suppose Eq (22) is the solution of following equation

$$h' = ah^{2-n} + bh + ch^n, \tag{25}$$

where a, b and n are constants that are calculated subsequently.

By Eq (25), we have

$$h'' = \left(nc^2h^{2n-2} - a^2(n-2)h^{2-2n} + bc(n+1)h^{n-1} - ab(n-3)h^{1-n} + b^2 + 2ac\right)h. \tag{26}$$

Substituting Eq (26) into (22), we get

$$\begin{aligned}&\left(nc^2h^{2n-2} - a^2(n-2)h^{2-2n} + bc(n+1)h^{n-1} - ab(n-3)h^{1-n} + b^2 + 2ac\right)h\times\\&\mu^2\left(1-\lambda^4\right) + \left(-\delta^2\lambda^4 + \alpha\lambda^2 + \delta^2 + 2A + \alpha\right)h + \beta\left(1-\lambda^2\right)h^3 = 0.\end{aligned} \tag{27}$$

Setting $n = 0$, Eq (27) becomes

$$\begin{aligned}&\mu^2\left(1-\lambda^4\right)\left(3abh^2 + 2a^2h^3 + bc + \left(2ac + b^2\right)h\right) +\\&\left(-\delta^2\lambda^4 + \alpha\lambda^2 + \delta^2 + 2A + \alpha\right)h + \beta\left(1-\lambda^2\right)h^3 = 0.\end{aligned} \tag{28}$$

Setting Eq (28) and each coefficient of $h^i (i = 0, 1, 2, 3)$ to zero, we get

$$\mu^2\left(1-\lambda^4\right)bc = 0, \tag{29a}$$

$$\mu^2\left(1-\lambda^4\right)\left(b^2+2ac\right) + \left(-\delta^2\lambda^4 + \alpha\lambda^2 + \delta^2 + 2A + \alpha\right) = 0, \tag{29b}$$

$$3\mu^2\left(1-\lambda^4\right)ab = 0, \tag{29c}$$

$$2\mu^2\left(1-\lambda^4\right)a^2 + \beta\left(1-\lambda^2\right) = 0. \tag{29d}$$

Solving Eq (29), we have

$$b = 0, \tag{30a}$$

$$ac = \frac{\delta^2\lambda^4 - \alpha\lambda^2 - \delta^2 - 2A - \alpha}{2\mu^2(1-\lambda^4)}, \tag{30b}$$

$$a = \pm\frac{1}{\sqrt{2}\mu}\sqrt{\frac{-\beta}{1+\lambda^2}}. \tag{30c}$$

Case II-1. When $ac > 0$, substituting Eq (30) and $n = 0$ into Eq (11), the solution of Eq (22) is

$$\begin{aligned}h(\eta) = \pm&\sqrt{\frac{-\delta^2\lambda^4 + \alpha\lambda^2 + \delta^2 + 2A + \alpha}{\beta(1-\lambda^2)}}\times\\&\tan\left(\frac{1}{\mu}\sqrt{\frac{-\delta^2\lambda^4 + \alpha\lambda^2 + \delta^2 + 2A + \alpha}{2(\lambda^4-1)}}(\mu + B)\right),\end{aligned} \tag{31}$$

where $\lambda(\lambda \neq \pm 1)$, B, μ, δ and A are arbitrary constants. Then, we have

$$\begin{aligned}u(x,t) = \pm&\sqrt{\frac{-\delta^2\lambda^4 + \alpha\lambda^2 + \delta^2 + 2A + \alpha}{\beta(1-\lambda^2)}}\times\\&\tan\left(\sqrt{\frac{-\delta^2\lambda^4 + \alpha\lambda^2 + \delta^2 + 2A + \alpha}{2(\lambda^4-1)}}(x + \lambda t + B)\right)e^{i\delta(\lambda x + t)},\end{aligned} \tag{32a}$$

$$v(x,t)=\frac{-\delta^2\lambda^4+\alpha\lambda^2+\delta^2+2A+\alpha}{1-\lambda^4}\times$$
$$\tan^2\left(\sqrt{\frac{-\delta^2\lambda^4+\alpha\lambda^2+\delta^2+2A+\alpha}{2(\lambda^4-1)}}(x+\lambda t+B)\right)e^{2i\delta(\lambda x+t)}+\frac{A}{1+\lambda^2}, \tag{32b}$$

where $\lambda(\lambda\neq\pm1)$, A, B and δ are arbitrary constants.

Case II-2. If $ac<0$, substituting Eq (30) and $n=0$ into Eq (12), the solution of Eq (22) is

$$h(\eta)=\pm\sqrt{\frac{-\delta^2\lambda^4+\alpha\lambda^2+\delta^2+2A+\alpha}{\beta(\lambda^2-1)}}\times\frac{1+Ce^{\frac{2\rho\eta}{\mu}}}{1-Ce^{\frac{2\rho\eta}{\mu}}}, \tag{33}$$

where $\lambda(\lambda\neq\pm1)$, C, μ, A and δ are arbitrary real constants and

$$\rho=\pm\sqrt{\frac{-\delta^2\lambda^4+\alpha\lambda^2+\delta^2+2A+\alpha}{2(1-\lambda^4)}}. \tag{34}$$

Then, we have

$$u(x,t)=\pm\sqrt{\frac{-\delta^2\lambda^4+\alpha\lambda^2+\delta^2+2A+\alpha}{\beta(\lambda^2-1)}}\times\frac{1+Ce^{2\rho(x+\lambda t)}}{1-Ce^{2\rho(x+\lambda t)}}e^{i\delta(\lambda x+t)}, \tag{35a}$$

$$v(x,t)=\frac{-\delta^2\lambda^4+\alpha\lambda^2+\delta^2+2A+\alpha}{\lambda^4-1}\times\left(\frac{1+Ce^{2\rho(x+\lambda t)}}{1-Ce^{2\rho(x+\lambda t)}}\right)^2e^{2i\delta(\lambda x+t)}+\frac{A}{1+\lambda^2}, \tag{35b}$$

where $\lambda(\lambda\neq\pm1)$, A, C, μ and δ are arbitrary constants. By taking $C>0$, Eq (35) becomes

$$u(x,t)=\pm\sqrt{\frac{-\delta^2\lambda^4+\alpha\lambda^2+\delta^2+2A+\alpha}{\beta(\lambda^2-1)}}\coth(\rho(x+\lambda t+B))e^{i\delta(\lambda x+t)}, \tag{36a}$$

$$v(x,t)=\frac{-\delta^2\lambda^4+\alpha\lambda^2+\delta^2+2A+\alpha}{\lambda^4-1}\coth^2(\rho(x+\lambda t+B))e^{2i\delta(\lambda x+t)}+\frac{A}{1+\lambda^2}, \tag{36b}$$

where $\lambda(\lambda\neq\pm1)$, A, B, μ and δ are arbitrary constants. If we choose $C<0$, Eq (35) becomes

$$u(x,t)=\pm\sqrt{\frac{-\delta^2\lambda^4+\alpha\lambda^2+\delta^2+2A+\alpha}{\beta(\lambda^2-1)}}\tanh(\rho(x+\lambda t+B))e^{i\delta(\lambda x+t)}, \tag{37a}$$

$$v(x,t)=\frac{-\delta^2\lambda^4+\alpha\lambda^2+\delta^2+2A+\alpha}{\lambda^4-1}\tanh^2(\rho(x+\lambda t+B))e^{2i\delta(\lambda x+t)}+\frac{A}{1+\lambda^2}, \tag{37b}$$

where $\lambda(\lambda\neq\pm1)$, A, B, μ and δ are arbitrary constants.

Case II-3. If $ac=0$, substituting Eq (30) and $n=0$ into Eq (13), the solution of Eq (22) is

$$h(\eta)=\frac{1}{\pm\frac{1}{\mu}\sqrt{-\frac{\beta}{2(1+\lambda^2)}}\eta+C}, \tag{38}$$

where $\lambda(\lambda \neq \pm 1)$, C and μ are arbitrary constants. Then, we have

$$u(x,t) = \frac{1}{\pm\sqrt{-\frac{\beta}{2(1+\lambda^2)}}(x+\lambda t) + C} e^{i\delta(\lambda x + t)}, \tag{39a}$$

$$v(x,t) = \frac{\beta}{(1+\lambda^2)\left(\pm\sqrt{-\frac{\beta}{2(1+\lambda^2)}}(x+\lambda t) + C\right)^2} e^{2i\delta(\lambda x + t)} + \frac{\delta^2\left(\lambda^2 - 1\right) - \alpha}{2}, \tag{39b}$$

where $\lambda(\lambda \neq \pm 1)$, C, A and δ are arbitrary constants. Eqs (24), (35) and (39) are a new type of exact traveling wave solutions to Eq (1). Eqs (36) and (37) are a new kind of envelope solitary solutions to Eq (1). Eq (32) is a new type of exact periodical wave solution to Eq (1). The solutions (24) and (39) could not be obtained by the method presented in [25,26]. The solutions (32) and (35)–(37) are identical to the results presented in [25].

4. Bäcklund transformation of the CHF equation

If $n = 0$, Eq (25) is the Riccati equation

$$h' = ah^2 + bh + c. \tag{40}$$

Supposing that the solution of Eq (40) is the form

$$h_2 = h_1 + h_0, \tag{41}$$

where $h_0 = h_0(\eta)$ is a given solution of Eq (40) and $h_1 = h_1(\eta)$ is a function to be determined later. For this reason, substituting Eq (41) into (40) yields

$$h_1' = ah_1^2 + (b + 2ah_0)h_1. \tag{42}$$

Solving Bernoulli equation Eq (42), we get h_1, then we get h_2. Thus, we have a Bäcklund transformation of Eq (40) as follows:

$$h_2 = h_1 + h_0,$$
$$h_1' = ah_1^2 + (b + 2ah_0)h_1.$$

Using the Bäcklund transformation in section three, infinitely new solutions of Eq (1) can be obtained. For example, choosing

$$\begin{aligned} h_0 = h_0(\eta) = &\sqrt{\frac{-\delta^2\lambda^4 + \alpha\lambda^2 + \delta^2 + 2A + \alpha}{\beta(1-\lambda^2)}} \times \\ &\tan\left(\frac{1}{\mu}\sqrt{\frac{-\delta^2\lambda^4 + \alpha\lambda^2 + \delta^2 + 2A + \alpha}{2(\lambda^4 - 1)}}\eta\right) \end{aligned} \tag{43}$$

and applying Eqs (41) and (42) to Case II-1, we get

$$h_1 = h_1(\eta) = \frac{\sec^2 E\eta}{C - \frac{a}{E}\tan E\eta}, \tag{44}$$

where

$$a=\frac{1}{\mu}\sqrt{-\frac{\beta}{2\left(1+\lambda^2\right)}},E=\frac{1}{\mu}\sqrt{-\frac{\rho}{2\left(1-\lambda^4\right)}},\eta=\mu\left(x+\lambda t\right),\rho=-\delta^2\lambda^4+\alpha\lambda^2+\delta^2+2A+\alpha$$

and μ, δ, λ and A are arbitrary constants. Then, we obtain a new solution of Eq (40)

$$h_2=h_2\left(\eta\right)=\frac{\sec^2 E\eta}{C-\frac{a}{E}\tan E\eta}+\frac{E}{a}\tan E\eta. \tag{45}$$

From Eq (21), we obtain

$$v\left(\eta\right)=\frac{\beta}{1+\lambda^2}\left(\frac{\sec^2 E\eta}{C-\frac{a}{E}\tan E\eta}+\frac{E}{a}\tan E\eta\right)^2+\frac{A}{1+\lambda^2}. \tag{46}$$

Then, we obtain new solutions; that is,

$$u\left(x,t\right)=h_2\left(\eta\right)e^{i\delta(\lambda x+t)}, \tag{47a}$$

$$v\left(x,t\right)=v\left(\eta\right). \tag{47b}$$

Similar to the above discussions, choosing

$$h_0=h_0\left(\eta\right)=\mu\sqrt{\frac{2\left(1+\lambda^2\right)}{-\beta}\frac{1}{\eta}} \tag{48}$$

and applying Eqs (41) and (42) to Case II-3, we obtain

$$h_1=h_1\left(\eta\right)=\frac{3\eta^2}{C-a\eta^3}, \tag{49}$$

where $a=\frac{1}{\mu}\sqrt{-\frac{\beta}{2\left(1+\lambda^2\right)}}$, $\eta=\mu\left(x+\lambda t\right)$ and μ, λ are arbitrary constants. Then, we obtain a new solution of Eq (40)

$$h_2=h_2\left(\eta\right)=\frac{3\eta^2}{C-a\eta^3}+\frac{1}{a\eta}. \tag{50}$$

From Eq (21), we get

$$v\left(\eta\right)=\frac{\beta}{1+\lambda^2}\left(\frac{3\eta^2}{C-a\eta^3}+\frac{1}{a\eta}\right)^2+\frac{A}{1+\lambda^2}, \tag{51}$$

where μ, λ and A are arbitrary constants. Then, we have new solutions; that is,

$$u\left(x,t\right)=h_2\left(\eta\right)e^{i\delta(\lambda x+t)}, \tag{52a}$$

$$v\left(x,t\right)=v\left(\eta\right). \tag{52b}$$

Similar to the above discussions, we can obtain a Bäcklund transformation of Eq (40) as follows:

$$h_{m+2}=h_{m+1}+h_m, \tag{53a}$$

$$h'_{m+1} = ah_{m+1}^2 + (b + 2ah_m)\, h_{m+1}, \tag{53b}$$

where $m = 1, 2, 3, ..., h_m = h_m(\eta)$ is a given solution of Eq (40) and $h_{m+1} = h_{m+1}(\eta)$ is a function to be determined later. Solving Eq (53b), we get h_{m+1}, thus, we get h_{m+2}, which is a new solution of Eq (40). Therefore, we get the Bäcklund transformations of Eq (1)

$$h'_{m+1} = ah_{m+1}^2 + (b + 2ah_m)\, h_{m+1}, \tag{54a}$$

$$h_{m+2} = h_{m+1} + h_m, \tag{54b}$$

$$u(x, t) = h_2(\eta)\, e^{i\delta(\lambda x + t)}, \tag{54c}$$

$$v(x, t) = v(\eta), \tag{54d}$$

where $h_m = h_m(\eta)$ is a given solution of Eq (40).

Remark 2. In general, we can get the Bäcklund transformations of Eq (5) as follows:

$$w_m = h_m^{1-n}, \tag{55a}$$

$$w'_{m+1} = (1 - n)\left(aw_{m+1}^2 + (b + 2aw_m)\, w_{m+1}\right), \tag{55b}$$

$$w_{m+2} = w_{m+1} + w_m, \tag{55c}$$

$$h_{m+1} = w_{m+2}^{\frac{1}{1-n}}, \tag{55d}$$

where $w_m, w_{m+1}, w_{m+2}, h_m$ and h_{m+1} are functions of η. Suppose that $h_m = h_m(\eta)$ is a given solution of Eq (5). From Eq (55a), we can get w_m. Solving Eq (55b), we obtain w_{m+1}. According to Eq (55c) and (55d), we obtain a new solution of Eq (5).

5. Conclusions

In this paper, we established exact traveling wave solutions of the CHF equation by using the RB method. Many new solutions of the CHF equation were obtained using the Bäcklund transformations. Many well-known NLPDEs can be processed in this way. We used computer software like Maple and Mathematica to facilitate the tedious algebraic calculations. Therefore, the RB method was a standard and computerizable approach. At the same time, the performance of this method was also found to be simple and efficient.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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