Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 31(11): 6763–6789.
DOI: 10.3934/era.2023341
Received: 31 July 2023
Revised: 21 September 2023
Accepted: 07 October 2023
Published: 19 October 2023

Theory article

Group invariant solutions for the planar Schrödinger-Poisson equations

Ganglong Zhou*

School of Mathematical Sciences, Key Laboratory of MEA (Ministry of Education) & Shanghai Key Laboratory of PMMP, East China Normal University, Shanghai 200241, China

* **Correspondence:** Email: 52195500021@stu.ecnu.edu.cn.

Abstract: This paper is concerned with the following planar Schrödinger-Poisson equations

$$-\Delta u + V(x)u + (\ln|\cdot| * |u|^p)\,|u|^{p-2}u = f(x,u), \quad x \in \mathbb{R}^2,$$

where $p \geq 2$ is a constant, and $V(x)$ and $f(x,u)$ are continuous, mirror symmetric or rotationally periodic functions. The nonlinear term $f(x,u)$ satisfies a certain monotonicity condition and has critical exponential growth in the Trudinger-Moser sense. We adopted a version of mountain pass theorem by constructing a Cerami sequence, which in turn leads to a ground state solution. Our method has two new insights. First, we observed that the integral $\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(|x-y|)|u(x)|^p|u(y)|^p dxdy$ is always negative if u belongs to a suitable space. Second, we built a new Moser type function to ensure the boundedness of the Cerami sequence, which further guarantees its compactness. In particular, by replacing the monotonicity condition with the Ambrosetti–Rabinowitz condition, our approach works also for the subcritical growth case.

Keywords: planar Schrödinger-Poisson equation; Cerami sequence; critical exponential growth; mirror symmetry/rotationally periodicity; nonlinear equations

1. Introduction

The present paper is concerned with the existence of solution to the planar Schrödinger-Poisson equations

$$-\Delta u + V(x)u + (\ln|\cdot| * |u|^p)\,|u|^{p-2}u = f(x,u), \quad x \in \mathbb{R}^2, \tag{1.1}$$

where $p \geq 2$, V, f are continuous, mirror symmetric or rotationally periodic functions, and $f(x,t)$ has exponential critical growth in the Trudinger-Moser sense ([1]).

In last decades, considerable attention has been paid to the following Schrödinger-Poisson equations:

$$\begin{cases}-\Delta u+V(x)u+K(x)\phi(x)|u|^{p-2}u=f(x,u), & x\in\mathbb{R}^3,\\ -\Delta\phi=K(x)|u|^p, & x\in\mathbb{R}^3\end{cases}\tag{1.2}$$

with various conditions on the parameters p, N and functions V, K, f. These kinds of equations arise in many contexts of physics, such as, in quantum mechanics [2–4] and semiconductor theory [5–8]. In [5], Eq (1.2) was introduced as a model describing solitary waves for nonlinear stationary equations of Schrödinger type interacting with an electrostatic field where the unknown functions u and ϕ wave function for particles and potential, respectively. Let $p\in(1,6]$ and $K\in L^\infty(\mathbb{R}^3)$. For each $u\in H^1(\mathbb{R}^3)$, the second equation in (1.2) determines the Newton potential ϕ_u in $D^{1,2}(\mathbb{R}^3)$, i.e.,

$$\phi_u(x):=\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{K(y)|u(y)|^p}{|x-y|}dy.$$

Many minimization techniques, such as minimizing on a constraint set [9, 10] and the Mountain Pass Theorem [11–15], were used in the Eq (1.2).

When $K(x)\equiv 0$, Eq (1.2) becomes the Schrödinger equation. In this case, there are many results to Eq (1.2) with the dual method if V and f satisfy some certain conditions, such as a positive lower bound on V or a monotonicity condition on f (see [16–20] and references therein).

In the following, let us focus on the two-dimensional case. Stubbe [21] considered the equations

$$\begin{cases}-\Delta u+\lambda u+\phi(x)u=0, & x\in\mathbb{R}^2,\\ \Delta\phi=u^2, & x\in\mathbb{R}^2,\end{cases}\tag{1.3}$$

where $\lambda\in\mathbb{R}$ is a constant. They set up a variational framework for Eq (1.3) with a subspace Z of $H^1(\mathbb{R}^2)$:

$$Z:=\left\{u\in H^1(\mathbb{R}^2):\int_{\mathbb{R}^2}\ln(1+|x|)u^2dx<\infty\right\}.$$

They proved that there exists a unique radial ground state solution for any $\lambda\geq 0$. In addition, they proved that there exists a negative number λ^*, such that for any $\lambda\in(\lambda^*,0)$ there are two radial ground states with different L^2 norms. Cigolani and Weth [22] considered Eq (1.1) with $p=2$ and $f(x,u)=b|u|^{\sigma-2}u$. Specifically, $V\in C(\mathbb{R}^2,(0,\infty))$ is $\mathbb{Z}^2$ periodic. Using the concentration-compactness theory, they proved that Eq (1.1) has a ground state $u\in X_2$ and a solution sequence $\{u_n\}_n\subset X_2$, such that $\lim_{n\to\infty}J(u_n)=\infty$. Here,

$$X_2:=\left\{u\in H^1(\mathbb{R}^2):\int_{\mathbb{R}^2}\left[|\nabla u|^2+V(x)u^2+\ln(1+|x|)u^2\right]dx<\infty\right\}$$

and J are the energy functionals associated with Eq (1.1).

Chen and Tang [23] considered Eq (1.1) with $p=2$, i.e.,

$$\begin{cases}-\Delta u+V(x)u+\phi(x)u=f(x,u), & x\in\mathbb{R}^2,\\ \Delta\phi=u^2, & x\in\mathbb{R}^2,\end{cases}\tag{1.4}$$

where $V\in C(\mathbb{R}^2,[0,\infty))$ is axially symmetrical and $f\in C(\mathbb{R}^2\times\mathbb{R})$ is of subcritical or critical exponential growth in the sense of Trudinger-Moser. More precisely, we say that $f(x,t)$ has subcritical exponential growth at $t=\pm\infty$ if it verifies

(F1') For every $A > 0$,

$$\sup_{x\in\mathbb{R}^2, |s|\le A} |f(x,s)| < +\infty \tag{1.5}$$

and

$$\lim_{|t|\to\infty} \frac{|f(x,t)|}{e^{\alpha t^2}} = 0, \text{ uniformly in } \mathbb{R}^2, \tag{1.6}$$

for any $\alpha > 0$;

and the function $f(x,t)$ is said to have the critical exponential growth at $t = \pm\infty$ if it verifies

(F1) The nonlinearity f satisfies (1.5) and there exists $\alpha_0 > 0$ such that, for any $\alpha > \alpha_0$, (1.6) holds but

$$\lim_{|t|\to\infty} \frac{|f(x,t)|}{e^{\alpha t^2}} = +\infty, \text{ uniformly in } \mathbb{R}^2 \text{ for all } \alpha < \alpha_0.$$

This notion of criticality can be referred to [24].

For the critical growth case, Chen and Tang [23] established the existence of a ground state solution for Eq (1.4) by assuming following conditions on V and f:

(V0) $V \in C(\mathbb{R}, [0,\infty))$ and $\liminf_{|x|\to\infty} V(x) > 0$;

(CF1) $V(x) := V(x_1, x_2) = V(|x_1|, |x_2|)$ for all $x \in \mathbb{R}^2$, $f(x,t) := f(x_1, x_2, t) = f(|x_1|, |x_2|, t)$ for all $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$;

(CF2) $f(x,t)t > 0$ for all $(x,t) \in \mathbb{R}^2 \times \mathbb{R}\backslash\{0\}$, and there exists $M_0 > 0$ and $t_0 > 0$ such that

$$F(x,t) \le M_0 |f(x,t)|, \quad \forall\ x \in \mathbb{R}^2,\ |t| \ge t_0,$$

where $F(x,t) := \int_0^t f(x,s)ds$;

(CF3) $\liminf_{|t|\to\infty} \frac{t^2 F(x,t)}{e^{\alpha_0 t^2}} > \frac{2}{\alpha_0^2 \rho^2}$ uniformly on $x \in \mathbb{R}^2$, where $\rho \in (0, 1/2)$ satisfying $\rho^2 \max_{|x|\le\rho} V(x) \le 1$;

(CF4) $\frac{f(x,t) - V(x)t}{|t|^3}$ is non-decreasing on $t \in \mathbb{R}\backslash\{0\}$.

Recently, Cao et al. [25] considered the equations

$$-\Delta u + V(x)u + (\ln|\cdot| * |u|^p)\, |u|^{p-2}u = b|u|^{\sigma-2}u, \quad x \in \mathbb{R}^2, \tag{1.7}$$

where $\sigma \ge 2p$, $b \ge 0$ and $V \in C(\mathbb{R}^2, (0,\infty))$ are $\mathbb{Z}^2$ periodic. With a similar method in [22], they obtained the existence of a positive ground state solution of Eq (1.7) in X_p, where

$$X_p := \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} \left[|\nabla u|^2 + V(x)u^2 + \ln(1+|x|)|u|^p \right] dx < \infty \right\}.$$

Here, we will prove the existence of a nontrivial solution to Eq (1.1), not only for all $p \ge 2$, but also for general nonlinearities f and potentials V.

To describe our main results, we introduce the following notations: Let us view $\mathbb{R}^2$ as $\mathbb{C}$, let $k \in \mathbb{N}, k \ge 2$ and we say that $v \in \mathcal{P}_k$ if $v(ze^{2\pi i/k}) = v(z)$ over $\mathbb{C}$. We define

$$E_{k,p} := X_p \cap \mathcal{P}_k, \qquad \mathcal{V}_{k,1} := C(\mathbb{C}) \cap \mathcal{P}_k$$

$$\mathcal{F}_{k,1} := \{f \in C(\mathbb{C} \times \mathbb{R}) : f(\cdot, t) \in \mathcal{P}_k, \forall\ t \in \mathbb{R}\}.$$

We say that v is mirror symmetric denoted by $v \in \mathcal{M}$ if $v(\bar{z}) = v(z)$ in $\mathbb{C}$. Let

$$T_{k,p} := E_{k,p} \cap \mathcal{M}, \qquad \mathcal{V}_{k,2} := \mathcal{V}_{k,1} \cap \mathcal{M},$$

$$\mathcal{F}_{k,2} := \{f \in \mathcal{F}_{k,1} : f(\cdot, t) \in \mathcal{M}, \forall\ t \in \mathbb{R}\}.$$

Finally, we write the associated functional of Eq (1.1) in the following form

$$\Phi(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4p\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(|x-y|)|u(x)|^p|u(y)|^p dxdy - \int_{\mathbb{R}^2} F(x,u)dx, \tag{1.8}$$

and the associated Nehari manifold of the functional (1.8) is

$$\begin{aligned} \mathcal{N}_1 &:= \{u \in E_{k,p}\backslash\{0\} : \langle \Phi'(u), u\rangle = 0\}, \\ \mathcal{N}_2 &:= \{u \in T_{k,p}\backslash\{0\} : \langle \Phi'(u), u\rangle = 0\}. \end{aligned} \tag{1.9}$$

Our main result is stated as follows.

Theorem 1.1. *Let $p \geq 2$, V and f satisfy* (V0), (F1) *and the following conditions*

(VF) *$V \in \mathcal{V}_{k,1}$ and $f \in \mathcal{F}_{k,1}$ with $k \geq 4$ or $V \in \mathcal{V}_{k,2}$ and $f \in \mathcal{F}_{k,2}$ with $k \geq 2$;*

(F2) *$f(x,t)t \geq 0$ for all $(x,t) \in \mathbb{R}^2 \times \mathbb{R}\backslash\{0\}$, and there exists $M_0 > 0$ and $t_0 > 0$ such that $F(x,t) \leq M_0|f(x,t)|$ for $x \in \mathbb{R}^2$, $|t| \geq t_0$;*

(F3) *There exists $q \in \mathbb{R}$ such that $\liminf_{|t|\to\infty} \frac{|t|^q F(x,t)}{e^{\alpha_0 t^2}} = +\infty$;*

(F4) *$g_p(x,t)$ is non-decreasing on $t \in (-\infty, 0)$ and $t \in (0,\infty)$, where*

$$g_p(x,t) := \begin{cases} \frac{f(x,t)-V(x)t}{|t|^{2p-1}}, & p = 2, \\ \frac{f(x,t)-\mu V(x)t}{|t|^{2p-1}}, & p > 2 \end{cases}$$

for some $\mu < 1$.

(F5) *If $p = 2$, $f(x,t) = o(t)$ as $t \to 0$ uniformly on $\mathbb{R}^2$; and if $p > 2$, $f(x,t) = O(t^{s_0})$ with $s_0 > 1$ as $t \to 0$ uniformly on $\mathbb{R}^2$.*

Then, Eq (1.1) *has a nontrivial solution $\bar{u}$. Moreover, if $V \in \mathcal{V}_{k,1}$ and $f \in \mathcal{F}_{k,1}$, then $\bar{u} \in E_{k,p}$ satisfies*

$$\Phi(\bar{u}) = \min_{\mathcal{N}_1} \Phi;$$

if $V \in \mathcal{V}_{k,2}$ and $f \in \mathcal{F}_{k,2}$, then $\bar{u} \in T_{k,p}$ satisfies

$$\Phi(\bar{u}) = \min_{\mathcal{N}_2} \Phi.$$

Remark 1.2. *Comparing to* [23, Theorem 4]*, we have weakened the assumptions* (CF1)–(CF3) *to* (VF) *and* (F2)–(F3)*, respectively. More precisely,*

- (CF1) *means $V \in \mathcal{V}_{2,2}$ and $f \in \mathcal{F}_{2,2}$, hence, it is a special case of* (VF)*;*

- *The condition* (F3) *is less restrictive than* (CF3) *for the behavior of f at infinity;*
- (F2) *improves slightly* (CF2) *where $f(x,t)t > 0$ is replaced by $f(x,t)t \geq 0$;*

Here is an example of f, which satisfies (VF) *and* (F1)–(F5)*, but not* (CF3)*. Let $\theta > 0, p_0 \geq p, q_0 > 2$ and*

$$f_0(x,t) = \begin{cases} \frac{\theta e^{\alpha_0 t^2}(2\alpha_0 t^2 - q_0)}{t^{q_0+1}}, & t \geq \sqrt{\frac{p_0+q_0}{\alpha_0}}, \\ \theta \frac{e^{p_0+q_0}(2p_0+q_0)\alpha_0^{p_0+q_0/2}}{(p_0+q_0)^{p_0+q_0/2}} t^{2p_0-1}, & 0 \leq t < \sqrt{\frac{p_0+q_0}{\alpha_0}} \end{cases}$$

with odd extension to $t < 0$. Finally, it seems that [23] *used implicitly $f(t) = o(t)$ as $t \to 0$ with $p = 2$ in* (F5) *(see the proof of Lemma 2.6 there).*

Our approach works also for the subcritical case.

Theorem 1.3. *Let $p \geq 2$, V and f satisfy* (V0), (VF), (F1'), (F5) and the following condition:

(F4') $f(x,t)t > 0$ for all $(x,t) \in \mathbb{R}^2 \times (\mathbb{R}\backslash\{0\})$ and there exists $\nu \in (2,\infty), t_1 \in (0,\infty)$ such that

$$f(x,t)t \geq \nu F(x,t), \quad \forall\, x \in \mathbb{R}^2, |t| \geq t_1;$$

Furthermore, if $p > 2$, we assume that

$$M_{t_1} < \left(\frac{1}{2} - \frac{1}{\mu}\right)\gamma^2,$$

where

$$M_{t_1} = \sup_{(x,t)\in\mathbb{R}^2\times[-t_1,t_1]\backslash\{0\}} \frac{F(x,t)}{t^2} \qquad \text{and} \qquad \gamma = \inf_{u\in X_p} \frac{\|u\|}{\|u\|_{H^1(\mathbb{R}^2)}} > 0.$$

Then, Eq (1.1) has a nontrivial solution $\bar{u}$. Moreover, $\bar{u} \in E_{k,p}$ if $V \in \mathcal{V}_{k,1}$ and $f \in \mathcal{F}_{k,1}$, and $\bar{u} \in T_{k,p}$ if $V \in \mathcal{V}_{k,2}$ and $f \in \mathcal{F}_{k,2}$.

This paper is organized as follows: In Section 2, we present some basic results; in particular we show that the energy functional corresponding to the nonlocal term is non positive, which is our key observation and different from the available results, see Lemma 2.3. In Section 3, we prove a mountain pass type theorem using a new test function, see Lemma 3.2 below. In Sections 4 and 5, we give the proof of Theorems 1.1 and 1.3, respectively.

2. Preliminaries

In this section, we will give some preliminary definitions and basic facts about inequalities, such as the Moser-Trudinger inequality, the energy estimate of the nonlocal term. In the following, the letter C denotes generic positive constants and $\|\cdot\|_q$ denotes the standard norm in $L^q(\mathbb{R}^2)$.

The function space X_p is a Banach space equipped with the norm

$$\|u\|_{X_p} := \|u\| + \|u\|_*, \tag{2.1}$$

where

$$\|u\|_* := \left(\int_{\mathbb{R}^2} \ln(1+|x|)|u|^p dx\right)^{\frac{1}{p}}; \tag{2.2}$$

while

$$\|u\| := \left(\int_{\mathbb{R}^2} \left[|\nabla u|^2 + V(x)u^2\right] dx\right)^{\frac{1}{2}} \tag{2.3}$$

is induced by the scalar product

$$\langle u, v\rangle := \int_{\mathbb{R}^2} (\nabla u \cdot \nabla v + V(x)uv)dx. \tag{2.4}$$

We will use the following bilinear functionals (see [21]):

$$A_1(u,v) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|)u(x)v(y)dxdy; \tag{2.5}$$

$$A_2(u,v) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1+\frac{1}{|x-y|}\right)u(x)v(y)dxdy; \tag{2.6}$$

$$A_0(u,v) := A_1(u,v) - A_2(u,v) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(|x-y|)u(x)v(y)dxdy. \tag{2.7}$$

By the Hardy-Littlewood-Sobolev inequality (see [26]), there exists $C > 0$ such that for any $u, v \in L^{4/3}(\mathbb{R}^2)$,

$$|A_2(u,v)| \le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{|x-y|}u(x)v(y)dxdy \le C\|u\|_{\frac{4}{3}}\|v\|_{\frac{4}{3}}. \tag{2.8}$$

Corresponding to (2.5)–(2.7), we define

$$I_i(u) := A_i(|u|^p, |u|^p), \quad i = 0, 1, 2. \tag{2.9}$$

The following bound for $I_2(u)$ is a direct consequence of (2.8):

$$|I_2(u)| \le C\|u\|_{\frac{4p}{3}}^{2p}, \quad \forall\, u \in L^{\frac{4p}{3}}(\mathbb{R}^2),\ \forall\, p \ge 1. \tag{2.10}$$

We can rewrite the associated functional of Eq (1.1) in the following form

$$\Phi(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4p\pi}I_0(u) - \int_{\mathbb{R}^2} F(x,u)dx. \tag{2.11}$$

Next, we state several lemmas.

Lemma 2.1. (i) *Let $u \in H^1(\mathbb{R}^2)$, then for any $\alpha > 0$,*

$$\int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) dx < \infty.$$

(ii) *Given $M > 0$, $\alpha \in (0, 4\pi)$, there exists a constant $C(M,\alpha)$ such that for all $u \in H^1(\mathbb{R}^2)$ satisfying $\|\nabla u\|_2 \le 1$, $\|u\|_2 \le M$, there holds*

$$\int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) dx < C(M,\alpha).$$

The statements (i) and (ii) of the above lemma were first established by [27, Lemma 1] and [1, Lemma 2.1], respectively (see also [28, 29]).

Lemma 2.2. *Assume that V and f satisfy* (V0), (F1)(*or* (F1')), (F5). *Then, $I_i, \Phi \in C^1(X_p, \mathbb{R})$ and*

$$\begin{aligned} \langle I_i'(u), v\rangle &= 2pA_i(|u|^p, |u|^{p-2}uv), \quad i = 1, 2 \\ \langle \Phi'(u), v\rangle &= \langle u, v\rangle + \frac{1}{2\pi}A_0(|u|^p, |u|^{p-2}uv) - \int_{\mathbb{R}^2} f(x, u)v dx. \end{aligned} \tag{2.12}$$

For the sake of completeness, we present a proof of Lemma 2.2 in the appendix. The following lemma is our first key observation.

Lemma 2.3. *For any $u \in X_p$, we have $I_0(u) \le 0$.*

Proof. First, let $u \in C_0^\infty(\mathbb{R}^2)$ with $\mathrm{supp}(u) \subset B_{\frac{1}{2}}(0)$. Then

$$I_0(u) = \int_{B_{\frac{1}{2}}(0)}\int_{B_{\frac{1}{2}}(0)} \ln(|x-y|)|u(x)|^p|u(y)|^p dxdy \le 0.$$

Consider now $u \in C_0^\infty(\mathbb{R}^2, \mathbb{R})$. Take $R > 0$ such that $\mathrm{supp}(u) \subset B_R(0)$ and let $w(x) = u(2Rx)$, so $\mathrm{supp}(w) \subset B_{\frac{1}{2}}(0)$ and $\phi_w(x) := \frac{1}{4R^2}\phi_u(2Rx)$. Hence,

$$\begin{aligned} \frac{1}{2\pi}I_0(u) =& \int_{B_R(0)}\int_{B_R(0)} \ln(|x-y|)|u(x)|^p|u(y)|^p dxdy \\ =& 16R^4 \int_{B_{\frac{1}{2}}(0)}\int_{B_{\frac{1}{2}}(0)} \ln(|x-y|)|w(x)|^p|w(y)|^p dxdy \le 0. \end{aligned}$$

We conclude by the density argument. For any $R > 0$, let $\varphi_R(r)$ be a C_0^∞ cut-off function such that $0 \le \varphi \le 1$, $\varphi_R \equiv 1$ on $[0, R]$ and $\varphi_R \equiv 0$ on $[R+1, \infty)$. Let η be the standard mollifier and $\eta_\delta(x) := \frac{1}{\delta^2}\eta(\frac{x}{\delta})$, where $\delta > 0$. Given $\epsilon > 0$, since $\sqrt{V}u \in L^2(\mathbb{R}^2)$, $[\ln(1 + |\cdot|)]^{1/p}u \in L^p(\mathbb{R}^2)$, [30, Pages 264 and 714], we can choose δ small enough such that

$$\left\|\varphi_{\frac{1}{\delta}}(|\cdot|)u - u\right\|_{X_p} < \epsilon, \qquad \left\|\eta_\delta * \left[\varphi_{\frac{1}{\delta}}(|\cdot|)u\right] - \varphi_{\frac{1}{\delta}}(|\cdot|)u\right\|_{X_p} < \epsilon.$$

Therefore, for any $u \in X_p$, there exists $\{u_n\}_n \subset C_0^\infty(\mathbb{R}^2)$ such that $\lim_{n\to\infty}\|u_n - u\|_{X_p} = 0$. By the fact $I_0 = I_1 - I_2$ and Lemma 2.2, we conclude that $I_0(u) \le 0$.

Corollary 2.4. *Assume that V and f satisfy* (V0) *and* (F1)(*or* (F4')). *Then*

$$\lim_{t\to\infty} \Phi(t\omega) = -\infty, \quad \forall\ \omega \in X_p\backslash\{0\}.$$

Proof. For any $\omega \in X_p\backslash\{0\}$, there exists $\delta > 0$ such that $m\{|\omega(x)| \ge \delta\} > 0$. For a critical case, by Lemma 2.3 and (F1), one has

$$\Phi(t\omega) = \frac{t^2}{2}\|\omega\|^2 + \frac{t^4}{4p\pi}I_0(\omega) - \int_{\mathbb{R}^2} F(x, t\omega)dx$$

$$\leq \frac{t^2}{2}\|\omega\|^2 - C\int_{\{|\omega(x)|\geq\delta\}} e^{\alpha_0\delta^2 t^2/2}dx \to -\infty, \quad as\ t \to \infty.$$

For a subcritical case, we choose large enough $R > 0$ such that $m(G) > 0$, where

$$G := \{|\omega(x)| \geq \delta\} \cap B_R(0).$$

By (F4') and choosing $M := \frac{\|\omega\|^2}{m(G)} > 0$, there exists $t_M > 0$ such that

$$|F(x, tw)| \geq Mt^2, \quad \forall x \in G, |t| \geq t_M,$$

which together with Lemma 2.3 implies

$$\begin{aligned}\Phi(t\omega) &= \frac{t^2}{2}\|\omega\|^2 + \frac{t^4}{4p\pi}I_0(\omega) - \int_{\mathbb{R}^2} F(x, t\omega)dx \\ &\leq \frac{t^2}{2}\|\omega\|^2 - t^2 Mm(G) = -\frac{t^2}{2}\|\omega\|^2 \to -\infty\end{aligned}$$

as $t \to \infty$, and we complete the proof.

The following lemma is inspired by [31, Lemma 2.2].

Lemma 2.5. *Assume that V and f satisfy* (V0) *and* (VF). *Then there exists $C_k > 0$ such that*

$$A_1(|u|^p, |v|^p) \geq C_k\|u\|_*^p\|v\|_p^p, \quad \forall\ u, v \in E_{k,p}. \tag{2.13}$$

In particular, since $T_{k,p} \subset E_{k,p}$, (2.13) *holds for $u, v \in T_{k,p}$.*

Proof. Let $\Omega_1 := \{(x_1, x_2) \in \mathbb{R}^2 : x_i \geq 0\}$, $\Omega_2 = -\Omega_1$. For any $x \in \Omega_1$ and $y \in \Omega_2$, one has

$$|x - y|^2 = |x|^2 + |y|^2 - 2x \cdot y \geq |x|^2 + |y|^2.$$

Then, it follows from the definition of $E_{k,p}$ and $k \geq 4$ that

$$\begin{aligned}A_1(|u|^p, |v|^p) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1 + |x - y|)|u(x)|^p|v(y)|^p dxdy \\ &\geq \int_{\Omega_2} |v(y)|^p dy \int_{\Omega_1} \ln(1 + |x - y|)|u(x)|^p dx \\ &\geq \int_{\Omega_2} |v(y)|^p dy \int_{\Omega_1} \ln(1 + |x|)|u(x)|^p dx \\ &\geq \frac{1}{k^2}\int_{\mathbb{R}^2} |v(y)|^p dy \int_{\mathbb{R}^2} \ln(1 + |x|)|u(x)|^p dx \\ &\geq C_k\|u\|_*^p\|v\|_p^p, \quad \forall\ u, v \in E_{k,p},\end{aligned}$$

so we obtain (2.13).

3. Variational framework

In this section, we will quote a version of Mountain Pass Theorem and prepare the proof of Theorems 1.1 and 1.3.

Lemma 3.1. *Let Y be a real Banach space and $I \in C^1(Y, \mathbb{R})$. Let S be a closed subset of Y, which disconnects Y into distinct connected Y_1 and Y_2. Suppose further that $I(0) = 0$ and*
(i) $0 \in Y_1$, *and there exists $\alpha > 0$ such that $I|_S \geq \alpha$,*
(ii) *There is $e \in Y_2$ such that $I(e) \leq 0$.*
Then, I possesses a Cerami sequence with $c \geq \alpha > 0$ given by

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = e\},$$

and a Cerami sequence means a sequence $\{u_n\} \subset X$ such that

$$I(u_n) \to c, \qquad \|I'(u_n)\|_{Y'}(1 + \|u_n\|_Y) \to 0.$$

The proof of the above lemma can be found in [32, Theorem 3]. We state another result that serves as a bridge between the mountain pass structure (see Lemma 3.3) and Theorem 1.1.

Lemma 3.2. *Assume that V and f satisfy* (V0), (F1) *and* (F3)–(F5). *Then there exists $n_0 \in \mathbb{N}$ such that*

$$\max_{t\geq 0} \Phi(t\omega_{n_0}) < \frac{2\pi}{\alpha_0}, \tag{3.1}$$

where

$$\omega_n(x) = \begin{cases} \frac{\sqrt{\ln n}}{\sqrt{2\pi}} - \frac{q\ln(\ln n)}{2\sqrt{2\pi\ln n}}, & 0 \leq |x| \leq (\ln n)^{q/2}/n; \\ \frac{\ln(1/|x|)}{\sqrt{2\pi\ln n}}, & (\ln n)^{q/2}/n \leq |x| \leq 1; \\ 0, & |x| \geq 1. \end{cases}$$

Proof. Without loss of generality, we can fix $q \geq 2$. Direct computation yields

$$\begin{aligned} \|\omega_n\|^2 \leq & \int_{B_1} |\nabla\omega_n|^2 dx + V_1 \int_{B_1} \omega_n^2 dx \\ =& 1 - \frac{q\ln(\ln n)}{2\ln n} + \delta_n, \end{aligned} \tag{3.2}$$

where $\delta_n = O(\frac{1}{\ln n})$ as $n \to \infty$. By (F3), there exists $t_0 > 0$ such that

$$\frac{|t|^q F(x, t)}{e^{\alpha_0 t^2}} \geq 1, \ \forall\, |t| \geq t_0. \tag{3.3}$$

There are three cases for the value of t.

Case (i): $0 \leq t \leq \sqrt{\frac{3\pi}{\alpha_0}}$. For large n, then it follows from (3.2) and Lemma 2.3 that

$$\Phi(t\omega_n) = \frac{t^2}{2}\|\omega_n\|^2 + \frac{t^{2p}}{2p} I_0(\omega_n) - \int_{\mathbb{R}^2} F(x, t\omega_n)dx \leq \frac{1+\delta_n}{2} t^2 \leq \frac{7\pi}{4\alpha_0}. \tag{3.4}$$

Case (ii): $\sqrt{\frac{3\pi}{\alpha_0}} \leq t \leq \sqrt{\frac{8\pi}{\alpha_0}}$. For large n, we have $t\omega_n(x) \geq t_0$ for $x \in B_{(\ln n)^{q/2}/n}$. Then it follows from (3.2), (3.3) and Lemma 2.3 that

$$\begin{aligned}\Phi(t\omega_n) &= \frac{t^2}{2}\|\omega_n\|^2 + \frac{t^{2p}}{2p}I_0(\omega_n) - \int_{\mathbb{R}^2} F(x, t\omega_n)dx \\ &\leq \frac{1+\delta_n}{2}t^2 - \frac{q\ln(\ln n)}{4\ln n}t^2 - \frac{2^{q/2}\pi^{1+q/2}(\ln n)^q}{n^2 t^q T_n^{q/2}} e^{\frac{\alpha_0}{2\pi}t^2 T_n} \\ &\leq \frac{1+\delta_n}{2}t^2 - \frac{q\ln(\ln n)}{4\ln n}t^2 - \frac{\alpha_0^{q/2}\pi(\ln n)^q}{2^q n^2 T_n^{q/2}} e^{\frac{\alpha_0}{2\pi}t^2 T_n} =: \varphi_n(t),\end{aligned} \tag{3.5}$$

where

$$T_n := \ln n - q\ln(\ln n) + \frac{q^2 \ln^2(\ln n)}{4\ln n}.$$

Let $t_n > 0$ be the unique maximum of φ_n in $\mathbb{R}_+$, then (as $n \to \infty$)

$$t_n^2 = \frac{4\pi}{\alpha_0}\left[1 + \frac{(q-1)\ln(\ln n)}{2\ln n} + O\left(\frac{1}{\ln n}\right)\right] \tag{3.6}$$

and

$$\varphi_n(t) \leq \varphi_n(t_n) = \frac{1+\delta_n}{2}t_n^2 - \frac{q\ln(\ln n)}{4\ln n}t_n^2 + O\left(\frac{1}{\ln n}\right). \tag{3.7}$$

Combining (3.5)–(3.7), one has

$$\Phi(t\omega_n) \leq \varphi_n(t_n) = \frac{2\pi}{\alpha_0} - \frac{\ln(\ln n)}{2\ln n} + O\left(\frac{1}{\ln n}\right). \tag{3.8}$$

Case (iii): $t \geq \sqrt{\frac{8\pi}{\alpha_0}}$. As in the above case (ii), we have

$$\begin{aligned}\Phi(t\omega_n) &\leq \frac{1+\delta_n}{2}t^2 - \frac{2^{q/2}\pi^{1+q/2}(\ln n)^q}{n^2 t^q T_n^{q/2}} e^{\frac{\alpha_0}{2\pi}t^2 T_n} \\ &\leq \frac{1+\delta_n}{2}t^2 - \frac{2^{q/2}\pi^{1+q/2}(\ln n)^q}{t^q T_n^{q/2}} \exp\left[2\left(\frac{\alpha_0}{4\pi}t^2 - 1\right)T_n\right] \\ &\leq \frac{4\pi(1+\delta_n)}{\alpha_0} - \frac{\alpha_0^{q/2}\pi(\ln n)^{q/2}}{2^q}n^2 \\ &\leq 0\end{aligned} \tag{3.9}$$

for large n. To get the third inequality, we used the fact that the function

$$\frac{1+\delta_n}{2}t^2 - \frac{2^{q/2}\pi^{1+q/2}(\ln n)^q}{t^q T_n^{q/2}} \exp\left[2\left(\frac{\alpha_0}{4\pi}t^2 - 1\right)T_n\right]$$

is decreasing on $t \geq \sqrt{\frac{8\pi}{\alpha_0}}$ when n is large enough. Combining the conclusions for cases (i)–(iii), the proof is completed.

Now we show the existence of the Cerami sequence.

Lemma 3.3. *Assume that V and f satisfy* (V0), (VF), (F1)(*or* (F1') *and* (F4')) *and* (F5). *Then there exists a constant $\tilde{c} \in (0, \sup_{t\geq 0} \Phi(t\omega_{n_0})]$ and a Cerami sequence $\{u_n\} \subset E_{k,p}$ such that*

$$\Phi(u_n) \to \tilde{c}, \qquad \|\Phi'(u_n)\|_{X_p'}(1 + \|u_n\|_{X_p}) \to 0. \tag{3.10}$$

Proof. Applying the Sobolev embedding theorem for given $s \in [2, \infty)$, there exists $\gamma_s > 0$ such that

$$\|u\|_s \leq \gamma_s \|u\|, \quad \forall\ u \in X_p. \tag{3.11}$$

By (F1) (or (F1')) and (F5) for any $\epsilon > 0$, there exists some constant $C_\epsilon > 0$ such that

$$|F(x,t)| \leq \epsilon t^2 + C_\epsilon (e^{3\alpha_0 t^2/2} - 1)|t|^3, \quad \forall\ (x,t) \in \mathbb{R}^2 \times \mathbb{R}. \tag{3.12}$$

On the other hand, in view of Lemma 2.1, one has

$$\int_{\mathbb{R}^2} \left(e^{3\alpha_0 u^2} - 1\right) dx \leq C, \quad \forall\ \|u\| \leq \sqrt{\frac{\pi}{\alpha_0}}. \tag{3.13}$$

Let $\epsilon = \frac{1}{4\gamma_2^2}$ from (3.11)–(3.13), and there holds

$$\int_{\mathbb{R}^2} F(x,u)dx \leq \frac{1}{4}\|u\|^2 + C_3\|u\|^3, \quad \forall\ \|u\| \leq \sqrt{\frac{\pi}{\alpha_0}}. \tag{3.14}$$

Hence, it follows from (2.11) and (3.14) that if $\|u\| \leq \sqrt{\frac{\pi}{\alpha_0}}$,

$$\begin{aligned} \Phi(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4p\pi}(I_1(u) - I_2(u)) - \int_{\mathbb{R}^2} F(x,u)dx \\ &\geq \frac{1}{4}\|u\|^2 - C_3\|u\|^3 - C_4\|u\|^{2p}. \end{aligned} \tag{3.15}$$

Therefore, there exists $\kappa_0 > 0$ and $0 < \rho < \sqrt{\frac{\pi}{\alpha_0}}$ such that

$$\Phi(u) \geq \kappa_0, \quad \forall\ u \in S := \{u \in E_{k,p} : \|u\| = \rho\}. \tag{3.16}$$

By (V0), (F1) (or (F4')) and Corollary 2.4, we have $\lim_{t\to\infty} \Phi(t\omega_{n_0}) = -\infty$, and then we can choose $t^* > 0$ such that $e = t^*\omega_{n_0} \in Y_2 := \{u \in E_{k,p} : \|u\| > \rho\}$ and $\Phi(e) < 0$. Let $Y_1 := \{u \in E_{k,p} : \|u\| \leq \rho\}$, then in view of Lemma 3.1, one deduces that there exists $\tilde{c} \in [\kappa_0, \sup_{t\geq 0} \Phi(t\omega_{n_0})]$ and a Cerami sequence $\{u_n\} \subset E_{k,p}$ satisfying (3.10).

4. Proof of Theorem 1.1

The proof of Theorem 1.1 is based on the following lemmas. As in Lemma 2.5, we only consider the $E_{k,p}$ case.

Lemma 4.1. *Assume that V and f satisfy* (V0), (VF), (F1), (F4) *and* (F5), *then we have*

(i) *Let* $m_1 := \inf_{\mathcal{N}_1} \Phi(u)$*, then there exists a constant* $c_* \in (0, m_1]$ *and a sequence* $\{u_n\} \subset E_{k,p}$ *satisfying*

$$\Phi(u_n) \to c_*, \qquad \|\Phi'(u_n)\|_{X_p'}(1 + \|u_n\|_{X_p}) \to 0. \tag{4.1}$$

(ii) *For any* $u \in E_{k,p}\backslash\{0\}$*, there exists a unique* $t_u > 0$ *such that* $t_u u \in \mathcal{N}_1$*. Moreover, we have*

$$m_1 = \inf_{E_{k,p}\backslash\{0\}} \max_{t\geq 0} \Phi(tu).$$

Proof. We will prove that

$$\Phi(u) = \max_{t\geq 0} \Phi(tu), \quad \forall\, u \in \mathcal{N}_1 \tag{4.2}$$

and then we can get the statement (i). Indeed, if (4.2) holds the same as [33, Lemma 3.2], we can choose $v_k \in \mathcal{N}_1$ such that

$$m_1 \leq \Phi(v_k) \leq m_1 + \frac{1}{k}, \quad k \in \mathbb{N}.$$

For any v_k, similarly to Lemma 3.3, we can obtain a Cerami sequence $\{u_{k,n}\}_n \subset E_{k,p}$ such that

$$\Phi(u_{k,n}) \to c_k, \qquad \|\Phi'(u_{k,n})\|_{X_p'}(1 + \|u_n\|_{X_p}) \to 0, \qquad \forall\, k \in \mathbb{N}$$

with $c_k \in (0, \sup_{t\geq 0} \Phi(tv_k)]$. By (4.2) and the diagonal rule, we can verify (4.1), and now we prove (4.2). By (2.11) and (2.12), one has

$$\begin{aligned}
\Phi(u) - \Phi(tu) &= \frac{1-t^2}{2}\|u\|^2 + \frac{1-t^{2p}}{4p\pi} I_0(u) + \int_{\mathbb{R}^2} [F(x,tu) - F(x,u)]dx \\
&= \frac{1-t^{2p}}{2p}\langle \Phi'(u), u\rangle + \frac{t^{2p} - pt^2 + p - 1}{2p}\|u\|^2 \\
&\quad + \int_{\mathbb{R}^2} \left[\frac{1-t^{2p}}{2p} f(x,u)u + F(x,tu) - F(x,u)\right] dx \\
&= \frac{1-t^{2p}}{2p}\langle \Phi'(u), u\rangle + \frac{t^{2p} - pt^2 + p - 1}{2p}\|u\|^2 \\
&\quad + \int_{\mathbb{R}^2}\int_t^1 \left[\frac{f(x,u) - V(x)u}{|u|^{2p-1}} - \frac{f(x,su) - V(x)su}{|su|^{2p-1}}\right] s^{2p-1}|u|^{2p-1} u\, ds\, dx \\
&\geq \frac{1-t^{2p}}{2p}\langle \Phi'(u), u\rangle + \frac{t^{2p} - pt^2 + p - 1}{2p}\|u\|^2.
\end{aligned} \tag{4.3}$$

According to the fact $u \in \mathcal{N}_1$ and $\min_{t\geq 0}(t^{2p} - pt^2 + p - 1)$ attained at $t = 1$, then (4.2) holds.

Next, we consider statement (ii). Let $u \in E_{k,p}\backslash\{0\}$ be fixed and $\zeta(t) := \Phi(tu)$ on $[0, \infty)$. By the definition (2.11),

$$\zeta'(t) = 0 \iff t^2\|u\|^2 + \frac{t^{2p}}{2\pi} I_0(u) - \int_{\mathbb{R}^2} f(x,tu)tu\,dx = 0 \iff tu \in \mathcal{N}_1.$$

Using (3.15), (F1) and Lemma 2.3, one has $\zeta(0) = 0$, $\zeta(t) > 0$ for $t > 0$ small and $\zeta(t) < 0$ for t large. Therefore $\max_{t\in(0,\infty)} \zeta(t)$ is achieved at some $t_u > 0$ so that $\zeta'(t_u) = 0$ and $t_u u \in \mathcal{N}_1$. Now, we claim that t_u is unique. In fact, for any given $u \in E_{k,p}\backslash\{0\}$, let $t_1, t_2 > 0$ such that $\zeta'(t_1) = \zeta'(t_2) = 0$. By (4.3), taking $t = \frac{t_2}{t_1}$ and $t = \frac{t_1}{t_2}$ respectively, it implies

$$\Phi(t_1 u) \geq \Phi(t_2 u) + \frac{t_1^2 g(t_2/t_1)}{2p}\|u\|^2 \qquad \text{and} \qquad \Phi(t_2 u) \geq \Phi(t_1 u) + \frac{t_2^2 g(t_1/t_2)}{2p}\|u\|^2,$$

where $g(t) := t^{2p} - pt^2 + p - 1$. Therefore, we must have $t_1 = t_2$, since $g(s) > 0$ for any $s > 0$, $s \neq 1$.

Lemma 4.2. *Assume that V and f satisfy* (V0), (VF), (F1), (F4) *and* (F5). *Then any sequence satisfying* (4.1) *is bounded w.r.t.* $\|\cdot\|$.

Proof. We only consider the case $p > 2$. The case $p = 2$ is obtained by [23, Lemma 2.11]. First, we prove that

$$\frac{1}{2p} f(x,t)t - F(x,t) \geq \frac{\mu(1-p)}{2p} V(x)t^2, \ \forall\, t \in \mathbb{R}. \tag{4.4}$$

Indeed, by (F4), there holds

$$\begin{aligned} F(x,t) - \frac{\mu}{2}V(x)t^2 &= \int_0^t [f(x,\tau) - \mu V(x)\tau]d\tau \\ &\leq \int_0^t \frac{f(x,t) - \mu V(x)t}{|t|^{2p-1}} |\tau|^{2p-2}\tau d\tau \\ &= \frac{f(x,t)t - \mu V(x)t^2}{2p}. \end{aligned}$$

By (4.4), one has

$$\begin{aligned} c_* + o(1) &= \Phi(u_n) - \frac{1}{2p}\langle \Phi'(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{2p}\right)\|u_n\|^2 + \int_{\mathbb{R}^2} \left(\frac{1}{2p} f(x,u_n)u_n - F(x,u_n)\right) dx \\ &\geq \left(\frac{1}{2} - \frac{1}{2p}\right)\|u_n\|^2 - \left(\frac{1}{2} - \frac{1}{2p}\right)\mu \int_{\mathbb{R}^2} V(x)u_n^2 dx \\ &\geq \frac{(p-1)(1-\mu)}{2p}\|u_n\|^2. \end{aligned} \tag{4.5}$$

Here, we also used (2.11), (2.12) and (4.1). Therefore, we complete the proof.

Proof of Theorem 1.1 completed. Applying Lemmas 4.1 and 4.2, we deduce that there exists a sequence $\{u_n\} \subset E_{k,p}$ satisfying (4.1) and $\|u_n\| \leq C < \infty$. Now, we prove

$$\int_{\mathbb{R}^2} f(x,u_n)u_n dx \leq C. \tag{4.6}$$

Indeed, let $p \geq 2$, and by (2.11), (2.12) and (4.1) there holds

$$c_* + o(1) = \Phi(u_n) - \frac{p + \mu(1-p)}{2p}\langle \Phi'(u_n), u_n\rangle$$

$$\geq \frac{(p-1)\mu}{2p}\|u_n\|^2 - \frac{(p-1)\mu}{2p}\int_{\mathbb{R}^2} V(x)u_n^2 dx + \frac{(1-p)(1-\mu)}{4p\pi} I_0(u_n)$$
$$+ \frac{(p-1)(1-\mu)}{2p}\int_{\mathbb{R}^2} f(x,u_n)u_n dx$$
$$\geq \frac{(p-1)(1-\mu)}{2p}\int_{\mathbb{R}^2} f(x,u_n)u_n dx,$$

hence (4.6) holds true. Next, we complete the proof of Theorem 1.1 in three steps.

Step 1: $\{u_n\}$ is bounded in $E_{k,p}$.

We first prove that $\delta_0 := \limsup_{n\to\infty}\|u_n\|_p > 0$. Suppose the contrary $\delta_0 = 0$, then from the Gagliardo-Nirenberg inequality (see [34, Page 125]):

$$\|u_n\|_s^s \leq C_s \|u_n\|_p^{\theta} \|\nabla u_n\|_2^{1-\theta}, \tag{4.7}$$

where $2 \leq p < t < \infty, \theta = \frac{p}{t}$. Hence, $u_n \to 0$ in $L^{\eta}(\mathbb{R}^2)$ for $\eta \in (2, +\infty)$. Given any $\varepsilon \in (0, M_0 C_{10}/t_2)$, we choose $M_\varepsilon > M_0 C_{10}/\varepsilon$, then it follows from (F2) and (4.6) that

$$\begin{aligned}\int_{|u_n|\geq M_\varepsilon} F(x,u_n)dx &\leq M_0 \int_{|u_n|\geq M_\varepsilon} |f(x,u_n)|dx \\ &\leq \frac{M_0}{M_\varepsilon}\int_{|u_n|\geq M_\varepsilon} f(x,u_n)u_n dx < \varepsilon.\end{aligned} \tag{4.8}$$

Applying (F5), one has

$$\int_{|u_n|\leq M_\varepsilon} F(x,u_n)dx \leq \begin{cases} C_\varepsilon \|u_n\|_2^2 = o(1), & p = 2, \\ C_\varepsilon \|u_n\|_{s+1}^{s+1} = o(1), & p > 2 \end{cases} \tag{4.9}$$

and

$$\int_{|u_n|\leq 1} f(x,u_n)u_n dx \leq \begin{cases} C\|u_n\|_2^2 = o(1), & p = 2, \\ C\|u_n\|_{s+1}^{s+1} = o(1), & p > 2. \end{cases} \tag{4.10}$$

By the arbitrariness of $\varepsilon > 0$, we deduce from (F2), (4.8) and (4.9) that

$$\int_{\mathbb{R}^2} F(x,u_n)dx = o(1). \tag{4.11}$$

Hence, by (2.10) we have

$$0 \leq I_2(u_n) \leq C\|u_n\|_{\frac{4p}{3}}^{2p} = o(1). \tag{4.12}$$

By Lemmas 3.2 and 3.3, we know that $\bar{\varepsilon} := \frac{1}{3}(1 - \frac{\alpha_0 \tilde{c}}{2\pi}) > 0$, which together with (2.11), (3.10), (4.11), (4.12) and the fact $I_1(u_n) \geq 0$ implies

$$\|u_n\|^2 = 2\tilde{c} - \frac{1}{2p\pi}I_1(u_n) + \frac{1}{2p\pi}I_2(u_n) + o(1) \leq 2\tilde{c} + o(1) = \frac{4\pi}{\alpha_0}(1 - 3\bar{\varepsilon}) + o(1). \tag{4.13}$$

Now, let $d \in (1, \frac{p}{p-1})$ satisfy

$$\frac{(1+\bar{\varepsilon})(1-3\bar{\varepsilon})d}{1-\bar{\varepsilon}} < 1. \tag{4.14}$$

By (F1), there exists $C > 0$ such that

$$|f(x,t)|^d \leq C\left[e^{\alpha_0(1+\bar{\varepsilon})dt^2} - 1\right], \quad \forall\ x \in \mathbb{R}^2, |t| \geq 1. \tag{4.15}$$

It follows from (4.13)–(4.15) and Lemma 2.1 that

$$\begin{aligned}\int_{|u_n|\geq 1} |f(x,u_n)|^d dx &\leq C\int_{\mathbb{R}^2}\left[e^{\alpha_0(1+\bar{\varepsilon})du_n^2} - 1\right]dx \\ &= C\int_{\mathbb{R}^2}\left[e^{\alpha_0(1+\bar{\varepsilon})d\|u_n\|^2(u_n/\|u_n\|)^2} - 1\right]dx \leq C.\end{aligned} \tag{4.16}$$

As $d' = \frac{d}{d-1} > p$, using (4.16) there holds

$$\int_{|u_n|\geq 1} f(x,u_n)u_n dx \leq \left[\int_{|u_n|\geq 1} |f(x,u_n)|^q dx\right]^{1/d} \|u_n\|_{d'} = o(1). \tag{4.17}$$

Combining (2.10)–(2.12), (3.10), (4.10) and (4.17), we arrive at

$$\begin{aligned}\tilde{c} + o(1) &= \Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle \\ &= -\left(\frac{1}{4\pi} - \frac{1}{4p\pi}\right)I_1(u_n) + \left(\frac{1}{4\pi} - \frac{1}{4p\pi}\right)I_2(u_n) \\ &\quad + \int_{\mathbb{R}^2}\left[\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right]dx \\ &\leq o(1).\end{aligned} \tag{4.18}$$

This contradiction shows that $\delta_0 > 0$. Now, from (2.10), (4.5) and Lemma 2.3, one has

$$I_1(u_n) \leq I_2(u_n) \leq C,$$

which, together with Lemma 2.5, implies that $\|u_n\|_*$ is bounded and $\{u_n\}$ is bounded in $E_{k,p}$.

Step 2: $\Phi'(\bar{u}) = 0$ in $E'_{k,p}$ and $\Phi(\bar{u}) = m_1$.

We can assume by [25, Lemma 2.3] and passing to a subsequence again if necessary, that $u_n \rightharpoonup \bar{u}$ in $E_{k,p}$, $u_n \to \bar{u}$ a.e. on $\mathbb{R}^2$ and

$$u_n \to \bar{u} \ \text{ in } \ L^s(\mathbb{R}^2),$$

where $s \in [2,\infty)$ if $p = 2$ and $s \in (2,\infty)$ if $p > 2$. First, we need prove that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} F(x,u_n)dx = \int_{\mathbb{R}^2} F(x,\bar{u})dx \qquad \text{and} \qquad \lim_{n\to\infty}\int_{\mathbb{R}^2} f(x,u_n)\bar{u}dx = \int_{\mathbb{R}^2} f(x,\bar{u})\bar{u}dx. \tag{4.19}$$

Since (4.6) and the condition (F1), (F2) and (F5) hold the same as [23, Assertions 2 and 3], (4.19) still holds. Next, we prove that

$$\lim_{n\to\infty} I_2(u_n) = I_2(\bar{u}). \tag{4.20}$$

Indeed, noting that $u_n \to \bar{u}$ in $L^{\frac{4p}{3}}(\mathbb{R}^2)$ by [35, Lemma A.1], there exists $w_0 \in L^{\frac{4p}{3}}(\mathbb{R}^2)$ such that

$$|u_n(x)| \leq w_0(x) \qquad \text{and} \qquad |\bar{u}(x)| \leq w_0(x),$$

a.e., for a subsequence if necessary, which together with the Lebesgue dominated convergence theorem and Hardy-Littlewood-Sobolev inequality implies

$$|I_2(u_n) - I_2(\bar{u})| \leq \left|A_2(|u_n|^p, |u_n|^p - |\bar{u}|^p)\right| + \left|A_2(|u_n|^p - |\bar{u}|^p, |\bar{u}|^p)\right| = o(1) \tag{4.21}$$

as $n \to \infty$ and (4.20) is proved. Now, we claim that

$$\Phi(\bar{u}) = m_1, \qquad \langle \Phi'(\bar{u}), \bar{u} \rangle_{\langle X'_p, X_p \rangle} = 0. \tag{4.22}$$

Indeed, similar as (4.20), we also have

$$A_2(|u_n|^p, |u_n|^p u_n \bar{u}) - A_2(|\bar{u}|^p, |\bar{u}|^p) = o(1). \tag{4.23}$$

By [25, Lemma 3.3], one has

$$A_1(|u_n|^p, |u_n|^{p-2}\bar{u}(u_n - \bar{u})) = o(1),$$

which together with (3.10), (4.19) and Fatou's Lemma implies

$$\begin{aligned} o(1) =& \langle \Phi'(u_n), \bar{u} \rangle_{\langle X'_p, X_p \rangle} \\ =& \langle u_n, \bar{u} \rangle + \frac{1}{2\pi} A_1(|u_n|^p, |u_n|^{p-2} u_n \bar{u}) - \frac{1}{2\pi} A_2(|u_n|^p, |u_n|^{p-2} \bar{u}^2) - \int_{\mathbb{R}^2} f(x, u_n)\bar{u}dx + o(1) \\ \geq& \langle \Phi'(\bar{u}), \bar{u} \rangle_{\langle X'_p, X_p \rangle} + o(1). \end{aligned} \tag{4.24}$$

Hence, we can obtain

$$\langle \Phi'(\bar{u}), \bar{u} \rangle_{\langle X'_p, X_p \rangle} \leq 0. \tag{4.25}$$

Since $\bar{u} \neq 0$, by Lemma 4.1 there exists $\bar{t} \in (0, 1]$ such that $\bar{t}\bar{u} \in \mathcal{N}_1$. By (3.10), (4.4) and (4.25), the weak lower semi-continuity of norm, Lemma 4.1, the condition (F4) and Fatou's Lemma, we have

$$\begin{aligned} m_1 \geq c_* =& \lim_{n\to\infty} \left[\Phi(u_n) - \frac{1}{2p} \langle \Phi'(u_n), u_n \rangle_{\langle X'_p, X_p \rangle} \right] \\ =& \lim_{n\to\infty} \left\{ \left(\frac{1}{2} - \frac{1}{2p} \right) \|\nabla u_n\|_2^2 + \int_{\mathbb{R}^2} \left[\frac{1}{2p} f(x, u_n)u_n - F(x, u_n) + \frac{p-1}{2p} V(x)u_n^2 \right] dx \right\} \\ \geq& \left(\frac{1}{2} - \frac{1}{2p} \right) \|\nabla \bar{u}\|_2^2 + \int_{\mathbb{R}^2} \left[\frac{1}{2p} f(x, \bar{u})\bar{u} - F(x, \bar{u}) + \frac{p-1}{2p} V(x)\bar{u}^2 \right] dx \\ =& \Phi(\bar{u}) - \frac{1}{2p} \langle \Phi'(\bar{u}), \bar{u} \rangle_{\langle X'_p, X_p \rangle} \\ \geq& \Phi(\bar{t}\bar{u}) - \frac{\bar{t}^{2p}}{2p} \langle \Phi'(\bar{u}), \bar{u} \rangle_{\langle X'_p, X_p \rangle} \\ \geq& m_1 - \frac{\bar{t}^{2p}}{2p} \langle \Phi'(\bar{u}), \bar{u} \rangle_{\langle X'_p, X_p \rangle} \geq m_1, \end{aligned} \tag{4.26}$$

which implies (4.22) and

$$\lim_{n\to\infty} \Phi(u_n) = m_1. \tag{4.27}$$

By (4.19), (4.20), (4.22) and (4.27) and the weak lower semi-continuity of norm and Fatou's Lemma, one has

$$o(1) = \Phi(u_n) - \Phi(\bar{u}) = \|u_n\|^2 - \|\bar{u}\|^2 + \frac{1}{4p\pi}[I_1(u_n) - I_1(\bar{u})] + o(1), \tag{4.28}$$

which implies

$$\lim_{n\to\infty} \|u_n - \bar{u}\| = 0 \qquad \text{and} \qquad \lim_{n\to\infty} I_1(u_n) = I_1(\bar{u}). \tag{4.29}$$

Hereafter, we claim that

$$A_1(|u_n|^p, |v_n|^p) \to 0, \quad as \;\; n \to \infty, \tag{4.30}$$

where $|v_n|^p := |u_n|^{p-2}|u_n - \bar{u}|^2$. Indeed, we have

$$A_1(|u_n|^p, |v_n|^p) = I_1(u_n) - 2A_1(|u_n|^p, |u_n|^{p-2}(u_n - \bar{u})\bar{u}) - A_1(|u_n|^p, |u_n|^{p-2}\bar{u}^2).$$

Then, we estimate

$$\begin{aligned} |A_1(|u_n|^p, |u_n|^{p-2}(u_n - \bar{u})\bar{u})| \le& \|u_n\|_p^p \int_{\mathbb{R}^2} |u_n|^{p-2}|u_n - \bar{u}||\bar{u}|dx \\ &+ \|u_n\|_*^p \|u_n\|_p^{p-2} \|u_n - \bar{u}\|_p \|\bar{u}\|_p \\ \le& \|u_n\|_p^p \int_{\mathbb{R}^2} |u_n|^{p-2}|u_n - \bar{u}||\bar{u}| \ln(1 + |x|)dx + o_n(1). \end{aligned} \tag{4.31}$$

For any $\epsilon > 0$, there exists $R_\epsilon > 0$ such that

$$h(R_\epsilon) := \left(\int_{\mathbb{R}^2 \setminus B_{R_\epsilon}(0)} |\bar{u}|^p \ln(1 + |x|)dx\right)^{1/p} < \epsilon.$$

Now, we split

$$\int_{\mathbb{R}^2} |u_n|^{p-2}|u_n - \bar{u}||\bar{u}| \ln(1 + |x|)dx = d_n(R_\epsilon) + e_n(R_\epsilon),$$

where

$$d_n(R_\epsilon) := \int_{B_{R_\epsilon}(0)} |u_n|^{p-2}|u_n - \bar{u}||\bar{u}| \ln(1 + |x|)dx \le R_\epsilon \|u_n\|_p^{p-2} \|u_n - \bar{u}\|_p \|\bar{u}\|_p \le \epsilon$$

for large enough n and

$$\begin{aligned} e_n(R_\epsilon) :=& \int_{\mathbb{R}^2 \setminus B_{R_\epsilon}(0)} |u_n|^{p-2}|u_n - \bar{u}||\bar{u}| \ln(1 + |x|)dx \\ \le& \|u_n - u\|_* \|u_n\|_*^{p-2} \left(\int_{\mathbb{R}^2 \setminus B_{R_\epsilon}(0)} |\bar{u}|^p \ln(1 + |x|)dx\right)^{1/p} \\ \le& Ch(R_\epsilon) \le C\epsilon, \end{aligned}$$

which together with (4.31) implies

$$\lim_{n\to\infty} A_1(|u_n|^p, |u_n|^{p-2}(u_n - \bar{u})u) = 0.$$

Hence, by Fatou's Lemma, we have

$$\limsup_{n\to\infty} A_1(|u_n|^p, |v_n|^p) \le \limsup_{n\to\infty} I_1(u_n) - \liminf_{n\to\infty} A_1(|u_n|^p, |u_n|^{p-2}\bar{u}^2) \le \limsup_{n\to\infty} I_1(u_n) - I_1(\bar{u}).$$

Since $I_1(u_n) \to I_1(\bar{u})$ and $A_1(|u_n|^p, |v_n|^p) \ge 0$, we conclude with (4.30). Finally, by [25, Lemma 3.2] we obtain $\|u_n - \bar{u}\|_* \to 0$, which together with (4.27) and (4.29) implies

$$\lim_{n\to\infty} \|u_n - \bar{u}\|_{X_p} = 0 \qquad \text{and} \qquad \Phi(\bar{u}) = m_1.$$

Step 3: $\Phi'(\bar{u}) = 0$ in X_p'.

By using the group action on the space X_p, we will conclude $\Phi'(\bar{u}) = 0$. Let $G \subset \mathcal{O}(2)$ be a finite group of transforms acting on X_p, where $\mathcal{O}(2)$ denotes the group of orthogonal transformations in $\mathbb{R}^2$. The action of G on the space X_p is a continuous map (see [35, Definition 1.27]):

$$G \times X_p \to X_p : [\tau, u] \to \tau(u) = u \circ \tau.$$

Assume that $\varphi \in C^1(X_p, \mathbb{R})$ is invariant by G; that is, $\varphi(w \circ \tau) = \varphi(w)$ for any $\tau \in G$, $w \in X_p$. Let u be a critical point of φ in $X_{p,G}$, where

$$X_{p,G} := \{u \in X_p : \tau u = u, \ \forall\, \tau \in G\}.$$

Then $\varphi'(u) = 0$ in X_p'. In fact, given any $v \in X_p$, we define

$$\bar{v} = \frac{1}{\#(G)} \sum_{\tau\in G} \tau v,$$

where $\#(G)$ denotes the cardinal of G. For any $\tau_0 \in G$, since $G \subset \mathcal{O}(2)$, we have

$$\tau_0 \bar{v} = \tau_0 \left[\frac{1}{\#(G)} \sum_{\tau\in G} \tau v \right] = \frac{1}{\#(G)} \sum_{\tau\in G} \tau_0 \tau v = \bar{v},$$

which implies $\bar{v} \in X_{p,G}$. Therefore, one has

$$\begin{aligned} 0 = \langle \varphi'(u), \bar{v} \rangle = \frac{1}{\#(G)} \sum_{\tau\in G} \langle \varphi'(u), \tau v \rangle &= \frac{1}{\#(G)} \sum_{\tau\in G} \langle \varphi'(u) \circ \tau^{-1}, v \rangle \\ &= \frac{1}{\#(G)} \sum_{\tau\in G} \langle \varphi'(u \circ \tau^{-1}) \circ \tau^{-1}, v \rangle \\ &= \langle \varphi'(u), v \rangle. \end{aligned}$$

For the second line, we used the fact $u \in X_{p,G}$, so we have $\varphi'(u) = 0$ in X_p'.

The two cases in Theorem 1.1 are direct consequences of the above discussion. Indeed, let G_1 be the subgroup of $\mathcal{O}(2)$ generated by $z \mapsto ze^{2\pi i/k}$, then $X_{p,G_1} = E_{k,p}$. If G_2 is generated by $z \mapsto ze^{2\pi i/k}$ and $z \mapsto \bar{z}$, $X_{p,G_2} = T_{k,p}$, so $\Phi'(\bar{u}) = 0$ in X_p'.

5. Proof of Theorem 1.3

We prove Theorem 1.3 in three steps for subcritical case and the Ambrosetti-Rabinowitz condition (F4'). As in the proof of Theorem 1.1, we only consider the function space $E_{k,p}$.

Step 1: $\|u_n\|$ is bounded.

We only consider the case $p > 2$. Same as critical case, the case $p = 2$ can be obtained by [23, Lemma 2.11]. Applying Lemma 3.3 and (5.1), there exists a sequence $\{u_n\} \subset E_{k,p}$ satisfying (3.10). By (3.10) and (F4'), we can choose a constant $\lambda_0 \in \left(\frac{1}{\nu}, \frac{1}{2} - \frac{M_{t_1}}{\gamma^2}\right)$ and then we have

$$\begin{aligned}
\tilde{c} + o(1) &= \Phi(u_n) - \lambda_0 \langle \Phi'(u_n), u_n \rangle \\
&= \left(\frac{1}{2} - \lambda_0\right) \|u_n\|^2 + \frac{1}{2\pi}\left(\frac{1}{2p} - \lambda_0\right) I_0(u_n) \\
&+ \int_{\mathbb{R}^2} (\lambda_0 f(x, u_n)u_n - F(x, u_n))dx \\
&\geq \left(\frac{1}{2} - \lambda_0\right) \|u_n\|^2 - \int_{\{|u_n| < t_1\}} F(x, u_n)dx \\
&+ \left(\lambda_0 - \frac{1}{\nu}\right) \int_{\|u_n\| \geq t_1} f(x, u_n)u_n dx \\
&\geq \left(\frac{1}{2} - \frac{M_{t_1}}{\gamma^2} - \lambda_0\right) \|u_n\|^2,
\end{aligned} \tag{5.1}$$

and then $\|u_n\|$ is bounded.

Step 2: $\{u_n\}$ is bounded in $E_{k,p}$.

As in the proof of the critical case, we first prove $\delta_0 := \limsup_{n\to\infty} \|u_n\|_p > 0$. Suppose the contrary $\delta_0 = 0$. Denoting $M_* := \sup_n \|u_n\|$ and $M_{**} := \sup_n \|u_n\|_2$. By (F1') and (F5), choosing $\alpha \in (0, \frac{p-1}{pM_*^2})$, one has

$$|f(x, t)| \leq \frac{\tilde{c}}{2M_{**}^2}|t| + C\left(e^{\alpha t^2} - 1\right), \quad \forall\ (x, t) \in \mathbb{R}^2 \times \mathbb{R}. \tag{5.2}$$

By (5.2) and Lemma 2.1, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} f(x, u_n)u_n dx &\leq \frac{\tilde{c}}{2M_{**}^2}\|u_n\|_2^2 + C\int_{\mathbb{R}^2} \left(e^{\alpha u_n^2} - 1\right)|u_n|dx \\
&\leq \frac{\tilde{c}}{2} + C\left[\int_{\mathbb{R}^2} \left(e^{\frac{p}{p-1}\alpha u_n^2} - 1\right)dx\right]^{\frac{p-1}{p}} \|u_n\|_p \\
&= \frac{\tilde{c}}{2} + C\left[\int_{\mathbb{R}^2} \left(e^{\frac{p}{p-1}\alpha \|u_n\|^2 (u_n^2/\|u_n\|^2)} - 1\right)dx\right]^{\frac{p-1}{p}} \|u_n\|_p \\
&\leq \frac{\tilde{c}}{2} + o(1).
\end{aligned} \tag{5.3}$$

Hence, by (5.3) and Lemma 3.3, we know

$$\begin{aligned}\tilde{c} + o(1) &= \Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle \\ &= -\frac{p-1}{4p\pi} I_1(u_n) + \frac{p-1}{4p\pi} I_2(u_n) + \int_{\mathbb{R}^2} \left[\frac{1}{2} f(x, u_n)u_n - F(x, u_n)\right] dx \\ &\leq \frac{\tilde{c}}{2} + o(1).\end{aligned} \tag{5.4}$$

This contradiction shows that $\delta_0 > 0$. Now, from (2.10), (5.1) and Lemma 2.3, one has

$$I_1(u_n) \leq I_2(u_n) \leq C,$$

which, together with Lemma 2.5, implies that $\|u_n\|_*$ is bounded and $\{u_n\}$ is bounded in $E_{k,p}$.

Step 3: $\Phi'(\bar{u}) = 0$ in X_p'.

We may assume by [25, Lemma 2.3] and passing to a subsequence again if necessary, that $u_n \rightharpoonup \bar{u}$ in $E_{k,p}$, $u_n \to \bar{u}$, a.e., on $\mathbb{R}^2$ and

$$u_n \to \bar{u} \ \text{ in } \ L^s(\mathbb{R}^2),$$

where $s \in [2, \infty)$ if $p = 2$ and $s \in (2, \infty)$ if $p > 2$. Let $M := \sup_n \|\nabla u_n\|_2$. By (F1'), we can choose $\alpha > 0$ small enough such that $M^2 < \frac{4\pi}{\alpha}$. Therefore, there is $\beta > p$ big enough such that $\frac{M^2\beta}{\beta-1} < \frac{4\pi}{\alpha}$. Without loss of generality, we may assume that $s_0 \in (1, 2)$ in (F5). Then, it follows (F5) and Lemma 2.1 that

$$\begin{aligned}&\int_{\mathbb{R}^2} |f(x, u_n)(u_n - \bar{u})| dx \\ &\leq \int_{\{|u_n|<1\}} |f(x, u_n)(u_n - \bar{u})| dx + \int_{\{|u_n|\geq 1\}} |f(x, u_n)(u_n - \bar{u})| dx \\ &\leq C\|u_n\|_2 \|u_n - \bar{u}\|_{\frac{2}{2-s_0}} + C\|u_n - \bar{u}\|_\beta \\ &= o(1).\end{aligned} \tag{5.5}$$

Similarly, one has

$$\int_{\mathbb{R}^2} |f(x, \bar{u})(u_n - \bar{u})| dx = o(1). \tag{5.6}$$

Furthermore, it follows from (2.9), (2.10) and the Hölder inequality that

$$A_2(|u_n|^p, |u_n|^{p-2}u_n(u_n - \bar{u})) = o(1), \qquad A_2(|\bar{u}|^p, |\bar{u}|^{p-2}\bar{u}(u_n - \bar{u})) = o(1). \tag{5.7}$$

By [25, Lemma 3.3], we have

$$A_1(|u_n|^p, |u_n|^{p-2}\bar{u}(u_n - \bar{u})) = o(1), \qquad A_1(|\bar{u}|^p, |\bar{u}|^{p-2}\bar{u}(u_n - \bar{u})) = o(1). \tag{5.8}$$

Combining (2.11), (2.12), (3.10), and (5.5)–(5.8), there holds

$$\begin{aligned} o(1) &= \langle \Phi'(u_n) - \Phi'(\bar{u}), u_n - \bar{u} \rangle_{\langle X'_p, X_p \rangle} \\ &= \|u_n - \bar{u}\|^2 + \frac{1}{2\pi} A_1(|u_n|^p, |u_n|^{p-2}(u_n - \bar{u})^2) \\ &\quad + \frac{1}{2\pi} A_1(|u_n|^p, |u_n|^{p-2}\bar{u}(u_n - \bar{u})) - \frac{1}{2\pi} A_1(|\bar{u}|^p, |\bar{u}|^{p-2}\bar{u}(u_n - \bar{u})) \\ &\quad + \frac{1}{2\pi} A_2(|\bar{u}|^p, |\bar{u}|^{p-2}\bar{u}(u_n - \bar{u})) - \frac{1}{2\pi} A_2(|u_n|^p, |u_n|^{p-2}u_n(u_n - \bar{u})) \\ &\quad + \int_{\mathbb{R}^2} f(x, \bar{u})(u_n - \bar{u}) dx - \int_{\mathbb{R}^2} f(x, u_n)(u_n - \bar{u}) dx + o(1) \\ &\geq \|u_n - \bar{u}\|^2 + o(1). \end{aligned} \tag{5.9}$$

By (2.10), (5.9) and Lemma 2.1, we have

$$A_2(|u_n|^p, |v_n|^p) = o(1), \qquad \langle u_n, u_n - \bar{u} \rangle = o(1), \qquad \int_{\mathbb{R}^2} f(x, u_n)(u_n - \bar{u}) dx = o(1), \tag{5.10}$$

where $|v_n|^p := |u_n|^{p-2}|u_n - \bar{u}|^2$ for every $n \in \mathbb{N}$. By (3.10) and (5.10), one has

$$\begin{aligned} o(1) = \langle \Phi'(u_n), u_n - \bar{u} \rangle_{\langle X'_p, X_p \rangle} &= \frac{1}{2\pi} A_1(|u_n|^p, |v_n|^p) - \frac{1}{2\pi} A_2(|u_n|^p, |v_n|^p) \\ &\quad + \langle u_n, u_n - \bar{u} \rangle + \int_{\mathbb{R}^2} f(x, u_n)(u_n - \bar{u}) dx \\ &= \frac{1}{2\pi} A_1(|u_n|^p, |v_n|^p) + o(1) \end{aligned}$$

which, together with Lemma 2.5, implies

$$\lim_{n\to\infty} (\|v_n\|_p + \|v_n\|_*) \to 0. \tag{5.11}$$

From (5.11) and [25, Lemma 3.3], one has

$$\begin{aligned} \|u_n - \bar{u}\|_*^p &= \int_{\mathbb{R}^2} \ln(1 + |x|)(|u_n - \bar{u}|^{p-2} - |u_n|^{p-2})|u_n - \bar{u}|^2 dx + o(1) \\ &\leq \frac{1}{2}\|u_n - \bar{u}\|_*^p + C \int_{\mathbb{R}^2} \ln(1 + |x|)|\bar{u}|^{p-2}|u_n - \bar{u}|^2 dx + o(1) \\ &= \frac{1}{2}\|u_n - \bar{u}\|_*^p + o(1), \end{aligned} \tag{5.12}$$

where we used the following inequality

$$\left| |a + b|^{p-2} - |b|^{p-2} \right| \leq \frac{1}{2}|b|^{p-2} + C|a|^{p-2},$$

and C is independent of $a, b \in \mathbb{R}$. Combining with (5.9), we have $u_n \to \bar{u}$ in $E_{k,p}$. Hence, $0 < \tilde{c} = \lim_{n\to\infty} \Phi(u_n) = \Phi(\bar{u})$ and $\Phi'(\bar{u}) = 0$ in $E'_{k,p}$. We conclude that $\Phi'(\bar{u}) = 0$ in X'_p, as in the proof of step 3 of Theorem 1.1.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is partially supported by National Natural Science Foundation of China, No. 12071189. Thanks to the reviewers for their valuable suggestions and comments. The author would like to express the sincere gratitude to his tutors, professors Dong Ye and Feng Zhou, for their valuable guidance and insights that helped to improve the paper.

Conflict of interest

The authors declare there are no conflicts of interest.

References

1. D. M. Cao, Nontrivial solution of semilinear elliptic equation with critical exponent in $\mathbb{R}^2$, *Commun. Partial Differ. Equations*, **17** (1992), 407–435. https://doi.org/10.1080/03605309208820848
2. R. Benguria, H. Brezis, E. H. Lieb, The Thomas-Fermi-von Weizsäcker theory of atoms and molecules, *Commun. Math. Phys.*, **79** (1981), 167–180. https://doi.org/10.1007/bf01942059
3. I. Catto, P. L. Lions, Binding of atoms and stability of molecules in Hartree and Thomas-Fermi type theories, *Commun. Partial Differ. Equations*, **18** (1993), 1149–1159. https://doi.org/10.1080/03605309308820967
4. E. H. Lieb, Thomas-Fermi and related theories of atoms and molecules, *Rev. Mod. Phys.*, **53** (1981), 603–641. https://doi.org/10.1103/revmodphys.53.603
5. V. Benci, D. Fortunato, An eigenvalue problem for the Schrödinger-Maxwell equations, *Topol. Methods Nonlinear Anal.*, **11** (1998), 283–293. https://doi.org/10.12775/tmna.1998.019
6. V. Benci, D. Fortunato, Solitary waves of the nonlinear Klein-Gordon equation coupled with Maxwell equations, *Rev. Math. Phys.*, **14** (2002), 409–420. https://doi.org/10.1142/s0129055x02001168
7. P. L. Lions, Solutions of Hartree-Fock equations for Coulomb systems, *Commun. Math. Phys.*, **109** (1987), 33–97. https://doi.org/10.1007/bf01205672
8. P. A. Markowich, C. A. Ringhofer, C. Schmeiser, *Semiconductor Equations*, 1^{nd} edition, Springer, 1990. https://doi.org/10.1007/978-3-7091-6961-2
9. G. Cerami, G. Vaira, Positive solutions for some non-autonomous Schrödinger-Poisson systems, *J. Differ. Equations*, **248** (2010), 521–543. https://doi.org/10.1016/j.jde.2009.06.017
10. D. Ruiz, The Schrödinger-Poisson equation under the effect of a nonlinear local term, *J. Funct. Anal.*, **237** (2006), 655–674. https://doi.org/10.1016/j.jfa.2006.04.005

11. R. P. Agarwal, A. M. Alghamdi, S. Gala, M. A. Ragusa, Regularity criteria via horizontal component of velocity for the Boussinesq equations in anisotropic Lorentz spaces, *Demonstr. Math.*, **56** (2023), 20220221. https://doi.org/10.1515/dema-2022-0221

12. A. Ambrosetti, D. Ruiz, Multiple bound states for the Schrödinger-Poisson problem, *Commun. Contemp. Math.*, **10** (2008), 391–404. https://doi.org/10.1142/s021919970800282x

13. A. El-Abed, A. A. B. Ali, M. Dammak, Schrödinger equation with asymptotically linear nonlinearities, *Filomat*, **36** (2022), 629–639. https://doi.org/10.2298/FIL2202629E

14. G. C. Yang, S. Z. Duan, Existence solutions for a class of Schrödinger-Maxwell systems with steep well potential, *J. Funct. Spaces*, **2022** (2022), 6791308. https://doi.org/10.1155/2022/6791308

15. L. G. Zhao, F. K. Zhao, On the existence of solutions for the Schrödinger-Poisson equations, *J. Math. Anal. Appl.*, **346** (2008), 155–169. https://doi.org/10.1016/j.jmaa.2008.04.053

16. M. Colin, L. Jeanjean, Solutions for a quasilinear Schrödinger equation: A dual approach, *Nonlinear Anal.*, **56** (2004), 213–226. https://doi.org/10.1016/j.na.2003.09.008

17. L. Zhang, Y. Q. Li, Z. Q. Wang, Multiple normalized solutions for a quasi-linear Schrödinger equation via dual approach, *Topol. Methods Nonlinear Anal.*, **61** (2023), 465–489. https://doi.org/10.12775/tmna.2022.052

18. X. G. Zhang, L. S. Liu, Y. H. Wu, Y. J. Cui, Existence of infinitely solutions for a modified nonlinear Schrödinger equation via dual approach, *Electron. J. Differ. Equations*, **2018** (2018), 1–15.

19. X. G. Zhang, L. S. Liu, Y. H. Wu, B. Wiwatanapataphee, Multiple solutions for a modified quasilinear Schrödinger elliptic equation with a nonsquare diffusion term, *Nonlinear Anal. Model. Control*, **26** (2021), 702–717. https://doi.org/10.15388/namc.2021.26.22520

20. X. G. Zhang, L. S. Liu, Y. H. Wu, B. Wiwatanapataphee, Y. J. Cui, Solvability and asymptotic properties for an elliptic geophysical fluid flows model in a planar exterior domain, *Nonlinear Anal. Model. Control*, **26** (2021), 315–333. https://doi.org/10.15388/namc.2021.26.21202

21. J. Stubbe, Bound states of two-dimensional Schrödinger-Newton equations, *arXiv preprint*, (2008), arXiv:0807.4059. https://doi.org/10.48550/arXiv.0807.4059

22. S. Cingolani, T. Weth, On the planar Schrödinger-Poisson system, in *Annales de l'Institut Henri Poincaré C, Analyse non Linéaire*, (2016), 169–197. https://doi.org/10.1016/j.anihpc.2014.09.008

23. S. T. Chen, X. H. Tang, Axially symmetric solutions for the planar Schrödinger-Poisson system with critical exponential growth, *J. Differ. Equations*, **269** (2020), 9144–9174. https://doi.org/10.1016/j.jde.2020.06.043

24. S. L. Yadava, Multiplicity results for semilinear elliptic equations in a bounded domain of $\mathbb{R}^2$ involving crtical exponents, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **17** (1990), 481–504.

25. D. M. Cao, W. Dai, Y. Zhang, Existence and symmetry of solutions to 2-D Schrödinger-Newton equations, *Dyn. Partial Differ. Equations*, **18** (2021), 113–156. https://doi.org/10.4310/dpde.2021.v18.n2.a3

26. E. H. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, *Ann. Math.*, **118** (1983), 349–374. https://doi.org/10.2307/2007032

27. J. M. B. do Ó, N-Laplacian equations in $\mathbb{R}^N$ with critical growth, *Abstr. Appl. Anal.*, **2** (1997), 301–315. https://doi.org/10.1155/s1085337597000419
28. S. Adachi, K. Tanka, Trudinger type inequalities in $\mathbb{R}^N$ and their best exponents, *Proc. Am. Math. Soc.*, **128** (2000), 2051–2057. https://doi.org/10.1090/s0002-9939-99-05180-1
29. D. Cassani, F. Sani, C. Tarsi, Equivalent Moser type inequalities in $\mathbb{R}^2$ and the zero mass cases, *J. Funct. Anal.*, **267** (2014), 4236–4263. https://doi.org/10.1016/j.jfa.2014.09.022
30. L. C. Evans, *Partial Differential Equations,* 2^{nd} edition, American Mathematical Society, 2010. https://doi.org/10.1090/gsm/019
31. S. T. Chen, X. H. Tang, On the planar Schrödinger-Poisson system with the axially symmetric potentials, *J. Differ. Equations*, **268** (2020), 945–976. https://doi.org/10.1016/j.jde.2019.08.036
32. E. A. B. Silva, G. F. Vieira, Quasilinear asymptotically periodic Schrödinger equations with critical growth, *Calc. Var. Partial Differ. Equations*, **39** (2010), 1–33. https://doi.org/10.1007/s00526-009-0299-1
33. S. T. Chen, J. P. Shi, X. H. Tang, Ground state solutions of Nehari-Pohozaev type for the planar Schrödinger-Poisson system with general nonlinearity, *Discrete Contin. Dyn. Syst.*, **39** (2019), 5867–5889. https://doi.org/10.3934/dcds.2019257
34. L. Nirenberg, On elliptic partial differential equations, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **13** (1959), 115–162.
35. M. Willem, *Minimax Theorems,* 1^{nd} edition, Springer, 1996. https://doi.org/10.1007/978-1-4612-4146-1
36. D. G. de Figueiredo, O. H. Miyagaki, B. Ruf, Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range, *Calc. Var. Partial Differ. Equations*, **3** (1995), 139–153. https://doi.org/10.1007/BF01205003

Appendix

In this section, we give the proof of Lemma 2.2. For any $u \in X_p$, we denote $\Psi(u) := \int_{\mathbb{R}^2} F(x,u)dx$. In fact, we just need to prove $\Psi \in C^1(X_p, \mathbb{R})$, and the readers can refer to [25, Lemma 2.3] for the rest. First, given any $u, v \in X_p$, for almost every $x \in \mathbb{R}^2$

$$\lim_{t\to 0} \frac{F(x,u(x)+tv(x)) - F(x,u(x))}{t} = f(x,u(x))v(x).$$

On the other hand, we can choose a large enough number $t_1 > 0$ such that

$$|f(x,t)| \le e^{(\alpha_0+1)t^2} - 1, \ \forall\, |t| \ge t_1.$$

By (F1), (F5) and Lemma 2.1, one has, for any $u \in X_p$,

$$\begin{aligned}\int_{\mathbb{R}^2} |f(x,u)|^2 dx &= \int_{\{|u|\le t_1\}} |f(x,u)|^2 dx + \int_{\{|u|\ge t_1\}} |f(x,u)|^2 dx \\ &\le C\|u\|_2^2 + \int_{\mathbb{R}^2} \left(e^{(\alpha_0+1)u^2} - 1\right) dx\end{aligned}$$

$$\leq C.$$

Then, for any $u \in X_p$, $f(x,u) \in L^2(\mathbb{R}^2)$, it implies that the Gateaux derivative $\Psi'_g(u)$ exists and $\Psi'_g(u) \in X'_p$.

Now let $\{u_n\} \subset X_p$, $\|u_n - \bar{u}\|_{X_p} \to 0$. Hence, $u_n \to \bar{u}$ in $H^1(\mathbb{R}^2)$. Let us prove $\Psi'(\bar{u}) = \lim_{n\to\infty} \Psi'(u_n)$. It suffices to prove

$$\lim_{n\to\infty} \sup_{\|v\|_{X_p}=1} \left| \int_{\mathbb{R}^2} [f(x,u_n) - f(x,\bar{u})] v dx \right| = 0.$$

Define that $M := \sup_n \|\nabla u_n\|_2$, then we prove this lemma in two cases.

Special case: $M \leq \sqrt{\frac{\pi}{2\alpha_0}}$.

For any given $\varepsilon \in (0,1)$, we choose large enough $R_\varepsilon > 0$ such that

$$\|v\|_{L^p(B_R^c)} \leq \|v\|_{L^p(B_{R_\varepsilon}^c)} \leq \varepsilon[\ln(1+R_\varepsilon)]^{1/p}\|v\|_{L^p(B_{R_\varepsilon}^c)} \leq \varepsilon\|v\|_{X_p}, \ \forall\, R \geq R_\varepsilon$$

and

$$\|\bar{u}\|_{L^2(B_R^c)} \leq \|\bar{u}\|_{L^2(B_{R_\varepsilon}^c)} \leq \varepsilon, \ \forall\, R \geq R_\varepsilon.$$

By Lemma 2.1 and $\alpha_0 \leq \frac{\pi}{2M^2}$, it is easy to verify that there is a $C_0 > 0$ such that $\|u\|_4 \leq C_0\|u\|_{X_p}$ for all $u \in X_p$ and

$$\int_{B_{R_\varepsilon}} |f(x,u_n)|^2 dx + \int_{B_{R_\varepsilon}} |f(x,u_n)|^2 |u_n| dx \leq C, \quad \forall\, n.$$

Now, we claim that

$$\limsup_{n\to\infty} \{\|f(x,u_n) - f(x,\bar{u})\|_2 + \|u_n - \bar{u}\|_2\} \leq C\varepsilon. \tag{A1}$$

The proof of (A1) is in spirit of [36, Lemma 2.1]. As $L^2(B_{R_\varepsilon})$ is a Hilbert space, we need only to prove

$$\limsup_{n\to\infty} \int_{B_{R_\varepsilon}} (|f(x,u_n)|^2 - |f(x,\bar{u})|^2) dx \leq C\varepsilon$$

for the first part of (A1). Let M' be large enough such that

$$\int_{\{|u_n|\geq M'\}\cap B_{R_\varepsilon}} |f(x,u_n)|^2 dx = \int_{\{|u_n|\geq M'\}\cap B_{R_\varepsilon}} \frac{|f(x,u_n)|^2|u_n|}{|u_n|} dx \leq \frac{C_0}{M'} \leq \varepsilon.$$

By the dominated convergence theorem and Fatou's Lemma, one has

$$\begin{aligned}
&\left| \int_{B_{R_\varepsilon}} (|f(x,u_n)|^2 - |f(x,\bar{u})|^2) \right| \\
&\leq \int_{\{|u_n|\geq M'\}\cap B_{R_\varepsilon}} \frac{|f(x,u_n)|^2|u_n|}{M'} dx + \int_{\{|u|\geq M'\}\cap B_{R_\varepsilon}} \frac{|f(x,u_n)|^2|\bar{u}|}{M'} dx \\
&+ \int_{B_{R_\varepsilon}} h_n(x) dx \\
&= 2\varepsilon + o_n(1),
\end{aligned}$$

where $h_n(x) := \left||f(x,u_n(x))|^2\chi_{\{|u_n|<M'\}\cap B_{R_\varepsilon}} - |f(x,\bar{u}(x))|^2\chi_{\{|u_n|<M'\}\cap B_{R_\varepsilon}}\right|$, and we use the fact

$$|h_n(x)| \le \begin{cases} |f(x,\bar{u}(x))|^2, & |u_n| \ge M', \\ \sup\{|f(x,t)| : x \in \overline{B_{R_\varepsilon}}, |t| < M'\} + |f(x,\bar{u}(x))|^2, & |u_n| < M'. \end{cases}$$

Therefore, we get (A1). By (A1), for large n, one has

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} [f(x,u_n) - f(x,\bar{u})]v dx\right| \\
\le& \int_{B_{R_\varepsilon}} |f(x,u_n) - f(x,\bar{u})||v|dx + \int_{B_{R_\varepsilon}^c} |f(x,u_n) - f(x,\bar{u})||v|dx \\
\le& \frac{1}{\gamma}\varepsilon\|v\|_{X_p} + \frac{C}{\gamma}(\|u_n - \bar{u}\|_2 + 2\|\bar{u}\|_{L^2(B_{R_\varepsilon}^c)})\|v\|_{X_p} \\
&+ 2^{2/p}C\left(\int_{\mathbb{R}^2}\left[\exp\left(p'\alpha u_n^2\right) - 1 + \exp\left(p'\alpha\bar{u}^2\right) - 1\right]dx\right)^{1/p'} \varepsilon\|v\|_{X_p} \\
\le& C\varepsilon\|v\|_{X_p},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\gamma := \inf_{u\in X} \frac{\|u\|}{\|u\|_{H^1(\mathbb{R}^2)}} > 0$ (see [31, Lemma 2.1]).

General case: $M > 0$.

For any $R > 0$, let $\varphi_R(r)$ be a C_0^∞ cut-off function such that $0 \le \varphi \le 1$, $\varphi_R \equiv 1$ on $[0, R]$ and $\varphi_R \equiv 0$ on $[R+1, \infty)$. Let $\delta > 0$ (to be determined later), we can choose large enough bounded domain $B_R(0)$ and its bounded open coverage $\{\Omega_\ell\}_{\ell \le N_c}$ which has a partition of unity w_ℓ $(1 \le \ell \le N_c)$ such that

$$\|\varphi_{R-1}(|x|)\bar{u}(x) - \bar{u}(x)\| \le \delta;$$

$$B_R(0) \subset \bigcup_{1\le\ell\le N_c} \Omega_\ell, \qquad \sum_{\ell=1}^{N_c} w_\ell(x) = 1, \qquad \forall\ x \in B_R(0);$$

$$w_\ell \in C_c^1(\Omega_\ell), \qquad |\nabla w_\ell| \le C, \qquad \forall\ \ell;$$

$$\int_{\Omega_\ell} |\psi_\ell|^2 dx \le \delta, \qquad \int_{\Omega_\ell} |\psi_{\ell,n}|^2 dx \le \delta, \qquad \forall\ n, \ell;$$

$$\int_{\Omega_\ell} |\nabla\psi_\ell|^2 dx \le \delta, \qquad \int_{\Omega_\ell} |\nabla\psi_{\ell,n}|^2 dx \le \delta, \qquad \forall\ n, \ell;$$

where

$$\psi_\ell(x) = \varphi_R(|x|)w_\ell\bar{u}, \qquad \psi_{\ell,n}(x) = \varphi_R(|x|)w_\ell u_n.$$

Choosing $\delta > 0$ small enough and repeating now the proof of the special case, we can prove

$$\int_{B_R^c(0)} \left|[f(x,u_n) - f(x,\bar{u})]v\right| dx \le C\varepsilon\|v\|_X$$

and

$$\int_{\Omega_\ell} |[f(x,u_n) - f(x,\bar{u})]v|\, dx \le C\varepsilon\|v\|_X, \quad \forall\ \ell.$$

Therefore, one has

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^2}[f(x,u_n)-f(x,\bar{u})]vdx\right| \\
\leq &\int_{B_R^c(0)}|(f(x,u_n)-f(x,\bar{u}))v|\,dx+\sum_{\ell=1}^{N_c}\int_{\Omega_i}|(f(x,u_n)-f(x,\bar{u}))v|\,dx \\
\leq &(N_c+1)C\varepsilon\|v\|_X.
\end{aligned}
$$

So we obtain Lemma 2.2.