Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 31(11): 6697–6724.
DOI: 10.3934/era.2023338
Received: 03 May 2023
Revised: 21 August 2023
Accepted: 28 September 2023
Published: 17 October 2023

Research article

Investigation of the global dynamics of two exponential-form difference equations systems

Merve Kara*

Department of Mathematics, Kamil Ozdag Science Faculty, Karamanoglu Mehmetbey University, Karaman 70100, Turkey

* **Correspondence:** Email:mervekara@kmu.edu.tr.

Abstract: In this study, we investigate the boundedness, persistence of positive solutions, local and global stability of the unique positive equilibrium point and rate of convergence of positive solutions of the following difference equations systems of exponential forms:

$$\Upsilon_{n+1} = \frac{\Gamma_1 + \delta_1 e^{-\Psi_{n-1}}}{\Theta_1 + \Psi_n},\ \Psi_{n+1} = \frac{\Gamma_2 + \delta_2 e^{-\Omega_{n-1}}}{\Theta_2 + \Omega_n},\ \Omega_{n+1} = \frac{\Gamma_3 + \delta_3 e^{-\Upsilon_{n-1}}}{\Theta_3 + \Upsilon_n},$$

$$\Upsilon_{n+1} = \frac{\Gamma_1 + \delta_1 e^{-\Psi_{n-1}}}{\Theta_1 + \Upsilon_n},\ \Psi_{n+1} = \frac{\Gamma_2 + \delta_2 e^{-\Omega_{n-1}}}{\Theta_2 + \Psi_n},\ \Omega_{n+1} = \frac{\Gamma_3 + \delta_3 e^{-\Upsilon_{n-1}}}{\Theta_3 + \Omega_n},$$

for $n \in \mathbb{N}_0$, where the initial conditions Υ_{-j}, Ψ_{-j}, Ω_{-j}, for $j \in \{0, 1\}$ and the parameters Γ_i, δ_i, Θ_i for $i \in \{1, 2, 3\}$ are positive constants.

Keywords: boundedness; system of difference equations; persistence; stability

1. Introduction and preliminaries

First of all, it is important to keep in mind that $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}^+ \cup \{0\}$, refer to the non-negative integer set, integer set, real set, positive real set and non-negative real set, respectively. If $\Phi, \Psi \in \mathbb{Z}$ and $\Phi \leq \Psi$, the notation $k = \overline{\Phi, \Psi}$ stands for $\{k \in \mathbb{Z} : \Phi \leq k \leq \Psi\}$.

It is of utmost importance to use different branches of science in tandem. Mathematics can also be deployed in research fields of other scientific areas. Especially, mathematical models of many sorts have been developed for many areas of study, such as ecology, medicine, population biology, biostatistics and molecular biology in [1]. In order to apply mathematical modeling in biology, both mathematical and biological information are required. As such, multidisciplinary or interdisciplinary research studies are very popular these days (see [2–15]).

Difference equations, which represent a topic of applied mathematics, can also be used in mathematical modeling. In addition, difference equations are used to define real discrete models in different areas of advanced science, such as control theory, biology, physics, economics and psychology. One of the applications of difference equations in biology is the population model. Some population models involve exponential-form difference equations. Difference equations of the mentioned type attract the attention of mathematicians. But, their stability analysis process can be complex. Therefore, studying these difference equations and systems is worthwhile. To date, a significant number of papers concerning difference equations have been published (see [16–27]).

As a model in the field of mathematical biology, several difference equations of the exponential-type were studied. For example, the following single-species population model

$$U_{n+1} = \kappa + \epsilon U_{n-1}e^{-U_n},\ n \in \mathbb{N}_0, \tag{1.1}$$

where $\kappa, \epsilon \in \mathbb{R}^+$ and $U_{-1}, U_0 \in \mathbb{R}^+ \cup \{0\}$, was investigated by El-Metwally et al. in [28]. κ was used as the migration rate and ϵ was taken as the population growth rate in Eq (1.1).

There is another example, in which the authors of [29] changed Eq (1.1) to a newer version as;

$$U_{n+1} = \kappa + \epsilon U_n e^{-U_{n-1}},\ n \in \mathbb{N}_0, \tag{1.2}$$

where $U_{-1}, U_0 \in \mathbb{R}^+ \cup \{0\}$ and $\kappa, \epsilon \in \mathbb{R}^+$. As a model in the field of mathematical biology, Eq (1.2) was investigated. In Eq (1.2), κ is the migration rate and ϵ is the population growth rate.

The authors of [30] researched the dynamical properties of the next exponential-type difference equation:

$$U_{n+1} = \frac{\kappa + \epsilon e^{-U_n}}{\zeta + U_{n-1}},\ n \in \mathbb{N}_0, \tag{1.3}$$

where $U_{-1}, U_0 \in \mathbb{R}^+ \cup \{0\}$ and $\kappa, \epsilon, \zeta \in \mathbb{R}^+$. Equation (1.3) represent a mathematical biology-purposed model, where κ is the migration rate, ϵ is the population growth proportion and ζ is the carrying capacity.

Comert et al. in [31], explored the global behavior of the next exponential-type difference equation:

$$U_{n+1} = \frac{\delta + \epsilon e^{-U_n}}{\zeta + U_{n-k}},\ n \in \mathbb{N}_0, \tag{1.4}$$

where k is an even number, the initial values $U_{-k}, U_{-k+1}, \ldots, U_0 \in \mathbb{R}^+ \cup \{0\}$ and the parameters $\delta, \epsilon, \zeta \in \mathbb{R}^+$. Moreover, there are numerous studies that can be viewed as a model for difference equations of the exponential-type in the literature [32–35].

Some authors extended Eq (1.3) to the two-dimensional exponential-type difference equations systems. For instance, Papaschinopoulos et al. examined the following systems of two-dimensional exponential-form difference equations:

$$\begin{aligned} T_{n+1} &= \frac{\kappa + \epsilon e^{-R_n}}{\zeta + R_{n-1}},\ R_{n+1} = \frac{\delta + \lambda e^{-T_n}}{\mu + T_{n-1}}, \\ T_{n+1} &= \frac{\kappa + \epsilon e^{-R_n}}{\zeta + T_{n-1}},\ R_{n+1} = \frac{\delta + \lambda e^{-T_n}}{\mu + R_{n-1}}, \\ T_{n+1} &= \frac{\kappa + \epsilon e^{-T_n}}{\zeta + R_{n-1}},\ R_{n+1} = \frac{\delta + \lambda e^{-R_n}}{\mu + T_{n-1}}, \end{aligned} \tag{1.5}$$

for $n \in \mathbb{N}_0$, where $T_{-1}, T_0, R_{-1}, R_0 \in \mathbb{R}^+$ and $\kappa, \epsilon, \zeta, \delta, \lambda, \mu \in \mathbb{R}^+$ in [36]. The systems given by in (1.5) are two-species population models.

In addition, Thai et al. in [37], studied the dynamical properties of the following systems:

$$\begin{aligned} T_{n+1} &= \frac{\alpha_1 + \beta_1 e^{-T_{n-1}}}{\gamma_1 + R_n}, \; R_{n+1} = \frac{\alpha_2 + \beta_2 e^{-R_{n-1}}}{\gamma_2 + T_n}, \\ T_{n+1} &= \frac{\alpha_1 + \beta_1 e^{-R_{n-1}}}{\gamma_1 + T_n}, \; R_{n+1} = \frac{\alpha_2 + \beta_2 e^{-T_{n-1}}}{\gamma_2 + R_n}, \end{aligned} \tag{1.6}$$

for $n \in \mathbb{N}_0$, where the parameters $\alpha_k, \beta_k, \gamma_k \in \mathbb{R}^+$, for $k \in \{1, 2\}$ and $T_{-1}, T_0, R_{-1}, R_0 \in \mathbb{R}^+$. Two-dimensional exponential-form difference equations systems have been studied by many authors in recent years (see [38–44]).

Our aim in this paper is to generalize the systems given by (1.6) to the following three-dimensional system of difference equations of exponential-form:

$$\Upsilon_{n+1} = \frac{\Gamma_1 + \delta_1 e^{-\Psi_{n-1}}}{\Theta_1 + \Psi_n}, \; \Psi_{n+1} = \frac{\Gamma_2 + \delta_2 e^{-\Omega_{n-1}}}{\Theta_2 + \Omega_n}, \Omega_{n+1} = \frac{\Gamma_3 + \delta_3 e^{-\Upsilon_{n-1}}}{\Theta_3 + \Upsilon_n}, \tag{1.7}$$

$$\Upsilon_{n+1} = \frac{\Gamma_1 + \delta_1 e^{-\Psi_{n-1}}}{\Theta_1 + \Upsilon_n}, \; \Psi_{n+1} = \frac{\Gamma_2 + \delta_2 e^{-\Omega_{n-1}}}{\Theta_2 + \Psi_n}, \Omega_{n+1} = \frac{\Gamma_3 + \delta_3 e^{-\Upsilon_{n-1}}}{\Theta_3 + \Omega_n}, \tag{1.8}$$

for $n \in \mathbb{N}_0$, where the parameters Γ_k, δ_k, Θ_k for $k = \overline{1,3}$ and the initial values $\Upsilon_{-\nu}$, $\Psi_{-\nu}$, $\Omega_{-\nu}$, for $\nu \in \{0, 1\}$ are positive constants. Another goal of this study is to contribute to the literature, since there are few studies on systems of three-dimensional exponential-form difference equations in the literature (see [45, 46]).

In mathematical biology, the systems given by (1.7) and (1.8) can be viewed as three-species population models. Also, the biological parameters of systems (1.7) and (1.8) are shown in Table 1.

Table 1. Biological presentations in systems (1.7) and (1.8).

Parameters	Biological presentations
Γ_1	Migration rate of species Υ_n
δ_1	Population growth rate of species Υ_n
Θ_1	The carrying capacity of species Υ_n
Γ_2	Migration rate of species Ψ_n
δ_2	Population growth rate of species Ψ_n
Θ_2	The carrying capacity of species Ψ_n
Γ_3	Migration rate of species Ω_n
δ_3	Population growth rate of species Ω_n
Θ_3	The carrying capacity of species Ω_n

Our paper is organized as follows: In the following section, we study the boundedness, local and global stability of the unique positive equilibrium point and rate of convergence of system (1.7). In the third section, we study the boundedness, local and global stability of the unique positive equilibrium point and rate of convergence of system (1.8). The conclusion is given in the last section.

Before we start our analysis, recall some lemmas and definitions which are used throughout this work. For more particulars, one can refer to the references [47–50].

$$\begin{aligned} \mho_{n+1} &= f\left(\mho_n, \mho_{n-1}, \rho_n, \rho_{n-1}, \sigma_n, \sigma_{n-1}\right), \\ \rho_{n+1} &= g\left(\mho_n, \mho_{n-1}, \rho_n, \rho_{n-1}, \sigma_n, \sigma_{n-1}\right), \ n \in \mathbb{N}_0, \\ \sigma_{n+1} &= h\left(\mho_n, \mho_{n-1}, \rho_n, \rho_{n-1}, \sigma_n, \sigma_{n-1}\right), \end{aligned} \tag{1.9}$$

where $f : U^2 \times V^2 \times W^2 \to U$, $g : U^2 \times V^2 \times W^2 \to V$, $h : U^2 \times V^2 \times W^2 \to W$ are continuous differentiable functions and U, V, W are some intervals of real numbers. Also, a solution $\{\mho_n, \rho_n, \sigma_n\}_{n=-1}^{\infty}$ of system (1.9) is uniquely defined by the initial values $\left(\mho_{-\mu}, \rho_{-\mu}, \sigma_{-\mu}\right) \in U \times V \times W$ for $\mu \in \{0, 1\}$. Along with system (1.9), we take into account the suitable vector map $\mathbb{F} = (f, \mho_n, g, \rho_n, h, \sigma_n)$. An equilibrium point of system (1.9) is a point $\left(\overline{\mho}, \overline{\rho}, \overline{\sigma}\right)$ that supplies

$$\overline{\mho} = f\left(\overline{\mho}, \overline{\mho}, \overline{\rho}, \overline{\rho}, \overline{\sigma}, \overline{\sigma}\right), \ \overline{\rho} = g\left(\overline{\mho}, \overline{\mho}, \overline{\rho}, \overline{\rho}, \overline{\sigma}, \overline{\sigma}\right), \ \overline{\sigma} = h\left(\overline{\mho}, \overline{\mho}, \overline{\rho}, \overline{\rho}, \overline{\sigma}, \overline{\sigma}\right).$$

The point $\left(\overline{\mho}, \overline{\mho}, \overline{\rho}, \overline{\rho}, \overline{\sigma}, \overline{\sigma}\right)$ is named a fixed point of the vector map $\mathbb{F}$.

Definition 1. *Assume that $\left(\overline{\mho}, \overline{\mho}, \overline{\rho}, \overline{\rho}, \overline{\sigma}, \overline{\sigma}\right)$ is a fixed point of the vector map $\mathbb{F} = (f, \mho_n, g, \rho_n, h, \sigma_n)$, where f, g and h are continuous differentiable functions at $\left(\overline{\mho}, \overline{\rho}, \overline{\sigma}\right)$. The linearized system given by system* (1.9) *about the equilibrium point $\left(\overline{\mho}, \overline{\rho}, \overline{\sigma}\right)$ is*

$$K_{n+1} = J_{\mathbb{F}} K_n,$$

where $K_n = \begin{pmatrix} \mho_n \\ \mho_{n-1} \\ \rho_n \\ \rho_{n-1} \\ \sigma_n \\ \sigma_{n-1} \end{pmatrix}$ *and $J_{\mathbb{F}}$ is the Jacobian matrix of system* (1.9) *about the equilibrium point $\left(\overline{\mho}, \overline{\rho}, \overline{\sigma}\right)$.*

Lemma 1. *Let $K_{n+1} = \mathbb{F}(K_n)$, $n \in \mathbb{N}_0$, be a system of difference equations where $\overline{K}$ is a fixed point of $\mathbb{F}$. If all eigenvalues of the Jacobian matrix $J_{\mathbb{F}}$ about $\overline{K}$ lie inside the open unit disk $|\lambda| < 1$, then $\overline{K}$ is locally asymptotically stable. If one of them has a modulus greater than one, then $\overline{K}$ is unstable.*

Definition 2. *If there exist positive constants t and T and an integer $N \geq -1$, the positive solution $\{\mho_n, \rho_n, \sigma_n\}_{n=-1}^{\infty}$ of system* (1.9) *is bounded and persists such that*

$$t \leq \mho_n, \rho_n, \sigma_n \leq T, \ n \geq N.$$

The following lemma gives the rate of convergence of solutions of the systems of difference equations.

Lemma 2. *([37])*

$$X_{n+1} = \left[\alpha + \beta(n)\right] X_n, \tag{1.10}$$

where X_n is a k-dimensional vector, $\alpha \in \mathbb{C}^{k \times k}$ is a constant matrix and $\beta : \mathbb{Z}^+ \to \mathbb{C}^{k \times k}$ is a matrix function satisfying

$$\|\beta_n\| \to 0, \text{when } n \to \infty, \tag{1.11}$$

where $\|.\|$ denotes any matrix norm which is associated with the vector norm

$$\|(x, y)\| = \sqrt{x^2 + y^2}.$$

2. Global behavior of system (1.7)

Lemma 3. *Every positive solution of system* (1.7) *is bounded and persists.*

Proof. Suppose that $\{(\Upsilon_n, \Psi_n, \Omega_n)\}$ is an arbitrary solution of system (1.7). We get

$$\Upsilon_n \leq \frac{\Gamma_1 + \delta_1}{\Theta_1} = \Pi_1,\ \Psi_n \leq \frac{\Gamma_2 + \delta_2}{\Theta_2} = \Pi_2,\ \Omega_n \leq \frac{\Gamma_3 + \delta_3}{\Theta_3} = \Pi_3,\ n \in \mathbb{N}. \tag{2.1}$$

Then, from system (1.7) and Eq (2.1), we have

$$\begin{aligned} \Upsilon_n &\geq \frac{\Gamma_1 + \delta_1 e^{-\left(\frac{\Gamma_2+\delta_2}{\Theta_2}\right)}}{\Theta_1 + \left(\frac{\Gamma_2+\delta_2}{\Theta_2}\right)} = \Lambda_1, \\ \Psi_n &\geq \frac{\Gamma_2 + \delta_2 e^{-\left(\frac{\Gamma_3+\delta_3}{\Theta_3}\right)}}{\Theta_2 + \left(\frac{\Gamma_3+\delta_3}{\Theta_3}\right)} = \Lambda_2, \\ \Omega_n &\geq \frac{\Gamma_3 + \delta_3 e^{-\left(\frac{\Gamma_1+\delta_1}{\Theta_1}\right)}}{\Theta_3 + \left(\frac{\Gamma_1+\delta_1}{\Theta_1}\right)} = \Lambda_3, \end{aligned} \tag{2.2}$$

for $n \in \mathbb{N}$.
Therefore, from Eqs (2.1) and (2.2), the proof of the lemma is complete. □

We can indicate the following lemma, which is useful for our study of system (1.7). We consider the following general system of difference equations:

$$x_{n+1} = f(y_n, y_{n-1}),\ y_{n+1} = g(z_n, z_{n-1}),\ z_{n+1} = h(x_n, x_{n-1}), \tag{2.3}$$

where f, g and h are continuous functions and the initial conditions x_{-1}, x_0, y_{-1}, y_0, z_{-1} and z_0 are positive numbers.

Lemma 4. *Let f, g, h, $f : \mathbb{R}^+ \to \mathbb{R}^+$, $g : \mathbb{R}^+ \to \mathbb{R}^+$ and $h : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous functions. Let a_1, b_1, a_2, b_2, a_3 and b_3 be positive numbers such that $a_1 < b_1$, $a_2 < b_2$, $a_3 < b_3$ and*

$$f : [a_2, b_2] \times [a_2, b_2] \to [a_1, b_1],\ g : [a_3, b_3] \times [a_3, b_3] \to [a_2, b_2],\ h : [a_1, b_1] \times [a_1, b_1] \to [a_3, b_3].$$

Suppose that the function $f(u, v)$ is a decreasing function with respect to u (resp. v) for every v (resp. u), $g(w, t)$ is a decreasing function with respect to w (resp. t) for all t (resp. w) and $h(p, s)$ is a decreasing function with respect to p (resp. s) for every s (resp. p). Finally, suppose that, if m, M, r, R, t and T are real numbers such that, if

$$M = f(r, r),\ m = f(R, R),\ R = g(t, t),\ r = g(T, T),\ T = h(m, m),\ t = h(M, M), \tag{2.4}$$

then $m = M$, $r = R$ and $t = T$. Then, system (2.3) *has a unique positive equilibrium $(\bar{x}, \bar{y}, \bar{z})$ and every positive solution of system* (2.3) *that satisfies*

$$x_{n_0} \in [a_1, b_1], x_{n_0+1} \in [a_1, b_1], y_{n_0} \in [a_2, b_2], y_{n_0+1} \in [a_2, b_2], z_{n_0} \in [a_3, b_3], z_{n_0+1} \in [a_3, b_3], n_0 \in \mathbb{N},$$

tends to the unique positive equilibrium of system (2.3).

Proof. System (2.3) is equivalent to the following system of separated equations:

$$
\begin{aligned}
x_{n+1} =& f(y_n, y_{n-1}) = f(g(z_{n-1}, z_{n-2}), g(z_{n-2}, z_{n-3})) \\
=& f(g(h(x_{n-2}, x_{n-3}), h(x_{n-3}, x_{n-4})), g(h(x_{n-3}, x_{n-4}), h(x_{n-4}, x_{n-5}))) \\
=& F(x_{n-2}, x_{n-3}, x_{n-4}, x_{n-5}), n \geq 4, \\
y_{n+1} =& g(z_n, z_{n-1}) = g(h(x_{n-1}, x_{n-2}), h(x_{n-2}, x_{n-3})) \\
=& g(h(f(y_{n-2}, y_{n-3}), f(y_{n-3}, y_{n-4})), h(f(y_{n-3}, y_{n-4}), f(y_{n-4}, y_{n-5}))) \\
=& G(y_{n-2}, y_{n-3}, y_{n-4}, y_{n-5}), n \geq 4, \\
z_{n+1} =& h(x_n, x_{n-1}) = h(f(y_{n-1}, y_{n-2}), f(y_{n-2}, y_{n-3})) \\
=& h(f(g(z_{n-2}, z_{n-3}), g(z_{n-3}, z_{n-4})), f(g(z_{n-3}, z_{n-4}), g(z_{n-4}, z_{n-5}))) \\
=& H(z_{n-2}, z_{n-3}, z_{n-4}, z_{n-5}), n \geq 4.
\end{aligned}
$$

We consider the equation

$$x_{n+1} = F(x_{n-2}, x_{n-3}, x_{n-4}, x_{n-5}). \tag{2.5}$$

From the conditions of f, g, h, we have that $F : [a_1, b_1] \times [a_1, b_1] \times [a_1, b_1] \to [a_1, b_1]$ and $F(\alpha, \beta, \gamma, \delta)$ is increasing in α for all β, γ, δ; increasing in β for all α, γ, δ; increasing in γ for all α, β, δ; increasing in δ for all α, β, γ. Now, let us take m and M as positive numbers so that

$$
\begin{aligned}
M =& F(M, M, M, M,) = f(g(h(M, M), h(M, M)), g(h(M, M), h(M, M))), \\
m =& F(m, m, m, m,) = f(g(h(m, m), h(m, m)), g(h(m, m), h(m, m))).
\end{aligned}
$$

By setting $h(m, m) = T$ and $h(M, M) = t$, we get

$$
\begin{aligned}
M =& f(g(t, t), g(t, t)), \\
m =& f(g(T, T), g(T, T)).
\end{aligned}
$$

By setting $g(t, t) = R$ and $g(T, T) = r$, we have that the relations given by Eq (2.4) are satisfied. Then, from a hypothesis of Lemma 4, we have that $m = M$. Equation (2.5) has a unique positive equilibrium $\overline{x}$, and every positive solution of Eq (2.5) tends to the unique positive equilibrium $\overline{x}$. Similarly, we can prove that the equation

$$y_{n+1} = G(y_{n-2}, y_{n-3}, y_{n-4}, y_{n-5}), \tag{2.6}$$

has a unique positive equilibrium $\overline{y}$ and every positive solution of Eq (2.6) tends to the unique positive equilibrium $\overline{y}$. Similarly, we can prove that the equation

$$z_{n+1} = H(z_{n-2}, z_{n-3}, z_{n-4}, z_{n-5}), \tag{2.7}$$

has a unique positive equilibrium $\overline{z}$ and every positive solution of Eq (2.7) tends to the unique positive equilibrium $\overline{z}$. This completes the proof of the lemma. □

2.1. Local and global asymptotic stability

Theorem 3. *For the local stability about* $\left(\overline{\Upsilon}, \overline{\Psi}, \overline{\Omega}\right) \in [\Lambda_1, \Pi_1] \times [\Lambda_2, \Pi_2] \times [\Lambda_3, \Pi_3]$*, i.e., the equilibrium point of system* (1.7)*, the next declarations are valid:*

(i) $\left(\overline{\Upsilon}, \overline{\Psi}, \overline{\Omega}\right)$ *is locally asymptotically stable if* $L_1 < 1$*,*

(ii) $\left(\overline{\Upsilon}, \overline{\Psi}, \overline{\Omega}\right)$ *is unstable if* $U_1 > 1$*,*

where

$$L_1 = \frac{1}{\Theta_1\Theta_2\Theta_3}\Big(\Lambda_1\Lambda_2\Lambda_3 + e^{-1}\Big(\delta_1\left(\Lambda_3 + \delta_2 e^{-\Lambda_2}\right) + \delta_2\left(\Lambda_1 + \delta_3 e^{-\Lambda_3}\right) + \delta_3\left(\Lambda_2 + \delta_1 e^{-\Lambda_1}\right)\Big) + \delta_1\delta_2\delta_3\Big), \tag{2.8}$$

$$\begin{aligned} U_1 = {} & \frac{1}{(\Theta_1+\Pi_2)(\Theta_2+\Pi_3)(\Theta_3+\Pi_1)}\left(\frac{\left(\Gamma_1+\delta_1 e^{-\Pi_2}\right)\left(\Gamma_2+\delta_2 e^{-\Pi_3}\right)\left(\Gamma_3+\delta_3 e^{-\Pi_1}\right)}{(\Theta_1+\Pi_2)(\Theta_2+\Pi_3)(\Theta_3+\Pi_1)} + \delta_1\delta_2\delta_3 e^{-\Pi_1-\Pi_2-\Pi_3}\right) \\ & + \frac{\left(\Gamma_2+\delta_2 e^{-\Pi_3}\right)}{(\Theta_1+\Pi_2)(\Theta_2+\Pi_3)^2(\Theta_3+\Pi_1)}\left(\frac{\left(\Gamma_1+\delta_1 e^{-\Pi_2}\right)\delta_3 e^{-\Pi_1}}{(\Theta_1+\Pi_2)} + \frac{\left(\Gamma_3+\delta_3 e^{-\Pi_1}\right)\delta_1 e^{-\Pi_2}}{(\Theta_3+\Pi_1)}\right) \\ & + \frac{\left(\Gamma_1+\delta_1 e^{-\Pi_2}\right)\delta_2 e^{-\Pi_3}}{(\Theta_1+\Pi_2)(\Theta_2+\Pi_3)(\Theta_3+\Pi_1)}\left(\frac{\left(\Gamma_3+\delta_3 e^{-\Pi_1}\right)}{(\Theta_1+\Pi_2)(\Theta_3+\Pi_1)} + \frac{\delta_3 e^{-\Pi_1}}{(\Theta_1+\Pi_2)}\right) \\ & + \frac{\delta_1 e^{-\Pi_2}}{(\Theta_1+\Pi_2)(\Theta_2+\Pi_3)(\Theta_3+\Pi_1)}\left(\frac{\left(\Gamma_2+\delta_2 e^{-\Pi_3}\right)\delta_3 e^{-\Pi_1}}{(\Theta_2+\Pi_3)} + \frac{\left(\Gamma_3+\delta_3 e^{-\Pi_1}\right)\delta_2 e^{-\Pi_3}}{(\Theta_3+\Pi_1)}\right). \end{aligned} \tag{2.9}$$

Proof. (i) From system (1.7), we obtain

$$\overline{\Upsilon} = \frac{\Gamma_1 + \delta_1 e^{-\overline{\Psi}}}{\Theta_1 + \overline{\Psi}},\ \overline{\Psi} = \frac{\Gamma_2 + \delta_2 e^{-\overline{\Omega}}}{\Theta_2 + \overline{\Omega}},\ \overline{\Omega} = \frac{\Gamma_3 + \delta_3 e^{-\overline{\Upsilon}}}{\Theta_3 + \overline{\Upsilon}}. \tag{2.10}$$

In order to construct the corresponding linearized form of system (1.7), we consider the following transformation:

$$(\Upsilon_{n+1}, \Upsilon_n, \Psi_{n+1}, \Psi_n, \Omega_{n+1}, \Omega_n) \to (f, f_1, g, g_1, h, h_1), \tag{2.11}$$

where

$$\begin{aligned} f &= \frac{\Gamma_1 + \delta_1 e^{-\Psi_{n-1}}}{\Theta_1 + \Psi_n},\ f_1 = \Upsilon_n, \\ g &= \frac{\Gamma_2 + \delta_2 e^{-\Omega_{n-1}}}{\Theta_2 + \Omega_n}, g_1 = \Psi_n, \\ h &= \frac{\Gamma_3 + \delta_3 e^{-\Upsilon_{n-1}}}{\Theta_3 + \Upsilon_n},\ h_1 = \Omega_n. \end{aligned} \tag{2.12}$$

By using the transformation given by Eq (2.11), we obtain

$$F_J\left(\overline{\Upsilon},\overline{\Psi},\overline{\Omega}\right)=\begin{pmatrix}0&0&s_1&t_1&0&0\\1&0&0&0&0&0\\0&0&0&0&s_2&t_2\\0&0&1&0&0&0\\s_3&t_3&0&0&0&0\\0&0&0&0&1&0\end{pmatrix},\tag{2.13}$$

where

$$\begin{aligned}s_1&=-\frac{\Gamma_1+\delta_1e^{-\overline{\Psi}}}{\left(\Theta_1+\overline{\Psi}\right)^2},\ t_1=-\frac{\delta_1e^{-\overline{\Psi}}}{\Theta_1+\overline{\Psi}},\\s_2&=-\frac{\Gamma_2+\delta_2e^{-\overline{\Omega}}}{\left(\Theta_2+\overline{\Omega}\right)^2},\ t_2=-\frac{\delta_2e^{-\overline{\Omega}}}{\Theta_2+\overline{\Omega}},\\s_3&=-\frac{\Gamma_3+\delta_3e^{-\overline{\Upsilon}}}{\left(\Theta_3+\overline{\Upsilon}\right)^2},\ t_3=-\frac{\delta_3e^{-\overline{\Upsilon}}}{\Theta_3+\overline{\Upsilon}}.\end{aligned}\tag{2.14}$$

The characteristic equation of $F_J\left(\overline{\Upsilon},\overline{\Psi},\overline{\Omega}\right)$ is below:

$$\lambda^6+m_1\lambda^3+m_2\lambda^2+m_3\lambda+m_4=0,\tag{2.15}$$

where

$$\begin{aligned}m_1&=-s_1s_2s_3,\\m_2&=-\left(s_1s_2t_3+s_1t_2s_3+t_1s_2s_3\right),\\m_3&=-\left(s_1t_2t_3+t_1s_2t_3+t_1t_2s_3\right),\\m_4&=-t_1t_2t_3.\end{aligned}\tag{2.16}$$

Because $xe^{-x}<e^{-1}$, for $x>0$, we have

$$\begin{aligned}\sum_{i=1}^{4}|m_i|=&\frac{\left(\Gamma_1+\delta_1e^{-\overline{\Psi}}\right)\left(\Gamma_2+\delta_2e^{-\overline{\Omega}}\right)\left(\Gamma_3+\delta_3e^{-\overline{\Upsilon}}\right)}{\left(\Theta_1+\overline{\Psi}\right)^2\left(\Theta_2+\overline{\Omega}\right)^2\left(\Theta_3+\overline{\Upsilon}\right)^2}\\&+\frac{\left(\Gamma_1+\delta_1e^{-\overline{\Psi}}\right)\left(\Gamma_2+\delta_2e^{-\overline{\Omega}}\right)\delta_3e^{-\overline{\Upsilon}}}{\left(\Theta_1+\overline{\Psi}\right)^2\left(\Theta_2+\overline{\Omega}\right)^2\left(\Theta_3+\overline{\Upsilon}\right)}+\frac{\left(\Gamma_1+\delta_1e^{-\overline{\Psi}}\right)\delta_2e^{-\overline{\Omega}}\left(\Gamma_3+\delta_3e^{-\overline{\Upsilon}}\right)}{\left(\Theta_1+\overline{\Psi}\right)^2\left(\Theta_2+\overline{\Omega}\right)\left(\Theta_3+\overline{\Upsilon}\right)^2}\\&+\frac{\delta_1e^{-\overline{\Psi}}\left(\Gamma_2+\delta_2e^{-\overline{\Omega}}\right)\left(\Gamma_3+\delta_3e^{-\overline{\Upsilon}}\right)}{\left(\Theta_1+\overline{\Psi}\right)\left(\Theta_2+\overline{\Omega}\right)^2\left(\Theta_3+\overline{\Upsilon}\right)^2}+\frac{\left(\Gamma_1+\delta_1e^{-\overline{\Psi}}\right)\delta_2\delta_3e^{-\overline{\Omega}-\overline{\Upsilon}}}{\left(\Theta_1+\overline{\Psi}\right)^2\left(\Theta_2+\overline{\Omega}\right)\left(\Theta_3+\overline{\Upsilon}\right)}\\&+\frac{\left(\Gamma_2+\delta_2e^{-\overline{\Omega}}\right)\delta_1\delta_3e^{-\overline{\Psi}-\overline{\Upsilon}}}{\left(\Theta_1+\overline{\Psi}\right)\left(\Theta_2+\overline{\Omega}\right)^2\left(\Theta_3+\overline{\Upsilon}\right)}+\frac{\left(\Gamma_3+\delta_3e^{-\overline{\Upsilon}}\right)\delta_1\delta_2e^{-\overline{\Psi}-\overline{\Omega}}}{\left(\Theta_1+\overline{\Psi}\right)\left(\Theta_2+\overline{\Omega}\right)\left(\Theta_3+\overline{\Upsilon}\right)^2}\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\delta_1\delta_2\delta_3 e^{-\overline{\Upsilon}-\overline{\Psi}-\overline{\Omega}}}{\left(\Theta_1+\overline{\Psi}\right)\left(\Theta_2+\overline{\Omega}\right)\left(\Theta_3+\overline{\Upsilon}\right)} \\
&= \frac{\overline{\Upsilon}\,\overline{\Psi}\,\overline{\Omega}}{\left(\Theta_1+\overline{\Psi}\right)\left(\Theta_2+\overline{\Omega}\right)\left(\Theta_3+\overline{\Upsilon}\right)} + \frac{\overline{\Psi}\delta_3 e^{-\overline{\Upsilon}}\overline{\Upsilon}}{\left(\Theta_1+\overline{\Psi}\right)\left(\Theta_2+\overline{\Omega}\right)\left(\Theta_3+\overline{\Upsilon}\right)} \\
&+ \frac{\overline{\Upsilon}\delta_2 e^{-\overline{\Omega}}\overline{\Omega}}{\left(\Theta_1+\overline{\Psi}\right)\left(\Theta_2+\overline{\Omega}\right)\left(\Theta_3+\overline{\Upsilon}\right)} + \frac{\overline{\Omega}\delta_1 e^{-\overline{\Psi}}\overline{\Psi}}{\left(\Theta_1+\overline{\Psi}\right)\left(\Theta_2+\overline{\Omega}\right)\left(\Theta_3+\overline{\Upsilon}\right)} \\
&+ \frac{\delta_3 e^{-\overline{\Upsilon}}\overline{\Upsilon}\delta_2 e^{-\overline{\Omega}}}{\left(\Theta_1+\overline{\Psi}\right)\left(\Theta_2+\overline{\Omega}\right)\left(\Theta_3+\overline{\Upsilon}\right)} + \frac{\delta_1 e^{-\overline{\Psi}}\overline{\Psi}\delta_3 e^{-\overline{\Upsilon}}}{\left(\Theta_1+\overline{\Psi}\right)\left(\Theta_2+\overline{\Omega}\right)\left(\Theta_3+\overline{\Upsilon}\right)} \\
&+ \frac{\delta_2 e^{-\overline{\Omega}}\overline{\Omega}\delta_1 e^{-\overline{\Psi}}}{\left(\Theta_1+\overline{\Psi}\right)\left(\Theta_2+\overline{\Omega}\right)\left(\Theta_3+\overline{\Upsilon}\right)} + \frac{\delta_1\delta_2\delta_3 e^{-\overline{\Upsilon}-\overline{\Psi}-\overline{\Omega}}}{\left(\Theta_1+\overline{\Psi}\right)\left(\Theta_2+\overline{\Omega}\right)\left(\Theta_3+\overline{\Upsilon}\right)} \\
&< \frac{\overline{\Upsilon}\,\overline{\Psi}\,\overline{\Omega}}{\Theta_1\Theta_2\Theta_3} + \frac{e^{-1}\left(\delta_3\overline{\Psi}+\delta_2\overline{\Upsilon}+\delta_1\overline{\Omega}+\delta_2\delta_3 e^{-\overline{\Omega}}+\delta_1\delta_3 e^{-\overline{\Upsilon}}+\delta_1\delta_2 e^{-\overline{\Psi}}\right)}{\Theta_1\Theta_2\Theta_3} \\
&+ \frac{\delta_1\delta_2\delta_3}{\Theta_1\Theta_2\Theta_3} \\
&< \frac{1}{\Theta_1\Theta_2\Theta_3}\Big(\Lambda_1\Lambda_2\Lambda_3 + e^{-1}\left(\delta_1\left(\Lambda_3+\delta_2 e^{-\Lambda_2}\right)+\delta_2\left(\Lambda_1+\delta_3 e^{-\Lambda_3}\right)+\delta_3\left(\Lambda_2+\delta_1 e^{-\Lambda_1}\right)\right) \\
&\qquad + \delta_1\delta_2\delta_3\Big) < 1.
\end{aligned} \tag{2.17}
$$

By supposing that $L_1 < 1$, from Eq (2.17), we have that $\sum_{i=1}^{4}|m_i| < 1$. According to the Rouche theorem, $\left(\overline{\Upsilon},\overline{\Psi},\overline{\Omega}\right)$ is locally asymptotically stable.

(ii) We get

$$
\begin{aligned}
\sum_{i=1}^{4}|m_i| &= \frac{\left(\Gamma_1+\delta_1 e^{-\overline{\Psi}}\right)\left(\Gamma_2+\delta_2 e^{-\overline{\Omega}}\right)\left(\Gamma_3+\delta_3 e^{-\overline{\Upsilon}}\right)}{\left(\Theta_1+\overline{\Psi}\right)^2\left(\Theta_2+\overline{\Omega}\right)^2\left(\Theta_3+\overline{\Upsilon}\right)^2} \\
&+ \frac{\left(\Gamma_1+\delta_1 e^{-\overline{\Psi}}\right)\left(\Gamma_2+\delta_2 e^{-\overline{\Omega}}\right)\delta_3 e^{-\overline{\Upsilon}}}{\left(\Theta_1+\overline{\Psi}\right)^2\left(\Theta_2+\overline{\Omega}\right)^2\left(\Theta_3+\overline{\Upsilon}\right)} + \frac{\left(\Gamma_1+\delta_1 e^{-\overline{\Psi}}\right)\delta_2 e^{-\overline{\Omega}}\left(\Gamma_3+\delta_3 e^{-\overline{\Upsilon}}\right)}{\left(\Theta_1+\overline{\Psi}\right)^2\left(\Theta_2+\overline{\Omega}\right)\left(\Theta_3+\overline{\Upsilon}\right)^2} \\
&+ \frac{\delta_1 e^{-\overline{\Psi}}\left(\Gamma_2+\delta_2 e^{-\overline{\Omega}}\right)\left(\Gamma_3+\delta_3 e^{-\overline{\Upsilon}}\right)}{\left(\Theta_1+\overline{\Psi}\right)\left(\Theta_2+\overline{\Omega}\right)^2\left(\Theta_3+\overline{\Upsilon}\right)^2} + \frac{\left(\Gamma_1+\delta_1 e^{-\overline{\Psi}}\right)\delta_2\delta_3 e^{-\overline{\Omega}-\overline{\Upsilon}}}{\left(\Theta_1+\overline{\Psi}\right)^2\left(\Theta_2+\overline{\Omega}\right)\left(\Theta_3+\overline{\Upsilon}\right)} \\
&+ \frac{\left(\Gamma_2+\delta_2 e^{-\overline{\Omega}}\right)\delta_1\delta_3 e^{-\overline{\Psi}-\overline{\Upsilon}}}{\left(\Theta_1+\overline{\Psi}\right)\left(\Theta_2+\overline{\Omega}\right)^2\left(\Theta_3+\overline{\Upsilon}\right)} + \frac{\left(\Gamma_3+\delta_3 e^{-\overline{\Upsilon}}\right)\delta_1\delta_2 e^{-\overline{\Psi}-\overline{\Omega}}}{\left(\Theta_1+\overline{\Psi}\right)\left(\Theta_2+\overline{\Omega}\right)\left(\Theta_3+\overline{\Upsilon}\right)^2} \\
&+ \frac{\delta_1\delta_2\delta_3 e^{-\overline{\Upsilon}-\overline{\Psi}-\overline{\Omega}}}{\left(\Theta_1+\overline{\Psi}\right)\left(\Theta_2+\overline{\Omega}\right)\left(\Theta_3+\overline{\Upsilon}\right)}
\end{aligned}
$$

$$
\begin{aligned}
&\geq \frac{\left(\Gamma_1+\delta_1 e^{-\Pi_2}\right)\left(\Gamma_2+\delta_2 e^{-\Pi_3}\right)\left(\Gamma_3+\delta_3 e^{-\Pi_1}\right)}{(\Theta_1+\Pi_2)^2(\Theta_2+\Pi_3)^2(\Theta_3+\Pi_1)^2} \\
&+\frac{\left(\Gamma_1+\delta_1 e^{-\Pi_2}\right)\left(\Gamma_2+\delta_2 e^{-\Pi_3}\right)\delta_3 e^{-\Pi_1}}{(\Theta_1+\Pi_2)^2(\Theta_2+\Pi_3)^2(\Theta_3+\Pi_1)}+\frac{\left(\Gamma_1+\delta_1 e^{-\Pi_2}\right)\delta_2 e^{-\Pi_3}\left(\Gamma_3+\delta_3 e^{-\Pi_1}\right)}{(\Theta_1+\Pi_2)^2(\Theta_2+\Pi_3)(\Theta_3+\Pi_1)^2} \\
&+\frac{\delta_1 e^{-\Pi_2}\left(\Gamma_2+\delta_2 e^{-\Pi_3}\right)\left(\Gamma_3+\delta_3 e^{-\Pi_1}\right)}{(\Theta_1+\Pi_2)(\Theta_2+\Pi_3)^2(\Theta_3+\Pi_1)^2}+\frac{\left(\Gamma_1+\delta_1 e^{-\Pi_2}\right)\delta_2\delta_3 e^{-\Pi_3-\Pi_1}}{(\Theta_1+\Pi_2)^2(\Theta_2+\Pi_3)(\Theta_3+\Pi_1)} \\
&+\frac{\left(\Gamma_2+\delta_2 e^{-\Pi_3}\right)\delta_1\delta_3 e^{-\Pi_2-\Pi_1}}{(\Theta_1+\Pi_2)(\Theta_2+\Pi_3)^2(\Theta_3+\Pi_1)}+\frac{\left(\Gamma_3+\delta_3 e^{-\Pi_1}\right)\delta_1\delta_2 e^{-\Pi_2-\Pi_3}}{(\Theta_1+\Pi_2)(\Theta_2+\Pi_3)(\Theta_3+\Pi_1)^2} \\
&+\frac{\delta_1\delta_2\delta_3 e^{-\Pi_1-\Pi_2-\Pi_3}}{(\Theta_1+\Pi_2)(\Theta_2+\Pi_3)(\Theta_3+\Pi_1)} \\
&=\frac{1}{(\Theta_1+\Pi_2)(\Theta_2+\Pi_3)(\Theta_3+\Pi_1)}\left(\frac{\left(\Gamma_1+\delta_1 e^{-\Pi_2}\right)\left(\Gamma_2+\delta_2 e^{-\Pi_3}\right)\left(\Gamma_3+\delta_3 e^{-\Pi_1}\right)}{(\Theta_1+\Pi_2)(\Theta_2+\Pi_3)(\Theta_3+\Pi_1)}+\delta_1\delta_2\delta_3 e^{-\Pi_1-\Pi_2-\Pi_3}\right) \\
&+\frac{\left(\Gamma_2+\delta_2 e^{-\Pi_3}\right)}{(\Theta_1+\Pi_2)(\Theta_2+\Pi_3)^2(\Theta_3+\Pi_1)}\left(\frac{\left(\Gamma_1+\delta_1 e^{-\Pi_2}\right)\delta_3 e^{-\Pi_1}}{(\Theta_1+\Pi_2)}+\frac{\left(\Gamma_3+\delta_3 e^{-\Pi_1}\right)\delta_1 e^{-\Pi_2}}{(\Theta_3+\Pi_1)}\right) \\
&+\frac{\left(\Gamma_1+\delta_1 e^{-\Pi_2}\right)\delta_2 e^{-\Pi_3}}{(\Theta_1+\Pi_2)(\Theta_2+\Pi_3)(\Theta_3+\Pi_1)}\left(\frac{\left(\Gamma_3+\delta_3 e^{-\Pi_1}\right)}{(\Theta_1+\Pi_2)(\Theta_3+\Pi_1)}+\frac{\delta_3 e^{-\Pi_1}}{(\Theta_1+\Pi_2)}\right) \\
&+\frac{\delta_1 e^{-\Pi_2}}{(\Theta_1+\Pi_2)(\Theta_2+\Pi_3)(\Theta_3+\Pi_1)}\left(\frac{\left(\Gamma_2+\delta_2 e^{-\Pi_3}\right)\delta_3 e^{-\Pi_1}}{(\Theta_2+\Pi_3)}+\frac{\left(\Gamma_3+\delta_3 e^{-\Pi_1}\right)\delta_2 e^{-\Pi_3}}{(\Theta_3+\Pi_1)}\right)>1.
\end{aligned}
\tag{2.18}
$$

By supposing that $U_1 > 1$, from Eq (2.18), we have that $\sum_{i=1}^{4} |m_i| > 1$. According to the Rouche theorem, $\left(\overline{\Upsilon}, \overline{\Psi}, \overline{\Omega}\right)$ is unstable.

□

Theorem 4. *System* (1.7) *has a unique positive equilibrium* $\left(\overline{\Upsilon}, \overline{\Psi}, \overline{\Omega}\right)$*, and every positive solution of system* (1.7) *tends to the unique positive equilibrium of system* (1.7) *as* $n \to \infty$ *if*

$$
\begin{aligned}
\delta_1, \delta_3 &< \Theta_2, \\
\delta_1, \delta_2 &< \Theta_3, \\
\delta_2, \delta_3 &< \Theta_1.
\end{aligned}
\tag{2.19}
$$

Proof. We consider the functions

$$
f(u,v) = \frac{\Gamma_1 + \delta_1 e^{-u}}{\Theta_1 + v},\ g(w,t) = \frac{\Gamma_2 + \delta_2 e^{-w}}{\Theta_2 + t}, h(k,l) = \frac{\Gamma_3 + \delta_3 e^{-k}}{\Theta_3 + l}, \tag{2.20}
$$

where

$$
\begin{aligned}
k, l \in [\Lambda_1, \Pi_1] &= \left[\frac{\Gamma_1 + \delta_1 e^{-\left(\frac{\Gamma_2+\delta_2}{\Theta_2}\right)}}{\Theta_1 + \left(\frac{\Gamma_2+\delta_2}{\Theta_2}\right)}, \frac{\Gamma_1 + \delta_1}{\Theta_1} \right], \\
u, v \in [\Lambda_2, \Pi_2] &= \left[\frac{\Gamma_2 + \delta_2 e^{-\left(\frac{\Gamma_3+\delta_3}{\Theta_3}\right)}}{\Theta_2 + \left(\frac{\Gamma_3+\delta_3}{\Theta_3}\right)}, \frac{\Gamma_2 + \delta_2}{\Theta_2} \right], \\
w, t \in [\Lambda_3, \Pi_3] &= \left[\frac{\Gamma_3 + \delta_3 e^{-\left(\frac{\Gamma_1+\delta_1}{\Theta_1}\right)}}{\Theta_3 + \left(\frac{\Gamma_1+\delta_1}{\Theta_1}\right)}, \frac{\Gamma_3 + \delta_3}{\Theta_3} \right].
\end{aligned} \tag{2.21}
$$

From Eqs (2.20) and (2.21) we get the following relations for $k, l \in [\Lambda_1, \Pi_1]$, $u, v \in [\Lambda_2, \Pi_2]$ and $w, t \in [\Lambda_3, \Pi_3]$:

$$
f(u, v) \in [\Lambda_1, \Pi_1], g(w, t) \in [\Lambda_2, \Pi_2], h(k, l) \in [\Lambda_3, \Pi_3];
$$

thus, $f : [\Lambda_2, \Pi_2] \times [\Lambda_2, \Pi_2] \to [\Lambda_1, \Pi_1]$, $g : [\Lambda_3, \Pi_3] \times [\Lambda_3, \Pi_3] \to [\Lambda_2, \Pi_2]$ and $h : [\Lambda_1, \Pi_1] \times [\Lambda_1, \Pi_1] \to [\Lambda_3, \Pi_3]$. Let $\{(\Upsilon_n, \Psi_n, \Omega_n)\}$ be an arbitrary solution of system (1.7). Therefore, from Lemma 3, for $n \in \mathbb{N}$, we get

$$
\Upsilon_n \in [\Lambda_1, \Pi_1], \Psi_n \in [\Lambda_2, \Pi_2], \Omega_n \in [\Lambda_3, \Pi_3].
$$

Now, let m, M, r, R, t and T be positive numbers such that

$$
\begin{aligned}
M &= \frac{\Gamma_1 + \delta_1 e^{-r}}{\Theta_1 + r}, \; m = \frac{\Gamma_1 + \delta_1 e^{-R}}{\Theta_1 + R}, \\
R &= \frac{\Gamma_2 + \delta_2 e^{-t}}{\Theta_2 + t}, \; r = \frac{\Gamma_2 + \delta_2 e^{-T}}{\Theta_2 + T}, \\
T &= \frac{\Gamma_3 + \delta_3 e^{-m}}{\Theta_3 + m}, \; t = \frac{\Gamma_3 + \delta_3 e^{-M}}{\Theta_3 + M}.
\end{aligned} \tag{2.22}
$$

Then, we consider the functions

$$
F(x) = \frac{\Gamma_1 + \delta_1 e^{-q_1(x)}}{\Theta_1 + q_1(x)} - x, \; q_1(x) = \frac{\Gamma_3 + \delta_3 e^{-x}}{\Theta_3 + x}, \; x \in [\Lambda_1, \Pi_1], \tag{2.23}
$$

$$
G(y) = \frac{\Gamma_2 + \delta_2 e^{-q_2(y)}}{\Theta_2 + q_2(y)} - y, \; q_2(y) = \frac{\Gamma_1 + \delta_1 e^{-y}}{\Theta_1 + y}, \; y \in [\Lambda_2, \Pi_2], \tag{2.24}
$$

$$
H(z) = \frac{\Gamma_3 + \delta_3 e^{-q_3(z)}}{\Theta_3 + q_3(z)} - z, \; q_3(z) = \frac{\Gamma_2 + \delta_2 e^{-z}}{\Theta_2 + z}, \; z \in [\Lambda_3, \Pi_3]. \tag{2.25}
$$

Note that F maps the interval $[\Lambda_1, \Pi_1]$ into itself. We claim that the equation $F(x) = 0$ has a unique solution in $[\Lambda_1, \Pi_1]$. From Eq (2.23), we get

$$F'(x) = -q_1'(x)\frac{\delta_1 e^{-q_1(x)}(\Theta_1 + q_1(x)) + \Gamma_1 + \delta_1 e^{-q_1(x)}}{(\Theta_1 + q_1(x))^2} - 1, \tag{2.26}$$
$$q_1'(x) = -\frac{\delta_3 e^{-x}(\Theta_3 + x) + (\Gamma_3 + \delta_3 e^{-x})}{(\Theta_3 + x)^2},\ x \in [\Lambda_1, \Pi_1].$$

Let $\overline{x}$, $\overline{x} \in [\Lambda_1, \Pi_1]$ be a solution of the equation $F(x) = 0$. Then, from Eq (2.23), we get

$$\overline{x}(\Theta_1 + q_1(\overline{x})) = \Gamma_1 + \delta_1 e^{-q_1(\overline{x})},\ q_1(\overline{x})(\Theta_3 + \overline{x}) = \Gamma_3 + \delta_3 e^{-\overline{x}}. \tag{2.27}$$

Now, observe that Eqs (2.26) and (2.27) imply that

$$q_1'(\overline{x}) = -\frac{\delta_3 e^{-\overline{x}} + q_1(\overline{x})}{\Theta_3 + \overline{x}},\ \frac{\delta_1 e^{-q_1(\overline{x})}(\Theta_1 + q_1(\overline{x})) + \Gamma_1 + \delta_1 e^{-q_1(\overline{x})}}{(\Theta_1 + q_1(\overline{x}))^2} = \frac{\delta_1 e^{-q_1(\overline{x})} + \overline{x}}{\Theta_1 + q_1(\overline{x})}. \tag{2.28}$$

Then, from Eqs (2.19), (2.26) and (2.28), we have

$$F'(\overline{x}) = \frac{\delta_3 e^{-\overline{x}} + q_1(\overline{x})}{\Theta_1 + q_1(\overline{x})} \times \frac{\delta_1 e^{-q_1(\overline{x})} + \overline{x}}{\Theta_3 + \overline{x}} - 1 < 0. \tag{2.29}$$

Therefore, from Eq (2.29), we see that the equation $F(x) = 0$ has a unique solution in $[\Lambda_1, \Pi_1]$.

Note that G maps the interval $[\Lambda_2, \Pi_2]$ into itself. We claim that the equation $G(y) = 0$ has a unique solution in $[\Lambda_2, \Pi_2]$. From Eq (2.24), we get

$$G'(y) = -q_2'(y)\frac{\delta_2 e^{-q_2(y)}(\Theta_2 + q_2(y)) + \Gamma_2 + \delta_2 e^{-q_2(y)}}{(\Theta_2 + q_2(y))^2} - 1, \tag{2.30}$$
$$q_2'(y) = -\frac{\delta_1 e^{-y}(\Theta_1 + y) + (\Gamma_1 + \delta_1 e^{-y})}{(\Theta_1 + y)^2},\ y \in [\Lambda_2, \Pi_2].$$

Let $\overline{y}$, $\overline{y} \in [\Lambda_2, \Pi_2]$ be a solution of the equation $G(y) = 0$. Then, from Eq (2.24), we get

$$\overline{y}(\Theta_2 + q_2(\overline{y})) = \Gamma_2 + \delta_2 e^{-q_2(\overline{y})},\ q_2(\overline{y})(\Theta_1 + \overline{y}) = \Gamma_1 + \delta_1 e^{-\overline{y}}. \tag{2.31}$$

Now, observe that Eqs (2.30) and (2.31) imply that

$$q_2'(\overline{y}) = -\frac{\delta_1 e^{-\overline{y}} + q_2(\overline{y})}{\Theta_1 + \overline{y}},\ \frac{\delta_2 e^{-q_2(\overline{y})}(\Theta_2 + q_2(\overline{y})) + \Gamma_2 + \delta_2 e^{-q_2(\overline{y})}}{(\Theta_2 + q_2(\overline{y}))^2} = \frac{\delta_2 e^{-q_2(\overline{y})} + \overline{y}}{\Theta_2 + q_2(\overline{y})}. \tag{2.32}$$

Then, from Eqs (2.19), (2.30) and (2.32), we have

$$G'(\overline{y}) = \frac{\delta_1 e^{-\overline{y}} + q_2(\overline{y})}{\Theta_2 + q_2(\overline{y})} \times \frac{\delta_2 e^{-q_2(\overline{y})} + \overline{y}}{\Theta_1 + \overline{y}} - 1 < 0. \tag{2.33}$$

Therefore, from Eq (2.33), we see that the equation $G(y) = 0$ has a unique solution in $[\Lambda_2, \Pi_2]$.

Note that H maps the interval $[\Lambda_3, \Pi_3]$ into itself. We claim that the equation $H(z) = 0$ has a unique solution in $[\Lambda_3, \Pi_3]$. From Eq (2.25), we get

$$H'(z) = -q_3'(z)\frac{\delta_3 e^{-q_3(z)}(\Theta_3 + q_3(z)) + \Gamma_3 + \delta_3 e^{-q_3(z)}}{(\Theta_3 + q_3(z))^2} - 1, \tag{2.34}$$
$$q_3'(z) = -\frac{\delta_2 e^{-z}(\Theta_2 + z) + (\Gamma_2 + \delta_2 e^{-z})}{(\Theta_2 + z)^2},\ z \in [\Lambda_3, \Pi_3].$$

Let $\overline{z}$, $\overline{z} \in [\Lambda_3, \Pi_3]$ be a solution of the equation $H(z) = 0$. Then, from Eq (2.25), we get

$$\overline{z}(\Theta_3 + q_3(\overline{z})) = \Gamma_3 + \delta_3 e^{-q_3(\overline{z})},\ q_3(\overline{z})(\Theta_2 + \overline{z}) = \Gamma_2 + \delta_2 e^{-\overline{z}}. \tag{2.35}$$

Now, observe that Eqs (2.34) and (2.35) imply that

$$q_3'(\overline{z}) = -\frac{\delta_2 e^{-\overline{z}} + q_3(\overline{z})}{\Theta_2 + \overline{z}},\ \frac{\delta_3 e^{-q_3(\overline{z})}(\Theta_3 + q_3(\overline{z})) + \Gamma_3 + \delta_3 e^{-q_3(\overline{z})}}{(\Theta_3 + q_3(\overline{z}))^2} = \frac{\delta_3 e^{-q_3(\overline{z})} + \overline{z}}{\Theta_3 + q_3(\overline{z})}. \tag{2.36}$$

Then, from Eqs (2.19), (2.34) and (2.36), we have

$$H'(\overline{z}) = \frac{\delta_2 e^{-\overline{z}} + q_3(\overline{z})}{\Theta_3 + q_3(\overline{z})} \times \frac{\delta_3 e^{-q_3(\overline{z})} + \overline{z}}{\Theta_2 + \overline{z}} - 1 < 0. \tag{2.37}$$

Therefore, from, Eq (2.37) we see that the equation $H(z) = 0$ has a unique solution in $[\Lambda_3, \Pi_3]$.

In addition, Eq (2.22) implies that m and M are roots of $F(x) = 0$. Hence, we get that $m = M$. Similarly, Eq (2.22) implies that r and R are roots of $G(y) = 0$. Hence, we get that $r = R$. Finally, Eq (2.22) implies that t and T are roots of $H(z) = 0$. Hence, we get that $t = T$. From Lemma 4, system (1.7) has a unique positive equilibrium $\left(\overline{\Upsilon}, \overline{\Psi}, \overline{\Omega}\right)$ and every positive solution of system (1.7) tends to the unique positive equilibrium as $n \to \infty$. This completes the proof of the theorem. □

2.2. Rate of convergence

Theorem 5. *If* $\{(\Upsilon_n, \Psi_n, \Omega_n,)\}_{n=-1}^{\infty}$ *is a positive solution of system* (1.7) *such that*

$$\lim_{n\to\infty} \Upsilon_n = \overline{\Upsilon},\ \lim_{n\to\infty} \Psi_n = \overline{\Psi},\ \lim_{n\to\infty} \Omega_n = \overline{\Omega}, \tag{2.38}$$

where

$$\overline{\Upsilon} \in [\Lambda_1, \Pi_1],\ \overline{\Psi} \in [\Lambda_2, \Pi_2],\ \overline{\Omega} \in [\Lambda_3, \Pi_3], \tag{2.39}$$

then the error vector $\Phi_n = \left(\Phi_n^1, \Phi_{n-1}^1, \Phi_n^2, \Phi_{n-1}^2, \Phi_n^3, \Phi_{n-1}^3\right)^T$ *of every solution of system* (1.7) *supplies both of the following asymptotic relations:*

$$\lim_{n\to\infty} (\|\Phi_n\|)^{\frac{1}{n}} = \left|\lambda_{1,2,3,4,5,6} F_J\left(\overline{\Upsilon}, \overline{\Psi}, \overline{\Omega}\right)\right|,$$
$$\lim_{n\to\infty} \frac{\|\Phi_{n+1}\|}{\|\Phi_n\|} = \left|\lambda_{1,2,3,4,5,6} F_J\left(\overline{\Upsilon}, \overline{\Psi}, \overline{\Omega}\right)\right|, \tag{2.40}$$

where $\left|\lambda_{1,2,3,4,5,6} F_J\left(\overline{\Upsilon}, \overline{\Psi}, \overline{\Omega}\right)\right|$ *denotes the characteristic root of* $F_J\left(\overline{\Upsilon}, \overline{\Psi}, \overline{\Omega}\right)$.

Proof. To find the error terms, from system (1.7), we obtain

$$\begin{aligned}
\Upsilon_{n+1}-\overline{\Upsilon}=&\frac{\Gamma_1+\delta_1 e^{-\Psi_{n-1}}}{\Theta_1+\Psi_n}-\frac{\Gamma_1+\delta_1 e^{-\overline{\Psi}}}{\Theta_1+\overline{\Psi}}\\
=&-\frac{\Gamma_1+\delta_1 e^{-\overline{\Psi}}}{(\Theta_1+\Psi_n)\left(\Theta_1+\overline{\Psi}\right)}\left(\Psi_n-\overline{\Psi}\right)+\frac{\delta_1\left(e^{-\Psi_{n-1}}-e^{-\overline{\Psi}}\right)}{(\Theta_1+\Psi_n)\left(\Psi_{n-1}-\overline{\Psi}\right)}\left(\Psi_{n-1}-\overline{\Psi}\right),\\
\Psi_{n+1}-\overline{\Psi}=&\frac{\Gamma_2+\delta_2 e^{-\Omega_{n-1}}}{\Theta_2+\Omega_n}-\frac{\Gamma_2+\delta_2 e^{-\overline{\Omega}}}{\Theta_2+\overline{\Omega}}\\
=&-\frac{\Gamma_2+\delta_2 e^{-\overline{\Omega}}}{(\Theta_2+\Omega_n)\left(\Theta_2+\overline{\Omega}\right)}\left(\Omega_n-\overline{\Omega}\right)+\frac{\delta_2\left(e^{-\Omega_{n-1}}-e^{-\overline{\Omega}}\right)}{(\Theta_2+\Omega_n)\left(\Omega_{n-1}-\overline{\Omega}\right)}\left(\Omega_{n-1}-\overline{\Omega}\right),\\
\Omega_{n+1}-\overline{\Omega}=&\frac{\Gamma_3+\delta_3 e^{-\Upsilon_{n-1}}}{\Theta_3+\Upsilon_n}-\frac{\Gamma_3+\delta_3 e^{-\overline{\Upsilon}}}{\Theta_3+\overline{\Upsilon}}\\
=&-\frac{\Gamma_3+\delta_3 e^{-\overline{\Upsilon}}}{(\Theta_3+\Upsilon_n)\left(\Theta_3+\overline{\Upsilon}\right)}\left(\Upsilon_n-\overline{\Upsilon}\right)+\frac{\delta_3\left(e^{-\Upsilon_{n-1}}-e^{-\overline{\Upsilon}}\right)}{(\Theta_3+\Upsilon_n)\left(\Upsilon_{n-1}-\overline{\Upsilon}\right)}\left(\Upsilon_{n-1}-\overline{\Upsilon}\right),
\end{aligned} \tag{2.41}$$

that is,

$$\begin{aligned}
\Upsilon_{n+1}-\overline{\Upsilon}=&-\frac{\Gamma_1+\delta_1 e^{-\overline{\Psi}}}{(\Theta_1+\Psi_n)\left(\Theta_1+\overline{\Psi}\right)}\left(\Psi_n-\overline{\Psi}\right)+\frac{\delta_1\left(e^{-\Psi_{n-1}}-e^{-\overline{\Psi}}\right)}{(\Theta_1+\Psi_n)\left(\Psi_{n-1}-\overline{\Psi}\right)}\left(\Psi_{n-1}-\overline{\Psi}\right),\\
\Psi_{n+1}-\overline{\Psi}=&-\frac{\Gamma_2+\delta_2 e^{-\overline{\Omega}}}{(\Theta_2+\Omega_n)\left(\Theta_2+\overline{\Omega}\right)}\left(\Omega_n-\overline{\Omega}\right)+\frac{\delta_2\left(e^{-\Omega_{n-1}}-e^{-\overline{\Omega}}\right)}{(\Theta_2+\Omega_n)\left(\Omega_{n-1}-\overline{\Omega}\right)}\left(\Omega_{n-1}-\overline{\Omega}\right),\\
\Omega_{n+1}-\overline{\Omega}=&-\frac{\Gamma_3+\delta_3 e^{-\overline{\Upsilon}}}{(\Theta_3+\Upsilon_n)\left(\Theta_3+\overline{\Upsilon}\right)}\left(\Upsilon_n-\overline{\Upsilon}\right)+\frac{\delta_3\left(e^{-\Upsilon_{n-1}}-e^{-\overline{\Upsilon}}\right)}{(\Theta_3+\Upsilon_n)\left(\Upsilon_{n-1}-\overline{\Upsilon}\right)}\left(\Upsilon_{n-1}-\overline{\Upsilon}\right).
\end{aligned} \tag{2.42}$$

Set

$$\Phi_n^1=\Upsilon_n-\overline{\Upsilon},\ \Phi_n^2=\Psi_n-\overline{\Psi},\ \Phi_n^3=\Omega_n-\overline{\Omega}. \tag{2.43}$$

By using Eq (2.43), Eq (2.42) can be written in the following form:

$$\begin{aligned}
\Phi_{n+1}^1=&d_n\Phi_n^2+e_n\Phi_{n-1}^2,\\
\Phi_{n+1}^2=&f_n\Phi_n^3+g_n\Phi_{n-1}^3,\\
\Phi_{n+1}^3=&h_n\Phi_n^1+j_n\Phi_{n-1}^1,
\end{aligned} \tag{2.44}$$

where

$$d_n = -\frac{\Gamma_1 + \delta_1 e^{-\overline{\Psi}}}{(\Theta_1 + \Psi_n)\left(\Theta_1 + \overline{\Psi}\right)},\ e_n = \frac{\delta_1\left(e^{-\Psi_{n-1}} - e^{-\overline{\Psi}}\right)}{(\Theta_1 + \Psi_n)\left(\Psi_{n-1} - \overline{\Psi}\right)},$$
$$f_n = -\frac{\Gamma_2 + \delta_2 e^{-\overline{\Omega}}}{(\Theta_2 + \Omega_n)\left(\Theta_2 + \overline{\Omega}\right)},\ g_n = \frac{\delta_2\left(e^{-\Omega_{n-1}} - e^{-\overline{\Omega}}\right)}{(\Theta_2 + \Omega_n)\left(\Omega_{n-1} - \overline{\Omega}\right)}, \tag{2.45}$$
$$h_n = -\frac{\Gamma_3 + \delta_3 e^{-\overline{\Upsilon}}}{(\Theta_3 + \Upsilon_n)\left(\Theta_3 + \overline{\Upsilon}\right)},\ j_n = \frac{\delta_3\left(e^{-\Upsilon_{n-1}} - e^{-\overline{\Upsilon}}\right)}{(\Theta_3 + \Upsilon_n)\left(\Upsilon_{n-1} - \overline{\Upsilon}\right)}.$$

By taking the limits of d_n, e_n, f_n, g_n, h_n and j_n, we respectively obtain

$$\lim_{n\to\infty} d_n = -\frac{\Gamma_1 + \delta_1 e^{-\overline{\Psi}}}{\left(\Theta_1 + \overline{\Psi}\right)^2},\ \lim_{n\to\infty} e_n = -\frac{\delta_1 e^{-\overline{\Psi}}}{\Theta_1 + \overline{\Psi}},$$
$$\lim_{n\to\infty} f_n = -\frac{\Gamma_2 + \delta_2 e^{-\overline{\Omega}}}{\left(\Theta_2 + \overline{\Omega}\right)^2},\ \lim_{n\to\infty} g_n = -\frac{\delta_2 e^{-\overline{\Omega}}}{\Theta_2 + \overline{\Omega}}, \tag{2.46}$$
$$\lim_{n\to\infty} h_n = -\frac{\Gamma_3 + \delta_3 e^{-\overline{\Upsilon}}}{\left(\Theta_3 + \overline{\Upsilon}\right)^2},\ \lim_{n\to\infty} j_n = -\frac{\delta_3 e^{-\overline{\Upsilon}}}{\Theta_3 + \overline{\Upsilon}},$$

that is,

$$d_n = -\frac{\Gamma_1 + \delta_1 e^{-\overline{\Psi}}}{\left(\Theta_1 + \overline{\Psi}\right)^2} + \xi 2_n,\ e_n = -\frac{\delta_1 e^{-\overline{\Psi}}}{\Theta_1 + \overline{\Psi}} + \xi 2_{n-1},$$
$$f_n = -\frac{\Gamma_2 + \delta_2 e^{-\overline{\Omega}}}{\left(\Theta_2 + \overline{\Omega}\right)^2} + \xi 3_n,\ g_n = -\frac{\delta_2 e^{-\overline{\Omega}}}{\Theta_2 + \overline{\Omega}} + \xi 3_{n-1}, \tag{2.47}$$
$$h_n = -\frac{\Gamma_3 + \delta_3 e^{-\overline{\Upsilon}}}{\left(\Theta_3 + \overline{\Upsilon}\right)^2} + \xi 1_n,\ j_n = -\frac{\delta_3 e^{-\overline{\Upsilon}}}{\Theta_3 + \overline{\Upsilon}} + \xi 1_{n-1},$$

where $\xi 1_n \to 0$, $\xi 1_{n-1} \to 0$, $\xi 2_n \to 0$, $\xi 2_{n-1} \to 0$, $\xi 3_n \to 0$ and $\xi 3_{n-1} \to 0$ as $n \to \infty$. Then, we get Poincare difference system (1.10) of [51], where

$$A = \begin{pmatrix} 0 & 0 & -\frac{\Gamma_1+\delta_1 e^{-\overline{\Psi}}}{\left(\Theta_1+\overline{\Psi}\right)^2} & -\frac{\delta_1 e^{-\overline{\Psi}}}{\Theta_1+\overline{\Psi}} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\Gamma_2+\delta_2 e^{-\overline{\Omega}}}{\left(\Theta_2+\overline{\Omega}\right)^2} & -\frac{\delta_2 e^{-\overline{\Omega}}}{\Theta_2+\overline{\Omega}} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{\Gamma_3+\delta_3 e^{-\overline{\Upsilon}}}{\left(\Theta_3+\overline{\Upsilon}\right)^2} & -\frac{\delta_3 e^{-\overline{\Upsilon}}}{\Theta_3+\overline{\Upsilon}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{2.48}$$

and

$$B_n = \begin{pmatrix} 0 & 0 & \xi 2_n & \xi 2_{n-1} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi 3_n & \xi 3_{n-1} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \xi 1_n & \xi 1_{n-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{2.49}$$

and $\|B_n\| \to 0$ as $n \to \infty$. Moreover, the limiting system of error terms turns into

$$\begin{pmatrix} \Phi^1_{n+1} \\ \Phi^1_n \\ \Phi^2_{n+1} \\ \Phi^2_n \\ \Phi^3_{n+1} \\ \Phi^3_n \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{\Gamma_1+\delta_1 e^{-\overline{\Psi}}}{\left(\Theta_1+\overline{\Psi}\right)^2} & -\frac{\delta_1 e^{-\overline{\Psi}}}{\Theta_1+\overline{\Psi}} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\Gamma_2+\delta_2 e^{-\overline{\Omega}}}{\left(\Theta_2+\overline{\Omega}\right)^2} & -\frac{\delta_2 e^{-\overline{\Omega}}}{\Theta_2+\overline{\Omega}} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{\Gamma_3+\delta_3 e^{-\overline{\Upsilon}}}{\left(\Theta_3+\overline{\Upsilon}\right)^2} & -\frac{\delta_3 e^{-\overline{\Upsilon}}}{\Theta_3+\overline{\Upsilon}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \Phi^1_n \\ \Phi^1_{n-1} \\ \Phi^2_n \\ \Phi^2_{n-1} \\ \Phi^3_n \\ \Phi^3_{n-1} \end{pmatrix}, \tag{2.50}$$

which is similar to the linearized system (1.7) about the equilibrium $\left(\overline{\Upsilon}, \overline{\Psi}, \overline{\Omega}\right)$. □

3. Global behavior of system (1.8)

Lemma 5. *Every positive solution of system* (1.8) *is bounded and persists.*

Proof. Suppose that $\{(\Upsilon_n, \Psi_n, \Omega_n)\}$ is an arbitrary solution of system (1.8). From Lemma 3, by using induction and applying $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \Upsilon_n \in I_4 &= \left[\frac{\Gamma_1 + \delta_1 e^{-\left(\frac{\Gamma_2+\delta_2}{\Theta_2}\right)}}{\Theta_1 + \left(\frac{\Gamma_1+\delta_1}{\Theta_1}\right)}, \frac{\Gamma_1+\delta_1}{\Theta_1}\right] = [\Lambda_4, \Pi_4], \\ \Psi_n \in I_5 &= \left[\frac{\Gamma_2 + \delta_2 e^{-\left(\frac{\Gamma_3+\delta_3}{\Theta_3}\right)}}{\Theta_2 + \left(\frac{\Gamma_2+\delta_2}{\Theta_2}\right)}, \frac{\Gamma_2+\delta_2}{\Theta_2}\right] = [\Lambda_5, \Pi_5], \\ \Omega_n \in I_6 &= \left[\frac{\Gamma_3 + \delta_3 e^{-\left(\frac{\Gamma_1+\delta_1}{\Theta_1}\right)}}{\Theta_3 + \left(\frac{\Gamma_3+\delta_3}{\Theta_3}\right)}, \frac{\Gamma_3+\delta_3}{\Theta_3}\right] = [\Lambda_6, \Pi_6]. \end{aligned} \tag{3.1}$$

The proof of the lemma is similar to Lemma 3, so it is omitted. □

3.1. Local and global asymptotic stability

Theorem 6. *For the local stability about* $\left(\overline{\Upsilon}, \overline{\Psi}, \overline{\Omega}\right) \in [\Lambda_4, \Pi_4] \times [\Lambda_5, \Pi_5] \times [\Lambda_6, \Pi_6]$, *i.e., the equilibrium point of system* (1.8)*, the next declarations are valid:*

(i) $\left(\overline{\Upsilon}, \overline{\Psi}, \overline{\Omega}\right)$ *is locally asymptotically stable if* $L_2 < 1$*,*

(ii) $\left(\overline{\Upsilon}, \overline{\Psi}, \overline{\Omega}\right)$ *is unstable if* $U_2 > 1$*,*

where

$$L_2 = \frac{1}{\Theta_1\Theta_2\Theta_3}\Big(\Theta_2\Lambda_4\left(\Theta_3 + 2\Lambda_6\right) + \Theta_3\Lambda_5\left(\Theta_1 + 2\Lambda_4\right) + \Theta_1\Lambda_6\left(\Theta_2 + 2\Lambda_5\right) + \Lambda_4\Lambda_5\Lambda_6 + \delta_1\delta_2\delta_3\Big), \tag{3.2}$$

$$\begin{aligned} U_2 = {} & \frac{\Pi_4\left(\Theta_2 + \Pi_5\right)}{\left(\Theta_1 + \Pi_4\right)\left(\Theta_2 + \Pi_5\right)\left(\Theta_3 + \Pi_6\right)}\left(\Theta_3 + 2\Pi_6\right) + \frac{\Pi_5\left(\Theta_3 + \Pi_6\right)}{\left(\Theta_1 + \Pi_4\right)\left(\Theta_2 + \Pi_5\right)\left(\Theta_3 + \Pi_6\right)}\left(\Theta_1 + 2\Pi_4\right) \\ & + \frac{\Pi_6\left(\Theta_1 + \Pi_4\right)}{\left(\Theta_1 + \Pi_4\right)\left(\Theta_2 + \Pi_5\right)\left(\Theta_3 + \Pi_6\right)}\left(\Theta_2 + 2\Pi_5\right) + \frac{1}{\left(\Theta_1 + \Pi_4\right)\left(\Theta_2 + \Pi_5\right)\left(\Theta_3 + \Pi_6\right)}\Big(\Pi_4\Pi_5\Pi_6 \\ & + \delta_1\delta_2\delta_3 e^{-\Pi_4-\Pi_5-\Pi_6}\Big). \end{aligned} \tag{3.3}$$

Proof. (i) From system (1.8), we have

$$\overline{\Upsilon} = \frac{\Gamma_1 + \delta_1 e^{-\overline{\Psi}}}{\Theta_1 + \overline{\Upsilon}},\ \overline{\Psi} = \frac{\Gamma_2 + \delta_2 e^{-\overline{\Omega}}}{\Theta_2 + \overline{\Psi}},\ \overline{\Omega} = \frac{\Gamma_3 + \delta_3 e^{-\overline{\Upsilon}}}{\Theta_3 + \overline{\Omega}}. \tag{3.4}$$

In order to construct the corresponding linearized form of system (1.8), we consider the following transformation:

$$\left(\Upsilon_{n+1}, \Upsilon_n, \Psi_{n+1}, \Psi_n, \Omega_{n+1}, \Omega_n\right) \to \left(f, f_1, g, g_1, h, h_1\right), \tag{3.5}$$

where

$$\begin{aligned} f &= \frac{\Gamma_1 + \delta_1 e^{-\Psi_{n-1}}}{\Theta_1 + \Upsilon_n},\ f_1 = \Upsilon_n, \\ g &= \frac{\Gamma_2 + \delta_2 e^{-\Omega_{n-1}}}{\Theta_2 + \Psi_n},\ g_1 = \Psi_n, \\ h &= \frac{\Gamma_3 + \delta_3 e^{-\Upsilon_{n-1}}}{\Theta_3 + \Omega_n},\ h_1 = \Omega_n. \end{aligned} \tag{3.6}$$

By using the transformation given by Eq (3.5), we have

$$F_J\left(\overline{\Upsilon}, \overline{\Psi}, \overline{\Omega}\right) = \begin{pmatrix} s_4 & 0 & 0 & t_4 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_5 & 0 & 0 & t_5 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & s_6 & 0 & 0 & t_6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{3.7}$$

where

$$\begin{aligned}
s_4 &= -\frac{\Gamma_1 + \delta_1 e^{-\overline{\Psi}}}{\left(\Theta_1 + \overline{\Upsilon}\right)^2},\ t_4 = -\frac{\delta_1 e^{-\overline{\Psi}}}{\Theta_1 + \overline{\Upsilon}},\\
s_5 &= -\frac{\Gamma_2 + \delta_2 e^{-\overline{\Omega}}}{\left(\Theta_2 + \overline{\Psi}\right)^2},\ t_5 = -\frac{\delta_2 e^{-\overline{\Omega}}}{\Theta_2 + \overline{\Psi}},\\
s_6 &= -\frac{\delta_3 e^{-\overline{\Upsilon}}}{\Theta_3 + \overline{\Omega}},\ t_6 = -\frac{\Gamma_3 + \delta_3 e^{-\overline{\Upsilon}}}{\left(\Theta_3 + \overline{\Omega}\right)^2}.
\end{aligned} \tag{3.8}$$

The characteristic equation of $F_J\left(\overline{\Upsilon}, \overline{\Psi}, \overline{\Omega}\right)$ is below:

$$\lambda^6 + \widehat{m}_1\lambda^5 + \widehat{m}_2\lambda^4 + \widehat{m}_3\lambda^3 + \widehat{m}_4 = 0, \tag{3.9}$$

where

$$\begin{aligned}
\widehat{m}_1 &= -\left(s_4 + s_5 + t_6\right),\\
\widehat{m}_2 &= s_4 s_5 + s_4 t_6 + s_5 t_6,\\
\widehat{m}_3 &= -s_4 s_5 t_6,\\
\widehat{m}_4 &= -s_6 t_4 t_5.
\end{aligned} \tag{3.10}$$

Now, we can compute $\sum_{i=1}^{4}\left|\widehat{m}_i\right|$ as follows:

$$\begin{aligned}
\sum_{i=1}^{4}\left|\widehat{m}_i\right| &= \frac{\left(\Gamma_1 + \delta_1 e^{-\overline{\Psi}}\right)}{\left(\Theta_1 + \overline{\Upsilon}\right)^2} + \frac{\left(\Gamma_2 + \delta_2 e^{-\overline{\Omega}}\right)}{\left(\Theta_2 + \overline{\Psi}\right)^2} + \frac{\left(\Gamma_3 + \delta_3 e^{-\overline{\Upsilon}}\right)}{\left(\Theta_3 + \overline{\Omega}\right)^2}\\
&+ \frac{\left(\Gamma_1 + \delta_1 e^{-\overline{\Psi}}\right)\left(\Gamma_2 + \delta_2 e^{-\overline{\Omega}}\right)}{\left(\Theta_1 + \overline{\Upsilon}\right)^2\left(\Theta_2 + \overline{\Psi}\right)^2} + \frac{\left(\Gamma_1 + \delta_1 e^{-\overline{\Psi}}\right)\left(\Gamma_3 + \delta_3 e^{-\overline{\Upsilon}}\right)}{\left(\Theta_1 + \overline{\Upsilon}\right)^2\left(\Theta_3 + \overline{\Omega}\right)^2}\\
&+ \frac{\left(\Gamma_2 + \delta_2 e^{-\overline{\Omega}}\right)\left(\Gamma_3 + \delta_3 e^{-\overline{\Upsilon}}\right)}{\left(\Theta_2 + \overline{\Psi}\right)^2\left(\Theta_3 + \overline{\Omega}\right)^2} + \frac{\left(\Gamma_1 + \delta_1 e^{-\overline{\Psi}}\right)\left(\Gamma_2 + \delta_2 e^{-\overline{\Omega}}\right)\left(\Gamma_3 + \delta_3 e^{-\overline{\Upsilon}}\right)}{\left(\Theta_1 + \overline{\Upsilon}\right)^2\left(\Theta_2 + \overline{\Psi}\right)^2\left(\Theta_3 + \overline{\Omega}\right)^2}\\
&+ \frac{\delta_1\delta_2\delta_3 e^{-\overline{\Upsilon}-\overline{\Psi}-\overline{\Omega}}}{\left(\Theta_1 + \overline{\Upsilon}\right)\left(\Theta_2 + \overline{\Psi}\right)\left(\Theta_3 + \overline{\Omega}\right)}\\
&= \frac{\overline{\Upsilon}}{\left(\Theta_1 + \overline{\Upsilon}\right)} + \frac{\overline{\Psi}}{\left(\Theta_2 + \overline{\Psi}\right)} + \frac{\overline{\Omega}}{\left(\Theta_3 + \overline{\Omega}\right)} + \frac{\overline{\Upsilon}.\overline{\Psi}}{\left(\Theta_1 + \overline{\Upsilon}\right)\left(\Theta_2 + \overline{\Psi}\right)} + \frac{\overline{\Upsilon}.\overline{\Omega}}{\left(\Theta_1 + \overline{\Upsilon}\right)\left(\Theta_3 + \overline{\Omega}\right)}\\
&+ \frac{\overline{\Psi}\,\overline{\Omega}}{\left(\Theta_2 + \overline{\Psi}\right)\left(\Theta_3 + \overline{\Omega}\right)} + \frac{\overline{\Upsilon}\,\overline{\Psi}\,\overline{\Omega}}{\left(\Theta_1 + \overline{\Upsilon}\right)\left(\Theta_2 + \overline{\Psi}\right)\left(\Theta_3 + \overline{\Omega}\right)}\\
&+ \frac{\delta_1\delta_2\delta_3 e^{-\overline{\Upsilon}-\overline{\Psi}-\overline{\Omega}}}{\left(\Theta_1 + \overline{\Upsilon}\right)\left(\Theta_2 + \overline{\Psi}\right)\left(\Theta_3 + \overline{\Omega}\right)}
\end{aligned} \tag{3.11}$$

$$= \frac{\overline{\Upsilon}}{\left(\Theta_1+\overline{\Upsilon}\right)\left(\Theta_3+\overline{\Omega}\right)}\left(\Theta_3+2\overline{\Omega}\right)+\frac{\overline{\Psi}}{\left(\Theta_1+\overline{\Upsilon}\right)\left(\Theta_2+\overline{\Psi}\right)}\left(\Theta_1+2\overline{\Upsilon}\right)$$
$$+\frac{\overline{\Omega}}{\left(\Theta_2+\overline{\Psi}\right)\left(\Theta_3+\overline{\Omega}\right)}\left(\Theta_2+2\overline{\Psi}\right)+\frac{1}{\left(\Theta_1+\overline{\Upsilon}\right)\left(\Theta_2+\overline{\Psi}\right)\left(\Theta_3+\overline{\Omega}\right)}\left(\overline{\Upsilon}\,\overline{\Psi}\,\overline{\Omega}+\delta_1\delta_2\delta_3 e^{-\overline{\Upsilon}-\overline{\Psi}-\overline{\Omega}}\right)$$
$$< \frac{\overline{\Upsilon}}{\Theta_1\Theta_3}\left(\Theta_3+2\overline{\Omega}\right)+\frac{\overline{\Psi}}{\Theta_1\Theta_2}\left(\Theta_1+2\overline{\Upsilon}\right)+\frac{\overline{\Omega}}{\Theta_2\Theta_3}\left(\Theta_2+2\overline{\Psi}\right)+\frac{1}{\Theta_1\Theta_2\Theta_3}\left(\overline{\Upsilon}\,\overline{\Psi}\,\overline{\Omega}+\delta_1\delta_2\delta_3\right)$$
$$= \frac{1}{\Theta_1\Theta_2\Theta_3}\left(\Theta_2\overline{\Upsilon}\left(\Theta_3+2\overline{\Omega}\right)+\Theta_3\overline{\Psi}\left(\Theta_1+2\overline{\Upsilon}\right)+\Theta_1\overline{\Omega}\left(\Theta_2+2\overline{\Psi}\right)+\overline{\Upsilon}\,\overline{\Psi}\,\overline{\Omega}+\delta_1\delta_2\delta_3\right)$$
$$< \frac{1}{\Theta_1\Theta_2\Theta_3}\Big(\Theta_2\Lambda_4\left(\Theta_3+2\Lambda_6\right)+\Theta_3\Lambda_5\left(\Theta_1+2\Lambda_4\right)+\Theta_1\Lambda_6\left(\Theta_2+2\Lambda_5\right)+\Lambda_4\Lambda_5\Lambda_6$$
$$+\delta_1\delta_2\delta_3\Big) < 1.$$

By supposing that $L_2 < 1$, from Eq (3.11), we have that $\sum_{i=1}^{4}\left|\widehat{m}_i\right| < 1$. According to the Rouche theorem, $\left(\overline{\Upsilon},\overline{\Psi},\overline{\Omega}\right)$ is locally asymptotically stable.

(ii) We have

$$\sum_{i=1}^{4}\left|\widehat{m}_i\right| = \frac{\left(\Gamma_1+\delta_1 e^{-\overline{\Psi}}\right)}{\left(\Theta_1+\overline{\Upsilon}\right)^2}+\frac{\left(\Gamma_2+\delta_2 e^{-\overline{\Omega}}\right)}{\left(\Theta_2+\overline{\Psi}\right)^2}+\frac{\left(\Gamma_3+\delta_3 e^{-\overline{\Upsilon}}\right)}{\left(\Theta_3+\overline{\Omega}\right)^2}$$
$$+\frac{\left(\Gamma_1+\delta_1 e^{-\overline{\Psi}}\right)\left(\Gamma_2+\delta_2 e^{-\overline{\Omega}}\right)}{\left(\Theta_1+\overline{\Upsilon}\right)^2\left(\Theta_2+\overline{\Psi}\right)^2}+\frac{\left(\Gamma_1+\delta_1 e^{-\overline{\Psi}}\right)\left(\Gamma_3+\delta_3 e^{-\overline{\Upsilon}}\right)}{\left(\Theta_1+\overline{\Upsilon}\right)^2\left(\Theta_3+\overline{\Omega}\right)^2}$$
$$+\frac{\left(\Gamma_2+\delta_2 e^{-\overline{\Omega}}\right)\left(\Gamma_3+\delta_3 e^{-\overline{\Upsilon}}\right)}{\left(\Theta_2+\overline{\Psi}\right)^2\left(\Theta_3+\overline{\Omega}\right)^2}+\frac{\left(\Gamma_1+\delta_1 e^{-\overline{\Psi}}\right)\left(\Gamma_2+\delta_2 e^{-\overline{\Omega}}\right)\left(\Gamma_3+\delta_3 e^{-\overline{\Upsilon}}\right)}{\left(\Theta_1+\overline{\Upsilon}\right)^2\left(\Theta_2+\overline{\Psi}\right)^2\left(\Theta_3+\overline{\Omega}\right)^2}$$
$$+\frac{\delta_1\delta_2\delta_3 e^{-\overline{\Upsilon}-\overline{\Psi}-\overline{\Omega}}}{\left(\Theta_1+\overline{\Upsilon}\right)\left(\Theta_2+\overline{\Psi}\right)\left(\Theta_3+\overline{\Omega}\right)}$$
$$= \frac{\overline{\Upsilon}}{\left(\Theta_1+\overline{\Upsilon}\right)\left(\Theta_3+\overline{\Omega}\right)}\left(\Theta_3+2\overline{\Omega}\right)+\frac{\overline{\Psi}}{\left(\Theta_1+\overline{\Upsilon}\right)\left(\Theta_2+\overline{\Psi}\right)}\left(\Theta_1+2\overline{\Upsilon}\right) \tag{3.12}$$
$$+\frac{\overline{\Omega}}{\left(\Theta_2+\overline{\Psi}\right)\left(\Theta_3+\overline{\Omega}\right)}\left(\Theta_2+2\overline{\Psi}\right)+\frac{1}{\left(\Theta_1+\overline{\Upsilon}\right)\left(\Theta_2+\overline{\Psi}\right)\left(\Theta_3+\overline{\Omega}\right)}\left(\overline{\Upsilon}.\overline{\Psi}.\overline{\Omega}+\delta_1\delta_2\delta_3 e^{-\overline{\Upsilon}-\overline{\Psi}-\overline{\Omega}}\right)$$
$$\geq \frac{\Pi_4\left(\Theta_2+\Pi_5\right)}{\left(\Theta_1+\Pi_4\right)\left(\Theta_2+\Pi_5\right)\left(\Theta_3+\Pi_6\right)}\left(\Theta_3+2\Pi_6\right)+\frac{\Pi_5\left(\Theta_3+\Pi_6\right)}{\left(\Theta_1+\Pi_4\right)\left(\Theta_2+\Pi_5\right)\left(\Theta_3+\Pi_6\right)}\left(\Theta_1+2\Pi_4\right)$$
$$+\frac{\Pi_6\left(\Theta_1+\Pi_4\right)}{\left(\Theta_1+\Pi_4\right)\left(\Theta_2+\Pi_5\right)\left(\Theta_3+\Pi_6\right)}\left(\Theta_2+2\Pi_5\right)+\frac{1}{\left(\Theta_1+\Pi_4\right)\left(\Theta_2+\Pi_5\right)\left(\Theta_3+\Pi_6\right)}\Big(\Pi_4\Pi_5\Pi_6$$
$$+\delta_1\delta_2\delta_3 e^{-\Pi_4-\Pi_5-\Pi_6}\Big) > 1.$$

By supposing that $U_2 > 1$, from Eq (3.12), we have that $\sum_{i=1}^{4} \left|\widehat{m}_i\right| > 1$. According to the Rouche theorem, $\left(\overline{\Upsilon}, \overline{\Psi}, \overline{\Omega}\right)$ is unstable. □

Theorem 7. *Consider system* (1.8). *Assume that the next relation is true:*

$$\delta_1\delta_2\delta_3 < \Theta_1\Theta_2\Theta_3. \tag{3.13}$$

Moreover, system (1.8) *has a unique positive equilibrium* $\left(\overline{\Upsilon}, \overline{\Psi}, \overline{\Omega}\right)$*, and every positive solution of system* (1.8) *tends to the unique positive equilibrium of system* (1.8) *as* $n \to \infty$.

Proof. We consider the functions

$$f_2(\Upsilon, \Psi) = \frac{\Gamma_1 + \delta_1 e^{-\Psi}}{\Theta_1 + \Upsilon},\ g_2(\Psi, \Omega) = \frac{\Gamma_2 + \delta_2 e^{-\Omega}}{\Theta_2 + \Psi},\ h_2(\Upsilon, \Omega) = \frac{\Gamma_3 + \delta_3 e^{-\Upsilon}}{\Theta_3 + \Omega}, \tag{3.14}$$

where

$$\Upsilon \in I_4,\ \Psi \in I_5,\ \Omega \in I_6, \tag{3.15}$$

and I_4, I_5, I_6 are defined in Eq (3.1). From Eqs (3.14) and (3.15), we see that, for $\Upsilon \in I_4$, $\Psi \in I_5$ and $\Omega \in I_6$,

$$f_2(\Upsilon, \Psi) \in I_4,\ g_2(\Psi, \Omega) \in I_5,\ h_2(\Upsilon, \Omega) \in I_6;$$

thus,

$$f_2 : I_4 \times I_5 \to I_4,\ g_2 : I_5 \times I_6 \to I_5,\ h_2 : I_4 \times I_6 \to I_6.$$

Let p, P, l, L, w and W be positive numbers such that

$$\begin{aligned} P &= \frac{\Gamma_1 + \delta_1 e^{-l}}{\Theta_1 + p},\ p = \frac{\Gamma_1 + \delta_1 e^{-L}}{\Theta_1 + P}, \\ L &= \frac{\Gamma_2 + \delta_2 e^{-w}}{\Theta_2 + l},\ l = \frac{\Gamma_2 + \delta_2 e^{-W}}{\Theta_2 + L}, \\ W &= \frac{\Gamma_3 + \delta_3 e^{-p}}{\Theta_3 + w},\ w = \frac{\Gamma_3 + \delta_3 e^{-P}}{\Theta_3 + W}. \end{aligned} \tag{3.16}$$

From Eq (3.16), we get

$$e^{-l} = \frac{P(\Theta_1 + p) - \Gamma_1}{\delta_1},\ e^{-L} = \frac{p(\Theta_1 + P) - \Gamma_1}{\delta_1},$$

$$e^{-w} = \frac{L(\Theta_2 + l) - \Gamma_2}{\delta_2},\ e^{-W} = \frac{l(\Theta_2 + L) - \Gamma_2}{\delta_2},$$

$$e^{-p} = \frac{W(\Theta_3 + w) - \Gamma_3}{\delta_3},\ e^{-P} = \frac{w(\Theta_3 + W) - \Gamma_3}{\delta_3},$$

which imply that

$$
\begin{aligned}
P-p &= \frac{\delta_1}{\Theta_1}\left(e^{-l}-e^{-L}\right)=\frac{\delta_1}{\Theta_1}e^{-l-L}\left(e^{L}-e^{l}\right),\\
L-l &= \frac{\delta_2}{\Theta_2}\left(e^{-w}-e^{-W}\right)=\frac{\delta_2}{\Theta_2}e^{-w-W}\left(e^{W}-e^{w}\right),\\
W-w &= \frac{\delta_3}{\Theta_3}\left(e^{-p}-e^{-P}\right)=\frac{\delta_3}{\Theta_3}e^{-p-P}\left(e^{P}-e^{p}\right).
\end{aligned}
\tag{3.17}
$$

Moreover, we get

$$
\begin{aligned}
e^{P}-e^{p} &= e^{d_1}\left(P-p\right),\ min\left\{P,p\right\}\leq d_1\leq max\left\{P,p\right\},\\
e^{L}-e^{l} &= e^{d_2}\left(L-l\right),\ min\left\{L,l\right\}\leq d_2\leq max\left\{L,l\right\},\\
e^{W}-e^{w} &= e^{d_3}\left(W-w\right),\ min\left\{W,w\right\}\leq d_3\leq max\left\{W,w\right\}.
\end{aligned}
\tag{3.18}
$$

Then Eqs (3.17) and (3.18) imply that

$$
\begin{aligned}
P-p =&\frac{\delta_1}{\Theta_1}e^{d_2-l-L}\left(L-l\right),\\
L-l =&\frac{\delta_2}{\Theta_2}e^{d_3-w-W}\left(W-w\right),\\
W-w =&\frac{\delta_3}{\Theta_3}e^{d_1-p-P}\left(P-p\right);
\end{aligned}
\tag{3.19}
$$

thus,

$$
\begin{aligned}
\left|P-p\right| &\leq \frac{\delta_1}{\Theta_1}\left|L-l\right|,\\
\left|L-l\right| &\leq \frac{\delta_2}{\Theta_2}\left|W-w\right|,\\
\left|W-w\right| &\leq \frac{\delta_3}{\Theta_3}\left|P-p\right|.
\end{aligned}
\tag{3.20}
$$

In addition, observe that Eqs (3.13) and (3.20) imply that

$$
\begin{aligned}
\left(1-\frac{\delta_1\delta_2\delta_3}{\Theta_1\Theta_2\Theta_3}\right)\left|P-p\right| &\leq 0,\\
\left(1-\frac{\delta_1\delta_2\delta_3}{\Theta_1\Theta_2\Theta_3}\right)\left|L-l\right| &\leq 0,\\
\left(1-\frac{\delta_1\delta_2\delta_3}{\Theta_1\Theta_2\Theta_3}\right)\left|W-w\right| &\leq 0,
\end{aligned}
$$

from which we have that $P = p$, $L = l$ and $W = w$. Thus from Eq (3.1), system (1.8) has a unique positive equilibrium $\left(\overline{\Upsilon},\overline{\Psi},\overline{\Omega}\right)$, and every positive solution of system (1.8) tends to the unique positive equilibrium as $n\to\infty$.

□

3.2. Rate of convergence

Theorem 8. *If* $\{(\Upsilon_n, \Psi_n, \Omega_n,)\}_{n=-1}^{\infty}$ *is a positive solution of system* (1.8) *such that*

$$\lim_{n\to\infty} \Upsilon_n = \overline{\Upsilon},\ \lim_{n\to\infty} \Psi_n = \overline{\Psi},\ \lim_{n\to\infty} \Omega_n = \overline{\Omega}, \tag{3.21}$$

where

$$\overline{\Upsilon} \in [\Lambda_4, \Pi_4]\,,\ \overline{\Psi} \in [\Lambda_5, \Pi_5]\,,\ \overline{\Omega} \in [\Lambda_6, \Pi_6]\,, \tag{3.22}$$

then the error vector $\Phi_n = \left(\Phi_n^4, \Phi_{n-1}^4, \Phi_n^5, \Phi_{n-1}^5, \Phi_n^6, \Phi_{n-1}^6\right)^T$ *of every solution of system* (1.8) *supplies both of the following asymptotic relations:*

$$\begin{aligned} &\lim_{n\to\infty} (\|\Phi_n\|)^{\frac{1}{n}} = \left|\lambda_{1,2,3,4,5,6} F_J\left(\overline{\Upsilon}, \overline{\Psi}, \overline{\Omega}\right)\right|, \\ &\lim_{n\to\infty} \frac{\|\Phi_{n+1}\|}{\|\Phi_n\|} = \left|\lambda_{1,2,3,4,5,6} F_J\left(\overline{\Upsilon}, \overline{\Psi}, \overline{\Omega}\right)\right|, \end{aligned} \tag{3.23}$$

where $\left|\lambda_{1,2,3,4,5,6} F_J\left(\overline{\Upsilon}, \overline{\Psi}, \overline{\Omega}\right)\right|$ *denotes the characteristic root of* $F_J\left(\overline{\Upsilon}, \overline{\Psi}, \overline{\Omega}\right)$.

Proof. To find the error terms, from system (1.8), we apply

$$\begin{aligned} \Upsilon_{n+1} - \overline{\Upsilon} =& \frac{\Gamma_1 + \delta_1 e^{-\Psi_{n-1}}}{\Theta_1 + \Upsilon_n} - \frac{\Gamma_1 + \delta_1 e^{-\overline{\Psi}}}{\Theta_1 + \overline{\Upsilon}} \\ =& -\frac{\Gamma_1 + \delta_1 e^{-\overline{\Psi}}}{(\Theta_1 + \Upsilon_n)\left(\Theta_1 + \overline{\Upsilon}\right)}\left(\Upsilon_n - \overline{\Upsilon}\right) + \frac{\delta_1\left(e^{-\Psi_{n-1}} - e^{-\overline{\Psi}}\right)}{(\Theta_1 + \Upsilon_n)\left(\Psi_{n-1} - \overline{\Psi}\right)}\left(\Psi_{n-1} - \overline{\Psi}\right), \\ \Psi_{n+1} - \overline{\Psi} =& \frac{\Gamma_2 + \delta_2 e^{-\Omega_{n-1}}}{\Theta_2 + \Psi_n} - \frac{\Gamma_2 + \delta_2 e^{-\overline{\Omega}}}{\Theta_2 + \overline{\Psi}} \\ =& -\frac{\Gamma_2 + \delta_2 e^{-\overline{\Omega}}}{(\Theta_2 + \Psi_n)\left(\Theta_2 + \overline{\Psi}\right)}\left(\Psi_n - \overline{\Psi}\right) + \frac{\delta_2\left(e^{-\Omega_{n-1}} - e^{-\overline{\Omega}}\right)}{(\Theta_2 + \Psi_n)\left(\Omega_{n-1} - \overline{\Omega}\right)}\left(\Omega_{n-1} - \overline{\Omega}\right), \\ \Omega_{n+1} - \overline{\Omega} =& \frac{\Gamma_3 + \delta_3 e^{-\Upsilon_{n-1}}}{\Theta_3 + \Omega_n} - \frac{\Gamma_3 + \delta_3 e^{-\overline{\Upsilon}}}{\Theta_3 + \overline{\Omega}} \\ =& -\frac{\Gamma_3 + \delta_3 e^{-\overline{\Upsilon}}}{(\Theta_3 + \Omega_n)\left(\Theta_3 + \overline{\Omega}\right)}\left(\Omega_n - \overline{\Omega}\right) + \frac{\delta_3\left(e^{-\Upsilon_{n-1}} - e^{-\overline{\Upsilon}}\right)}{(\Theta_3 + \Omega_n)\left(\Upsilon_{n-1} - \overline{\Upsilon}\right)}\left(\Upsilon_{n-1} - \overline{\Upsilon}\right), \end{aligned} \tag{3.24}$$

that is,

$$\begin{aligned} \Upsilon_{n+1} - \overline{\Upsilon} =& -\frac{\Gamma_1 + \delta_1 e^{-\overline{\Psi}}}{(\Theta_1 + \Upsilon_n)\left(\Theta_1 + \overline{\Upsilon}\right)}\left(\Upsilon_n - \overline{\Upsilon}\right) + \frac{\delta_1\left(e^{-\Psi_{n-1}} - e^{-\overline{\Psi}}\right)}{(\Theta_1 + \Upsilon_n)\left(\Psi_{n-1} - \overline{\Psi}\right)}\left(\Psi_{n-1} - \overline{\Psi}\right), \\ \Psi_{n+1} - \overline{\Psi} =& -\frac{\Gamma_2 + \delta_2 e^{-\overline{\Omega}}}{(\Theta_2 + \Psi_n)\left(\Theta_2 + \overline{\Psi}\right)}\left(\Psi_n - \overline{\Psi}\right) + \frac{\delta_2\left(e^{-\Omega_{n-1}} - e^{-\overline{\Omega}}\right)}{(\Theta_2 + \Psi_n)\left(\Omega_{n-1} - \overline{\Omega}\right)}\left(\Omega_{n-1} - \overline{\Omega}\right), \end{aligned} \tag{3.25}$$

$$\Omega_{n+1}-\overline{\Omega}=-\frac{\Gamma_3+\delta_3 e^{-\overline{\Upsilon}}}{(\Theta_3+\Omega_n)\left(\Theta_3+\overline{\Omega}\right)}\left(\Omega_n-\overline{\Omega}\right)+\frac{\delta_3\left(e^{-\Upsilon_{n-1}}-e^{-\overline{\Upsilon}}\right)}{(\Theta_3+\Omega_n)\left(\Upsilon_{n-1}-\overline{\Upsilon}\right)}\left(\Upsilon_{n-1}-\overline{\Upsilon}\right).$$

Set

$$\Phi_n^4=\Upsilon_n-\overline{\Upsilon},\ \Phi_n^5=\Psi_n-\overline{\Psi},\ \Phi_n^6=\Omega_n-\overline{\Omega}. \tag{3.26}$$

By using Eq (3.26), Eq (3.25) can be written in the following form:

$$\begin{aligned}\Phi_{n+1}^4&=\widehat{d}_n\Phi_n^4+\widehat{e}_n\Phi_{n-1}^5,\\ \Phi_{n+1}^5&=\widehat{f}_n\Phi_n^5+\widehat{g}_n\Phi_{n-1}^6,\\ \Phi_{n+1}^6&=\widehat{h}_n\Phi_n^6+\widehat{j}_n\Phi_{n-1}^4,\end{aligned} \tag{3.27}$$

where

$$\begin{aligned}\widehat{d}_n&=-\frac{\Gamma_1+\delta_1 e^{-\overline{\Psi}}}{(\Theta_1+\Upsilon_n)\left(\Theta_1+\overline{\Upsilon}\right)},\ \widehat{e}_n=\frac{\delta_1\left(e^{-\Psi_{n-1}}-e^{-\overline{\Psi}}\right)}{(\Theta_1+\Upsilon_n)\left(\Psi_{n-1}-\overline{\Psi}\right)},\\ \widehat{f}_n&=-\frac{\Gamma_2+\delta_2 e^{-\overline{\Omega}}}{(\Theta_2+\Psi_n)\left(\Theta_2+\overline{\Psi}\right)},\ \widehat{g}_n=\frac{\delta_2\left(e^{-\Omega_{n-1}}-e^{-\overline{\Omega}}\right)}{(\Theta_2+\Psi_n)\left(\Omega_{n-1}-\overline{\Omega}\right)},\\ \widehat{h}_n&=-\frac{\Gamma_3+\delta_3 e^{-\overline{\Upsilon}}}{(\Theta_3+\Omega_n)\left(\Theta_3+\overline{\Omega}\right)},\ \widehat{j}_n=\frac{\delta_3\left(e^{-\Upsilon_{n-1}}-e^{-\overline{\Upsilon}}\right)}{(\Theta_3+\Omega_n)\left(\Upsilon_{n-1}-\overline{\Upsilon}\right)}.\end{aligned} \tag{3.28}$$

By taking the limits of $\widehat{d}_n, \widehat{e}_n, \widehat{f}_n, \widehat{g}_n, \widehat{h}_n$ and $\widehat{j}_n$, we respectively obtain

$$\begin{aligned}\lim_{n\to\infty}\widehat{d}_n&=-\frac{\Gamma_1+\delta_1 e^{-\overline{\Psi}}}{\left(\Theta_1+\overline{\Upsilon}\right)^2},\ \lim_{n\to\infty}\widehat{e}_n=-\frac{\delta_1 e^{-\overline{\Psi}}}{\Theta_1+\overline{\Upsilon}},\\ \lim_{n\to\infty}\widehat{f}_n&=-\frac{\Gamma_2+\delta_2 e^{-\overline{\Omega}}}{\left(\Theta_2+\overline{\Psi}\right)^2},\ \lim_{n\to\infty}\widehat{g}_n=-\frac{\delta_2 e^{-\overline{\Omega}}}{\Theta_2+\overline{\Psi}},\\ \lim_{n\to\infty}\widehat{h}_n&=-\frac{\Gamma_3+\delta_3 e^{-\overline{\Upsilon}}}{\left(\Theta_3+\overline{\Omega}\right)^2},\ \lim_{n\to\infty}\widehat{j}_n=-\frac{\delta_3 e^{-\overline{\Upsilon}}}{\Theta_3+\overline{\Omega}},\end{aligned} \tag{3.29}$$

that is,

$$\begin{aligned}\widehat{d}_n&=-\frac{\Gamma_1+\delta_1 e^{-\overline{\Psi}}}{\left(\Theta_1+\overline{\Upsilon}\right)^2}+\xi 4_n,\ \widehat{e}_n=-\frac{\delta_1 e^{-\overline{\Psi}}}{\Theta_1+\overline{\Upsilon}}+\xi 5_{n-1},\\ \widehat{f}_n&=-\frac{\Gamma_2+\delta_2 e^{-\overline{\Omega}}}{\left(\Theta_2+\overline{\Psi}\right)^2}+\xi 5_n,\ \widehat{g}_n=-\frac{\delta_2 e^{-\overline{\Omega}}}{\Theta_2+\overline{\Psi}}+\xi 6_{n-1},\\ \widehat{h}_n&=-\frac{\Gamma_3+\delta_3 e^{-\overline{\Upsilon}}}{\left(\Theta_3+\overline{\Omega}\right)^2}+\xi 6_n,\ \widehat{j}_n=-\frac{\delta_3 e^{-\overline{\Upsilon}}}{\Theta_3+\overline{\Omega}}+\xi 4_{n-1},\end{aligned} \tag{3.30}$$

where $\xi 4_n \to 0$, $\xi 4_{n-1} \to 0$, $\xi 5_n \to 0$, $\xi 5_{n-1} \to 0$, $\xi 6_n \to 0$ and $\xi 6_{n-1} \to 0$ as $n \to \infty$. Then, we obtain Poincare difference system (1.10) of [51], where

$$\widehat{A} = \begin{pmatrix} -\frac{\Gamma_1+\delta_1 e^{-\overline{\Psi}}}{\left(\Theta_1+\overline{\Upsilon}\right)^2} & 0 & 0 & -\frac{\delta_1 e^{-\overline{\Psi}}}{\Theta_1+\overline{\Upsilon}} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\Gamma_2+\Psi_2 e^{-\overline{\Omega}}}{\left(\Theta_2+\overline{\Psi}\right)^2} & 0 & 0 & -\frac{\delta_2 e^{-\overline{\Omega}}}{\Theta_2+\overline{\Psi}} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{\delta_3 e^{-\overline{\Upsilon}}}{\Theta_3+\overline{\Omega}} & 0 & 0 & -\frac{\Gamma_3+\delta_3 e^{-\overline{\Upsilon}}}{\left(\Theta_3+\overline{\Omega}\right)^2} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{3.31}$$

and

$$\widehat{B}_n = \begin{pmatrix} \xi 4_n & 0 & 0 & \xi 5_{n-1} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi 5_n & 0 & 0 & \xi 6_{n-1} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \xi 4_{n-1} & 0 & 0 & \xi 6_n & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{3.32}$$

and $\left\|\widehat{B}_n\right\| \to 0$ as $n \to \infty$. Moreover, the limiting system of error terms turns into

$$\begin{pmatrix} \Phi^4_{n+1} \\ \Phi^4_n \\ \Phi^5_{n+1} \\ \Phi^5_n \\ \Phi^6_{n+1} \\ \Phi^6_n \end{pmatrix} = \begin{pmatrix} -\frac{\Gamma_1+\delta_1 e^{-\overline{\Psi}}}{\left(\Theta_1+\overline{\Upsilon}\right)^2} & 0 & 0 & -\frac{\delta_1 e^{-\overline{\Psi}}}{\Theta_1+\overline{\Upsilon}} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\Gamma_2+\delta_2 e^{-\overline{\Omega}}}{\left(\Theta_2+\overline{\Psi}\right)^2} & 0 & 0 & -\frac{\delta_2 e^{-\overline{\Omega}}}{\Theta_2+\overline{\Psi}} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{\delta_3 e^{-\overline{\Upsilon}}}{\Theta_3+\overline{\Omega}} & 0 & 0 & -\frac{\Gamma_3+\delta_3 e^{-\overline{\Upsilon}}}{\left(\Theta_3+\overline{\Omega}\right)^2} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \Phi^4_n \\ \Phi^4_{n-1} \\ \Phi^5_n \\ \Phi^5_{n-1} \\ \Phi^6_n \\ \Phi^6_{n-1} \end{pmatrix}, \tag{3.33}$$

which is similar to the linearized system (1.8) about the equilibrium $\left(\overline{\Upsilon}, \overline{\Psi}, \overline{\Omega}\right)$. □

4. Conclusions

In this paper, the global dynamics of two exponential-type systems of difference equations are investigated. The main results are as follows:

(i) All positive solutions of systems (1.7) and (1.8) have been shown to persist and to be bounded.

(ii) It has been shown that the equilibrium points of systems (1.7) and (1.8) are locally asymptotically stable or unstable based on the parameters L_1, L_2, U_1 and U_2.

(iii) It has been explained, by using both increasing and decreasing functions and a well-known mean-value theorem, that the equilibrium points of systems (1.7) and (1.8) are globally asymptotically stable when the conditions given by Eqs (2.19) and (3.13) are valid.

(iv) Information has been given regarding the rates of convergence of systems (1.7) and (1.8).

Use of AI tools declarations

The author declares that she has not used artificial intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest that may influence the publication of this paper.

References

1. F. Brauer, C. C. Chavez, *Mathematical Models in Population Biology and Epidemiology*, Springer-Verlag, New York, 2001.
2. E. M. Elsayed, F. Alzahrani, I. Abbas, N. H. Alotaibi, Dynamical behavior and solution of nonlinear difference equation via Fibonacci sequence, *J. Appl. Anal. Comput.*, **10** (2020), 282–296. https://doi.org/10.11948/20190143
3. Y. Ke, J. Zheng, An optimal result for global existence in a three-dimensional Keller–Segel–Navier–Stokes system involving tensor-valued sensitivity with saturation, *Calc. Var.*, **58** (2019), 1–27. https://doi.org/10.1007/s00526-019-1568-2
4. N. Taskara, D. T. Tollu, Y. Yazlik, Solutions of rational difference system of order three in terms of Padovan numbers, *J. Adv. Res. Appl. Math.*, **7** (2015), 18–29.
5. D. T. Tollu, Y. Yazlik, N. Taskara, The solutions of four Riccati difference equations associated with Fibonacci numbers, *Balkan J. Math.*, **2** (2014), 163–172.
6. N. Touafek, On a general system of difference equations defined by homogeneous functions, *Math. Slov.*, **71** (2021), 697–720. https://dx.doi.org/10.1515/ms-2021-0014
7. Z. A. Wang, J. Zheng, Global boundedness of the fully parabolic Keller-Segel system with signal-dependent motilities, *Acta Appl. Math.*, **171** (2021), 1–19. https://doi.org/10.1007/s10440-021-00392-8
8. M. Winkler, How far do chemotaxis-driven forces in uence regularity in the Navier-Stokes system?, *Trans. Am. Math. Soc.*, **369** (2017), 3067–3125. https://doi.org/10.1090/tran/6733
9. M. Winkler, Global mass-preserving solutions in a two-dimensional chemotaxis-Stokes system with rotational flux components, *J. Evol. Equation*, **18** (2018), 1267–1289. https://doi.org/10.1007/s00028-018-0440-8
10. Y. Yazlik, D. T. Tollu, N. Taskara, On the solutions of difference equation systems with Padovan numbers, *Appl. Math.*, **4** (2013), 15–20.
11. J. Zheng, Boundedness of solutions to a quasilinear parabolic–elliptic Keller–Segel system with logistic source, *J. Differ. Equation*, **259** (2015), 120–140. https://doi.org/10.1016/j.jde.2015.02.003
12. J. Zheng, An optimal result for global existence and boundedness in a three-dimensional Keller-Segel-Stokes system with nonlinear diffusion, *J. Differ. Equation*, **267** (2019), 2385–2415. https://doi.org/10.1016/j.jde.2019.03.013

13. J. Zheng, A new result for the global existence (and boundedness) and regularity of a three-dimensional Keller-Segel-Navier-Stokes system modeling coral fertilization, *J. Differ. Equation*, **272** (2021), 164–202. https://doi.org/10.1016/j.jde.2020.09.029

14. J. Zheng, Eventual smoothness and stabilization in a three-dimensional Keller-Segel-Navier-Stokes system with rotational flux, *Calc. Var.*, **61** (2022), 1–34. https://doi.org/10.1007/s00526-021-02164-6

15. X. Liu, J. Zheng, Convergence rates of solutions in apredator-prey system with indirect pursuit-evasion interaction in domains of arbitrary dimension, *Discrete Contin. Dyn. Syst. B.*, **28** (2023), 2269–2293. https://doi.org/10.3934/dcdsb.2022168

16. R. Abo-Zeid, H. Kamal, On the solutions of a third order rational difference equation, *Thai J. Math.*, **18** (2020), 1865–1874.

17. R. Abo-Zeid, Global behavior and oscillation of a third order difference equation, *Quaest. Math.*, **44** (2021), 1261–1280. https://doi.org/ 10.2989/16073606.2020.1787537

18. I. Dekkar, N. Touafek, Y. Yazlik, Global stability of a third-order nonlinear system of difference equations with period-two coefficients, *RACSAM*, **111** (2017), 325–347. https://doi.org/10.1007/s13398-016-0297-z

19. E. M. Elsayed, Expression and behavior of the solutions of some rational recursive sequences, *Math. Meth. Appl. Sci.*, **18** (2016), 5682–5694. https://doi.org/10.1002/mma.3953

20. Y. Halim, N. Touafek, Y. Yazlik, Dynamic behavior of a second-order nonlinear rational difference equation, *Turkish J. Math.*, **39** (2015), 1004–1018. https://doi.org/10.3906/mat-1503-80

21. T. F. Ibrahim, N. Touafek, On a third order rational difference equation with variable coefficients, *Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms*, **20** (2013), 251–264.

22. N. Taskara, D. T. Tollu, N. Touafek, Y. Yazlik, A solvable system of difference equations, *Commun. Korean Math. Soc.*, **35** (2020), 301–319. https://doi.org/10.4134/CKMS.c180472

23. D. T. Tollu, Y. Yazlik, N. Taskara, Behavior of positive solutions of a difference equation, *J. Appl. Math. Inform.*, **35** (2017), 217–230. https://dx.doi.org/10.14317/jami.2017.217

24. D. T. Tollu, I. Yalcinkaya, H. Ahmad, S. W. Yao, A detailed study on a solvable system related to the linear fractional difference equation, *Math. Biosci. Eng.*, **18** (2021), 5392–5408. https://doi.org/10.3934/mbe.2021273

25. I. Yalcinkaya, D. T. Tollu, Global behavior of a second order system of difference equations, *Adv. Stud. Contemp. Math.*, **26** (2016), 653–667.

26. I. Yalcinkaya, H. Ahmad, D. T. Tollu, Y. Li, On a system of $k-$difference equations of order three, *Math. Probl. Eng.*, **2020** (2020), 1–11. https://doi.org/10.1155/2020/6638700

27. Y. Yazlik, D. T. Tollu, N. Taskara, On the solutions of a three-dimensional system of difference equations, *Kuwait J. Sci.*, **43** (2016), 95–111.

28. H. El-Metwally, E. A. Grove, G. Ladas, R. Levins, On the difference equation $x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n}$, *Nonlinear Anal. Theor. Meth. Appl.*, **47** (2001), 4623–4634.

29. W. Wang, H. Feng, On the dynamics of positive solutions for the difference equation in a new population model, *J. Nonlinear Sci. Appl.*, **9** (2016), 1748–1754.

30. I. Ozturk, F. Bozkurt, S. Ozen, On the difference equation $y_{n+1} = \frac{\alpha+\beta e^{-y_n}}{\gamma+y_{n-1}}$, *Appl. Math. Comput.*, **181** (2006), 1387–1393. https://doi.org/10.1016/j.amc.2006.03.007
31. T. Comert, I. Yalcinkaya, D. T. Tollu, A study on the positive solutions of an exponential type difference equation, *Electron. J. Math. Anal. Appl.*, **6** (2018), 276–286.
32. F. Bozkurt, Stability analysis of a nonlinear difference equation, *Int. J. Mod. Nonlinear Theor. Appl.*, **2** (2013), 1–6. https://doi.org/10.4236/ijmnta.2013.21001
33. E. A. Grove, G. Ladas, N. R. Prokup, R. Levins, On the global behavior of solutions of a biological model, *Commun. Appl. Nonlinear Anal.*, **7** (2000), 33–46.
34. I. Ozturk, F. Bozkurt, S. Ozen, Global asymptotic behavior of the difference equations: $y_{n+1} = \frac{\alpha e^{-(ny_n+(n-k)y_{n-k})}}{\beta+ny_n+(n-k)y_{n-k}}$, *Appl. Math. Lett.*, **22** (2009), 595–599. https://doi.org/10.1016/j.aml.2008.06.037
35. G. Papaschinopoulos, N. Fotiades, C. J. Schinas, On a system of difference equations including negative exponential terms, *J. Differ. Equations Appl.*, **20** (2014), 717–732. https://doi.org/10.1080/10236198.2013.814647
36. G. Papaschinopoluos, M. A. Radin, C. J. Schinas, Study of the asymptotic behavior of the solutions of three systems of difference equations of exponential form, *Appl. Math. Comput.*, **218** (2012), 5310–5318. https://doi.org/10.1016/j.amc.2011.11.014
37. T. H. Thai, N. A. Dai, P. T. Anh, Global dynamics of some system of second-order difference equations, *Electron. Res. Arch.*, **29** (2021), 4159–4175. https://doi.org/10.3934/era.2021077
38. A. Q. Khan, M. N. Qureshi, Behavior of an exponential system of difference equations, *Discrete Dyn. Nat. Soc.*, **2014** (2014), 1–9. https://doi.org/10.1155/2014/607281
39. A. Q. Khan, M. S. M. Noorani, H. S. Alayachi, Global dynamics of higher-order exponential systems of difference equations, *Discrete Dyn. Nat. Soc.*, **2019** (2019), 1–19. https://doi.org/10.1155/2019/3825927
40. C. Mylona, N. Psarros, G. Papaschinopoulos, C. J. Schinas, Stability of the non-hyperbolic zero equilibrium of two closeto-symmetric systems of difference equations with exponential terms, *Symmetry*, **10** (2018), 188. https://doi.org/10.3390/sym10060188
41. G. Papaschinopoluos, M. A. Radin, C. J. Schinas, On the system of two difference equations of exponential form: $x_{n+1} = a + bx_{n-1}e^{-y_n}$, $y_{n+1} = c + dy_{n-1}e^{-x_n}$, *Math. Comput. Model.*, **54** (2011), 2969–2977. https://doi.org/10.1016/j.mcm.2011.07.019
42. G. Papaschinopoluos, C. J. Schinas, On the dynamics of two exponential type systems of difference equations, *Comput. Math. Appl.*, **64** (2012), 2326–2334. https://doi.org/10.1016/j.camwa.2012.04.002
43. G. Papaschinopoulos, G. Ellina, K. B. Papadopoulos, Asymptotic behavior of the positive solutions of an exponential type system of difference equations, *Appl. Math. Comput.*, **245** (2014), 181–190. https://doi.org/10.1016/j.amc.2014.07.074
44. N. Psarros, G. Papaschinopoulos, C. J. Schinas, On the stability of some systems of exponential difference equations, *Opuscula Math.*, **38** (2018), 95–115. https://doi.org/10.7494/OpMath.2018.38.1.95
45. A. Q. Khan, A. Sharif, Global dynamics of some 3×6 systems of exponential difference equations, *Discrete Dyn. Nat. Soc.*, **2018** (2018), 1–35. https://doi.org/10.1155/2018/8362837

46. C. Mylona, N. Psarros, G. Papaschinopoulos, C. J. Schinas, Stability of two 3×3 close-to-cyclic systems of exponential difference equations, *Math. Meth. Appl. Sci.*, **41** (2018), 7936–7948. https://doi.org/10.1002/mma.5256

47. S. Elaydi, *An Introduction to Difference Equations*, Springer, New York, 2005.

48. E. A. Grove, G. Ladas, *Advances in Discrete Mathematics and Applications*, Chapman and hall/CRC, Boca Raton, 2005.

49. V. L. Kocic, G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Springer, Dordrecht, 1993.

50. H. Sedaghat, *Nonlinear Difference Equations: Theory with Applications to Social Science Models*, Kluwer Academic Publishers, Dordreacht, 2013.

51. M. Pituk, More on Poincare's and Perron's theorems for difference equations, *J. Differ. Equation Appl.*, **8** (2002), 201–216. https://doi.org/10.1080/10236190211954

AIMS Press

© 2023 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)