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Research article

Localization and calculation for C-eigenvalues of a piezoelectric-type tensor

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Abstract: The largest C-eigenvalue of a piezoelectric tensor determines the highest piezoelectric coupling constant. In this paper, we first provide a new C-eigenvalue localization set for a piezoelectric-type tensor and prove that it is tighter than some existing sets. And then, we present a direct method to find all C-eigentriples of a piezoelectric-type tensor of dimension 3. Finally, we show the effectiveness of the direct method by numerical examples.

Keywords: piezoelectric tensors; piezoelectric-type tensors; C-eigenvalues; C-eigenvectors; localization

1. Introduction

A piezoelectric-type tensor is a third-order tensor, which plays an important role in physics [1–6] and engineering [7–11]. In particular, the largest C-eigenvalue of a piezoelectric tensor and its associated left and right C-eigenvectors play an important role in the piezoelectric effect and the converse piezoelectric effect in the solid crystal [12, 13]. Moreover, in the process of manufacturing and developing micro/nano-electromechanical devices, the development of new multifunctional intelligent structures needs consideration piezoelectric effect [14, 15]. In order to explore more information about piezoelectric-type tensors, Chen et al. [12] introduced piezoelectric-type tensors and their C-eigentriples.

Definition 1.1. [12, Definition 2.1] *Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{n\times n\times n}$ be a third-order n-dimensional tensor. If the latter two indices of $\mathcal{A}$ are symmetric, i.e., $a_{ijk} = a_{ikj}$, where $i, j, k \in [n] := \{1, 2, \dots, n\}$, then $\mathcal{A}$ is called a piezoelectric-type tensor. When $n = 3$, $\mathcal{A}$ is called a piezoelectric tensor.*

Definition 1.2. [12, Definition 2.2] *Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{n\times n\times n}$ be a piezoelectric-type tensor. If there is a real number $\lambda \in \mathbb{R}$, two vectors $x := (x_1, x_2, \dots, x_n)^\top \in \mathbb{R}^n\backslash\{0\}$ and $y := (y_1, y_2, \dots, y_n)^\top \in \mathbb{R}^n\backslash\{0\}$ such* *that*

$$
\begin{cases}
\mathcal{A}yy = \lambda x, & (1.1)\\
x\mathcal{A}y = \lambda y, & (1.2)\\
x^\top x = 1, & (1.3)\\
y^\top y = 1, & (1.4)
\end{cases}
$$

where

$$(\mathcal{A}yy)_i := \sum_{j,k\in[n]} a_{ijk}y_jy_k, \quad (x\mathcal{A}y)_k := \sum_{i,j\in[n]} a_{ijk}x_iy_j,$$

then λ is called a C-eigenvalue of $\mathcal{A}$, and x and y are its associated left and right C-eigenvectors, respectively. Then, (λ, x, y) is called a C-eigentriple of $\mathcal{A}$ and $\sigma(\mathcal{A})$ is used to denote the spectrum of $\mathcal{A}$, which is a set of all C-eigenvalues of $\mathcal{A}$.

Chen et al. provided the following results for the C-eigenvalues and C-eigenvectors of a piezoelectric-type tensor.

Property 1.1. [12, Theorem 2.3] *Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{n\times n\times n}$ be a piezoelectric-type tensor.*
(a) The C-eigenvalues of $\mathcal{A}$ and their associated left and right C-eigenvectors always exist.
(b) Let λ be a C-eigenvalue and (x, y) be its associated left and right C-eigenvectors. Then

$$\lambda = x\mathcal{A}yy := \sum_{i,j,k\in[n]} a_{ijk}x_iy_jy_k.$$

Furthermore, $(\lambda, x, -y)$, $(-\lambda, -x, y)$ and $(-\lambda, -x, -y)$ are also C-eigentriples of $\mathcal{A}$.
(c) Let λ^* be the largest C-eigenvalue of $\mathcal{A}$. Then

$$\lambda^* = \max\{x\mathcal{A}yy : x^\top x = 1, y^\top y = 1\}.$$

As we all know, the largest C-eigenvalue of a piezoelectric-type tensor and its associated C-eigenvectors constitute the best rank-one piezoelectric-type approximation. In view of this, Liang and Yang [16, 17] designed two methods to calculate the largest C-eigenvalue of a piezoelectric-type tensor. Later, Zhao and Luo [18] provided a method to calculate all C-eigentriples of a piezoelectric-type tensor by converting the C-eigenvalue problem to the Z-eigenvalue problem of tensors. Moreover, many researchers considered the C-eigenvalue localization problem and provided many C-eigenvalue localization sets [19–24]. For instance, Che et al. [25] presented the following Geršgorin-type C-eigenvalue localization set.

Theorem 1.1. [25, Theorem 2.1] *Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{n\times n\times n}$ be a piezoelectric-type tensor. Then*

$$\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) := \bigcup_{j\in[n]} \Gamma_j(\mathcal{A}),$$

where

$$\Gamma_j(\mathcal{A}) := \{z \in \mathbb{R} : |z| \le R_j(\mathcal{A})\} \quad and \quad R_j(\mathcal{A}) := \sum_{l,k\in[n]} |a_{lkj}|.$$

Definition 1.2 and Property 1.1 indicate that a C-eigenvalue λ is real and both λ and $-\lambda$ are C-eigenvalues, which implies that a C-eigenvalue localization set is always symmetric with respect to the origin. Therefore, the result of Theorem 1.1 is equivalent to $\sigma(\mathcal{A}) \subseteq [-\rho_\Gamma, \rho_\Gamma]$, where $\rho_\Gamma = \max_{j\in[n]}\{R_j(\mathcal{A})\}$.

The remainder of the paper is organized as follows. In Section 2, we construct a new C-eigenvalue localization set and prove that it is sharper than some existing sets. In Section 3, we provide a direct method to find all C-eigentriples when $n = 3$. In Section 4, we reviewed the practical application of C-eigentriples of a piezoelectric-type tensor to the piezoelectric effect and converse piezoelectric effect. In Section 5, we verify the effectiveness of obtained results by numerical examples. In Section 6, we give a summary of this paper.

2. A new C-eigenvalue localization set

In this section, we provide a new C-eigenvalue localization set and prove that it is sharper than the set in Theorem 1.1. Before that, the following lemma is needed.

Lemma 2.1. [26, pp. 10, Cauchy-Schwarz inequality] *Let* $x = (x_1, x_2, \dots, x_n)^\top \in \mathbb{R}^n$ *and* $y = (y_1, y_2, \dots, y_n)^\top \in \mathbb{R}^n$. *Then*

$$\Big(\sum_{i\in[n]} x_i y_i\Big)^2 \le \sum_{i\in[n]} x_i^2 \sum_{i\in[n]} y_i^2.$$

Theorem 2.1. *Let* $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{n\times n\times n}$ *be a piezoelectric-type tensor. Then*

$$\sigma(\mathcal{A}) \subseteq \Omega(\mathcal{A}) := \Bigg(\bigcup_{i\in[n]} \widehat{\Omega}_i(\mathcal{A})\Bigg) \cup \Bigg(\bigcup_{i,j\in[n], i\neq j} \Big(\widetilde{\Omega}_{i,j}(\mathcal{A}) \cap \mathcal{K}_i(\mathcal{A})\Big)\Bigg),$$

where

$$\begin{aligned}
\widehat{\Omega}_i(\mathcal{A}) &:= \{z \in \mathbb{R} : |z| \le \overline{r}_i^i(\mathcal{A})\},\\
\widetilde{\Omega}_{i,j}(\mathcal{A}) &:= \{z \in \mathbb{R} : (|z| - \overline{r}_i^i(\mathcal{A}))(|z| - \widetilde{r}_j^i(\mathcal{A})) \le \widetilde{r}_i^i(\mathcal{A})\overline{r}_j^i(\mathcal{A})\},\\
\mathcal{K}_i(\mathcal{A}) &:= \{z \in \mathbb{R} : |z| \le r_i(\mathcal{A})\},
\end{aligned}$$

and

$$\begin{aligned}
\overline{r}_j^i(\mathcal{A}) &:= \sqrt{\sum_{l\in[n]} a_{lij}^2}, \quad \widetilde{r}_j^i(\mathcal{A}) := \sqrt{\sum_{l\in[n]} \Big(\sum_{k\in[n], k\neq i} |a_{lkj}|\Big)^2},\\
r_i(\mathcal{A}) &:= \sqrt{\sum_{l\in[n]} \Big(\sum_{k\in[n]} |a_{lki}|\Big)^2}, \quad i, j \in [n].
\end{aligned}$$

Proof. Let (λ, x, y) be a C-eigentriple of $\mathcal{A}$. Let $|y_t| \ge |y_s| \ge \max_{k\in[n], k\neq t, k\neq s}\{|y_k|\}$. Then $0 \le |y_t| \le 1$. From (1.2), we have

$$\lambda y_t = \sum_{l,k\in[n]} a_{lkt} x_l y_k = \sum_{l\in[n]} a_{ltt} x_l y_t + \sum_{l,k\in[n], k\neq t} a_{lkt} x_l y_k.$$

Taking the modulus in above equation and using the triangle inequality and Lemma 2.1, we have

$$\begin{aligned}|\lambda||y_t| &\le \sum_{l,k\in[n]}|a_{lkt}||x_l||y_t| = |y_t|\sum_{l\in[n]}\Big(|x_l|\sum_{k\in[n]}|a_{lkt}|\Big)\\ &\le |y_t|\sqrt{\sum_{l\in[n]}|x_l|^2}\sqrt{\sum_{l\in[n]}\Big(\sum_{k\in[n]}|a_{lkt}|\Big)^2} = |y_t|\sqrt{\sum_{l\in[n]}\Big(\sum_{k\in[n]}|a_{lkt}|\Big)^2},\end{aligned}$$

i.e.,

$$|\lambda| \le \sqrt{\sum_{l\in[n]}\Big(\sum_{k\in[n]}|a_{lkt}|\Big)^2} = r_t(\mathcal{A}),$$

which implies that $\lambda \in \mathcal{K}_t(\mathcal{A})$, and

$$\begin{aligned}|\lambda||y_t| &\le \sum_{l\in[n]}|a_{ltt}||x_l||y_t| + \sum_{l,k\in[n],k\neq t}|a_{lkt}||x_l||y_k|\\ &\le \sum_{l\in[n]}|a_{ltt}||x_l||y_t| + \sum_{l,k\in[n],k\neq t}|a_{lkt}||x_l||y_s|\\ &= |y_t|\sum_{l\in[n]}|a_{ltt}||x_l| + |y_s|\sum_{l\in[n]}\Big(|x_l|\sum_{k\in[n],k\neq t}|a_{lkt}|\Big)\\ &\le |y_t|\sqrt{\sum_{l\in[n]}a_{ltt}^2}\sqrt{\sum_{l\in[n]}x_l^2} + |y_s|\sqrt{\sum_{l\in[n]}x_l^2}\sqrt{\sum_{l\in[n]}\Big(\sum_{k\in[n],k\neq t}|a_{lkt}|\Big)^2}\\ &= |y_t|\sqrt{\sum_{l\in[n]}a_{ltt}^2} + |y_s|\sqrt{\sum_{l\in[n]}\Big(\sum_{k\in[n],k\neq t}|a_{lkt}|\Big)^2}\\ &= \overline{r}_t^t(\mathcal{A})|y_t| + \widetilde{r}_t^t(\mathcal{A})|y_s|,\end{aligned}$$

i.e.,

$$(|\lambda| - \overline{r}_t^t(\mathcal{A}))|y_t| \le \widetilde{r}_t^t(\mathcal{A})|y_s|. \tag{2.1}$$

If $|y_s| = 0$ in (2.1), we have $|\lambda| \le \overline{r}_t^t(\mathcal{A})$, which implies that $\lambda \in \widehat{\Omega}_t(\mathcal{A}) \subseteq \Omega(\mathcal{A})$.

If $|y_s| > 0$ in (2.1), then $|\lambda| \le \overline{r}_t^t(\mathcal{A}) + \widetilde{r}_t^t(\mathcal{A}) \le r_t(\mathcal{A})$, which implies that $\lambda \in \mathcal{K}_t(\mathcal{A})$. Now, suppose that $\lambda \notin \widehat{\Omega}_t(\mathcal{A})$, i.e., $|\lambda| > \overline{r}_t^t(\mathcal{A})$. The s-th equation of (1.2) is

$$\lambda y_s = \sum_{l,k\in[n]}a_{lks}x_l y_k = \sum_{l\in[n]}a_{lts}x_l y_t + \sum_{l,k\in[n],k\neq t}a_{lks}x_l y_k,$$

which implies that

$$\begin{aligned}|\lambda||y_s| &\le \sum_{l\in[n]}|a_{lts}||x_l||y_t| + \sum_{l,k\in[n],k\neq t}|a_{lks}||x_l||y_k|\\ &\le \sum_{l\in[n]}|a_{lts}||x_l||y_t| + \sum_{l,k\in[n],k\neq t}|a_{lks}||x_l||y_s|\end{aligned}$$

$$
\begin{aligned}
&= |y_t| \sum_{l\in[n]} |a_{lts}||x_l| + |y_s| \sum_{l\in[n]} \Big(|x_l| \sum_{k\in[n],k\neq t} |a_{lks}|\Big) \\
&\leq |y_t| \sqrt{\sum_{l\in[n]} a_{lts}^2} \sqrt{\sum_{l\in[n]} x_l^2} + |y_s| \sqrt{\sum_{l\in[n]} x_l^2} \sqrt{\sum_{l\in[n]} \Big(\sum_{k\in[n],k\neq t} |a_{lks}|\Big)^2} \\
&= |y_t| \sqrt{\sum_{l\in[n]} a_{lts}^2} + |y_s| \sqrt{\sum_{l\in[n]} \Big(\sum_{k\in[n],k\neq t} |a_{lks}|\Big)^2} \\
&= \overline{r}_s^t(\mathcal{A})|y_t| + \widetilde{r}_s^t(\mathcal{A})|y_s|,
\end{aligned}
$$

i.e.,

$$
(|\lambda| - \widetilde{r}_s^t(\mathcal{A}))|y_s| \leq \overline{r}_s^t(\mathcal{A})|y_t|. \tag{2.2}
$$

By multiplying (2.1) and (2.2) and eliminating $|y_s||y_t| > 0$, we have

$$
(|\lambda| - \overline{r}_t^t(\mathcal{A}))(|\lambda| - \widetilde{r}_s^t(\mathcal{A})) \leq \widetilde{r}_t^t(\mathcal{A})\overline{r}_s^t(\mathcal{A}),
$$

which implies that $\lambda \in \Big(\widetilde{\Omega}_{t,s}(\mathcal{A}) \cap \mathcal{K}_t(\mathcal{A})\Big) \subseteq \Omega(\mathcal{A})$, and consequently, $\sigma(\mathcal{A}) \subseteq \Omega(\mathcal{A})$ by the arbitrariness of λ.

Next, we discuss the relationship between the set $\Omega(\mathcal{A})$ in Theorem 2.1 and the set $\Gamma(\mathcal{A})$ in Theorem 1.1.

Theorem 2.2. *Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{n\times n\times n}$ be a piezoelectric-type tensor. Then*

$$
\Omega(\mathcal{A}) \subseteq \Gamma(\mathcal{A}).
$$

Proof. Let $z \in \Omega(\mathcal{A})$. By Theorem 2.1, there exists $i, j \in [n]$ such that $z \in \widehat{\Omega}_i(\mathcal{A})$ or $z \in \Big(\widetilde{\Omega}_{i,j}(\mathcal{A}) \cap \mathcal{K}_i(\mathcal{A})\Big)$.

Case I. If $z \in \widehat{\Omega}_i(\mathcal{A})$, then

$$
|z| \leq \overline{r}_i^i(\mathcal{A}) \leq R_i(\mathcal{A}),
$$

which implies that $z \in \Gamma(\mathcal{A})$.

Case II. If $z \in \Big(\widetilde{\Omega}_{i,j}(\mathcal{A}) \cap \mathcal{K}_i(\mathcal{A})\Big)$, then $z \in \widetilde{\Omega}_{i,j}(\mathcal{A})$ and $z \in \mathcal{K}_i(\mathcal{A})$. If $z \in \widetilde{\Omega}_{i,j}(\mathcal{A})$, then

$$
(|z| - \overline{r}_i^i(\mathcal{A}))(|z| - \widetilde{r}_j^i(\mathcal{A})) \leq \widetilde{r}_i^i(\mathcal{A})\overline{r}_j^i(\mathcal{A}).
$$

If $\widetilde{r}_i^i(\mathcal{A})\overline{r}_j^i(\mathcal{A}) = 0$, then

$$
|z| \leq \overline{r}_i^i(\mathcal{A}) \leq R_i(\mathcal{A}) \quad \text{or} \quad |z| \leq \widetilde{r}_j^i(\mathcal{A}) \leq R_j(\mathcal{A}),
$$

which implies that $z \in \Gamma(\mathcal{A})$.

If $\widetilde{r}_i^i(\mathcal{A})\overline{r}_j^i(\mathcal{A}) > 0$, then

$$
\frac{|z| - \overline{r}_i^i(\mathcal{A})}{\widetilde{r}_i^i(\mathcal{A})} \frac{|z| - \widetilde{r}_j^i(\mathcal{A})}{\overline{r}_j^i(\mathcal{A})} \leq 1.
$$

Therefore,

$$\frac{|z| - \overline{r}_i^i(\mathcal{A})}{\widetilde{r}_i^i(\mathcal{A})} \le 1 \quad \text{or} \quad \frac{|z| - \widetilde{r}_j^i(\mathcal{A})}{\overline{r}_j^i(\mathcal{A})} \le 1,$$

i.e.,

$$|z| \le \overline{r}_i^i(\mathcal{A}) + \widetilde{r}_i^i(\mathcal{A}) \le R_i(\mathcal{A}) \quad \text{or} \quad |z| \le \widetilde{r}_j^i(\mathcal{A}) + \overline{r}_j^i(\mathcal{A}) \le R_j(\mathcal{A}),$$

and consequently $z \in \Gamma(\mathcal{A})$.

When $z \in \mathcal{K}_i(\mathcal{A})$, we have

$$|z| \le r_i(\mathcal{A}) \le R_i(\mathcal{A}),$$

then $z \in \Gamma(\mathcal{A})$. Hence, $\Omega(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$.

Similarly, we can write the set $\Omega(\mathcal{A})$ as $[-\rho_\Omega, \rho_\Omega]$, where

$$\rho_\Omega := \max\Big\{ \max_{i\in[n]}\{\overline{r}_i^i(\mathcal{A})\}, \max_{i,j\in[n], i\neq j} \min\Big\{\frac{1}{2}\nu_{i,j}(\mathcal{A}), r_i(\mathcal{A})\Big\}\Big\}, \tag{2.3}$$

and

$$\nu_{i,j}(\mathcal{A}) := \widetilde{r}_j^i(\mathcal{A}) + \widetilde{r}_i^i(\mathcal{A}) + \sqrt{(\widetilde{r}_j^i(\mathcal{A}) - \widetilde{r}_i^i(\mathcal{A}))^2 + 4\widetilde{r}_i^i(\mathcal{A})\overline{r}_j^i(\mathcal{A})}.$$

Theorem 2.2 indicates that $\rho_\Omega \le \rho_\Gamma$ and $[-\rho_\Omega, \rho_\Omega] \subseteq [-\rho_\Gamma, \rho_\Gamma]$.

3. A direct method to calculate all C-eigentriples when $n = 3$

By the idea to find all M-eigenpairs of a fourth-order tensor in Theorem 7 of [27], we in this section present a direct method to find all C-eigenpair when $n = 3$.

Theorem 3.1. *Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{n\times n\times n}$ be a piezoelectric-type tensor. Then all C-eigentriples are given as follows:*

(a) If $a_{211} = a_{311} = a_{112} = a_{113} = 0$, then $(a_{111}, (1,0,0)^\top, (\pm1,0,0)^\top)$ and $(-a_{111}, (-1,0,0)^\top, (\pm1,0,0)^\top)$ are four C-eigentriples of $\mathcal{A}$.

(b) (λ, x, y) is a C-eigentriple of $\mathcal{A}$, where

$$\lambda = a_{111}x_1 + a_{211}x_2 + a_{311}x_3,$$

and

$$x = (x_1, x_2, x_3)^\top, \quad y = (\pm1, 0, 0)^\top,$$

x_1, x_2 and x_3 are the real roots of the following equations:

$$\begin{cases} a_{211}x_1 - a_{111}x_2 = 0, & (3.1)\\ a_{311}x_1 - a_{111}x_3 = 0, & (3.2)\\ a_{112}x_1 + a_{212}x_2 + a_{312}x_3 = 0, \\ a_{113}x_1 + a_{213}x_2 + a_{313}x_3 = 0, \\ x_1^2 + x_2^2 + x_3^2 = 1. \end{cases}$$

(c) (λ, x, y) *and* $(-\lambda, -x, y)$ *are two C-eigentriples of* $\mathcal{A}$*, where*

$$\lambda = a_{111}y_1^2 + a_{122}y_2^2 + a_{133}y_3^2 + 2a_{112}y_1y_2 + 2a_{113}y_1y_3 + 2a_{123}y_2y_3,$$

and

$$x = (1,0,0)^\top, \quad y = (y_1, y_2, y_3)^\top,$$

y_1*,* y_2 *and* y_3 *are the real roots of the following equations:*

$$\begin{cases} a_{222}y_2^2 + a_{211}y_1^2 + a_{233}y_3^2 + 2a_{221}y_2y_1 + 2a_{232}y_3y_2 + 2a_{231}y_3y_1 = 0, \\ a_{333}y_3^2 + a_{311}y_1^2 + a_{322}y_2^2 + 2a_{331}y_3y_1 + 2a_{323}y_2y_3 + 2a_{321}y_2y_1 = 0, \\ a_{112}y_1^2 + (a_{122} - a_{111})y_1y_2 + a_{132}y_1y_3 - a_{121}y_2^2 - a_{131}y_2y_3 = 0, & (3.3) \\ a_{113}y_1^2 + (a_{133} - a_{111})y_1y_3 + a_{123}y_1y_2 - a_{121}y_2y_3 - a_{131}y_3^2 = 0, & (3.4) \\ y_1^2 + y_2^2 + y_3^2 = 1. \end{cases}$$

(d) (λ, x, y) *is a C-eigentriple of* $\mathcal{A}$*, where*

$$\lambda = a_{112}x_1t + a_{122}x_1 + a_{222}x_2 + a_{212}x_2t + a_{322}x_3 + a_{312}x_3t,$$

and

$$x = (x_1, x_2, x_3)^\top, \quad y = \pm\frac{(t,1,0)^\top}{\sqrt{t^2+1}},$$

x_1*,* x_2 *and* x_3 *and t are the real roots of the following equations:*

$$\begin{cases} a_{222}x_1 + a_{211}x_1t^2 + 2a_{221}x_1t - a_{111}x_2t^2 - a_{122}x_2 - 2a_{112}tx_2 = 0, & (3.5) \\ a_{311}t^2x_1 + a_{322}x_1 + 2a_{321}x_1t - a_{111}x_3t^2 - a_{122}x_3 - 2a_{112}x_3t = 0, & (3.6) \\ a_{112}x_1t^2 + (a_{122} - a_{111})x_1t + (a_{222} - a_{211})x_2t + (a_{322} - a_{311})x_3t \\ \quad +a_{212}x_2t^2 + a_{312}x_3t^2 - a_{321}x_3 - a_{221}x_2 - a_{121}x_1 = 0, & (3.7) \\ a_{113}x_1t + a_{123}x_1 + a_{223}x_2 + a_{213}x_2t + a_{313}x_3t + a_{323}x_3 = 0, \\ x_1^2 + x_2^2 + x_3^2 = 1. \end{cases}$$

(e) (λ, x, y) *and* $(-\lambda, -x, y)$ *are two C-eigentriples of* $\mathcal{A}$*, where*

$$\lambda = \sqrt{t^2+1}(a_{222}y_2^2 + a_{211}y_2^2 + a_{233}y_3^2 + 2a_{221}y_1y_2 + 2a_{232}y_2y_3 + 2a_{231}y_1y_3), \quad (3.8)$$

and

$$x = \frac{(t,1,0)^\top}{\sqrt{t^2+1}}, \quad y = (y_1, y_2, y_3)^\top,$$

y_1*,* y_2*,* y_3 *and t are the real roots of the following equations:*

$$\begin{cases} a_{222}y_2^2t + a_{211}y_1^2t + a_{233}y_3^2t + 2a_{221}y_2y_1t + 2a_{232}y_3y_2t + 2a_{231}y_3y_1t \\ \quad -a_{111}y_1^2 - a_{122}y_2^2 - a_{133}y_3^2 - 2a_{112}y_1y_2 - 2a_{113}y_1y_3 - 2a_{123}y_2y_3 = 0, & (3.9) \\ a_{333}y_3^2 + a_{311}y_1^2 + a_{322}y_2^2 + 2a_{331}y_3y_1 + 2a_{323}y_2y_3 + 2a_{321}y_2y_1 = 0, \\ a_{112}ty_1^2 + (a_{122} - a_{111})ty_1y_2 + a_{132}ty_1y_3 + (a_{222} - a_{211})y_2y_1 \\ \quad +a_{212}y_1^2 + a_{232}y_1y_3 - a_{121}ty_2^2 - a_{131}y_2ty_3 - a_{221}y_2^2 - a_{231}y_2y_3 = 0, & (3.10) \\ a_{113}ty_1^2 + (a_{133} - a_{111})ty_1y_3 + a_{123}ty_1y_2 + a_{223}y_1y_2 + a_{213}y_1^2 \\ \quad +(a_{233} - a_{211})y_1y_3 - a_{121}ty_2y_3 - a_{131}ty_3^2 - a_{221}y_2y_3 - a_{231}y_3^2 = 0, & (3.11) \\ y_1^2 + y_2^2 + y_3^2 = 1. \end{cases}$$

(f) (λ, x, y) *is a C-eigentriple of* $\mathcal{A}$*, where*

$$\lambda = \pm(a_{113}u_1v_1 + a_{123}u_1v_2 + a_{133}u_1 + a_{213}u_2v_1 + a_{223}u_2v_2 + a_{233}u_2 + a_{313}v_1 + a_{323}v_2 + a_{333})/\sqrt{u_1^2 + u_2^2 + 1}, \tag{3.12}$$

and

$$x = \pm\frac{(u_1, u_2, 1)^\top}{\sqrt{u_1^2 + u_2^2 + 1}}, \quad y = \pm\frac{(v_1, v_2, 1)^\top}{\sqrt{v_1^2 + v_2^2 + 1}}, \tag{3.13}$$

u_1*,* u_2*,* v_1 *and* v_2 *are the real roots of the following equations:*

$$\begin{cases} a_{222}u_1v_2^2 + a_{211}v_1^2u_1 + a_{233}u_1 + 2a_{221}v_1u_1v_2 + 2a_{232}u_1v_2 \\ \quad +2a_{231}v_1u_1 - a_{111}u_2v_1^2 - a_{122}u_2v_2^2 - a_{133}u_2 - 2a_{112}u_2v_1v_2 \\ \quad -2a_{113}u_2v_1 - 2a_{123}u_2v_2 = 0, & (3.14) \\ a_{333}u_1 + a_{311}u_1v_1^2 + a_{322}u_1v_2^2 + 2a_{331}u_1v_1 + 2a_{323}u_1v_2 + 2a_{321}u_1v_1v_2 \\ \quad -a_{111}v_1^2 - a_{122}v_2^2 - a_{133} - 2a_{112}v_1v_2 - 2a_{113}v_1 - 2a_{123}v_2 = 0, & (3.15) \\ a_{112}u_1v_1^2 + (a_{122} - a_{111})u_1v_1v_2 + a_{132}v_1u_1 + (a_{222} - a_{211})u_2v_1v_2 \\ \quad +a_{212}u_2v_1^2 + a_{232}v_1u_2 + a_{332}v_1 + (a_{322} - a_{311})v_1v_2 + a_{312}v_1^2 \\ \quad -a_{121}u_1v_2^2 - a_{131}u_1v_2 - a_{221}u_2v_2^2 - a_{231}v_2u_2 - a_{331}v_2 - a_{321}v_2^2 = 0, & (3.16) \\ a_{113}u_1v_1^2 + (a_{133} - a_{111})u_1v_1 + a_{123}v_1u_1v_2 + a_{223}v_1u_2v_2 + a_{323}v_1v_2 \\ \quad +(a_{233} - a_{211})u_2v_1 + a_{213}u_2v_1^2 + (a_{333} - a_{311})v_1 + a_{313}v_1^2 \\ \quad -a_{121}u_1v_2 - a_{131}u_1 - a_{221}u_2v_2 - a_{231}u_2 - a_{331} - a_{321}v_2 = 0. & (3.17) \end{cases}$$

Proof. When $n = 3$, the specific form of (1.1)–(1.4) is

$$\begin{cases} a_{111}y_1^2 + a_{122}y_2^2 + a_{133}y_3^2 + 2a_{112}y_1y_2 + 2a_{113}y_1y_3 + 2a_{123}y_2y_3 = \lambda x_1, & (3.18) \\ a_{222}y_2^2 + a_{211}y_1^2 + a_{233}y_3^2 + 2a_{221}y_2y_1 + 2a_{232}y_3y_2 + 2a_{231}y_3y_1 = \lambda x_2, & (3.19) \\ a_{333}y_3^2 + a_{311}y_1^2 + a_{322}y_2^2 + 2a_{331}y_3y_1 + 2a_{323}y_2y_3 + 2a_{321}y_2y_1 = \lambda x_3, & (3.20) \\ a_{111}x_1y_1 + a_{121}x_1y_2 + a_{131}x_1y_3 + a_{221}x_2y_2 + a_{211}x_2y_1 + a_{231}x_2y_3 \\ \quad +a_{331}x_3y_3 + a_{311}x_3y_1 + a_{321}x_3y_2 = \lambda y_1, & (3.21) \\ a_{112}x_1y_1 + a_{122}x_1y_2 + a_{132}x_1y_3 + a_{222}x_2y_2 + a_{212}x_2y_1 + a_{232}x_2y_3 \\ \quad +a_{332}x_3y_3 + a_{322}x_3y_2 + a_{312}x_3y_1 = \lambda y_2, & (3.22) \\ a_{113}x_1y_1 + a_{133}x_1y_3 + a_{123}x_1y_2 + a_{223}x_2y_2 + a_{233}x_2y_3 + a_{213}x_2y_1 \\ \quad +a_{333}x_3y_3 + a_{313}x_3y_1 + a_{323}x_3y_2 = \lambda y_3, & (3.23) \\ x_1^2 + x_2^2 + x_3^2 = 1, & (3.24) \\ y_1^2 + y_2^2 + y_3^2 = 1. & (3.25) \end{cases}$$

To proceed, we break the arguments into six Cases.

(a) Assume that $x = (1, 0, 0)^\top$ and $y = (\pm 1, 0, 0)^\top$. Then (3.18)–(3.25) becomes $\lambda = a_{111}$ and $a_{211} = a_{311} = a_{112} = a_{113} = 0$.

Assume that $x = (-1,0,0)^\top$ and $y = (\pm 1,0,0)^\top$. Then (3.18)–(3.25) becomes $\lambda = -a_{111}$ and $a_{211} = a_{311} = a_{112} = a_{113} = 0$.

Hence, if $a_{211} = a_{311} = a_{112} = a_{113} = 0$, then $(a_{111}, (1,0,0)^\top, (\pm 1,0,0)^\top)$ and $(-a_{111}, (-1,0,0)^\top, (\pm 1, 0,0)^\top)$ are four C-eigentriples of $\mathcal{A}$.

(b) Assume that $y = (\pm 1,0,0)^\top$. Then (3.18)–(3.25) becomes

$$\begin{cases} a_{111} = \lambda x_1, & (3.26)\\ a_{211} = \lambda x_2, & (3.27)\\ a_{311} = \lambda x_3, & (3.28)\\ a_{111}x_1 + a_{211}x_2 + a_{311}x_3 = \lambda, & (3.29)\\ a_{112}x_1 + a_{212}x_2 + a_{312}x_3 = 0, & (3.30)\\ a_{113}x_1 + a_{213}x_2 + a_{313}x_3 = 0, & (3.31)\\ x_1^2 + x_2^2 + x_3^2 = 1. \end{cases}$$

By (3.26) and (3.27), we have (3.1). By (3.26) and (3.28), we have (3.2). Next, solving (3.1), (3.2), (3.30), (3.31) and (3.24), we can obtain x_1, x_2 and x_3, which implies that $x = (x_1, x_2, x_3)^\top$ and $y = (\pm 1,0,0)^\top$ are a pair of C-eigenvectors. Furthermore, by (3.29), we can get a C-eigenvalue λ of $\mathcal{A}$.

(c) Assume that $x = (1,0,0)^\top$. Then (3.18)–(3.25) becomes

$$\begin{cases} a_{111}y_1^2 + a_{122}y_2^2 + a_{133}y_3^2 + 2a_{112}y_1y_2 + 2a_{113}y_1y_3 + 2a_{123}y_2y_3 = \lambda, & (3.32)\\ a_{222}y_2^2 + a_{211}y_1^2 + a_{233}y_3^2 + 2a_{221}y_2y_1 + 2a_{232}y_3y_2 + 2a_{231}y_3y_1 = 0, & (3.33)\\ a_{333}y_3^2 + a_{311}y_1^2 + a_{322}y_2^2 + 2a_{331}y_3y_1 + 2a_{323}y_2y_3 + 2a_{321}y_2y_1 = 0, & (3.34)\\ a_{111}y_1 + a_{121}y_2 + a_{131}y_3 = \lambda y_1, & (3.35)\\ a_{112}y_1 + a_{122}y_2 + a_{132}y_3 = \lambda y_2, & (3.36)\\ a_{113}y_1 + a_{133}y_3 + a_{123}y_2 = \lambda y_3, & (3.37)\\ y_1^2 + y_2^2 + y_3^2 = 1. \end{cases}$$

By (3.35) and (3.36), we have (3.3). By (3.35) and (3.37), we have (3.4). Next, solving (3.3), (3.4), (3.25), (3.33) and (3.34), we can obtain y_1, y_2 and y_3, which implies that $x = (1,0,0)^\top$ and $y = (y_1, y_2, y_3)^\top$ are a pair of C-eigenvectors. Furthermore, by (3.32), we can get a C-eigenvalue λ of $\mathcal{A}$.

Assume that $x = (-1,0,0)^\top$. Similarly, we have

$$\lambda = -(a_{111}y_1^2 + a_{122}y_2^2 + a_{133}y_3^2 + 2a_{112}y_1y_2 + 2a_{113}y_1y_3 + 2a_{123}y_2y_3),$$

with its a pair of C-eigenvectors are $x = (-1,0,0)^\top$ and $y = (y_1, y_2, y_3)^\top$, which also satisfies (3.3), (3.4), (3.25), (3.33) and (3.34).

(d) Assume that $y = (y_1, y_2, 0)^\top$, where $y_2 \neq 0$. Then (3.18)–(3.25) becomes

$$\begin{cases} a_{111}y_1^2 + a_{122}y_2^2 + 2a_{112}y_1y_2 = \lambda x_1, \\ a_{222}y_2^2 + a_{211}y_1^2 + 2a_{221}y_2y_1 = \lambda x_2, \\ a_{311}y_1^2 + a_{322}y_2^2 + 2a_{321}y_2y_1 = \lambda x_3, \\ a_{111}x_1y_1 + a_{121}x_1y_2 + a_{221}x_2y_2 + a_{211}x_2y_1 + a_{311}x_3y_1 + a_{321}x_3y_2 = \lambda y_1, \\ a_{112}x_1y_1 + a_{122}x_1y_2 + a_{222}x_2y_2 + a_{212}x_2y_1 + a_{322}x_3y_2 + a_{312}x_3y_1 = \lambda y_2, \\ a_{113}x_1y_1 + a_{123}x_1y_2 + a_{223}x_2y_2 + a_{213}x_2y_1 + a_{313}x_3y_1 + a_{323}x_3y_2 = 0, \\ x_1^2 + x_2^2 + x_3^2 = 1, \\ y_1^2 + y_2^2 = 1. \end{cases}$$

Let $t = \frac{y_1}{y_2}$. Then the above equations become

$$\begin{cases} (a_{111}t^2 + a_{122} + 2a_{112}t)y_2^2 = \lambda x_1, & (3.38) \\ (a_{222} + a_{211}t^2 + 2a_{221}t)y_2^2 = \lambda x_2, & (3.39) \\ (a_{311}t^2 + a_{322} + 2a_{321}t)y_2^2 = \lambda x_3, & (3.40) \\ a_{111}x_1t + a_{121}x_1 + a_{221}x_2 + a_{211}x_2t + a_{311}x_3t + a_{321}x_3 = \lambda t, & (3.41) \\ a_{112}x_1t + a_{122}x_1 + a_{222}x_2 + a_{212}x_2t + a_{322}x_3 + a_{312}x_3t = \lambda, & (3.42) \\ a_{113}x_1t + a_{123}x_1 + a_{223}x_2 + a_{213}x_2t + a_{313}x_3t + a_{323}x_3 = 0, & (3.43) \\ x_1^2 + x_2^2 + x_3^2 = 1. \end{cases}$$

By (3.38) and (3.39), we have (3.5). By (3.38) and (3.40), we have (3.6). By (3.41) and (3.42), we have (3.7). Next, solving (3.5)–(3.7), (3.24) and (3.43), we can obtain x_1, x_2, x_3 and t, which leads to its a pair of C-eigenvectors $x = (x_1, x_2, x_3)^\top$ and $y = \pm\frac{(t,1,0)^\top}{\sqrt{t^2+1}}$. Furthermore, by (3.42), we can get a C-eigenvalue λ of $\mathcal{A}$.

(e) Assume that $x = (x_1, x_2, 0)^\top$, where $x_2 \neq 0$. Then (3.18)–(3.25) becomes

$$\begin{cases} a_{111}y_1^2 + a_{122}y_2^2 + a_{133}y_3^2 + 2a_{112}y_1y_2 + 2a_{113}y_1y_3 + 2a_{123}y_2y_3 = \lambda x_1, \\ a_{222}y_2^2 + a_{211}y_1^2 + a_{233}y_3^2 + 2a_{221}y_2y_1 + 2a_{232}y_3y_2 + 2a_{231}y_3y_1 = \lambda x_2, \\ a_{333}y_3^2 + a_{311}y_1^2 + a_{322}y_2^2 + 2a_{331}y_3y_1 + 2a_{323}y_2y_3 + 2a_{321}y_2y_1 = 0, \\ a_{111}x_1y_1 + a_{121}x_1y_2 + a_{131}x_1y_3 + a_{221}x_2y_2 + a_{211}x_2y_1 + a_{231}x_2y_3 = \lambda y_1, \\ a_{112}x_1y_1 + a_{122}x_1y_2 + a_{132}x_1y_3 + a_{222}x_2y_2 + a_{212}x_2y_1 + a_{232}x_2y_3 = \lambda y_2, \\ a_{113}x_1y_1 + a_{133}x_1y_3 + a_{123}x_1y_2 + a_{223}x_2y_2 + a_{233}x_2y_3 + a_{213}x_2y_1 = \lambda y_3, \\ x_1^2 + x_2^2 = 1, \\ y_1^2 + y_2^2 + y_3^2 = 1. \end{cases}$$

Let $t = \frac{x_1}{x_2}$. Then the above equations become

$$\begin{cases} a_{111}y_1^2 + a_{122}y_2^2 + a_{133}y_3^2 + 2a_{112}y_1y_2 + 2a_{113}y_1y_3 + 2a_{123}y_2y_3 = \lambda t x_2, & (3.44) \\ a_{222}y_2^2 + a_{211}y_1^2 + a_{233}y_3^2 + 2a_{221}y_2y_1 + 2a_{232}y_3y_2 + 2a_{231}y_3y_1 = \lambda x_2, & (3.45) \\ a_{333}y_3^2 + a_{311}y_1^2 + a_{322}y_2^2 + 2a_{331}y_3y_1 + 2a_{323}y_2y_3 + 2a_{321}y_2y_1 = 0, & (3.46) \\ a_{111}tx_2y_1 + a_{121}tx_2y_2 + a_{131}tx_2y_3 + a_{221}x_2y_2 + a_{211}x_2y_1 + a_{231}x_2y_3 = \lambda y_1, & (3.47) \\ a_{112}tx_2y_1 + a_{122}tx_2y_2 + a_{132}tx_2y_3 + a_{222}x_2y_2 + a_{212}x_2y_1 + a_{232}x_2y_3 = \lambda y_2, & (3.48) \\ a_{113}tx_2y_1 + a_{133}tx_2y_3 + a_{123}tx_2y_2 + a_{223}x_2y_2 + a_{233}x_2y_3 + a_{213}x_2y_1 = \lambda y_3, & (3.49) \\ y_1^2 + y_2^2 + y_3^2 = 1. \end{cases}$$

By (3.44) and (3.45), we have (3.9). By (3.47) and (3.48), we have (3.10). By (3.47) and (3.49), we have (3.11). Next, solving (3.9)–(3.11), (3.25) and (3.46), we can obtain y_1, y_2, y_3 and t. And then, by $x_1^2 + x_2^2 = 1$, we have $x_2 = \pm\frac{1}{\sqrt{t^2+1}}$. Furthermore, by (3.45), we can get a C-eigenvalue λ of $\mathcal{A}$ in (3.8) and its a pair of C-eigenvectors $x = \pm\frac{(t,1,0)^\top}{\sqrt{t^2+1}}$ and $y = (y_1, y_2, y_3)^\top$.

(f) Assume that $x = (x_1, x_2, x_3)^\top$ and $y = (y_1, y_2, y_3)^\top$, where $x_3 \neq 0$ and $y_3 \neq 0$. Let

$$u_1 = \frac{x_1}{x_3}, \quad u_2 = \frac{x_2}{x_3}, \quad v_1 = \frac{y_1}{y_3}, \quad v_2 = \frac{y_2}{y_3}.$$

Then (3.18)–(3.25) becomes

$$\begin{cases} a_{111}v_1^2y_3^2 + a_{122}v_2^2y_3^2 + a_{133}y_3^2 + 2a_{112}v_1v_2y_3^2 + 2a_{113}v_1y_3^2 + 2a_{123}v_2y_3^2 = \lambda u_1x_3, & (3.50) \\ a_{222}v_2^2y_3^2 + a_{211}v_1^2y_3^2 + a_{233}y_3^2 + 2a_{221}v_1v_2y_3^2 + 2a_{232}v_2y_3^2 + 2a_{231}v_1y_3^2 = \lambda u_2x_3, & (3.51) \\ a_{333}y_3^2 + a_{311}v_1^2y_3^2 + a_{322}v_2^2y_3^2 + 2a_{331}v_1y_3^2 + 2a_{323}v_2y_3^2 + 2a_{321}v_1v_2y_3^2 = \lambda x_3, & (3.52) \\ a_{111}u_1v_1x_3 + a_{121}u_1v_2x_3 + a_{131}u_1x_3 + a_{221}u_2v_2x_3 + a_{211}u_2v_1x_3 + a_{231}u_2x_3 \\ \quad +a_{331}x_3 + a_{311}v_1x_3 + a_{321}v_2x_3 = \lambda v_1, & (3.53) \\ a_{112}u_1v_1x_3 + a_{122}u_1v_2x_3 + a_{132}u_1x_3 + a_{222}u_2v_2x_3 + a_{212}u_2v_1x_3 + a_{232}u_2x_3 \\ \quad +a_{332}x_3 + a_{322}v_2x_3 + a_{312}v_1x_3 = \lambda v_2, & (3.54) \\ a_{113}u_1v_1x_3 + a_{133}u_1x_3 + a_{123}u_1v_2x_3 + a_{223}u_2v_2x_3 + a_{233}u_2x_3 + a_{213}u_2v_1x_3 \\ \quad +a_{333}x_3 + a_{313}v_1x_3 + a_{323}v_2x_3 = \lambda. & (3.55) \end{cases}$$

By (3.50) and (3.51), we have (3.14). By (3.50) and (3.52), we have (3.15). By (3.53) and (3.54), we have (3.16). By (3.53) and (3.55), we have (3.17). Next, solving (3.14)–(3.17), we can obtain u_1, u_2, v_1 and v_2. Furthermore, by $x_1^2 + x_2^2 + x_3^2 = 1$ and $y_1^2 + y_2^2 + y_3^2 = 1$, we can get x and y in (3.13); by (3.55), we can get a C-eigenvalue λ of $\mathcal{A}$ in (3.12).

Remark 3.1. i) Solving the system of nonlinear equations in (b) of Theorem 3.1, we can choose any three of the first four equations to form a system of linear equations. If the determinant of the coefficient matrix of the system of linear equations is not equal to 0, then the system of linear equations has only zero solution and obviously it is not the solution of the last equation in the system of nonlinear equations, which implies that the system of nonlinear equations in (b) has no solution. If the determinant is equal to 0, then we need to verify that whether this solution satisfies the other two equations.

ii) Solving these system of nonlinear equations in (c)–(f) of Theorem 3.1, we can use the resultant method in algebraic geometry [28] to find their solutions, which has been verified and is feasible in finding all M-eigenpairs of an order 4 dimension 2 tensor in Theorem 7 of [27].

Given two polynomials $f(x)$ and $g(x)$, where

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n, \;\; a_0 \neq 0, \;\; n > 0.$$
$$g(x) = b_0x^m + b_1x^{m-1} + \cdots + b_m, \;\; b_0 \neq 0, \;\; m > 0.$$

Then the resultant of $f(x)$ and $g(x)$ is represented by $\mathrm{Res}_x(f, g)$, that is, the determinant of $(n+m)\times(n+m)$ is as follows:

$$\mathrm{Res}_x(f,g) = \det \underbrace{\begin{pmatrix} a_0 & & & & d_0 & & & \\ a_1 & a_0 & & & d_1 & d_0 & & \\ a_2 & a_1 & \ddots & & d_2 & d_1 & \ddots & \\ \vdots & a_2 & \ddots & a_0 & \vdots & d_2 & \ddots & b_0 \\ a_m & \vdots & \ddots & a_1 & d_n & \vdots & \ddots & b_1 \\ & a_m & & a_2 & & d_n & & b_2 \\ & & \ddots & \vdots & & & \ddots & \vdots \\ & & & a_m & & & & b_n \end{pmatrix}}_{n \text{ columns} \quad\quad m \text{ columns}},$$

where the blank spaces are filled with zeros. It is showed in [28, pp. 78] that over the complex field $f(x)$ and $g(x)$ have a common root if and only if their resultant is zero.

Solving the systems (3.14)–(3.17) with four variables u_1, u_2, v_1 and v_2, we can obtain its solutions by successive elimination of variables as follows:

Step 1: Mark the four equations in the systems (3.14)–(3.17) as $f_1(u_1, u_2, v_1, v_2)$, $f_2(u_1, u_2, v_1, v_2)$, $f_3(u_1, u_2, v_1, v_2)$ and $f_4(u_1, u_2, v_1, v_2)$ in turn, and regard f_1, f_2, f_3 and f_4 as functions with u_1 as a variable and u_2, v_1 and v_2 as their coefficients. If $\mathrm{Res}_{u_1}(f_1, f_2) = 0$, which is a function with u_2, v_1 and v_2 as three variables, then f_1 and f_2 have a common root. Similarly, $\mathrm{Res}_{u_1}(f_1, f_3) = 0$ and $\mathrm{Res}_{u_1}(f_1, f_4) = 0$ can be obtained and they are functions with u_2, v_1 and v_2 as three variables.

Step 2: Let

$$g_1(u_2, v_1, v_2) := \mathrm{Res}_{u_1}(f_1, f_2) = 0, \quad g_2(u_2, v_1, v_2) := \mathrm{Res}_{u_1}(f_1, f_3) = 0,$$

and

$$g_3(u_2, v_1, v_2) := \mathrm{Res}_{u_1}(f_1, f_4) = 0,$$

and g_1, g_2 and g_3 be regarded as functions with u_2 as a variable and v_1 and v_2 as their coefficients. If $\mathrm{Res}_{u_2}(g_1, g_2) = 0$, which is a function with v_1 and v_2 as two variables, then g_1 and g_2 have a common root. Similarly, $\mathrm{Res}_{u_2}(g_1, g_3) = 0$ can be obtained and it is a function with v_1 and v_2 as two variables.

Step 3: Let

$$h_1(v_1, v_2) := \mathrm{Res}_{u_2}(g_1, g_2) = 0 \quad \text{and} \quad h_2(v_1, v_2) := \mathrm{Res}_{u_2}(g_1, g_3) = 0.$$

and h_1 and h_2 be regarded as functions with v_1 as a variable and v_2 as a coefficient. If $\text{Res}_{v_1}(h_1, h_2) = 0$, which is a function with v_2 as a variable, then h_1 and h_2 have a common root.

Step 4: Solving the function $\text{Res}_{v_1}(h_1, h_2) = 0$ with v_2 as a variable by Matlab command '`solve`', its all real solutions v_2 can be obtained.

Step 5: Substituting v_2 to $h_1(v_1, v_2) = 0$ and $h_2(v_1, v_2) = 0$ to find all their real solutions v_1. And then, substituting v_2 and v_1 to $g_1(u_2, v_1, v_2) = 0$, $g_2(u_2, v_1, v_2) = 0$ and $g_3(u_2, v_1, v_2) = 0$ to find all their real solutions u_2. Furthermore, substituting v_2, v_1 and u_2 to $f_i(u_1, u_2, v_1, v_2) = 0$ for $i \in [4]$ to find all their real solutions u_1. Then, all real roots u_1, u_2, v_1 and v_2 of (3.14)–(3.17) are obtained. Finally, by (3.12) and (3.13), we can find all C-eigentriples of $\mathcal{A}$ in the Case (f) in Theorem 3.1.

4. Applications

It is shown in [12, 29] that the largest C-eigenvalue λ^* of a piezoelectric tensor determines the highest piezoelectric coupling constant, and its corresponding C-eigenvector y^* is the corresponding direction of the stress where this appears. In this section, let's review its physical background, which is shown in [29].

In physics, for non-centrosymmetric materials, we can write the linear piezoelectric equation as

$$p_i = \sum_{j,k \in [3]} a_{ijk} T_{jk},$$

where $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{3\times3\times3}$ is a piezoelectric tensor, $T = (T_{jk}) \in \mathbb{R}^{3\times3}$ is the stress tensor, and $p = (p_i) \in \mathbb{R}^3$ is the electric change density displacement.

Now, it is worth considering, under what conditions can the maximal piezoelectricity be triggered under a unit uniaxial stress? In this case, the stress tensor T can be rewritten as $T = yy^\top$ with $y^\top y = 1$. Then, this maximal piezoelectricity problem can be formulated into an optimization model

$$\begin{cases} \max & \|p\|_2 \\ s.t. & p = \mathcal{A}yy, \\ & y^\top y = 1. \end{cases}$$

By a dual norm, $\|p\|_2 = \max\limits_{x^\top x=1} x^\top p = \max\limits_{x^\top x=1} x\mathcal{A}yy$ is derived and hence the above optimization model is converted to the following optimization problem

$$\max \quad x\mathcal{A}yy \qquad \text{s.t. } x^\top x = 1, \ \ y^\top y = 1.$$

If (x^*, y^*) is an optimal solution of the above optimization problem, then $\lambda^* = x^*\mathcal{A}y^*y^*$ is the largest C-eigenvalue of $\mathcal{A}$ and y^* is the unit uniaxial direction that the maximal piezoelectric effect take place along.

Theorem 4.1. [29, Theorem 7.12] Let λ^* be the largest C-eigenvalue, x^* and y^* be the associated C-eigenvectors of a piezoelectric tensor $\mathcal{A}$. Then, λ^* is the maximum value of the 2-norm of the electric polarization under a unit uniaxial stress along the optimal axial direction y^*.

Moreover, the linear equation of the inverse piezoelectric effect is

$$S_{jk} = \sum_i a_{ijk} e_i,$$

where $S = (S_{jk}) \in \mathbb{R}^{3\times 3}$ is the strain tensor and $e = (e_i) \in \mathbb{R}^3$ is the electric field strength. Next, the following maximization problem is considered:

$$\begin{cases} \max & \|S\|_2 := \max_{y^\top y=1} y^\top S y \\ s.t. & S_{jk} = \sum_{i\in[3]} e_i a_{ijk}, \quad \forall j,k \in [3], \\ & e^\top e = 1. \end{cases}$$

By $\|S\|_2 = \max_{y^\top y=1} y^\top S y = \max_{y^\top y=1} e\mathcal{A}yy$, the above maximization problem is rewritten as

$$\max\{e\mathcal{A}yy : \quad e^\top e = 1, \quad y^\top y = 1\}.$$

If (e^*, y^*) is an optimal solution of the above optimization problem, then $\lambda^* = e^*\mathcal{A}y^*y^*$ is the largest C-eigenvalue of $\mathcal{A}$, e^* and y^* are its associated C-eigenvectors.

Theorem 4.2. [29, Theorem 7.13] Let λ^* be the largest C-eigenvalue and x^* and y^* be its associated C-eigenvectors of a piezoelectric tensor $\mathcal{A}$. Then, λ^* is the largest spectral norm of a strain tensor generated by the converse piezoelectric effect under unit electric field strength $\|x^*\| = 1$.

5. Numerical examples

In this section, numerical examples are given to verify the obtained theoretical results.

Example 1. Consider the eight piezoelectric tensors in [12, Examples 1–8].

(a) The first piezoelectric tensor is $\mathcal{A}_{\mathrm{VFeSb}}$ with its nonzero entries

$$a_{123} = a_{213} = a_{312} = -3.68180667.$$

(b) The second piezoelectric tensor $\mathcal{A}_{\mathrm{SiO_2}}$ with its nonzero entries

$$a_{111} = -a_{122} = -a_{212} = -0.13685, \quad a_{123} = -a_{213} = -0.009715.$$

(c) The third piezoelectric tensor $\mathcal{A}_{\mathrm{Cr_2AgBiO_8}}$ with its nonzero entries

$$a_{123} = a_{213} = -0.22163, \quad a_{113} = -a_{223} = 2.608665,$$
$$a_{311} = -a_{322} = 0.152485, \quad a_{312} = -0.37153.$$

(d) The fourth piezoelectric tensor $\mathcal{A}_{\mathrm{RbTaO_3}}$ with its nonzero entries

$$a_{113} = a_{223} = -8.40955, \quad a_{311} = a_{322} = -4.3031,$$
$$a_{222} = -a_{212} = -a_{211} = -5.412525, \quad a_{333} = -5.14766.$$

(e) The fifth piezoelectric tensor $\mathcal{A}_{\mathrm{NaBiS_2}}$ with its nonzero entries

$$a_{113} = -8.90808, \quad a_{223} = -0.00842, \quad a_{311} = -7.11526,$$
$$a_{322} = -0.6222, \quad a_{333} = -7.93831.$$

(f) The sixth piezoelectric tensor $\mathcal{A}_{\mathrm{LiBiB_2O_5}}$ with its nonzero entries

$$a_{112} = 0.34929, \quad a_{211} = 0.16101, \quad a_{222} = 0.12562, \quad a_{312} = 2.57812,$$
$$a_{123} = 2.35682, \quad a_{213} = -0.05587, \quad a_{233} = 0.1361, \quad a_{323} = 6.91074.$$

(g) The seventh piezoelectric tensor $\mathcal{A}_{\mathrm{KBi_2F_7}}$ with its nonzero entries

$$a_{111} = 12.64393, \quad a_{211} = 2.59187, \quad a_{311} = 1.51254, \quad a_{123} = 1.59052,$$
$$a_{122} = 1.08802, \quad a_{212} = 0.10570, \quad a_{312} = 0.08381, \quad a_{233} = 0.81041,$$
$$a_{113} = 1.96801, \quad a_{213} = 0.71432, \quad a_{313} = 0.39030, \quad a_{333} = -0.23019,$$
$$a_{112} = 0.22465, \quad a_{222} = 0.08263, \quad a_{322} = 0.68235, \quad a_{323} = 0.19013,$$
$$a_{133} = 4.14350, \quad a_{223} = 0.51165.$$

(h) The eigth piezoelectric tensor $\mathcal{A}_{\mathrm{BaNiO_3}}$ with its nonzero entries

$$a_{113} = 0.038385, \ a_{223} = 0.038385, \ a_{311} = a_{322} = 6.89822, \ a_{333} = 27.4628.$$

I. Localization for all C-eigenvalues of the above eight piezoelectric tensors.

Now, we use these C-eigenvalues intervals in Theorems 2.1 and 1.1, Theorems 1 and 2 of [20], Theorem 2.1 of [13], Theorems 2.2 and 2.4 of [25], Theorem 2.1 of [24], Theorem 2.1 of [23], Theorem 5 of [19], Theorem 7 of [21], Theorems 2.3–2.5 of [18] and Theorem 2.1 of [22] to locate all C-eigenvalues of the above eight piezoelectric tensors. Numerical results are shown in Table 1. Since these intervals are symmetric about the origin, only their right boundaries are listed in Table 1.

In Table 1, λ^* is the largest C-eigenvalue of a piezoelectric tensor; ϱ and $\varrho_{\min}$ are respectively the right boundaries of the interval $[-\varrho, \varrho]$ and $[-\varrho_{\min}, \varrho_{\min}]$ obtained by Theorems 1 and 2 in [20]; $\widetilde{\varrho}_{\min}$ are respectively the right boundary of the interval $[-\widetilde{\varrho}_{\min}, \widetilde{\varrho}_{\min}]$ obtained by Theorem 2.1 of [13]; $\rho_{\mathcal{L}}$ and $\rho_{\mathcal{M}}$ are respectively the right boundaries of the intervals $[-\rho_{\mathcal{L}}, \rho_{\mathcal{L}}]$ and $[-\rho_{\mathcal{M}}, \rho_{\mathcal{M}}]$ obtained by 2.2 and 2.4 in [25]; ρ_{Υ} is the right boundary of the interval $[-\rho_{\Upsilon}, \rho_{\Upsilon}]$ obtained by Theorem 2.1 of [24]; ρ_{γ} is the right boundary of the interval $[-\rho_{\gamma}, \rho_{\gamma}]$ obtained by Theorem 2.1 of [23]; ρ_{Ω^S} is the right boundary of the interval $[-\rho_{\Omega^S}, \rho_{\Omega^S}]$ obtained by Theorem 5 of [19]; ρ_C is the right boundary of the interval $[-\rho_C, \rho_C]$ obtained by Theorem 7 of [21]; ρ_G, ρ_B and $\rho_{\min}$ are respectively the right boundaries of the intervals $[-\rho_G, \rho_G]$, $[-\rho_B, \rho_B]$ and $[-\rho_{\min}, \rho_{\min}]$ obtained by Theorems 2.3–2.5 in [18]; ρ_{Ψ} is the right boundary of the interval $[-\rho_{\Psi}, \rho_{\Psi}]$ obtained by Theorem 2.1 of [22]; ρ_{Γ} is the right boundary of the interval $[-\rho_{\Gamma}, \rho_{\Gamma}]$ obtained by Theorem 1.1; ρ_{Ω} is the right boundary of the interval $[-\rho_{\Omega}, \rho_{\Omega}]$ obtained by Theorem 2.1.

From Table 1, it can be seen that:

i) ρ_{Ω} is smaller than ϱ, $\varrho_{\min}$, $\widetilde{\varrho}_{\min}$, ρ_{Γ}, $\rho_{\mathcal{L}}$, $\rho_{\mathcal{M}}$, ρ_{Υ}, ρ_{γ} for the eight piezoelectric tensors.

ii) $\rho_{\Omega} \le \rho_{\Omega^S}$, $\rho_{\Omega} \le \rho_B$ for the eight piezoelectric tensors.

iii) For some tensors, ρ_{Ω} is smaller than ρ_C, ρ_G, $\rho_{\min}$ and ρ_{Ψ}. For the other tensors, ρ_{Ω} is bigger than or equal to ρ_C, ρ_G, $\rho_{\min}$ and ρ_{Ψ}. For examples, for $\mathcal{A}_{\mathrm{VFeSb}}$, $\rho_{\Omega} < \rho_C$, $\rho_{\Omega} < \rho_G$, $\rho_{\Omega} < \rho_{\min}$ and $\rho_{\Omega} < \rho_{\Psi}$; For $\mathcal{A}_{\mathrm{SiO_2}}$, $\rho_{\Omega} > \rho_C$, $\rho_{\Omega} > \rho_{\min}$ and $\rho_{\Omega} > \rho_{\Psi}$; For $\mathcal{A}_{\mathrm{BaNiO_3}}$, $\rho_{\Omega} > \rho_G$.

II. Calculation for all C-eigentriples of the seventh piezoelectric tensor $\mathcal{A}_{\mathrm{KBi_2F_7}}$ by Theorem 3.1.

All C-eigentriples of $\mathcal{A}_{\mathrm{KBi_2F_7}}$ are obtained by Theorem 3.1 and are showen in Table 2. And the calculation process is shown in Appendix.

Table 1. Comparison among ϱ, $\varrho_{\min}$, $\widetilde{\varrho}_{\min}$, ρ_Γ, $\rho_{\mathcal{L}}$, $\rho_{\mathcal{M}}$, ρ_Υ, ρ_γ, ρ_{Ω^S}, ρ_C, ρ_G, ρ_B, $\rho_{\min}$, ρ_Ψ, ρ_Ω and λ^*.

	$\mathcal{A}_{VFeSb}$	$\mathcal{A}_{SiO_2}$	$\mathcal{A}_{Cr_2AgBiO_8}$	$\mathcal{A}_{RbTaO_3}$	$\mathcal{A}_{NaBiS_2}$	$\mathcal{A}_{LiBiB_2O_5}$	$\mathcal{A}_{KBi_2F_7}$	$\mathcal{A}_{BaNiO_3}$
ϱ	7.3636	0.2882	5.6606	30.0911	17.3288	15.2911	22.6896	33.7085
$\varrho_{\min}$	7.3636	0.2834	5.6606	23.5377	16.8548	12.3206	20.2351	27.5396
$\widetilde{\varrho}_{\min}$	7.3636	0.2393	4.6717	22.7163	14.5723	12.1694	18.7025	27.5396
ρ_Γ	7.3636	0.2834	5.6606	23.5377	16.8548	12.3206	20.2351	27.5396
$\rho_{\mathcal{L}}$	7.3636	0.2744	4.8058	23.5377	16.5640	11.0127	18.8793	27.5109
$\rho_{\mathcal{M}}$	7.3636	0.2834	4.7861	23.5377	16.8464	11.0038	19.8830	27.5013
ρ_Υ	7.3636	0.2834	4.7335	23.5377	16.8464	10.9998	19.8319	27.5013
ρ_γ	7.3636	0.2744	4.7860	23.0353	16.4488	10.2581	18.4090	27.5013
ρ_{Ω^S}	7.3636	0.2744	4.2732	23.0353	16.4486	10.2581	17.7874	27.4629
ρ_C	6.3771	0.1943	3.7242	16.0259	11.9319	7.7540	13.5113	27.4629
ρ_G	6.3771	0.2506	4.0455	21.5313	13.9063	9.8718	14.2574	29.1441
ρ_B	5.2069	0.2345	4.0026	19.4558	13.4158	10.0289	15.3869	27.5396
$\rho_{\min}$	6.5906	0.1942	3.5097	18.0991	11.9324	8.1373	14.3299	27.4725
ρ_Ψ	6.5906	0.1942	4.2909	18.9140	11.9319	8.1501	14.0690	27.4629
ρ_Ω	5.2069	0.2005	3.5097	19.2688	11.9319	8.6469	13.6514	27.4629
λ^*	4.2514	0.1375	2.6258	12.4234	11.6674	7.7376	13.5021	27.4628

Table 2. All C-eigentriples of $\mathcal{A}_{KBi_2F_7}$.

λ	x_1	x_2	x_3	y_1	y_2	y_3
13.50214	0.97050	0.20974	0.11890	0.97226	0.05065	0.22836
13.50214	0.97050	0.20974	0.11890	−0.97226	−0.05065	−0.22836
4.46957	0.98196	0.18905	−0.00362	0.22771	−0.41491	−0.88091
4.46957	0.98196	0.18905	−0.00362	−0.22771	0.41491	0.88091
0.54486	0.75981	−0.36879	0.53544	0.06168	0.87047	−0.48833
0.54486	0.75981	−0.36879	0.53544	−0.06168	−0.87047	0.48833
−0.54486	−0.75981	0.36879	−0.53544	0.06168	0.87047	−0.48833
−0.54486	−0.75981	0.36879	−0.53544	−0.06168	−0.87047	0.48833
−4.46957	−0.98196	−0.18905	0.00362	0.22771	−0.41491	−0.88091
−4.46957	−0.98196	−0.18905	0.00362	−0.22771	0.41491	0.88091
−13.50214	−0.97050	−0.20974	−0.11890	0.97226	0.05065	0.22836
−13.50214	−0.97050	−0.20974	−0.11890	−0.97226	−0.05065	−0.22836

6. Conclusions

Let $\mathcal{A} \in \mathbb{R}^{n\times n\times n}$ be a piezoelectric-type tensor. In this paper, we in Theorem 2.1 constructed a C-eigenvalue interval $\Omega(\mathcal{A})$ to locate all C-eigenvalues of $\mathcal{A}$ and proved that it is tighter than that in [25, Theorem 2.1]. Subsequently, we in Theorem 3.1 provided a direct method to find all C-eigentriples of $\mathcal{A}$ when $n = 3$. Although the method in Theorem 3.1 is divided into six Cases, it is indeed a little complicated, but it can be seen from Example 1 that this method is feasible.

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Conflict of interest

The author declares there is no confict of interest.

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Appendix

The following is the calculation process for all C-eigentriples of $\mathcal{A}_{\mathrm{KBi_2F_7}}$ by Theorem 3.1.

(a) Because $a_{211} \neq 0$, Case (a) in Theorem 3.1 does not holds.

(b) The system in Case (b) of Theorem 3.1 is

$$\begin{cases} 2.59187x_1 - 12.64393x_2 = 0, & \text{(A.1)} \\ 1.51254x_1 - 12.64393x_3 = 0, & \text{(A.2)} \\ 1.08802x_1 + 0.10570x_2 + 0.08381x_3 = 0, & \text{(A.3)} \\ 1.96801x_1 + 0.71432x_2 + 0.39030x_3 = 0, \\ x_1^2 + x_2^2 + x_3^2 = 1. \end{cases}$$

The three Eqs (A.1)–(A.3) yield a linear system of equation $Ax = 0$, where $x = (x_1, x_2, x_3)^\top$ and

$$A = \begin{pmatrix} 2.59187 & -12.64393 & 0 \\ 1.51254 & 0 & -12.64393 \\ 1.08802 & 0.10570 & 0.08381 \end{pmatrix}.$$

From $\det(A) = 179.0074 \neq 0$, the solution of $Ax = 0$ is $x = (x_1, x_2, x_3)^\top = (0, 0, 0)^\top$, which contradicts with $x_1^2 + x_2^2 + x_3^2 = 1$. Hence, the system in Case (b) of Theorem 3.1 has no solution.

(c) The system in Case (c) of Theorem 3.1 is

$$\begin{cases} f_1(y_1, y_2, y_3) = 0.08263y_2^2 + 2.59187y_1^2 + 0.81041y_3^2 + 0.2114y_1y_2 \\ \qquad\qquad +1.0233y_2y_3 + 1.42864y_1y_3 = 0, \\ f_2(y_1, y_2, y_3) = -0.23019y_3^2 + 1.51254y_1^2 + 0.68235y_2^2 + 0.7806y_1y_3 \\ \qquad\qquad +0.38026y_2y_3 + 0.16762y_1y_2 = 0, \\ f_3(y_1, y_2, y_3) = 0.22465y_1^2 - 0.22465y_2^2 - 11.55591y_1y_2 + 1.59052y_1y_3 \\ \qquad\qquad -1.96801y_2y_3 = 0, \\ f_4(y_1, y_2, y_3) = 1.96801y_1^2 - 1.96801y_3^2 - 8.50043y_1y_3 + a_{123}y_1y_2 \\ \qquad\qquad -0.22465y_2y_3 = 0, \\ f_5(y_1, y_2, y_3) = y_1^2 + y_2^2 + y_3^2 = 1. \end{cases}$$

We now regard $f_i(y_1, y_2, y_3)$, $i \in [5]$ as a function with y_1 as a variable and y_2 and y_3 as two coefficients and obtain their resultants as follows:

$$\mathrm{Res}_{y_1}(f_1, f_2) = 2.68640y_2^4 - 1.92404y_3y_2^3 - 5.51122y_2^2y_3^2 + 2.09880y_3^3y_2 + 3.18879y_3^4,$$
$$\mathrm{Res}_{y_1}(f_1, f_3) = 27.58122y_2^4 + 331.64703y_3y_2^3 + 134.75405y_3^2y_2^2 - 55.45236y_3^3y_2 + 4.93315y_3^4.$$

Let $g_1(y_2, y_3) := \mathrm{Res}_{y_1}(f_1, f_2)$, $g_2(y_2, y_3) := \mathrm{Res}_{y_1}(f_1, f_3)$, and its resultant

$$\mathrm{Res}_{y_2}(g_1, g_2) = 95376348203.97653y_3^{16} = 0.$$

Then $y_3 = 0$. Substituting $y_3 = 0$ into g_1 and g_2, we have

$$g_1(y_2, y_3) = 2.68640y_2^4, \quad g_2(y_2, y_3) = 27.58122y_2^4.$$

Let $g_1(y_2, y_3) = 0$ and $g_2(y_2, y_3) = 0$. We have $y_2 = 0$. Substituting $y_2 = 0$ and $y_3 = 0$ into f_1, we have $f_1(y_1, y_2, y_3) = 2.59187y_1^2$. Solving $f_1(y_1, y_2, y_3) = 0$, we have $y_1 = 0$. However, $y_1 = y_2 = y_3 = 0$ is not solution of $y_1^2 + y_2^2 + y_3^2 = 1$. Hence, the system in Case (c) of Theorem 3.1 has no solution.

(d) Similar to solution for Case (c), the system in Case (d) of Theorem 3.1 has no solution.

(e) Similar to solution for Case (c), the system in Case (e) of Theorem 3.1 has no solution.

(f) The system in Case (f) of Theorem 3.1 is

$$\begin{cases} f_1(u_1, u_2, v_1, v_2) := 0.08263u_1v_2^2 + 2.59187v_1^2u_1 + 0.81041u_1 + 0.21140v_1u_1v_2 \\ \qquad +1.02330u_1v_2 + 1.42864v_1u_1 - 12.64393u_2v_1^2 - 1.08802u_2v_2^2 \\ \qquad -4.14350u_2 - 0.44930u_2v_1v_2 - 3.93602u_2v_1 - 3.18104u_2v_2 = 0, \\ f_2(u_1, u_2, v_1, v_2) := -0.23019u_1 + 1.51254u_1v_1^2 + 0.68235u_1v_2^2 + 0.78060u_1v_1 \\ \qquad +0.38026u_1v_2 + 0.16762u_1v_1v_2 - 12.64393v_1^2 - 1.08802v_2^2 \\ \qquad -0.44930v_1v_2 - 3.93602v_1 - 3.18104v_2 - 4.14350 = 0, \\ f_3(u_1, u_2, v_1, v_2) := 0.22465u_1v_1^2 - 11.55591u_1v_1v_2 + 1.59052v_1u_1 \\ \qquad -2.50924u_2v_1v_2 + 0.10570u_2v_1^2 + 0.51165v_1u_2 + 0.19013v_1 \\ \qquad -0.83019v_1v_2 + 0.08381v_1^2 - 0.22465u_1v_2^2 - 1.96801u_1v_2 \\ \qquad -0.10570u_2v_2^2 - 0.71432v_2u_2 - 0.39030v_2 - 0.08381v_2^2 = 0, \\ f_4(u_1, u_2, v_1, v_2) := 1.96801u_1v_1^2 - 8.50043u_1v_1 + 1.59052v_1u_1v_2 + 0.51165v_1u_2v_2 \\ \qquad +0.19013v_1v_2 - 1.78146u_2v_1 + 0.71432u_2v_1^2 - 1.742730v_1 \\ \qquad +0.39030v_1^2 - 0.22465u_1v_2 - 1.96801u_1 - 0.10570u_2v_2 \\ \qquad -0.71432u_2 - 0.08381v_2 - 0.39030 = 0. \end{cases}$$

We now regard $f_i(u_1, u_2, v_1, v_2)$, $i \in [4]$ as a function with u_1 as a variable and u_2, v_1 and v_2 as three coefficients and obtain their resultants as follows:

$$\begin{aligned} \mathrm{Res}_{u_1}(f_1, f_2) =&(0.84336v_2 - 0.95379 + 2.32838v_1 + 3.78649v_2^2 + 2.58431v_2^3 \\ &+ 0.74241v_2^4 + 6.42916v_1^2 + 15.82324v_1^3 + 19.12445v_1^4 + 10.62991v_2v_1^2 \\ &+ 4.23911v_2^2v_1 + 2.79896v_2v_1^3 + 0.48895v_2^3v_1 + 10.34857v_2^2v_1^2 \\ &+ 4.57094v_2v_1)u_2 - 9.10936v_1 - 3.95976v_2^2v_1^2 - 6.81799v_2 \\ &- 9.81234v_2v_1 - 22.65734v_2v_1^2 - 3.01186v_2^2v_1 - 3.83745v_2v_1^3 \\ &- 0.26713v_2^3v_1 - 4.47928v_2^2 - 1.37622v_2^3 - 0.08990v_2^4 - 26.60934v_1^2 \\ &- 28.26528v_1^3 - 32.771423v_1^4 - 3.35793 = 0, \end{aligned}$$

$$\begin{aligned}\mathrm{Res}_{u_1}(f_1,f_3) =&(-8.73334v_2 + 7.00497v_1 - 8.00778v_2^2 - 3.02304v_2^3 - 0.25316v_2^4\\ &+ 8.00778v_1^2 + 22.47179v_1^3 + 3.11442v_1^4 - 74.15833v_2v_1^2 - 39.62519v_2^2v_1\\ &- 152.49246v_2v_1^3 - 12.90368v_2^3v_1 - 8.58379v_2^2v_1^2 - 53.09898v_2v_1)u_2\\ &+ 0.15408v_1 - 0.31630v_2 - 0.38580v_2^2v_1^2 - 1.03583v_2v_1 - 2.07169v_2v_1^2\\ &- 1.03607v_2^2v_1 - 2.13403v_2v_1^3 - 0.08632v_2^3v_1 - 0.46731v_2^2 - 0.11801v_2^3\\ &- 0.00693v_2^4 + 0.33955v_1^2 + 0.61253v_1^3 + 0.21722v_1^4 = 0,\end{aligned}$$

and

$$\begin{aligned}\mathrm{Res}_{u_1}(f_1,f_4) =&(-8.73334 - 8.00778v_2 - 45.43188v_1 - 3.02304v_2^2 - 0.25316v_2^3\\ &- 54.00439v_1^2 - 103.32952v_1^3 + 26.73481v_1^4 + 6.67230v_2v_1^2\\ &- 3.93604v_2^2v_1 + 22.47179v_2v_1^3 + 1.77280v_2^3v_1 + 3.02304v_2^2v_1^2\\ &- 23.92868v_2v_1)u_2 + 0.07244v_2^2v_1^2 - 1.96992v_1 - 0.46731v_2\\ &- 1.83150v_2v_1 + 0.08538v_2v_1^2 + 0.03284v_2^2v_1 + 0.57530v_2v_1^3\\ &+ 0.01571v_2^3v_1 - 0.11801v_2^2 - 0.00693v_2^3 - 3.18504v_1^2 - 3.95933v_1^3\\ &+ 1.01161v_1^4 - 0.31630 = 0.\end{aligned}$$

Let

$$\begin{aligned}&g_1(u_2,v_1,v_2) := \mathrm{Res}_{u_1}(f_1,f_2), \quad g_2(u_2,v_1,v_2) := \mathrm{Res}_{u_1}(f_1,f_3),\\ &g_3(u_2,v_1,v_2) := \mathrm{Res}_{u_1}(f_1,f_4).\end{aligned}$$

Then their resultants are

$$\begin{aligned}\mathrm{Res}_{u_2}(g_1,g_2) =&106.21826v_1^8 + (-5025.64756v_2 + 839.61383)v_1^7 + (-865.25666v_2^2\\ &- 6653.08038v_2 + 998.05266)v_1^6 + (-1083.52137v_2^3 - 5229.62871v_2^2\\ &- 7381.37411v_2 + 1094.99786)v_1^5 + (-137.50780v^4 - 1593.23550v_2^3\\ &- 4933.36887v_2^2 - 4733.12663v_2 + 632.08073)v_1^4 + (-70.75506v_2^5\\ &- 741.31965v_2^4 - 2567.50800v_2^3 - 4243.14169v_2^2 - 2467.61827v_2\\ &+ 335.99902)v_1^3 + (-5.62145v_2^6 - 90.33840v_2^5 - 524.86290v_2^4\\ &- 1366.09759v_2^3 - 1783.36316v_2^2 - 843.25156v_2 + 90.73526)v_1^2\\ &+ (-1.29518v_2^7 - 23.97008v_2^6 - 135.36791v_2^5 - 405.58564v_2^4\\ &- 682.18237v_2^3 - 624.18604v_2^2 - 209.71678v_2 + 23.37525)v_1\\ &+ (-0.02790v_2^8 - 0.72569v_2^7 - 6.69240v_2^6 - 28.96790v_2^5\\ &- 72.02906v_2^4 - 105.34644v_2^3 - 86.25447v_2^2 - 29.02430v_2) = 0\end{aligned}$$

and

$$\mathrm{Res}_{u_2}(g_1,g_3) =895.48403v_1^8 + (852.85983v_2 - 2690.30193)v_1^7 + (304.63154v_2^2$$

$$+ 1073.45667v_2 - 4096.10056)v_1^6 + (172.77376v_2^3 + 134.01848v_2^2 - 2360.05799v_2 - 5632.48044)v_1^5 + (29.00602v_2^4 + 203.47367v_2^3 - 304.10992v_2^2 - 2878.73753v_2 - 3926.74222)v_1^4 + (10.47279v_2^5 + 55.51707v_2^4 - 189.59056v_2^3 - 1392.51379v_2^2 - 3083.36023v_2 - 2315.99105)v_1^3 + (0.80682v_2^6 + 8.30736v_2^5 - 4.53864v_2^4 - 259.96426v_2^3 - 927.90427v_2^2 - 1442.92349v_2 - 831.16935)v_1^2 + (0.17104v_2^7 + 2.07986v_2^6 - 2.49973v_2^5 - 63.10600v_2^4 - 263.71529v_2^3 - 525.06788v_2^2 - 551.19369v_2 - 230.96997)v_1 + (-0.02790v_2^7 - 0.72569v_2^6 - 6.69240v_2^5 - 28.96790v_2^4 - 72.02906v_2^3 - 105.34644v_2^2 - 86.25447v_2 - 29.02430) = 0.$$

Let

$$h_1(v_1, v_2) := \mathrm{Res}_{u_2}(g_1, g_2) \quad \text{and} \quad h_2(v_1, v_2) := \mathrm{Res}_{u_2}(g_1, g_3).$$

Their results are

$$\begin{aligned}\mathrm{Res}_{v_1}(h_1, h_2) = &- 1.62711 \times 10^{-46} v_2^{64} - 1.18415 \times 10^{-44} v_2^{63} - 4.48673 \times 10^{-43} v_2^{62} \\ &- 1.18681 \times 10^{-41} v_2^{61} - 2.43052 \times 10^{-40} v_2^{60} - 4.06317 \times 10^{-39} v_2^{59} \\ &- 5.7302 \times 10^{-38} v_2^{58} - 6.93518 \times 10^{-37} v_2^{57} - 7.24748 \times 10^{-36} v_2^{56} \\ &- 6.49253 \times 10^{-35} v_2^{55} - 4.87257 \times 10^{-34} v_2^{54} - 2.98951 \times 10^{-33} v_2^{53} \\ &- 1.49477 \times 10^{-32} v_2^{52} - 6.3372 \times 10^{-32} v_2^{51} - 2.51516 \times 10^{-31} v_2^{50} \\ &- 1.06401 \times 10^{-30} v_2^{49} - 5.02246 \times 10^{-30} v_2^{48} - 2.43069 \times 10^{-29} v_2^{47} \\ &- 1.09655 \times 10^{-28} v_2^{46} - 4.44371 \times 10^{-28} v_2^{45} - 1.60392 \times 10^{-27} v_2^{44} \\ &- 5.08938 \times 10^{-27} v_2^{43} - 1.39566 \times 10^{-26} v_2^{42} - 3.2426 \times 10^{-26} v_2^{41} \\ &- 6.05752 \times 10^{-26} v_2^{40} - 7.48945 \times 10^{-26} v_2^{39} + 1.29902 \times 10^{-26} v_2^{38} \\ &+ 4.01056 \times 10^{-25} v_2^{37} + 1.382 \times 10^{-24} v_2^{36} + 3.13265 \times 10^{-24} v_2^{35} \\ &+ 5.06222 \times 10^{-24} v_2^{34} + 4.80914 \times 10^{-24} v_2^{33} - 1.80398 \times 10^{-24} v_2^{32} \\ &- 1.82876 \times 10^{-23} v_2^{31} - 4.38298 \times 10^{-23} v_2^{30} - 7.14302 \times 10^{-23} v_2^{29} \\ &- 8.82265 \times 10^{-23} v_2^{28} - 7.81232 \times 10^{-23} v_2^{27} - 3.29361 \times 10^{-23} v_2^{26} \\ &+ 1.21659 \times 10^{-23} v_2^{25} - 5.53171 \times 10^{-23} v_2^{24} - 2.91829 \times 10^{-22} v_2^{23} \\ &- 3.73689 \times 10^{-22} v_2^{22} + 1.93914 \times 10^{-22} v_2^{21} + 1.01905 \times 10^{-21} v_2^{20} \\ &+ 8.28494 \times 10^{-22} v_2^{19} - 4.6185 \times 10^{-22} v_2^{18} - 1.27125 \times 10^{-21} v_2^{17} \\ &- 9.35242 \times 10^{-22} v_2^{16} - 3.19849 \times 10^{-22} v_2^{15} + 7.28108 \times 10^{-24} v_2^{14} \\ &+ 6.11682 \times 10^{-23} v_2^{13} + 6.78336 \times 10^{-24} v_2^{12} + 1.47291 \times 10^{-24} v_2^{11} \\ &+ 8.0039 \times 10^{-24} v_2^{10} - 2.71136 \times 10^{-24} v_2^{9} - 1.27876 \times 10^{-24} v_2^{8} \\ &+ 1.35623 \times 10^{-24} v_2^{7} + 3.48519 \times 10^{-25} v_2^{6} + 2.48651 \times 10^{-26} v_2^{5} \\ &- 2.61144 \times 10^{-26} v_2^{4} - 1.87784 \times 10^{-27} v_2^{3} - 6.6019 \times 10^{-29} v_2^{2} \\ &+ 2.43536 \times 10^{-29} v_2 - 6.98911 \times 10^{-31}.\end{aligned}$$

Next, we obtain the solution of the system (f) by the following steps:

Step 1. Solving $\mathrm{Res}_{v_1}(h_1, h_2) = 0$, we have

$$v_2 = -0.85035, \quad 0.22179, \quad -7.79725, \quad -1.78254, \quad -11.53378, \quad -0.75907, \quad -0.70532, \quad 0.47100.$$

Step 2. Substituting $v_2 = -0.85035$ into $h_1(v_1, v_2)$, and letting $h_1(v_1, v_2) = 0$, it all real roots are $v_1 = -0.43450$ or $v_1 = 4.25751$. Substituting $v_2 = -0.85035$ into $h_2(v_1, v_2)$, and letting $h_2(v_1, v_2) = 0$, it all real roots are $v_1 = -0.48184$, -0.17296, 0.0000000000038876, or 5.04918. It is easy to see that $h_1(v_1, v_2) = 0$ and $h_2(v_1, v_2) = 0$ have no common solution, which implies that $v_2 = -0.85035$ is not a solution of the system (f).

Step 3. Substituting $v_2 = 0.22179$ into $h_1(v_1, v_2)$, and letting $h_1(v_1, v_2) = 0$, it all real roots are $v_1 = -0.43450$ or 4.25751. substituting $v_2 = 0.22179$ into $h_2(v_1, v_2)$, and letting $h_2(v_1, v_2) = 0$, it all real roots are $v_1 = -0.24034$ or 4.25751. It is easy to see that $v_1 = 4.25751$ is a common solution of $h_1(v_1, v_2) = 0$ and $h_2(v_1, v_2) = 0$.

Step 4. Substituting $v_2 = 0.22179$ and $v_1 = 4.25751$ into $g_1(u_2, v_1, v_2)$, $g_2(u_2, v_1, v_2)$ and $g_3(u_2, v_1, v_2)$, and letting $g_1(u_2, v_1, v_2) = 0$, its all real roots are $u_2 = 1.76393$; letting $g_2(u_2, v_1, v_2) = 0$, its all real roots are $u_2 = 1.76393$; letting $g_3(u_2, v_1, v_2) = 0$, its all real roots are $u_2 = 1.76393$. Hence, the common solution of $g_1(u_2, v_1, v_2) = 0$, $g_2(u_2, v_1, v_2) = 0$ and $g_3(u_2, v_1, v_2) = 0$ is $u_2 = 1.76393$.

Step 5. Substituting $v_2 = 0.22179$, $v_1 = 4.25751$ and $u_2 = 1.76393$ into f_1, and letting $f_1(u_1, u_2, v_1, v_2) = 0$, we can get its all real roots $u_1 = 8.16186$.

Step 6. By $v_2 = 0.22179$, $v_1 = 4.25751$, $u_2 = 1.76393$, $u_1 = 8.16186$, (3.12) and (3.13), we can get the corresponding C-eigentriples as follows:

- $\lambda = 13.50214$ and its C-eigenvectors are

$$x = (0.97050, 0.20974, 0.11890)^\top, \quad y = \pm(0.97226, 0.05065, 0.22836)^\top.$$

- $\lambda = -13.50214$ and its C-eigenvectors are

$$x = (-0.97050, -0.20974, -0.11890)^\top, \quad y = \pm(0.97226, 0.05065, 0.22836)^\top.$$

Step 7. For other values of v_2, such as, -7.79725, -1.78254, -11.53378, -0.75907, -0.70532, 0.47100, we can also obtain their corresponding C-eigentriples by using the method similar to Steps 3–6.

Finally, we find all C-eigentriples, which is listed in Table 2.

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