Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 30(2): 633–643.
DOI: 10.3934/era.2022033
Received: 16 November 2021
Revised: 25 January 2022
Accepted: 15 February 2022
Published: 17 February 2022

Research article

Global solution in a weak energy class for Klein-Gordon-Schrödinger system

Qihong Shi[1,*], **Yaqian Jia**[1] **and Xunyang Wang**[1,2]

[1] Department of Mathematics, Lanzhou University of Technology, Lanzhou 730050, China
[2] State Grid Gansu Electric Power Research Institute, Lanzhou 730070, China

* **Correspondence:** Email: shiqh03@163.com.

Abstract: Based on the possible singularity of stationary state, we revisit the initial boundary value problem of the classical Klein-Gordon-Schrödinger (KGS) system in one space dimension. The well-posedness is established in a class of Sobolev NLS solutions together with exponentially growing KG solutions.

Keywords: KGS system; weak solution; global wellposedness; energy space

1. Introduction and results

In this paper we consider the global existence and uniqueness of solutions in the energy class for the one dimensional Klein-Gordon-Schrödinger(KGS) equations with partial initial data nonvanishing at $x = \pm\infty$:

$$\partial_t^2\varphi - \partial_x^2\varphi + \varphi = |\psi|^2, \quad t \in \mathbb{R}, x \in \mathbb{R}, \tag{1.1}$$

$$i\partial_t\psi + \partial_x^2\psi = -\psi\varphi, \quad t \in \mathbb{R}, x \in \mathbb{R}, \tag{1.2}$$

$$\varphi(0, x) = \varphi_0(x), \quad \partial_t\varphi(0, x) = \varphi_1(x), \quad \psi(0, x) = \psi_0(x), \tag{1.3}$$

$$\varphi(t, x) \to e^{|x|}, \quad \psi(t, x) \to 0, \quad (|x| \to \infty), \tag{1.4}$$

where $\partial_t = \partial/\partial t, \partial_x = \partial/\partial x$ and $i = \sqrt{-1}$. $\bar{\psi}$ denotes the complex conjugate of ψ. This system describes the interaction between nucleons and mesons. Here ψ is a complex scalar nucleon field while φ is a real scalar meson one. In order to obtain more physical details, one can refer to [1, 2]. Formally, the solution (ϕ, ψ) of (1.1)–(1.3) satisfies the following conservation laws:

$$L^2\text{-norm}: \quad Q(\psi(t)) = \|\psi(t)\|_{L^2}^2 = \|\psi_0\|_{L^2}^2, \tag{1.5}$$

$$\text{energy}: \quad E(\phi(t), \psi(t)) = \frac{1}{2}(\|\phi\|_{L^2}^2 + \|\partial_t\phi\|_{L^2}^2 + \|\partial_x\phi\|_{L^2}^2) + \|\partial_x\psi\|_{L^2}^2$$

$$-\int_{\mathbb{R}} \phi(x)|\psi(x)|^2 dx = E(\phi_0, \psi_0). \tag{1.6}$$

There is a lot of literature concerning the wellposedness for the classical KGS system. I. Fukuda and M. Tsutsumi [3] established the global wellposedness in energy space for the cauchy problem by the projection method of the Galerkin's type and the compactness argument. They also investigated the existence and uniqueness of the solutions for the initial-boundary value problem in [4,5]. The authors of [6,7] proved the existence of global strong solutions to the initial value problem. B. Wang [8] studied the classical solutions for this equations with nonlinear terms and proved the global existence of small solutions. Moreover, some low regularity wellposedness results were reported. J. Colliander, J. Holmer and N. Tzirakis [9] pointed out the 3d KGS system is global wellposed in $L^2 \times H^{-1/2}$. H. Pecher [10] developed this result in $H^{>-1/4} \times H^{>-1/2}$. In addition, some works associated with asymptotic limit were invoked in [11,12].

However, we note that most works are devoted to the high dimensional case, and they may not have so much interest in one dimensional case. In one space dimension, it does not seem to be extremely necessary outcome that the following boundary condition at $x = \pm\infty$ is imposed on the solution φ of (1.1): $\varphi(t, x) \to 0(|x| \to \infty)$. Indeed, without the loss of generality, we assume $x > 0$ and consider the stationary state

$$-\partial_x^2 \varphi(x) + \varphi(x) = |\psi(x)|^2.$$

Integrating (1.1) from 0 to x yields

$$-\int_0^x \partial_x^2 \varphi(x) dx + \int_0^x \varphi(x) dx = \int_0^x |\psi(x)|^2 dx < \frac{1}{2} Q(\psi_0),$$

which implies $\varphi(x)$ may increase at the level of e^x with respect to the spatial variable. The regime $x < 0$ can be treated analogously.

This fact inspired us to consider KGS system (1.1)–(1.3) with restriction (1.4). To our knowledge, this issue has never been investigated in previous literature.

In order to study the existence of this kind of solutions satisfying (1.4), we introduce a function $M(x)$, which is a real-valued function in $C^\infty(\mathbb{R})$ and satisfies for some $R > 0$:

$$M(x) = \begin{cases} e^{|x|}, & |x| > R, \\ \geq 1, & |x| \leq R. \end{cases} \tag{A}$$

Moreover, we put $\phi(t, x) = \varphi(t, x) - M(x)$, $\phi_0(x) = \varphi_0(x) - M(x)$ and $\phi_1(x) = \varphi_1(x)$. Then (1.1)–(1.4) can be rewritten as follows:

$$\partial_t^2 \phi - \partial_x^2 \phi + \phi = |\psi|^2 + \partial_x^2 M - M, \quad t \in \mathbb{R}, x \in \mathbb{R}, \tag{1.7}$$

$$i\partial_t \psi + \partial_x^2 \psi = -\psi\phi - \psi M, \quad t \in \mathbb{R}, x \in \mathbb{R}, \tag{1.8}$$

$$\phi(0, x) = \phi_0(x), \quad \partial_t \phi(0, x) = \phi_1(x), \quad \psi(0, x) = \psi_0(x), \tag{1.9}$$

$$\phi(t, x), \quad \psi(t, x) \to 0, \quad (|x| \to \infty). \tag{1.10}$$

The solution (ϕ, ψ) of (1.7)–(1.10) formally conserves the L^2-norm Q and the energy $\mathcal{E}$ along the flow as well:

$$Q(\psi(t)) = \|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}, t \in \mathbb{R},$$

$$\mathcal{E}(\phi(t), \psi(t)) = \mathcal{E}(\phi_0, \psi_0), t \in \mathbb{R},$$

where

$$\begin{aligned}\mathcal{E}(\phi, \psi) =& \frac{1}{2}(\|\phi\|_{L^2}^2 + \|\partial_t\phi\|_{L^2}^2 + \|\partial_x\phi\|_{L^2}^2) + \|\partial_x\psi\|_{L^2}^2 \\ &- \int_{\mathbb{R}} \phi(x)|\psi(x)|^2 dx - \int_{\mathbb{R}} M(x)|\psi(x)|^2 dx \\ &+ \int_{\mathbb{R}} \phi(x)[M(x) - \partial_x^2 M(x)]dx. \end{aligned} \tag{1.11}$$

Generally, we call (ϕ, ψ) the energy solution, if it is the solution of (1.7)–(1.10) and belongs to the class $H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times H^1(\mathbb{R})$. However, when we try to prove the global existence and the uniqueness of the solutions (1.7)–(1.10), due to the boundlessness of $M(x)$, (ϕ, ψ) can not be bounded uniformly in this class, which leads that the compactness argument can not be applied to the system (1.7)–(1.10). For this, we consider a new weak topology $H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times (H^1(\mathbb{R}) \cap L^2(\mathbb{R}; e^{|x|}dx))$ as energy class, indeed, it is the weakest function space in which the energy identity (1.11) makes sense, in which by utilizing the method in [3], we also can establish the local wellposedness. In addition, with the help of Gronwall type estimates, the nonlinear term $\int_{\mathbb{R}} M(x)|\psi(x)|^2 dx$ can be bounded by a function depending on time, which leads to the possibility of global existence.

Before stating results, we first introduce several basic notations used throughout this paper. For $m \in \mathbb{N}$, we denote the standard L^2 Sobolev space by $H^m(\mathbb{R})$ and its dual space by $H^{-m}(\mathbb{R})$, where we put $H^0(\mathbb{R}) = L^2(\mathbb{R})$. For $s \in \mathbb{R}$, we denote the function space $L^2(\mathbb{R}; e^{s|x|}dx)$ by $\Pi(s)$. Let $\mathcal{H}^1$ denote the space $H^1(\mathbb{R}) \cap \Pi(\frac{1}{2})$, and $\mathcal{H}^{-1}$ denote its dual space. For $m \in \mathbb{N}$, $1 \le p \le \infty$, $I \subset \mathbb{R}$ denote an interval and X denote a Banach space, we define the Banach space $W^{m,p}(I; X)$ by

$$W^{m,p}(I; X) = \{f(t) \in L^p(I; X); \frac{d^j}{dt^j} f(t) \in L^p(I; X), \quad 1 \le j \le m\},$$

with the norm

$$\|f\|_{W^{m,p}(I;X)} = (\sum_{j=0}^{m} \|\frac{d^j}{dt^j} f\|_{L^p(I;X)}^p)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

$$\|f\|_{W^{m,\infty}(I;X)} = \max_{0 \le j \le m} \|\frac{d^j}{dt^j} f\|_{L^\infty(I;X)}, \quad p = \infty.$$

Now we state our main results in this paper.

Theorem 1.1. *Assume that* $(\phi_0, \phi_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, *where* ϕ_0 *and* ϕ_1 *are real-valued,* $\psi_0 \in \mathcal{H}^1$ *and* M *is real-valued smoothing function satisfying* (*A*). *Let* I *be a bounded open interval in* $\mathbb{R}$ *with* $0 \in I$. *Then,* (1.7)–(1.10) *is uniquely solvable in the following class:*

$$\phi \in \bigcap_{j=0}^{1} W^{j,\infty}(I; H^{1-j}(\mathbb{R})), \tag{1.12}$$

$$\psi \in L^\infty(I; \mathcal{H}^1). \tag{1.13}$$

Remark 1. If the function (ϕ, ψ) satisfies (1.12), (1.13) and (1.7), (1.8) in the distribution sense, then we have

$$\phi \in \bigcap_{j=0}^{1} C_w^j(I; H^{1-j}(\mathbb{R})), \tag{1.14}$$

$$\partial_t^2 \phi \in L^\infty(I; H^{-1}(\mathbb{R})), \tag{1.15}$$

$$\psi \in C_w(I; \mathcal{H}^1), \tag{1.16}$$

$$\partial_t \psi \in L^\infty(I; \mathcal{H}^{-1}). \tag{1.17}$$

Here, $C_w^m(I; X)$ denotes the set of all m-time weakly continuously differentiable functions from I to X .

Moreover, we can actually construct the energy solution globally in time. We have the following stronger wellposedness result in the weak energy class $H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times \mathcal{H}^1(\mathbb{R})$.

Theorem 1.2. *Under the assumptions of Theorem 1.1. There exists a unique solution* (ϕ, ψ) *of* (1.7)–(1.10) *such that*

$$\phi \in \bigcap_{j=0}^{2} C^j(\mathbb{R}; H^{1-j}(\mathbb{R})), \tag{1.18}$$

$$\psi \in \bigcap_{j=0}^{1} C^j(\mathbb{R}; \mathcal{H}^{1-2j}). \tag{1.19}$$

Remark 2. From L^2 conservation and stationary equation, it only can yield φ in systems (1.1)–(1.4) admits the singularity, i.e., the exponential spatial growth form, this property cannot be extended to ψ in Schrödinger equation. It turns out that ψ decays exponentially, while φ increases exponentially.

The rest of this paper is organized as follows. In Section 2, we prepare some basic lemmas which are used to prove our main theorems. Section 3 is devoted to proving the existence and the uniqueness for the weak solutions.

2. Preliminaries

In this section, we state two lemmas for the proof of Theorems 1.1 and 1.2. We begin with the estimates for the linear perturbed Schrödinger equation.

Lemma 2.1. We put $H(t) = (\partial_x + it(\partial_x M))^2$. Let M satisfy (A). Then, the operator $iH(t)$ generates the evolution operator $U(t, s)$, $-\infty < s \le t < +\infty$ associated with the linear Schrödinger equation:

$$i\partial_t \psi + H(t)\psi = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}.$$

The evolution operator $U(t, s)$ satisfies the following Strichartz estimates for any $T > 0$.

(i) Let $I = (-T, T)$. Assume that $f \in L^1(I; L^2)$, q and r are positive constants such that $2 \le q \le \infty$ and $(\frac{1}{2} - \frac{1}{q})r = 2$. Then,

$$\| \int_0^t U(t, s) f(s) ds \|_{L^r(I;L^q)} \le C \|f\|_{L^1(I;L^2)},$$

where C depends only on q and T.

(ii) Let $I = (-T, T)$, q and r be two positive constants such that $1 \leq q \leq 2$ and $\frac{1}{q} + \frac{2}{r} = \frac{5}{2}$. Assume that $f \in L^r(I; L^q)$. Then,

$$\| \int_0^t U(t, s) f(s) ds \|_{L^\infty(I;L^2)} \leq C \|f\|_{L^r(I;L^q)},$$

where C depends only on q and T.

Proof. The Lemma 2.1 is variant of the Strichartz estimates. For the proof, one can see [13–16].

Next, let us recall the Gagliardo-Nirenberg inequality [17] in $\mathbb{R}$, which will be used to treat the nonlinear terms.

Lemma 2.2. Let $1 \leq p, q, r \leq \infty$ and let j, m be two integers, $0 \leq j < m$. If

$$\frac{1}{p} = j + a\left(\frac{1}{r} - \frac{m}{M}\right) + \frac{(1-a)}{q}$$

for some $a \in [\frac{j}{m}, 1]$ ($a < 1$ if $r > 1$ and $m - j - \frac{1}{r} = 0$), then there exists $C(m, j, a, q, r)$ such that

$$\|D^j u\|_{L^p} \leq C \|D^m u\|_{L^r}^a \|u\|_{L^q}^{1-a}$$

for every $u \in \mathcal{D}(\mathbb{R})$.

3. Proof of the main results

In this part, we will prove Theorems 1.1 and 1.2. We start with the uniqueness of the weak energy solutions.

Proof of Theorem 1.1. We assume that (ϕ, ψ) is the solution of (1.7)–(1.10) satisfying (1.12)–(1.13), and put

$$u(t, x) = \psi(t, x) \exp(-itM(x)), \tag{3.1}$$

$$u_0(x) = \psi_0(x). \tag{3.2}$$

By (1.19) and (A), for a.e., $t \in I$, we note that $u \in L^\infty(I; H^1)$ and $\partial_t u \in L^\infty(I; H^{-1})$. Then, the function u satisfies

$$i\partial_t u + (\partial_x + 2it(\partial_x M))^2 u = -\phi u + it(\partial_x^2 M)u - 3t^2(\partial_x M)^2 u \quad \text{in} \quad \mathcal{H}^{-1}. \tag{3.3}$$

Since all the terms except $i\partial_t u$ belong to H^{-1} in (3.3), we can conclude that $\partial_t u \in H^{-1}$ and (3.3) holds true in H^{-1}. Let $U(t, s)$ be the evolution operator generated by $i(\partial_x + 2it(\partial_x M))^2$ (see Lemma 2.1). By the Duhamel principle, (3.2) and (3.3), we have

$$u(t) = U(t, 0)u_0 - i \int_0^t U(t, s)[-\phi u + is(\partial_x^2 M)u - 3s^2(\partial_x M)^2 u] ds, \quad t \in I. \tag{3.4}$$

Let (ϕ, ψ) and $(\tilde{\phi}, \tilde{\psi})$ be the two different solutions to (1.7)–(1.10) with the same initial data such that ϕ and $\tilde{\phi}$ are real-valued, and (ϕ, ψ) and $(\tilde{\phi}, \tilde{\psi})$ satisfy (1.12)–(1.13). And let v be defined as in (3.1) for $\tilde{\psi}$. Then, u and v satisfy (3.4). Set

$$w = u - v, \quad \alpha = \phi - \tilde{\phi}.$$

Then, by (3.4), we have

$$w(t) = -i \int_0^t U(t,s)[-\phi w - \alpha v + is(\partial_x^2 M)w - 3s^2(\partial_x M)^2 w]ds, \quad t \in I. \tag{3.5}$$

Let T be a positive constant with $[0, T] \subset I$ to be determined later. Denote $I_T = [0, T]$, and take the $L^\infty(I_T; L^2)$ norm of (3.5), by Lemma 2.1, (1.12) and (1.13), we can obtain

$$\begin{aligned} \|w\|_{L^\infty(I_T;L^2)} \leq & CT[\|\phi\|_{L^\infty(I_T\times\mathbb{R})}\|w\|_{L^\infty(I_T;L^2)} \\ & + \|\alpha\|_{L^\infty(I_T;L^2)}\|v\|_{L^\infty(I_T\times\mathbb{R})}] \\ & + C\int_0^t [s(1+s)\|w\|_{L^\infty(I_T;L^2)}]ds \\ \leq & CT[\|\alpha\|_{L^\infty(I_T;L^2)} + \|w\|_{L^\infty(I_T;L^2)}], \end{aligned} \tag{3.6}$$

where C is a positive constant. Moreover, from the definition of α, it follows that $\alpha(t)$ satisfies

$$\partial_t^2\alpha - \partial_x^2\alpha + \alpha = |\psi|^2 - |\tilde{\psi}|^2, \quad t \in I_T, \tag{3.7}$$

$$\alpha(0, x) = \partial_t\alpha(0, x) = 0. \tag{3.8}$$

If we put $\omega = (1 - \partial_x^2)^{\frac{1}{2}}$, by (3.7), we then have

$$\alpha(t) = \int_0^t \omega^{-1}\sin[(t-s)\omega](|\psi|^2 - |\tilde{\psi}|^2)ds, \quad t \in I_T. \tag{3.9}$$

Using Sobolev embedding $L^1 \hookrightarrow H^{-1}$, we can deduce,

$$\begin{aligned} \|\alpha(t)\|_{L^2} \leq & \int_0^t \|\omega^{-1}(|\psi|^2 - |\tilde{\psi}|^2)\|_{L^2}ds \\ \leq & \int_0^t \||\psi|^2 - |\tilde{\psi}|^2\|_{L^1}ds \\ \leq & \int_0^t (\|\psi\|_{L^2}\|\bar{\psi} - \bar{\tilde{\psi}}\|_{L^2} + \|\bar{\tilde{\psi}}\|_{L^2}\|\psi - \tilde{\psi}\|_{L^2})ds \\ \leq & C\int_0^t \|\psi\|_{L^2}\|\psi - \tilde{\psi}\|_{L^2}ds. \end{aligned} \tag{3.10}$$

And, among them

$$\|\psi(t) - \tilde{\psi}(t)\|_{L^2} \leq C\|w(t)\|_{L^2}. \tag{3.11}$$

Accordingly, we conclude with the estimates (3.6) and (3.10)–(3.11)

$$\|w\|_{L^\infty(I_T;L^2)} + \|\alpha(t)\|_{L^\infty(I_T;L^2)}$$

$$\leq C(T)\left(\|w\|_{L^\infty(I_T;L^2)} + \|\alpha(t)\|_{L^\infty(I_T;L^2)}\right), \tag{3.12}$$

where $C(T)$ is a positive constant depending increasingly on T. If we choose $T > 0$ so small that $C(T) \leq \frac{1}{2}$, then (3.12) implies that

$$\phi(t) = \tilde{\phi}(t), \quad \psi(t) = \tilde{\psi}(t), \quad t \in [0, T]. \tag{3.13}$$

We repeat the above procedure to obtain (3.13) for all $t \in I$. This shows Theorem 1.1.

Next we prove Theorem 1.2 by using the results stated in Theorem 1.1.

Proof of Theorem 1.2. Let $\{(\phi_{0n}, \phi_{1n})\}$, $\{\psi_{0n}\}$ and $\{v_n\}$ be the sequences in $C_0^\infty(\mathbb{R})$ such that $\phi_{0n} \to \phi_0$ in H^1, $\phi_{1n} \to \phi_1$ in L^2 and $\psi_{0n} \to \psi_0$ in $\mathcal{H}^1$. Actually, we can choose these sequences as follows. Let χ be a function in $C_0^\infty(\mathbb{R})$ with compact support on $[0, 2]$ such that $\chi(x) = 1$ for $|x| \leq 1$, and let ρ be a function in $C_0^\infty(\mathbb{R})$ such that $\rho \geq 0$ and $\int_{\mathbb{R}} \rho(x)dx = 1$. For $\eta, \epsilon > 0$, we define $\chi_\eta(x) = \chi(\eta x)$ and $\rho_\epsilon(x) = \epsilon^{-1}\rho(x/\epsilon)$. Moreover, we define

$$\begin{aligned}
\phi_{j\eta\varepsilon} &= \rho_\epsilon * (\chi_\eta \phi_j), \quad j = 0, 1 \\
\psi_{0\eta\varepsilon} &= \rho_\epsilon * (\chi_\eta \psi_0), \\
m_{\eta\varepsilon} &= \rho_\epsilon * [\chi_\eta(\partial_x^2 M - M)] - |\rho_\varepsilon * (\chi_\eta \psi_0)|^2 + \rho_\varepsilon * (\chi_\eta |\psi_0|^2) \\
&\quad + \rho_\varepsilon * [\chi_\eta \phi_0 + \eta^2 (\partial_x^2 \chi)_\eta \phi_0 + 2\eta(\partial_x \chi)_\eta \partial_x \phi_0)],
\end{aligned}$$

where $*$ denotes the convolution with respect to the spatial variable x. With the suitable choice of $\eta, \varepsilon > 0$, we may obtain the desired sequences.

We consider the Cauchy problem of (1.7)–(1.10) with $\partial_x^2 M - M$ replaced by m_n in (1.8). Then, for each pair of the initial data $(\phi_{0n}, \phi_{1n}, \psi_{0n})$, we have the unique local solutions (ϕ_n, ψ_n) of the initial value problem of (1.7)–(1.10) belonging to

$$\left\{\cap_{j=0}^1 C^j([-T, T]; H^{3-j})\right\} \times C([-T, T]; H^3 \cap \Pi(1/2))$$

with ϕ_n real-valued, where the existence time $T > 0$ depends only on $\|\phi_{0n}\|_{H^3}$, $\|\phi_{1n}\|_{H^2}$ and $\|\psi_{0n}\|_{H^3 \cap \Pi(1/2)}$. For the local existence of smooth solutions, one can also refer to Theorem 2.1 in [18].

We multiply (1.7) by $\partial_t \phi_n$ and multiply (1.8) by $\partial_t \bar{\psi}_n$, respectively, take the real part to obtain the energy identity:

$$\mathcal{E}(\phi_n, \psi_n) = \mathcal{E}(\phi_{0n}, \psi_{0n}), \quad t \in [-T, T], \tag{3.14}$$

where the energy functional $\mathcal{E}$ is defined as in (1.11). Moreover, multiplying (1.8) by $\bar{\psi}_n$, integrating the resulting equation in x over $\mathbb{R}$ and taking the imaginary part, we can obtain

$$\|\psi_n(t)\|_{L^2} = \|\psi_{0n}\|_{L^2}, \quad t \in [-T, T]. \tag{3.15}$$

Next we estimate $\int_{\mathbb{R}} |\psi(x)|^2 \phi(x)dx$. Applying Cauchy inequality, Hölder inequality and Lemma 2.2, we get

$$\||\psi_n(t)|^2 \phi_n(t)\|_{L^1} \leq \|\phi_n(t)\|_{L^2} \|\psi_n(t)\|_{L^4}^2$$

$$\leq \frac{1}{4}\|\phi_n(t)\|_{L^2}^2 + \|\psi_n(t)\|_{L^2}^3\|\partial_x\psi_n\|_{L^2}$$
$$\leq \frac{1}{4}\|\phi_n(t)\|_{L^2}^2 + \|\psi_n(t)\|_{L^2}^6 + \frac{1}{4}\|\partial_x\psi_n\|_{L^2}^2, \quad t \in [-T, T], \tag{3.16}$$

which implies

$$\begin{aligned}
&\frac{1}{4}\|\phi\|_{L^2}^2 + \frac{1}{2}\|\partial_t\phi\|_{L^2}^2 + \frac{1}{2}\|\partial_x\phi\|_{L^2}^2 + \frac{3}{4}\|\partial_x\psi\|_{L^2}^2 \\
&\quad \leq E(\phi_{0n}, \psi_{0n}) + \|\psi_n(t)\|_{L^2}^6 - \int_{B(0,R)} C(R)\phi dx + \int_{\mathbb{R}} M(x)|\psi(x)|^2 dx \\
&\quad \leq C + \int_{B(0,R)} |C(R)\phi| dx + \int_{\mathbb{R}} M(x)|\psi(x)|^2 dx \\
&\quad \leq C(R) + \frac{1}{8}\|\phi\|_{L^2}^2 + \int_{\mathbb{R}} M(x)|\psi(x)|^2 dx,
\end{aligned} \tag{3.17}$$

where $C(R)$ is a constant which is only depending on R given in (A). Therefore,

$$\frac{1}{8}\|\phi\|_{L^2}^2 + \frac{1}{2}\|\partial_t\phi\|_{L^2}^2 + \frac{1}{2}\|\partial_x\phi\|_{L^2}^2 + \frac{3}{4}\|\partial_x\psi\|_{L^2}^2 \leq C + \int_{\mathbb{R}} M(x)|\psi(x)|^2 dx. \tag{3.18}$$

Inequality (3.18) implies that

$$\|\partial_x\psi\|_{L^2}^2 \leq C + \int_{\mathbb{R}} M(x)|\psi(x)|^2 dx. \tag{3.19}$$

Then, we estimate $\int_{\mathbb{R}} M(x)|\psi(x)|^2 dx$. We multiply (1.8) by $M\bar{\psi}_n$, integrate the resulting equation in x over $\mathbb{R}$ and take the imaginary part to get

$$\begin{aligned}
\|M^{\frac{1}{2}}\psi_n(t)\|_{L^2}^2 &\leq \|M^{\frac{1}{2}}\psi_{0n}\|_{L^2}^2 + C\int_0^t \|\partial_x\psi_n(s)\|_{L^2}\|\psi_n(s)\|_{L^2} ds \\
&\leq \|M^{\frac{1}{2}}\psi_{0n}\|_{L^2}^2 + C|t| + C\int_0^t \|\partial_x\psi_n(s)\|_{L^2} ds \\
&\leq \|M^{\frac{1}{2}}\psi_{0n}\|_{L^2}^2 + C|t| + C\int_0^t \int_{\mathbb{R}} M(x)|\psi(x)|^2 dx ds.
\end{aligned} \tag{3.20}$$

Using Gronwall inequality and (3.20), we get

$$\|M^{\frac{1}{2}}\psi_n(t)\|_{L^2}^2 \leq (\|M^{\frac{1}{2}}\psi_{0n}\|_{L^2}^2 + C|t|)e^{C|t|}. \tag{3.21}$$

Therefore,

$$\|\partial_x\psi_n(t)\|_{L^2}^2 \leq Ce^{C|t|}, \quad t \in [-T, T], \tag{3.22}$$

where C does not depend on n and T. (3.18), (3.21), (3.22) and the linear hyperbolic theory show that

$$\|\phi_n(t)\|_{H^1} + \|\partial_t\phi_n(t)\|_{L^2} + \|\partial_x\phi_n(t)\|_{L^2} \leq Ce^{C|t|}, \quad t \in [-T, T], \tag{3.23}$$

where C does not depend on n and T.

Similarly, it is not difficult to obtain the a priori estimates (dependent on n) for the higher order derivatives of (ϕ_n, ψ_n), by which we can extend the above local solutions (ϕ_n, ψ_n) globally in time for each $n \geq 1$. Accordingly, (3.15)–(3.23) hold true for any time $t \in \mathbb{R}$. Thereafter, applying the standard compactness analysis, one can derive that there exists at least one global energy solution $(\phi(t), \psi(t))$ of (1.7)–(1.10) satisfying (1.18)–(1.19) and

$$\|M^{\frac{1}{2}}\psi(t)\|_{L^2}^2 \leq (\|M^{\frac{1}{2}}\psi_0\|_{L^2}^2 + C|t|)e^{C|t|}, \quad t \in \mathbb{R}, \tag{3.24}$$

$$\begin{aligned}
&\frac{1}{2}(\|\phi\|_{L^2}^2 + \|\partial_t\phi\|_{L^2}^2 + \|\partial_x\phi\|_{L^2}^2) + \|\partial_x\psi\|_{L^2}^2 \\
&\quad - \int_{\mathbb{R}} \phi(x)|\psi(x)|^2 dx + \int_{\mathbb{R}} \phi(x)[M(x) - \partial_x^2 M(x)]dx \\
&\leq \frac{1}{2}(\|\phi_0\|_{L^2}^2 + \|\partial_t\phi_0\|_{L^2}^2 + \|\partial_x\phi_0\|_{L^2}^2) + \|\partial_x\psi_0\|_{L^2}^2 \\
&\quad + \|M^{\frac{1}{2}}\psi_0\|_{L^2}^2(e^{C|t|} - 1) + C|t|e^{C|t|} \\
&\quad - \int_{\mathbb{R}} \phi_0(x)|\psi_0(x)|^2 dx + \int_{\mathbb{R}} \phi_0(x)[M(x) - \partial_x^2 M(x)]dx, \quad t \in \mathbb{R}.
\end{aligned} \tag{3.25}$$

Moreover, the equation (1.7) and the theory of linear hyperbolic equations, together with (1.13), imply that

$$\phi \in \bigcap_{j=0}^{1} C^j(\mathbb{R}; H^{1-j}). \tag{3.26}$$

With that, we claim

$$\limsup_{t\to 0} \|M^{\frac{1}{2}}\psi(t)\|_{L^2}^2 \leq \|M^{\frac{1}{2}}\psi_0\|_{L^2}^2, \tag{3.27}$$

$$\limsup_{t\to 0} \|\partial_x\psi(t)\|_{L^2}^2 \leq \|\partial_x\psi_0\|_{L^2}^2. \tag{3.28}$$

Since

$$\left|\int_{\mathbb{R}} \phi(x)|\psi(x)|^2 - \phi_0(x)|\psi_0(x)|^2 dx\right| \leq \|\phi(x) - \phi_0(x)\|_{H^1}\|\psi(x)\|_{L^2}, \tag{3.29}$$

$$\left|\int_{\mathbb{R}} (\phi(x) - \phi_0(x))(\partial_x^2 M - M)dx\right| \leq C(R)\|\phi(x) - \phi_0(x)\|_{H^1}, \tag{3.30}$$

it follows

$$\lim_{t\to 0} \int_{\mathbb{R}} \phi(x)|\psi(x)|^2 dx = \int_{\mathbb{R}} \phi_0(x)|\psi_0(x)|^2 dx, \tag{3.31}$$

$$\lim_{t\to 0} \int_{\mathbb{R}} \phi(x)(\partial_x^2 M - M)dx = \int_{\mathbb{R}} \phi_0(x)(\partial_x^2 M - M)dx. \tag{3.32}$$

Notice that $\psi(t) \in C(\mathbb{R}; L^2) \cap C_w(\mathbb{R}; H^1) \cap C_w(\mathbb{R}; \Pi(1/2))$(see Remark 1), by (3.24) and (3.25), we conclude the claim. Moreover, Let t_0 be an arbitrary real constant with $t_0 \neq 0$. In view of the uniqueness in Theorem 1.1, we then have

$$(\phi(t), \partial_t\phi(t)) \to (\phi(t_0), \partial_t\phi(t_0)) \quad \text{in} \quad H^1 \times L^2 \quad (t \to t_0),$$

$$\psi(t) \to \psi(t_0) \quad \text{in} \quad \mathcal{H}^1 \quad (t \to t_0).$$

A replacement of the initial time and the initial data by t_0 and $(\phi(t_0), \partial_t\phi(t_0), \psi(t_0))$ will follows the strong continuity of solution in energy class $H^1 \times L^2 \times \mathcal{H}^1$ at any time. This closes the proof.

Remark 3. Due to the chaotic structure inside the atom, the generation of this singular solution is not difficult to understand. It should be pointed out that we can also consider the existence of the solutions of this system in the higher-dimensional space $\mathbb{R}^{n+1}(n \geq 2)$ by compactness argument. However, it does not guarantee that the solution increases exponentially in all spatial directions. In addition, due to the breakdown of the Sobolev embedding $H^1 \hookrightarrow L^\infty$, we can not obtain the uniqueness of this kind of solutions.

Acknowledgments

The authors are grateful to the anonymous referees for valuable suggestions. The authors are supported by NNSF of China (Nos. 12061040, 11701244) and NSF of Gansu Province(CN) (Nos. 20JR5RA460, 21JR7RA217).

Conflict of interest

The authors have no conflicts to disclose.

References

1. H. Yukawa, On the interaction of elementary particles I, *Proc. Phys. Math. Soc. Japan*, **17** (1935), 48–57. https://doi.org/10.1143/PTPS.1.1
2. Q. Shi, C. Peng, Q. Wang, Blowup results for the fractional Schrödinger equation without gauge invariance, *Discrete Contin. Dyn. Syst. B*, (2021), In press, https://doi.org/10.3934/dcdsb.2021304
3. I. Fukuda, M. Tsutsumi, On coupled Klein-Gordon-Schrödinger equations I, *Bull. Sci. Engrg. Res. Lab. Waseda Univ.*, **69** (1975), 51–62.
4. I. Fukuda, M. Tsutsumi, On the Yukawa-coupled Klein-Gordon-Schrödinger equations in three space dimensions, *Proc. Jpn. Acad.*, **51** (1975), 402–405. https://doi.org/10.3792/pja/1195518563
5. I. Fukuda, M. Tsutsumi, On coupled Klein-Gordon-Schrödinger equations II, *J. Math. Anal. Appl.*, **66** (1978), 358–378. https://doi.org/10.1016/0022-247X(78)90239-1
6. J. Baillon, J. Chadam, The Cauchy problem for the coupled Schrödinger-Klein-Gordon equations, *Contemporary Developments in Continuum Mechanics and PDE* (eds. Guilherme M. De La Penha and Luiz Adauto J. Medeiros), North-Holland Mathematics Studies, (1978), 37–44. https://doi.org/10.1016/S0304-0208(08)70857-0
7. H. Nakao, W. Wolf, On the global strong solutions of coupled Klein-Gordon-Schrödinger equations, *J. Math. Soc. Jpn.*, **39** (1987), 489–497. https://doi.org/10.2969/jmsj/03930489

8. B. Wang, Classical global solutions for non-linear Klein-Gordon-Schrödinger equations, *Math. Method. Appl. Sci.*, **20** (1997), 599–616. https://doi.org/10.1002/(SICI)1099–1476 (19970510)20:7<599::AID-MMA866>3.0.CO;2-7

9. J. Colliander, J. Holmer, N. Tzirakis, Low regularity global Well-Posedness for the Zakharov and Klein-Gordon-Schrödinger system, *T. Am. Math. Soc.*, **360** (2008), 4619–4638. https://doi.org/10.1090/S0002-9947-08-04295-5

10. H. Pecher, Low regularity global well-posedness for the 3D Klein-Gordon-Schrödinger system, *Commun. Pur. Appl. Anal.*, **11** (2012), 1081–1096. https://doi.org/10.3934/cpaa.2012.11.1081

11. W. Bao, X. Zhao, A uniformly accurate (UA) multiscale time integrator Fourier pseudospectral method for the Klein CGordon CSchr odinger equations in the nonrelativistic limit regime, *Numer. Math.*, **135** (2017), 833–873. https://doi.org/10.1007/s00211-016-0818-x

12. Q. Shi, S. Wang, Nonrelativistic approximation in the energy space for KGS system, *J. Math. Anal. Appl.*, **462** (2018), 1242–1253. https://doi.org/10.1016/j.jmaa.2018.02.039

13. J. Ginibre, G. Velo, The global cauchy problem for the non linear Schrödinger equation revisited, *Ann. Inst. Henri Poincaré, Analyse non linéaire*, **2** (1985), 309–327. https://doi.org/10.1016/S0294-1449(16)30399-7

14. T. Kato, On nonlinear Schrödinger equations, *Ann. Inst. H. Poincaré, Phys. Théor.*, **46** (1987), 113–129. http://eudml.org/doc/76348

15. K. Yajima, Schrödinger evolution equations with magnetic fields, *J. Anal. Math.*, **56** (1991), 29–76. https://doi.org/10.1007/BF02820459

16. K. Yajima, Existence of solutions for Schrödinger evolution equations, *Commun. Math. Phys.*, **110** (1987), 415–426. https://doi.org/10.1007/BF01212420

17. M. Agueh, Gagliardo–Nirenberg inequalities involving the gradient L^2-norm, *C. R. Acad. Sci. Paris, Ser., I*, **346** (2008), 757–762. https://doi.org/10.1016/j.crma.2008.05.015

18. K. Nakamitsu, M. Tsutsumi, The Cauchy problem for the coupled Maxwell-Schrödinger equations, *J. Math. Phys.*, **27** (1986), 211–216. https://doi.org/10.1063/1.527363