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ERA, 30(2): 585–614.
DOI: 10.3934/era.2022031
Received: 21 September 2021
Revised: 25 November 2021
Accepted: 25 November 2021
Published: 15 February 2022

Research article

Multi-peak semiclassical bound states for Fractional Schrödinger Equations with fast decaying potentials

Xiaoming An[1] **and Shuangjie Peng**[2,*]

[1] School of Mathematics and Statistics & Guizhou University of Finance and Economics, Guiyang, 550025, China
[2] School of Mathematics and Statistics & Hubei Key Laboratory of Mathematical Sciences, Central China Normal University, Wuhan, 430079, China

* **Correspondence:** Email: sjpeng@mail.ccnu.edu.cn.

Abstract: We study the following fractional Schrödinger equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(u), \;\; x \in \mathbb{R}^N,$$

where $s \in (0,1)$. Under some conditions on $f(u)$, we show that the problem has a family of solutions concentrating at any finite given local minima of V provided that $V \in C(\mathbb{R}^N, [0,+\infty))$. All decay rates of V are admissible. Especially, V can be compactly supported. Different from the local case $s = 1$ or the case of single-peak solutions, the nonlocal effect of the operator $(-\Delta)^s$ makes the peaks of the candidate solutions affect mutually, which causes more difficulties in finding solutions with multiple bumps. The methods in this paper are penalized technique and variational method.

Keywords: variational method; fractional Schrödinger; multi-peak; compactly supported; penalized technique

1. Introduction and main results

In this paper, we consider the fractional Schrödinger equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(u), \;\; x \in \mathbb{R}^N, \tag{1.1}$$

where $N > 2s$, $s \in (0,1)$, V is a continuous function, $\varepsilon > 0$ is a small parameter, $f : \mathbb{R}^N \to \mathbb{R}$ is a nonlinear function. Problem (1.1) is derived from the study of time-independent waves $\psi(x,t) = e^{-iEt}u(x)$ of the following nonlinear fractional Schrödinger equation

$$i\varepsilon\frac{\partial\psi}{\partial t} = \varepsilon^{2s}(-\Delta)^s\psi + U(x)\psi - f(\psi) \; x \in \mathbb{R}^N. \tag{NLFS}$$

For example, letting $f(t) = |t|^{p-2}t$, $V(x) = U(x) - E$ and inserting $\psi(x,t) = e^{-iEt}u(x)$ into $(NLFS)$, one can show that $(NLFS)$ is

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = |u|^{p-2}u. \tag{1.2}$$

In physics, Eq (1.1) can be used to describe some properties of Einstein's theory of relativity and also has been derived as models of many physical phenomena, such as phase transition, conservation laws, especially in fractional quantum mechanics, etc., [1]. $(NLFS)$ was introduced by Laskin [2,3] as an extension of the classical nonlinear Schrödinger equations $s = 1$ in which the Brownian motion of the quantum paths is replaced by a Lèvy flight. To see more physical backgrounds, we refer to [4].

In this paper, we are interesting in semiclassical analysis of (1.1). From a mathematical point of view, the transition from quantum to classical mechanics can be formally performed by letting $\varepsilon \to 0$. For small $\varepsilon > 0$, solutions u_ε are usually referred to as semiclassical bound states.

In the local case $s = 1$, the study of the nonlinear Schrödinger equation

$$-\varepsilon^2\Delta u + V(x)u = f(u) \tag{NLS}$$

has been extensively investigated in the semiclassical regime and a considerable amount of work has been done, showing that existence and concentration phenomena of single- and multi-bump solutions occur at critical points of the electric potential V when $\varepsilon \to 0$, see [5–19] and the references therein for example.

To our best knowledge, there are few results on the semiclassical bound states to problem (1.1) in the nonlocal case $s \in (0,1)$. Basing on the well-known non-degenerate results in [20,21] and the mathematical reduction method, it was proved in [22–24] that problem (1.2) has solutions concentrating at the prescribed non-degenerate critical points of V when $\varepsilon \to 0$. When $\inf_{x\in\mathbb{R}^N} V(x) > 0$ and V has local minimum which may be degenerate, Alves et al. in [25] used the penalized method developed by del Pino et al. in [10] and the extension method developed by Caffarelli et al. in [26] to construct solutions concentrating at a local minimum of V when $\varepsilon \to 0$. Successively, assuming more weakly that $\liminf_{|x|\to\infty} V(x)|x|^{2s} \geq 0$, in [27,28], solutions concentrating at a local minimum of V were also obtained. We point out here that the solutions found in [25] and [27] have exactly one local maximum and hence are single-peaked.

However, concerning (1.1), up to now there are no research on the multi-bump solutions in the case that the potentials $V(x)$ vanish at infinity and critical points of $V(x)$ are degenerate. The main difficulty lies in that for a suitable function $u : \mathbb{R}^N \to \mathbb{R}$, under the nonlocal effects of $(-\Delta)^s$, one can not compute $(-\Delta)^s u$ as precisely as $-\Delta u$. Moreover, the nonlocal operator $(-\Delta)^s$ makes the peaks of the candidate solutions affect mutually, which causes more difficulties in finding solutions with multiple bumps (see the estimates of (2.23), (2.26) and (2.29) in Lemma A.2 for example).

This paper devotes to finding solutions with multiple bumps for more general potentials including fast decaying potentials, i.e.,

$$\liminf_{|x|\to\infty} V(x)|x|^{2s} = 0,$$

in which, a typical case is that V is compactly supported.

In order to state our main result, we need to introduce some notations and assumptions. For $s \in (0,1)$, the fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined as

$$H^s(\mathbb{R}^N) = \Big\{u \in L^2(\mathbb{R}^N) : \frac{u(x) - u(y)}{|x-y|^{N/2+s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N)\Big\},$$

endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + u^2 \, \mathrm{d}x \Big)^{\frac{1}{2}},$$

where

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \mathrm{d}x = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y.$$

Like the classical case, we define the space $\dot{H}^s(\mathbb{R}^N)$ as the completion of $C_c^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \mathrm{d}x = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y.$$

Define the following fractional Sobolev space

$$W^{s,2}(\Omega) = \Big\{ u \in L^2(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{2}+s}} \in L^2(\Omega \times \Omega) \Big\}.$$

It is easy to check that with the inner product

$$(u, v) = \int_\Omega \int_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dxdy + \int_\Omega uvdx \ \ \forall u, v \in W^{s,2}(\Omega),$$

$W^{s,2}(\Omega)$ is a Hilbert space (see [4] for details). According to [4], the fractional Laplacian is defined as

$$\begin{aligned} (-\Delta)^s u(x) \ &= \ C(N, s) P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, \mathrm{d}y \\ &= \ C(N, s) \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, \mathrm{d}y. \end{aligned}$$

For the sake of simplicity, we define for every $u \in \dot{H}^s(\mathbb{R}^N)$ the fractional $(-\Delta)^s u$ as

$$(-\Delta)^s u(x) = \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, \mathrm{d}y.$$

Our solutions will be found in the following weighted fractional Sobolev space:

$$\mathcal{D}^s_{V,\varepsilon}(\mathbb{R}^N) = \Big\{ u \in \dot{H}^s(\mathbb{R}^N) : \ u \in L^2(\mathbb{R}^N, V(x) \, \mathrm{d}x) \Big\},$$

endowed with the norm

$$\|u\|_{\mathcal{D}^s_{V,\varepsilon}(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} \varepsilon^{2s} |(-\Delta)^{s/2} u|^2 + V u^2 \, \mathrm{d}x \Big)^{\frac{1}{2}}.$$

For the nonlinear term $f(u)$, we assume

(f_1) $f(t)$ is an odd function and $f(t) = o(t^{1+\tilde{\kappa}})$ as $t \to 0^+$, where $\tilde{\kappa} = \dfrac{2s + 2\kappa}{N - 2s - \tilde{\nu}} > 0$ with $\tilde{\nu}, \kappa > 0$ are small parameters.

(f_2) $\displaystyle\lim_{t\to\infty} \frac{f(t)}{t^p} = 0$ for some $1 < p < 2_s^* - 1$. (1.3)

(f_3) There exists $2 < \theta \le p + 1$ such that $0 \le \theta F(t) < f(t)t$ for all $t > 0$, where

$$F(t) = \int_0^t f(\alpha)d\alpha.$$

(f_4) The map $t \mapsto \dfrac{f(t)}{t}$ is increasing on $(0, +\infty)$.

A typical case of $f(t)$ is: $f(t) = |t|^{p-2}t$ with $2 + \frac{2s}{N-2s} < p < 2_s^*$.
For the potential term V, we assume that $V \in C(\mathbb{R}^N, [0, \infty))$ and
(**V**) There exist open bounded sets $\Lambda_i \subset\subset S_i \subset\subset U_i$ with smooth boundaries, such that

$$0 < \lambda_i = \inf_{\Lambda_i} V < \inf_{U_i \setminus \Lambda_i} V, \ \overline{U_i} \cap \overline{U_j} = \emptyset \text{ if } 1 \leq i \neq j \leq k. \tag{1.4}$$

Denote $\Lambda = \bigcup_{i=1}^k \Lambda_i$, $S = \bigcup_{i=1}^k S_i$ and $U = \bigcup_{i=1}^k U_i$. Without loss of generality, we assume that $0 \in \Lambda$.

Theorem 1.1. *Let $N > 2s$, $s \in (0,1)$, V satisfy (**V**) and f satisfy the assumptions $(f_1) - (f_4)$. Then problem (1.1) has a positive solution $u_\varepsilon \in \mathcal{D}^s_{V,\varepsilon}(\mathbb{R}^N)$ if $\varepsilon > 0$ is small enough. Moreover, there exists k families of points $\{\{x^i_\varepsilon\} : 1 \leq i \leq k\}$ and an α close to $N - 2s$, such that*

$$(i) \ \lim_{\varepsilon \to 0} V(x^i_\varepsilon) = \lambda_i,$$
$$(ii) \ \liminf_{\varepsilon \to 0} \|u_\varepsilon\|_{L^\infty\left(B_{\varepsilon\rho}(x^i_\varepsilon)\right)} > 0$$
$$(iii) \ u_\varepsilon(x) \leq \sum_{i=1}^k \frac{C\varepsilon^\alpha}{\varepsilon^\alpha + |x - x^i_\varepsilon|^\alpha},$$

where C and ρ are positive constants.

Now we introduce the main idea of the proof. For the local case $s = 1$, certain penalized functional like

$$K_\varepsilon(u) = M_1 \sum_{j=1}^k \left(((L^j_\varepsilon(u))_+^{1/2} - \varepsilon^{N/2}(c_j + \sigma_j)^{1/2})_+ \right)^2 \tag{1.5}$$

was usually employed to prove that the penalized solution u_ε has exactly one peak in each Λ_i, see [17, 18, 29] for example. But, the key step of this argument is to eliminate the effect of $K_\varepsilon(u)$ to the equation, which needs a type of isolated property of the least energy of $-\Delta u + u = g(u)$. However, for our case $0 < s < 1$, this type of isolated property is still unknown. To overcome this difficulty, we use the method developed by Byeon and Jeanjean in [30], which proves the existence of multi-peak solutions of following equation

$$-\varepsilon^2 \Delta u + V(x)u = g(u) \tag{1.6}$$

by using only the compactness of the set consisting of the radial positive least energy solutions of the following limiting problem of (1.6):

$$-\Delta u + au = g(u),$$

where $a > 0$ is a constant and g is a nonlinear term satisfying some subcritical conditions. For more application of this methods, see [31]. Roughly speaking, by the compact property, we use the deformation ideas of Lemma 2.2 in [32] to construct a $(PS)_c$ sequence near the least energy solutions of the following k problems:

$$(-\Delta)^s u + \lambda_i u = f(u), \text{in } \mathbb{R}^N, \ i = 1, \cdots, k.$$

It is worth mentioning that the compact property can be obtained by the decay estimates of positive radial least energy solutions(see Proposition 2.4 below). However, the vanishing of V and the nonlocal effect of $(-\Delta)^s$ makes the construction of multi-peak solutions more difficult than the classical case $s = 1$, the non-vanishing case [30] and the single peak case [28]. Firstly, an elementary(but tedious) calculations show that when $V(x)$ vanishes faster than $|x|^{-2s}$, the natural functional $I_\varepsilon : \mathcal{D}^s_{V,\varepsilon}(\mathbb{R}^N) \to \mathbb{R}$ corresponding to (1.1) defined as

$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N}(\varepsilon^{2s}|(-\Delta)^{s/2}u|^2 + Vu^2)dx - \int_{\mathbb{R}^N} F(u)dx,$$

whose critical points are solutions of Eq (1.1), is not well-defined in $\mathcal{D}^s_{V,\varepsilon}(\mathbb{R}^N)$, where $F(t) = \int_0^t f(s)ds$. Moreover, the fact that $V(x)$ may be compactly supported makes it impossible that V can dominated the nonlinear term $|u|^{p-2}u$ like [27]. Hence we have to introduce a different penalized idea from [30] to cut-off the nonlinear term. More precisely, we will first use the nonlocal part $(-\Delta)^s$ to modify the problem by the following fractional Hardy inequality

$$\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}}\,\mathrm{d}x \le C_{N,s}\|(-\Delta)^{s/2}u\|_2^2 \tag{1.7}$$

for all $u \in \dot{H}^s(\mathbb{R}^N)$(see [35]), and then construct a sup-solution and estimate the energy of multi-peak solutions.

The celebrated paper [26] provides an easy way to understand the nonlocal problem (see [25] for example), by which, one can convert the nonlocal problem (1.1) into a local problem. But we do not use this method in our paper. Indeed, if problem (1.1) becomes a local problem, the vanishing of V and the added variable "$t > 0$" (which comes from extending the problem into $\mathbb{R}^{N+1}_+$, see [25] for instance) will make it difficult to construct precise penalized functions.

The paper is organized as follows: in Section 2, we establish the penalized scheme. By using the compact property of the set consisting of positive radial least energy solutions and the deformation idea in Lemma 2.2 of [32], we construct a $(PS)_c$ sequence with k-peaks in Λ, and then get a penalized multi-peak solution. In Section 3, we construct a penalized function to prove that the penalized solution is indeed a solution of the original equation (1.1). In the Appendix we will give some tedious energy estimates caused by the nonlocal operator.

2. The penalized problem

In this section, we first establish a penalized problem by using the fractional Hardy inequality (1.7) to cut off the nonlinear term f. A well-defined smooth penalized functional in $\mathcal{D}^s_{V,\varepsilon}(\mathbb{R}^N)$ will be obtained. Secondly, we use the compact property of set consisting of least energy solutions and the deformation lemma [32, Lemma 2.2] to construct a $(PS)_c$ sequence near the least energy solutions. A penalized solution with k peaks for the penalized problem will be obtained by passing limit on the $(PS)_c$ sequence.

2.1. The Penalized Functional

The following inequality exposes the relationship between $H^s(\mathbb{R}^N)$ and the Banach space $L^q(\mathbb{R}^N)$.

Proposition 2.1. *(Fractional version of the Gagliardo–Nirenberg inequality) [37] For every $u \in H^s(\mathbb{R}^N)$,*

$$\|u\|_q \leq C\|(-\Delta)^{s/2}u\|_2^{\beta}\|u\|_2^{1-\beta},$$

where $q \in [2, 2_s^*]$ and β satisfies $\frac{\beta}{2_s^*} + \frac{(1-\beta)}{2} = \frac{1}{q}$.

The above inequality implies that $H^s(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$ for $q \in [2, 2_s^*]$. Moreover, on bounded set, the embedding is compact (see [4]), i.e.,

$$H^s(\mathbb{R}^N) \subset\subset L^q_{loc}(\mathbb{R}^N) \text{ compactly, if } q \in [1, 2_s^*).$$

2.2. *The penalized functional*

Now we are going to modify the original problem (1.1). According to the fractional Hardy inequality (1.7), we choose a family of penalized potentials $\mathcal{P}_\varepsilon \in L^\infty(\mathbb{R}^N, [0, \infty))$ for $\varepsilon > 0$ small in such a way that

$$\begin{cases} \mathcal{P}_\varepsilon(x) = 0, \ x \in \Lambda, \\ \lim\limits_{\varepsilon\to 0} \sup_{\mathbb{R}^N\setminus\Lambda} \mathcal{P}_\varepsilon(x)\varepsilon^{-(2s+3\kappa/2)}|x|^{2s+\kappa} = 0, \end{cases} \tag{2.1}$$

where $\kappa > 0$ is the same parameter in (f_1). Noting that by (1.7), when $\varepsilon > 0$ is small enough, it holds that for any $A \subset \mathbb{R}^N$,

$$\int_A \mathcal{P}_\varepsilon(x)|u|^2 \leq C_{N,s}\frac{\varepsilon^{2s+\frac{3\kappa}{2}}}{\inf_{x\in(\mathbb{R}^N\setminus\Lambda)\cap A}|x|^\kappa}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u|^2 \ \text{ for all } u \in \mathcal{D}^s_{V,\varepsilon} \tag{2.2}$$

where $C_{N,s}$ is the constant in (1.7). This type of estimate plays a key role in the paper(see (2.10) below for example).

Now we give the penalized problem according to the choice of $\mathcal{P}_\varepsilon$:

$$\varepsilon^{2s}(-\Delta)^s u + Vu = \chi_\Lambda f(s_+) + \chi_{\mathbb{R}^N\setminus\Lambda}\min\{f(s_+), \mathcal{P}_\varepsilon(x)s_+\}. \tag{2.3}$$

It is easy to check that if a solution u_ε of (2.3) satisfies

$$f(u_\varepsilon) \leq \mathcal{P}_\varepsilon u_\varepsilon \ \text{ on } \mathbb{R}^N\setminus\Lambda,$$

then u_ε is a solution of (1.1).

Given a penalized potential $\mathcal{P}_\varepsilon$ that satisfies (2.1), we define the penalized nonlinearity $g_\varepsilon : \mathbb{R}^N\times\mathbb{R} \to \mathbb{R}$ as

$$g_\varepsilon(x, s) := \chi_\Lambda f(s_+) + \chi_{\mathbb{R}^N\setminus\Lambda}\min\{f(s_+), \mathcal{P}_\varepsilon(x)s_+\}.$$

We denote $G_\varepsilon(x, t) = \int_0^t g_\varepsilon(x, s)ds$.

Accordingly, the penalized superposition operators $\mathfrak{g}_\varepsilon$ and $\mathfrak{G}_\varepsilon$ are given by

$$\mathfrak{g}_\varepsilon(u)(x) = g_\varepsilon(x, u(x)) \text{ and } \mathfrak{G}_\varepsilon(u)(x) = G_\varepsilon(x, u(x)).$$

Following, we define the penalized functional $J_\varepsilon : \mathcal{D}^s_{V,\varepsilon}(\mathbb{R}^N) \to \mathbb{R}$ as

$$J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N}(\varepsilon^{2s}|(-\Delta)^{s/2}u|^2 + V(x)|u|^2) - \int_{\mathbb{R}^N}\mathfrak{G}_\varepsilon(u).$$

The strong assumption (2.1) can help to check that J_ε is C^1 and satisfies (P.S.) condition.

Lemma 2.2. *(1) If* $2 < p < 2_s^*$ *and* (2.1) *hold, then* $J_\varepsilon \in C^1(\mathcal{D}^s_{V,\varepsilon}(\mathbb{R}^N), \mathbb{R})$ *and for* $u \in \mathcal{D}^s_{V,\varepsilon}(\mathbb{R}^N)$, $\varphi \in \mathcal{D}^s_{V,\varepsilon}(\mathbb{R}^N)$,

$$\langle J'_\varepsilon(u), \varphi\rangle = \int_{\mathbb{R}^N} \varepsilon^{2s}(-\Delta)^{s/2}u(-\Delta)^{s/2}\varphi + Vu\varphi - \int_{\mathbb{R}^N} \mathfrak{g}_\varepsilon(u)\varphi.$$

Here $\langle\cdot,\cdot\rangle$ *denotes the duality product between the dual space* $\mathcal{D}^s_{V,\varepsilon}(\mathbb{R}^N)'$ *and the space* $\mathcal{D}^s_{V,\varepsilon}(\mathbb{R}^N)$. *In particular,* $u \in \mathcal{D}^s_{V,\varepsilon}(\mathbb{R}^N)$ *is a critical point of* J_ε *if and only if* u *is a weak solution of the penalized equation*

$$\varepsilon^{2s}(-\Delta)^s u + Vu = \mathfrak{g}_\varepsilon(u). \tag{2.4}$$

(2) ((P.S.) condition) If $2 < p < 2_s^*$ *and* (2.1) *holds, then* J_ε *owns the mountain pass geometry and satisfies the Palais-Smale condition.*

Proof. We omit the proof since it is quite similar to that in [28, Lemma 2.4]. □

2.3. Construction of solutions with k peaks

Definition 2.3. For $a > 0$, we define the value c_a as

$$c_a = \inf_{\gamma\in\Gamma_a} \max_{t\in[0,1]} L_a(\gamma(t)),$$

where $L_a : H^s(\mathbb{R}^N) \to \mathbb{R}$ and Γ_a are given by

$$L_a(u) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dy + \frac{1}{2}\int_{\mathbb{R}^N} a|u|^2 - \int_{\mathbb{R}^N} F(u)$$

and

$$\Gamma_a := \{\gamma \in (C[0,1], H^s(\mathbb{R}^N)) : \gamma(0) = 0,\ L_a(\gamma(1)) < 0\},$$

where $F(t) = \int_0^t f(s)ds$. From [1, 36], we know that c_a is continuous, increasing on a and can be achieved by a positive radial solution U_a which satisfies the following limiting problem

$$(-\Delta)^s u + au = f(u),\ x \in \mathbb{R}^N.$$

Moreover, there exist two positive constants $\tilde{c}_a$, $\widetilde{C}_a$ such that

$$\frac{\tilde{c}_a}{1+|x|^{N+2s}} \le U_a(|x|) \le \frac{\widetilde{C}_a}{1+|x|^{N+2s}},\ \ x \in \mathbb{R}^N. \tag{2.5}$$

Then, letting $S_a = \{U_a : U_a$ is positive radial and achieves $c_a\}$, by the decay estimate (2.5), we have

Proposition 2.4. *The set* S_a *is compact in* $H^s(\mathbb{R}^N)$.

Proof. If S_a contains finitely many elements, then it is compact. Otherwise, taking a sequence $\{U_n\} \subset S_a$, since $\{U_n\}$ is bounded in $H^s(\mathbb{R}^N)$, there exists a $\overline{U} \in H^s(\mathbb{R}^N)$ such that

$$\begin{cases} U_n \rightharpoonup \overline{U} \text{weakly in } H^s(\mathbb{R}^N),\\ U_n \to \overline{U} \text{ a.e. in } \mathbb{R}^N,\\ U_n \to \overline{U} \text{ strongly in } L^q_{loc}(\mathbb{R}^N),\ 1 < q < 2_s^* - 1. \end{cases}$$

Then, by (2.5), we have $U_n \to \overline{U}$ strongly in $L^p(\mathbb{R}^N)$. Obviously, $\overline{U}$ is nonnegative and satisfies

$$(-\Delta)^s\overline{U} + a\overline{U} = f(\overline{U}).$$

Furthermore, by standard regularity argument(see Appendix D in [21] for example), we have $\overline{U} > 0$. Then, by Definition 2.3, we have $\liminf_{n\to\infty} L_a(U_n) \geq L_a(\overline{U}) \geq c_a$. Then $L_a(\overline{U}) = c_a$, $\overline{U} \in S_a$ and

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2}U_n|^2 + a|U_n|^2 \to \int_{\mathbb{R}^N} |(-\Delta)^{s/2}\overline{U}|^2 + a|\overline{U}|^2$$

as $n \to \infty$. This completes the proof. □

From now on we define

$$\mathcal{M}_i = \{x \in \Lambda_i : V(x) = \lambda_i\} \text{ and } \mathcal{M} = \bigcup_{i=1}^{k} \mathcal{M}_i.$$

Let $\eta(x) = \eta(|x|) \in C_c^\infty(\mathbb{R}^N)$ satisfy $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $\overline{B}_\beta(0)$ and $\eta \equiv 0$ on $\mathbb{R}^N \backslash B_{2\beta}(0)$, where $\beta > 0$ is a small parameter satisfying $\mathcal{M}^{2\beta} \subset \Lambda$. For each $p_i \in \mathcal{M}_i$ and $U_{\lambda_i} \in S_{\lambda_i}$ given by Definition 2.3, we define

$$U_\varepsilon^{p_1,\ldots,p_k}(x) = \sum_{i=1}^{k} \eta(x - p_i)U_{\lambda_i}\Big(\frac{x - p_i}{\varepsilon}\Big), \quad x \in \mathbb{R}^N.$$

We will find a solution to (2.4), for sufficiently small $\varepsilon > 0$, near the set

$$X_\varepsilon = \{U_\varepsilon^{p_1,\ldots,p_k} : U_{\lambda_i} \in S_{\lambda_i},\ p_i \in \mathcal{M}_i,\ 1 \leq i \leq k\}.$$

For each $1 \leq i \leq k$, we also define

$$W_\varepsilon^i(x) = \eta(x - p_i)U_{\lambda_i}\Big(\frac{x - p_i}{\varepsilon}\Big).$$

We have:

Proposition 2.5. *For each $i \in \{1, \ldots, k\}$, it holds*

$$J_\varepsilon\Big(\sum_{j=1}^{k} t_j W_\varepsilon^j\Big) < 0$$

if $t_i > T$ for some $T \in (0, +\infty)$.

Proof. By the choice of W_ε^i, there exists a positive constant C such that

$$\begin{aligned} J_\varepsilon\Big(\sum_{i=1}^{k} t_i W_\varepsilon^i\Big) = &\sum_{\substack{i=1,k=1\\ i\neq j}}^{k} \frac{\varepsilon^{2s}}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{t_i t_j (W_\varepsilon^i(x) - W_\varepsilon^i(y))(W_\varepsilon^j(x) - W_\varepsilon^j(y))}{|x - y|^{N+2s}} dxdy \\ &+ \sum_{i=1}^{k}\Big(\frac{t_i^2}{2}\|W_\varepsilon^i\|_\varepsilon^2 - \int_{\mathbb{R}^N} F(t_i W_\varepsilon^i)\Big) \end{aligned}$$

$$\leq \sum_{i=1}^{k}\Big(Ct_i^2\|W_\varepsilon^i\|_\varepsilon^2 - \int_{\mathbb{R}^N} F(t_iW_\varepsilon^i)\Big)$$
$$= \varepsilon^N \sum_{i=1}^{k}\Big(Ct_i^2\|\eta_\varepsilon(x)U_{\lambda_i}(x)\|^2 - \int_{\mathbb{R}^N} F(t_i\eta_\varepsilon(x)U_{\lambda_i}(x))\Big).$$

By decomposition, we have

$$\begin{aligned}
&\|\eta_\varepsilon(x)U_{\lambda_i}(x)\|^2\\
&= \|U_{\lambda_i}(x)\|^2 + \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(\eta_\varepsilon^2(x)-1)|U_{\lambda_i}(x)-U_{\lambda_i}(y)|^2}{|x-y|^{N+2s}}dxdy\\
&\quad + \sum_{i=1}^{k} t_i^2 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\eta_\varepsilon(x)(U_{\lambda_i}(x)-U_{\lambda_i}(y))(\eta_\varepsilon(x)-\eta_\varepsilon(y))U_{\lambda_i}(y)}{|x-y|^{N+2s}}dxdy\\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\eta_\varepsilon(x)-\eta_\varepsilon(y))^2U_{\lambda_i}^2(y)}{|x-y|^{N+2s}}dxdy.
\end{aligned} \tag{2.6}$$

But, arguing as done in the proof of the following (2.23), (2.26) and (2.29) in Lemma A.2, we know that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\eta_\varepsilon(x)-\eta_\varepsilon(y))^2U_{\lambda_i}^2(y)}{|x-y|^{N+2s}}dxdy = o_\varepsilon(1).$$

Hence

$$J_\varepsilon(\sum_{i=1}^{k} t_iW_\varepsilon^i) \leq \varepsilon^N \sum_{i=1}^{k}\Big(Ct_i^2\|U_{\lambda_i}(x)\|^2 - \int_{B_1(0)} F(t_iU_{\lambda_i}(x))\Big).$$

Then, by the assumption on f and $\max\limits_{\substack{t>0\\1\leq i\leq k}} (Ct_i^2\|U_{\lambda_i}(x)\|^2 - \int_{B_1(0)} F(t_iU_{\lambda_i}(x))) < +\infty$, we get the conclusion.

□

As a result of Proposition 2.5, we know that the following definition is reasonable: for $\tau = (t_1,\dots,t_k) \in [0,T]^k$, let $\gamma_\varepsilon(\tau) = \sum_{i=1}^k t_iW_\varepsilon^i$ and define

$$\mathcal{D}_\varepsilon = \max_{\tau\in[0,T]^k} J_\varepsilon(\gamma_\varepsilon(\tau)).$$

We have the following estimate for $\mathcal{D}_\varepsilon$.

Proposition 2.6. *(i)* $\lim\limits_{\varepsilon\to 0} \frac{\mathcal{D}_\varepsilon}{\varepsilon^N} = \sum_{i=1}^k c_{\lambda_i}$.

(ii) $\limsup\limits_{\varepsilon\to 0} \frac{\max_{\tau\in\partial[0,T]^k} J_\varepsilon(\gamma_\varepsilon(\tau))}{\varepsilon^N} \leq \sum_{i=1}^k c_{\lambda_i} - \min\limits_{1\leq i\leq k} c_{\lambda_i}$.

(iii) For each $\delta > 0$, *there exists* $\alpha > 0$ *such that for sufficiently small* $\varepsilon > 0$,

$$\frac{J_\varepsilon(\gamma_\varepsilon(\tau))}{\varepsilon^N} \geq \frac{\mathcal{D}_\varepsilon}{\varepsilon^N} - \alpha$$

implies that $\gamma_\varepsilon(\tau) \in \mathcal{X}_\varepsilon^{\frac{\delta\varepsilon^{N/2}}{2}}$.

Proof. By the decay rates of U_{λ_i} and the analysis of (2.6), we have

$$
\begin{aligned}
J_\varepsilon(\gamma_\varepsilon(\tau))/\varepsilon^N &= \sum_{i=1}^{k} L_{\lambda_i}(t_i U_{\lambda_i}) + o_\varepsilon(1) \\
&+ \sum_{1\le i\neq j\le k} \frac{t_i t_j}{2} \int_{\mathbb{R}^N\times\mathbb{R}^N} |x-y|^{-N-2s}(\eta_\varepsilon(x)U_{\lambda_i}(x) - \eta_\varepsilon(y)U_{\lambda_i}(y)) \\
&\Big(\eta(\varepsilon x + p_i - p_j)U_{\lambda_j}(x + \frac{p_i - p_j}{\varepsilon}) - \eta(\varepsilon y + p_i - p_j)U_{\lambda_j}(y + \frac{p_i - p_j}{\varepsilon})\Big)dxdy \\
&+ \sum_{i=1}^{k} \frac{t_i^2}{2} \int_{\mathbb{R}^N} (\eta_\varepsilon^2(x)V(\varepsilon x + p_i) - \lambda_i)U_{\lambda_i}^2(x)dx \\
&+ \sum_{i=1}^{k} \int_{\mathbb{R}^N} \Big(F(t_i U_{\lambda_i}(x)) - F(t_i \eta_\varepsilon(x)U_{\lambda_i}(x))\Big),
\end{aligned}
$$

where $\eta_\varepsilon(x) = \eta(\varepsilon x)$. Choosing $\varepsilon > 0$ be small enough such that $supp\eta_\varepsilon \cap supp\eta_\varepsilon(\,\cdot + \frac{p_i - p_j}{\varepsilon}) = \emptyset$, we have

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^N\times\mathbb{R}^N} |x-y|^{-N-2s}(\eta_\varepsilon(x)U_{\lambda_i}(x) - \eta_\varepsilon(y)U_{\lambda_i}(y)) \\
&\quad \Big(\eta(\varepsilon x + p_i - p_j)U_{\lambda_j}(x + \frac{p_i - p_j}{\varepsilon}) - \eta(\varepsilon y + p_i - p_j)U_{\lambda_j}(y + \frac{p_i - p_j}{\varepsilon})\Big)dxdy\Big| \\
&= 2\int_{B_{\frac{2\beta}{\varepsilon}}(0)} dx \int_{B_{\frac{2\beta}{\varepsilon}}\left(\frac{p_j - p_i}{\varepsilon}\right)} \frac{\eta_\varepsilon(x)\eta_\varepsilon(y + \frac{p_i - p_j}{\varepsilon})U_{\lambda_i}(x)U_{\lambda_j}\Big(y + \frac{p_i - p_j}{\varepsilon}\Big)}{|x-y|^{N+2s}} dy \\
&\le C\Big(\frac{\min\limits_{\substack{i\neq j\\ 1\le i,j\le k}} (|p_i - p_j| - 4\beta)}{\varepsilon}\Big)^{-N-2s} \\
&= o_\varepsilon(1).
\end{aligned}
$$

Then by the fact that $p_i \in \mathcal{M}_i$ and $t_i \le T$, $1 \le i \le k$, we have

$$
\frac{J_\varepsilon(\gamma_\varepsilon(\tau))}{\varepsilon^N} = \sum_{i=1}^{k} L_{\lambda_i}(t_i U_{\lambda_i}) + o_\varepsilon(1). \tag{2.7}
$$

Hence we get (i) and obviously (ii) is true.

Finally, (2.7) implies that if $\tau_\varepsilon \in [0,T]^k$ satisfies $\lim\limits_{\varepsilon\to 0}\Big(\frac{J_\varepsilon\big(\gamma_\varepsilon(\tau_\varepsilon)\big)}{\varepsilon^N} - \frac{\mathcal{D}_\varepsilon}{\varepsilon^N}\Big) = 0$, then it must hold

$$
\lim_{\varepsilon\to 0} \tau_\varepsilon = (1, \dots, 1),
$$

which implies (iii).

Consequently, we complete the proof. □

Next, we define

$$
C_\varepsilon = \inf_{\psi\in\Psi_\varepsilon} \max_{\tau\in[0,T]^k} J_\varepsilon(\psi(\tau)),
$$

where

$$\Psi_\varepsilon := \{\psi_\varepsilon \in C(([0,T]^k, \mathcal{D}^s_{V,\varepsilon}(\mathbb{R}^N) \cap \mathcal{X}_\varepsilon^{\nu\varepsilon^{N/2}})|\psi_\varepsilon(\tau) = \gamma_\varepsilon(\tau) \text{ for } \tau \in \partial[0,T]^k\}, \tag{2.8}$$

where $\nu > 0$ is large positive constant. Obviously, Ψ_ε is nonempty since $\gamma_\varepsilon \in \Psi_\varepsilon$. We now prove the following property of C_ε.

Lemma 2.7.

$$\lim_{\varepsilon\to 0} \frac{C_\varepsilon}{\varepsilon^N} = \sum_{j=1}^k c_{\lambda_j}.$$

The proof will rely on the following lemma, whose proof, for the sake of continuity, is postponed to the appendix. We define for every $i \in \{1,\dots,k\}$, the functional $J^i_\varepsilon : W^{s,2}(S_i) \to \mathbb{R}$ as

$$J^i_\varepsilon(u) = \frac{\varepsilon^{2s}}{2}\int_{S_i}\int_{S_i} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dy + \frac{1}{2}\int_{S_i} V(x)|u|^2 - \int_{S_i} \mathfrak{G}_\varepsilon(u).$$

We have

Lemma 2.8. *The mountain pass value*

$$c^i_\varepsilon := \inf_{\gamma^i_\varepsilon\in\Gamma^i_\varepsilon} \max_{t\in[0,1]} J^i_\varepsilon(\gamma^i_\varepsilon(t)), \ \ i \in \{1,\dots,k\}$$

can be achieved, where

$$\Gamma^i_\varepsilon := \{\gamma^i_\varepsilon \in (C[0,1], W^{s,2}(S_i)) : \gamma^i_\varepsilon(0) = 0, \ J^i_\varepsilon(\gamma^i_\varepsilon(1)) < 0\}.$$

Moreover,

$$\lim_{\varepsilon\to 0} \frac{c^i_\varepsilon}{\varepsilon^N} = c_{\lambda_i}. \tag{2.9}$$

Now we prove Lemma 2.7:

Proof of Lemma 2.7. By Proposition (2.6), we have the upper bounds

$$\limsup_{\varepsilon\to 0} \frac{C_\varepsilon}{\varepsilon^N} \le \sum_{j=1}^k c_{\lambda_j}.$$

It remains to prove the lower estimate, i.e.,

$$\liminf_{\varepsilon\to 0} \frac{C_\varepsilon}{\varepsilon^N} \ge \sum_{j=1}^k c_{\lambda_j}.$$

We first observe that given any $\psi_\varepsilon \in \Psi_\varepsilon$ and any continuous curve $c : [0,1] \to [0,T]^k$ with $c(0) \in \{0\} \times [0,T]^{k-1}$ and $c(1) \in \{T\} \times [0,T]^{k-1}$, we have $\gamma^1_\varepsilon = \psi_\varepsilon \circ c|_{S_1} \in \Gamma^1_\varepsilon$. In fact, by the definition of Ψ_ε, we have

$$\gamma^1_\varepsilon(0) = 0, \ J^1_\varepsilon(\gamma^1_\varepsilon(1)) \le J_\varepsilon(TW^1_\varepsilon + 0 \cdot \sum_{i=2}^k W^i_\varepsilon) < 0.$$

Lemma 2.8 implies that

$$\sup_{t\in[0,1]} J_\varepsilon^1(\gamma_\varepsilon^1(t)) \geq \varepsilon^N(c_{\lambda_1} + o_\varepsilon(1)).$$

Similarly, for every $\gamma_\varepsilon^j = \gamma \circ c|_{S_j}$ belongs to Γ_ε^j, where c is arbitrary continuous path which joint $[0,T]^{j-1} \times \{0\} \times [0,T]^{k-j}$ with $[0,T]^{j-1} \times \{T\} \times [0,T]^{k-j}$, it holds

$$\sup_{t\in[0,1]} J_\varepsilon^j(\gamma_\varepsilon^j(t)) \geq \varepsilon^N(c_{\lambda_j} + o_\varepsilon(1)).$$

Thus we can repeat the argument of Coti-Zetali and Rabinowitz in [34] to prove, for every path $\psi_\varepsilon \in \Gamma$, the existence of a point $\hat{\tau} \in [0,1]^k$ satisfying

$$J_\varepsilon^i(\psi_\varepsilon(\hat{\tau})) \geq \varepsilon^N(c_{\lambda_i} + o_\varepsilon(1)) \text{ for } j = 1, \ldots, k.$$

Consequently, by (2.1), (2.2) and the fact that $\psi_\varepsilon(\tau) \in \mathcal{X}_\varepsilon^{\gamma\varepsilon^{N/2}}$, we get

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^N} \sup_{\tau\in[0,1]^k} J_\varepsilon(\psi_\varepsilon(\tau)) \\
\geq\ &\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^N} J_\varepsilon(\psi_\varepsilon(\hat{\tau})) \\
\geq\ &\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^N}\Big(\sum_{i=1}^k J_\varepsilon^i(\psi_\varepsilon(\hat{\tau})) - \varepsilon^{\kappa+2s} \int_{\mathbb{R}^N} |(-\Delta)^{s/2}\psi_\varepsilon(\hat{\tau})|^2 dx\Big) \\
\geq\ &\sum_{i=1}^k c_{\lambda_i},
\end{aligned} \tag{2.10}$$

which is exactly the required lower estimate. □

Next, we are going to construct a penalized solution for the penalized problem (2.3). We first prove that the limit of a $(PS)_c$ sequence near the set $\mathcal{X}_\varepsilon$ must own k-peaks.

Proposition 2.9. *Let* $\{\varepsilon_j\}_j$ *with* $\lim_{j\to\infty} \varepsilon_j = 0$ *and* $\{u_{\varepsilon_j}\} \subset \mathcal{X}_{\varepsilon_j}^{d\varepsilon_j^{N/2}}$ *satisfy*

$$\lim_{j\to\infty} \frac{J_{\varepsilon_j}(u_{\varepsilon_j})}{\varepsilon_j^N} \leq \sum_{i=1}^k c_{\lambda_i}, \ \lim_{j\to\infty} \frac{\|J'_{\varepsilon_j}(u_{\varepsilon_j})\|}{\varepsilon_j^{N/2}} = 0.$$

Then for sufficiently small $d > 0$, *there exist, up to subsequence,* $\{x_j^i\}_j \subset \mathbb{R}^3$, $i = 1, \ldots, k$, $x_i \in \mathcal{M}_i$, $\overline{U}_{\lambda_i} \in S_{\lambda_i}$ *such that*

$$\lim_{j\to\infty} x_j^i = x_i \tag{2.11}$$

and

$$\lim_{j\to\infty} \Big\| u_{\varepsilon_j}(\cdot) - \sum_{i=1}^k \eta(\cdot - x_j^i)\overline{U}_{\lambda_i}\Big(\frac{\cdot - x_j^i}{\varepsilon_j}\Big)\Big\|_{\mathcal{D}^s_{V,\varepsilon_j}} / \varepsilon_j^{N/2} = 0. \tag{2.12}$$

Proof. For the sake of convenience, we write ε for ε_j. Since S_{λ_i}, $i=1,\dots,k$ are compact in $H^s(\mathbb{R}^N)$, there exist $U_{\lambda_i}\in S_{\lambda_i}$ and $p_\varepsilon^i\in\mathcal{M}_i$ such that

$$\Big\|u_\varepsilon(x)-\sum_{i=1}^k\eta(x-p_\varepsilon^i)U_{\lambda_i}\Big(\frac{x-p_\varepsilon^i}{\varepsilon}\Big)\Big\|_{\mathcal{D}^s_{V,\varepsilon}}\le 2d\varepsilon^{N/2}.$$

Letting $R_0\ge 1$ be a fixed positive constant and $\varepsilon R_0\le\beta$, for each $i=1,\dots,k$, we have

$$\int_{B_{R_0}}|u_\varepsilon(\varepsilon x+p_\varepsilon^i)-U_{\lambda_i}(x)|^2\le\frac{4d^2}{\lambda_i}.$$

As a result, we can let $d>0$ be small enough so that

$$\liminf_{\varepsilon\to 0}\int_{B_{R_0}}|u_\varepsilon(\varepsilon x+p_\varepsilon^i)|^2>0 \text{ and } \liminf_{\varepsilon\to 0}\|u_\varepsilon\|_{L^\infty(B_{\varepsilon R_0}(p_\varepsilon^i))}>0, \tag{2.13}$$

for all $1\le i\le k$.

Denote $u_\varepsilon^{1,i}(x)=\eta(x-p_\varepsilon^i)u_\varepsilon(x)$, $u_\varepsilon^1(x)=\sum_{i=1}^k u_\varepsilon^{1,i}(x)$ and $u_\varepsilon^2(x)=u_\varepsilon(x)-u_\varepsilon^1(x)$. Denote $v_\varepsilon^{1,i}(x)=u_\varepsilon^{1,i}(\varepsilon x+p_\varepsilon^i)$ and $v_\varepsilon^{2,i}(x)=v_\varepsilon^i(x)-v_\varepsilon^{1,i}(x)$, where $v_\varepsilon^i(x)=u_\varepsilon(\varepsilon x+p_\varepsilon^i)$. Fix arbitrarily an $i\in\{1,\dots,k\}$. Obviously, by assumption, for each $\varphi\in C_c^\infty(\mathbb{R}^N)$ and ε small enough, testing $J'_\varepsilon(u_\varepsilon)$ with $\varphi\big(\frac{x-p_\varepsilon^i}{\varepsilon}\big)$, we find

$$o_\varepsilon(1)=\int_{\mathbb{R}^N}((-\Delta)^s v_\varepsilon^{1,i})\varphi+V_\varepsilon^i(x)v_\varepsilon^{1,i}\varphi-g_\varepsilon(\varepsilon x+p_\varepsilon^i,v_\varepsilon^{1,i})\varphi+\int_{\mathbb{R}^N}((-\Delta)^s v_\varepsilon^{2,i})\varphi. \tag{2.14}$$

Since $\{u_\varepsilon\}\subset\mathcal{X}_\varepsilon^{d\varepsilon^{N/2}}$, by fractional Hardy inequality (1.7), we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N}((-\Delta)^s v_\varepsilon^{2,i})\varphi\Big|\\
&=\Big|\int_{supp\varphi}dx\int_{B^c_{\beta/\varepsilon}(0)}\frac{\varphi(x)v_\varepsilon^{2,i}(y)}{|x-y|^{N+2s}}dy\Big|\\
&\le\int_{supp\varphi}(\varphi(x))^2dx\int_{B^c_{\beta/\varepsilon}(0)}\frac{1}{|x-y|^{N+2s}}dy\\
&\quad+\int_{supp\varphi}dx\int_{B^c_{\beta/\varepsilon}(0)}\frac{(v_\varepsilon^{2,i}(y))^2}{|x-y|^{N+2s}}dy\\
&=o_\varepsilon(1)+\int_{supp\varphi}dx\int_{B^c_{\beta/\varepsilon}(0)}\frac{(v_\varepsilon^{2,i}(y))^2}{|y|^{2s}}\frac{|y|^{2s}}{|x-y|^{N+2s}}dy\\
&=o_\varepsilon(1).
\end{aligned}\tag{2.15}$$

Then, since $\{v_\varepsilon^{1,i}\}$ is bounded in $H^s(\mathbb{R}^N)$, by the Liouville type Theorem 3.3 of [27], we have

$$(-\Delta)^s v_*^{1,i}+V(p_*^i)v_*^{1,i}=f((v_*^{1,i})_+)\ \text{ in }\mathbb{R}^N, \tag{2.16}$$

where $v_*^{1,i}$ is the weak limit of some subsequence of $v_\varepsilon^{1,i}$ in $H^s(\mathbb{R}^N)$ and $p_*^i\in\mathcal{M}_i$ is limit of p_ε^i. Consequently, according to the argument of Proposition 3.4 in [28], we have for every $R>0$ that

$$\liminf_{\varepsilon\to 0}\frac{J_\varepsilon(u_\varepsilon)}{\varepsilon^N}$$

$$= o_R(1) + \liminf_{\varepsilon\to 0} \sum_{i=1}^{k} \int_{B_{\varepsilon R}(p_\varepsilon^i)} \Big(\frac{1}{2}(|(-\Delta)^{s/2}u_\varepsilon|^2 + V(x)|u_\varepsilon(x)|^2) - \mathfrak{G}_\varepsilon(u_\varepsilon)\Big)/\varepsilon^N$$

$$\geq \sum_{i=1}^{k} L_{V(p_*^i)}(v_*^{1,i}) + o_R(1) \geq \sum_{i=1}^{k} c_{\lambda_i} + o_R(1). \tag{2.17}$$

Consequently, by Lemma 2.8, we have $\lambda_i = V(p_*^i)$, $p_*^i \in \mathcal{M}_i$ and $v_*^{1,i}(\cdot + z_i) \in S_{\lambda_i}$ for some $z_i \in \mathbb{R}^N$. Denote

$$v_*^{1,i}(\cdot + z_i) = \overline{U}_{\lambda_i}.$$

In the following we show that

$$v_\varepsilon^{1,i}(\cdot) \to \overline{U}_{\lambda_i}(\cdot - z_i) \text{ strongly in } H^s(\mathbb{R}^N). \tag{2.18}$$

By the same argument of Lemma 3.4 in [28], we can conclude that

$$\lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \|u_\varepsilon\|_{L^\infty\left(U\setminus\bigcup_{i=1}^k B_{R\varepsilon}(p_\varepsilon^i)\right)} = 0 \tag{2.19}$$

and for any $r > 0$, $y_\varepsilon \in \mathbb{R}^N$ with $\lim_{\varepsilon\to 0} \frac{|p_\varepsilon^i - y_\varepsilon|}{\varepsilon} = +\infty$, it holds

$$\limsup_{\varepsilon\to 0} \int_{B_r(y_\varepsilon)} |v_\varepsilon^{1,i}|^2 = 0. \tag{2.20}$$

Then according to Proposition 2.1 and the Concentration-Compactness Lemma 1.21 of [32], we have

$$v_\varepsilon^{1,i} \to v_*^{1,i} \text{ strongly in } L^q(\mathbb{R}^N),\ 2 < q < 2_s^* - 1. \tag{2.21}$$

By decomposition, one find

$$\begin{aligned} J_\varepsilon(u_\varepsilon) = J_\varepsilon(u_\varepsilon^1) &+ \frac{\varepsilon^{2s}}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u_\varepsilon^2|^2 \\ &+ \varepsilon^{2s}\int_{\mathbb{R}^N} dx \int_{R^N} \frac{(u_\varepsilon^1(x) - u_\varepsilon^1(y))(u_\varepsilon^2(x) - u_\varepsilon^2(y))}{|x-y|^{N+2s}} dy \\ &+ \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u_\varepsilon^2|^2 + \int_{\mathbb{R}^N} V(x)u_\varepsilon^1 u_\varepsilon^2 + \int_{\mathbb{R}^N} G_\varepsilon(u_\varepsilon^1) - \int_{\mathbb{R}^N} \mathcal{G}_\varepsilon(u_\varepsilon). \end{aligned}$$

But, with (2.19) at hand, we can use the same method in the proof of (2.24)(which needs only (2.25)) to show that

$$\frac{\varepsilon^{2s}}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u_\varepsilon^2|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u_\varepsilon^2|^2 = \varepsilon^N o_\varepsilon(1), \tag{2.22}$$

which and (2.2) imply that

$$J_\varepsilon(u_\varepsilon) = J_\varepsilon(u_\varepsilon^1) + \int_{\mathbb{R}^N} F(u_\varepsilon^1) - \int_\Lambda F(u_\varepsilon) + \varepsilon^N o_\varepsilon(1).$$

From (2.19), we have

$$\Big|\int_{\mathbb{R}^N} F(u_\varepsilon^1) - \int_\Lambda F(u_\varepsilon)\Big| \leq \|u_\varepsilon\|^{\tilde\kappa}_{L^\infty\left(\Lambda\setminus\bigcup_{i=1}^k B_\beta(p_\varepsilon^i)\right)} \int_{\Lambda\setminus\bigcup_{i=1}^k B_\beta(p_\varepsilon^i)} |u_\varepsilon|^2 = \varepsilon^N o_\varepsilon(1).$$

Hence, by the analysis above, we have

$$J_\varepsilon(u_\varepsilon) = J_\varepsilon(u_\varepsilon^1) + \varepsilon^N o_\varepsilon(1).$$

Decomposing again, we find

$$\frac{J_\varepsilon(u_\varepsilon)}{\varepsilon^N} = \sum_{i=1}^{k} J_\varepsilon(v_\varepsilon^{1,i}) + \varepsilon^{-N} T_\varepsilon^1(\tilde{\eta}_\varepsilon) + o_\varepsilon(1),$$

where

$$T_\varepsilon^1(\tilde{\eta}_\varepsilon) := \varepsilon^{2s} \sum_{1\le i\ne j\le k} \int_{\mathbb{R}^N} \frac{(u_\varepsilon^{1,i}(x) - u_\varepsilon^{1,i}(y))(u_\varepsilon^{1,j}(x) - u_\varepsilon^{1,j}(y))}{|x-y|^{N+2s}} dy.$$

But, it has been proved in Appendix that

$$T_\varepsilon^1(\tilde{\eta}_\varepsilon) := \varepsilon^N o_\varepsilon(1). \tag{2.23}$$

Hence, it holds

$$\frac{J_\varepsilon(u_\varepsilon)}{\varepsilon^N} = \sum_{i=1}^{k} J_\varepsilon(v_\varepsilon^{1,i}) + o_\varepsilon(1).$$

So

$$\lim_{\varepsilon\to 0} \sum_{i=1}^{k} J_\varepsilon(v_\varepsilon^{1,i}) = \sum_{i=1}^{k} c_{\lambda_i},$$

which combining with the analysis of (2.17) yields

$$\lim_{\varepsilon\to 0} J_\varepsilon(v_\varepsilon^{1,i}) = c_{\lambda_i},\ i = 1, \dots, k.$$

Consequently, by (2.21), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |(-\Delta)^{s/2} U_{\lambda_i}(\cdot - z_i)|^2 + \lambda_i |U_{\lambda_i}(\cdot - z_i)|^2 \\
\ge& \limsup_{\varepsilon\to 0} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} v_\varepsilon^{1,i}|^2 + V_\varepsilon^i(x)|v_\varepsilon^{1,i}|^2 \\
\ge& \limsup_{\varepsilon\to 0} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} v_\varepsilon^{1,i}|^2 + \lambda_i|v_\varepsilon^{1,i}|^2 \\
\ge& \int_{\mathbb{R}^N} |(-\Delta)^{s/2} U_{\lambda_i}(\cdot - z_i)|^2 + \lambda_i |U_{\lambda_i}(\cdot - z_i)|^2,
\end{aligned}$$

which gives (2.18).

Now from (2.18), we have

$$\varepsilon^{-N} \Big\| u_\varepsilon - \sum_{i=1}^{k} \eta(x - p_\varepsilon^i - \varepsilon z_i) U_{\lambda_i}\Big(\frac{x - p_\varepsilon^i - \varepsilon z_i}{\varepsilon}\Big)\Big\|_{\mathcal{D}_{V,\varepsilon}^s}^2$$

$$
\begin{aligned}
&\leq 2\varepsilon^{-N}\Big\|\sum_{i=1}^{k}\eta(x-p_\varepsilon^i-\varepsilon z_i)\Big(u_\varepsilon-U_{\lambda_i}\Big(\frac{x-p_\varepsilon^i-\varepsilon z_i}{\varepsilon}\Big)\Big)\Big\|^2_{\mathcal{D}^s_{V,\varepsilon}}\\
&\quad+2\varepsilon^{-N}\Big\|u_\varepsilon-\sum_{i=1}^{k}\eta(x-p_\varepsilon^i-\varepsilon z_i)u_\varepsilon\Big\|^2_{\mathcal{D}^s_{V,\varepsilon}}\\
&\leq 2k\varepsilon^{-N}\sum_{i=1}^{k}\Big\|\eta(x-p_\varepsilon^i-\varepsilon z_i)\Big(u_\varepsilon-U_{\lambda_i}\Big(\frac{x-p_\varepsilon^i-\varepsilon z_i}{\varepsilon}\Big)\Big)\Big\|^2_{\mathcal{D}^s_{V,\varepsilon}}\\
&\quad+2\varepsilon^{-N}\Big\|u_\varepsilon-\sum_{i=1}^{k}\eta(x-p_\varepsilon^i-\varepsilon z_i)u_\varepsilon\Big\|^2_{\mathcal{D}^s_{V,\varepsilon}}\\
&:= o_\varepsilon(1)+I_\varepsilon.
\end{aligned}
$$

It remains to show that

$$I_\varepsilon = o_\varepsilon(1). \tag{2.24}$$

By the same blow-up analysis of lemmas 3.3 and 3.4 in [28], it holds

$$\lim_{\substack{\varepsilon\to 0\\ R\to\infty}}\|u_\varepsilon\|_{L^\infty\left(U\setminus\bigcup_{i=1}^k B_{R\varepsilon}(p_\varepsilon^i+\varepsilon z_i)\right)}=0. \tag{2.25}$$

Consequently, denoting $\tilde\eta_\varepsilon = 1-\sum_{i=1}^k\eta(2(x-p_\varepsilon^i-\varepsilon z_i))$ and testing $J_\varepsilon'(u_\varepsilon)$ against with $\bar\eta_\varepsilon u_\varepsilon$, we have, for $\varepsilon>0$ small enough,

$$
\begin{aligned}
\tilde I_\varepsilon :=\ & \frac{\varepsilon^{2s}}{2}\int_{\mathbb{R}^N}\tilde\eta_\varepsilon(x)dx\int_{\mathbb{R}^N}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{N+2s}}dy+\int_{\mathbb{R}^N}V(x)|\tilde\eta_\varepsilon(x)||u_\varepsilon|^2dx\\
\leq\ & \int_{\mathbb{R}^N}g_\varepsilon(u_\varepsilon)\tilde\eta_\varepsilon u_\varepsilon+\frac{\varepsilon^{2s}}{2}\int_{\mathbb{R}^N}dx\int_{\mathbb{R}^N}\frac{(u_\varepsilon(x)-u_\varepsilon(y))(\tilde\eta_\varepsilon(y)-\tilde\eta_\varepsilon(x))u_\varepsilon(y)}{|x-y|^{N+2s}}dy\\
&+o_\varepsilon(1)\varepsilon^{N/2}\|\tilde\eta_\varepsilon u_\varepsilon\|_{\mathcal{D}^s_{V,\varepsilon}}\\
:=\ & \int_{\mathbb{R}^N}g_\varepsilon(u_\varepsilon)\tilde\eta_\varepsilon u_\varepsilon+T_\varepsilon^2(\tilde\eta)+o_\varepsilon(1)\varepsilon^{N/2}\|\tilde\eta_\varepsilon u_\varepsilon\|_{\mathcal{D}^s_{V,\varepsilon}}\\
\leq\ & \|u_\varepsilon\|^{\tilde\kappa}_{L^\infty\left(\Lambda\setminus\bigcup_{i=1}^k B_{R\varepsilon}(p_\varepsilon^i+\varepsilon z_i)\right)}\int_{\mathbb{R}^N}V(x)\tilde\eta_\varepsilon(x)|u_\varepsilon|^2dx+\int_{\mathbb{R}^N\setminus\Lambda}\mathcal{P}_\varepsilon|u_\varepsilon|^2\\
&+T_\varepsilon^2(\tilde\eta_\varepsilon)+o_\varepsilon(1)\varepsilon^{N/2}\|\tilde\eta_\varepsilon u_\varepsilon\|_{\mathcal{D}^s_{V,\varepsilon}},
\end{aligned}
$$

which implies

$$\tilde I_\varepsilon\leq C\Big(\int_{\mathbb{R}^N\setminus\Lambda}\mathcal{P}_\varepsilon|u_\varepsilon|^2+T_\varepsilon^2(\tilde\eta_\varepsilon)\Big)+o_\varepsilon(1)\varepsilon^{N/2}\|\tilde\eta_\varepsilon u_\varepsilon\|_{\mathcal{D}^s_{V,\varepsilon}}.$$

However, we have proved in the Appendix that

$$\limsup_{\varepsilon\to 0}\frac{T_\varepsilon^2(\tilde\eta_\varepsilon)}{\varepsilon^N}\leq 0 \tag{2.26}$$

and

$$\|\tilde{\eta}_\varepsilon u_\varepsilon\|_{\mathcal{D}^s_{V,\varepsilon}} \le C\varepsilon^{N/2}. \tag{2.27}$$

Hence, by the choice of $\mathcal{P}_\varepsilon$ and fractional Hardy inequality (1.7), it holds

$$\lim_{\varepsilon\to 0} \frac{\tilde{I}_\varepsilon}{\varepsilon^N} = 0. \tag{2.28}$$

Noting the following estimate proved in the Appendix

$$T^3_\varepsilon(\breve{\eta}_\varepsilon) = \varepsilon^{2s} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\breve{\eta}_\varepsilon(x)u_\varepsilon(x) - \breve{\eta}_\varepsilon(y)u_\varepsilon(y)|^2}{|x-y|^{N+2s}} dxdy = \varepsilon^N o_\varepsilon(1), \tag{2.29}$$

where $\breve{\eta}_\varepsilon(x) = 1 - \sum_{i=1}^k \eta(x - p^i_\varepsilon - \varepsilon z_i)$, we find

$$I_\varepsilon \le \frac{T^3_\varepsilon(\breve{\eta}_\varepsilon)}{\varepsilon^N} + \frac{\tilde{I}_\varepsilon}{\varepsilon^N} = o_\varepsilon(1),$$

which is exactly (2.24). Letting $x^i_\varepsilon = p^i_\varepsilon + \varepsilon z_i$, we get

$$\lim_{\varepsilon\to 0} \varepsilon^{-N} \Big\| u_\varepsilon - \sum_{i=1}^k \eta(x - x^i_\varepsilon) U_{\lambda_i}\Big(\frac{x - x^i_\varepsilon}{\varepsilon}\Big)\Big\|^2_{\mathcal{D}^s_{V,\varepsilon}} = 0.$$

Hence we complete the proof.

□

Proposition 2.10. *For $d > 0$ sufficiently small, there exist constants $\sigma > 0$ and $\varepsilon_0 > 0$, such that*

$$\|J'_\varepsilon(u)\|_{\mathcal{D}^s_{V,\varepsilon}(\mathbb{R}^N)} \ge \varepsilon^{N/2}\sigma \text{ for } J^{\mathcal{D}_\varepsilon}_\varepsilon \cap (\mathcal{X}^{d\varepsilon^{N/2}}_\varepsilon \backslash \mathcal{X}^{d\varepsilon^{N/2}/2}_\varepsilon) \text{ and } \varepsilon \in (0, \varepsilon_0),$$

where $J^{\mathcal{D}_\varepsilon}_\varepsilon = \{u \in \mathcal{D}^s_{V,\varepsilon}(\mathbb{R}^N) : J_\varepsilon(u) \le \mathcal{D}_\varepsilon\}$.

Proof. To the contrary, suppose that for small $d_1 > d_2 > 0$, there exist $\{\varepsilon_j\}_{j=1}^\infty$ with $\lim_{j\to\infty} \varepsilon_j = 0$ and $u_{\varepsilon_j} \in \mathcal{X}^{d_1\varepsilon_j^{N/2}}_{\varepsilon_j} \backslash \mathcal{X}^{d_2\varepsilon_j^{N/2}}_{\varepsilon_j}$ satisfying $\lim_{j\to\infty} J_{\varepsilon_j}(u_{\varepsilon_j})/\varepsilon_j^N \le \sum_{i=1}^k c_{\lambda_i}$ and $\lim_{j\to\infty} \frac{J'_{\varepsilon_j}(u_{\varepsilon_j})}{\varepsilon_j^{N/2}} = 0$. By Proposition 2.9, there exists $\{x^i_j\}_{j=1}^\infty \subset \mathbb{R}^N$, $i = 1, \ldots, k$, $x_i \in \mathcal{M}_i$, such that

$$\lim_{j\to\infty} |x^i_j - x_i| = 0 \text{ and } \lim_{j\to\infty} \Big\| u_{\varepsilon_j}(\cdot) - \sum_{i=1}^k \eta(\cdot - x^i_j) U_{\lambda_i}\Big(\frac{\cdot - x^i_j}{\varepsilon_j}\Big)\Big\|_{\mathcal{D}^s_{V,\varepsilon_j}} /\varepsilon_j^{N/2} = 0.$$

Hence, by the definition of $\mathcal{X}_\varepsilon$, we see that $\lim_{j\to\infty} \mathrm{dist}(u_{\varepsilon_j}, \mathcal{X}_{\varepsilon_j})/\varepsilon_j^{N/2} = 0$. This is a contradiction to $u_{\varepsilon_j} \notin \mathcal{X}^{d_2\varepsilon_j^{N/2}/2}_{\varepsilon_j}$. □

Now, we use Proposition 2.10 and the Deformation Lemma 2.2 in [32] to construct a $(PS)_c$ sequence near the set $\mathcal{X}_\varepsilon$.

Define

$$\mu := \varepsilon^{-N} \inf_{u\in\mathcal{X}_\varepsilon} \{\|u\|_{\varepsilon,S_i}, i = 1, \ldots, k\}.$$

Fix $d_0 \in (0, \frac{\mu}{2})$ such that Propositions 2.9 and 2.10 hold for $d \in (0, d_0]$.

Proposition 2.11. *For sufficiently small fixed $\varepsilon > 0$, there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset J_\varepsilon^{\mathcal{D}_\varepsilon} \cap \mathcal{X}_\varepsilon^{d\varepsilon^{N/2}}$ such that $J'_\varepsilon(u_n) \to 0$ as $n \to \infty$.*

Proof. By Proposition 2.10, there exists a constant $\sigma \in (0, 1)$, such that

$$\|J'_\varepsilon(u)\|_{\mathcal{D}^s_{V,\varepsilon}(\mathbb{R}^N)} \geq \varepsilon^{N/2}\sigma \text{ for } u \in J_\varepsilon^{\mathcal{D}_\varepsilon} \cap (\mathcal{X}_\varepsilon^{d\varepsilon^{N/2}} \backslash \mathcal{X}_\varepsilon^{d\varepsilon^{N/2}/2}) \text{ and } \varepsilon \in (0, \varepsilon_0).$$

From Proposition 2.6(iii), there exist constants $\alpha > 0$, $\varepsilon_1(\alpha) > 0$ such that for $\varepsilon \in (0, \varepsilon_1]$ and $d \in (0, d_0]$, that

$$J_\varepsilon(\gamma_\varepsilon(\tau))/\varepsilon^N \geq \mathcal{D}_\varepsilon/\varepsilon^N - \alpha \Rightarrow \gamma_\varepsilon(\tau) \in \mathcal{X}_\varepsilon^{\varepsilon^{N/2}d/2}. \tag{2.30}$$

Now, set

$$\alpha_0 := \min\{\frac{\alpha}{2}, \frac{1}{8}\sigma^2 d_0, \frac{\rho}{2}\},$$

where $\rho = \min_{1\leq i\leq k} c_{\lambda_i}$. We choose $0 < \bar{\varepsilon} < \min\{\varepsilon_0, \varepsilon_1\}$ such that for $\varepsilon \in (0, \bar{\varepsilon}]$

$$|\mathcal{D}_\varepsilon/\varepsilon^N - \sum_{i=1}^{k} c_{\lambda_i}| < \alpha_0,\ |C_\varepsilon/\varepsilon^N - \sum_{i=1}^{k} c_{\lambda_i}| < \alpha_0 \text{ and } |\mathcal{D}_\varepsilon/\varepsilon^N - C_\varepsilon/\varepsilon^N| < \alpha_0.$$

We assume to the contrary that for some $\varepsilon \in (0, \bar{\varepsilon}]$, $d \in (0, d_0)$, there exist $\beta = \beta(\varepsilon) \in (0, 1)$ such that

$$\|J'_\varepsilon(u)\|/\varepsilon^{N/2} \geq \beta > 0 \text{ for } u \in J_\varepsilon^{\mathcal{D}_\varepsilon} \cap \mathcal{X}_\varepsilon^{d\varepsilon^{N/2}}.$$

By Lemma 2.2 in [32], we can choose g_ε be a pseudo-gradient vector field for J'_ε on a neighbourhood N_ε of $J_\varepsilon^{\mathcal{D}_\varepsilon} \cap \mathcal{X}_\varepsilon^{d\varepsilon^{N/2}}$, which satisfies

$$\|g_\varepsilon(u)\| \leq 2\min\{\varepsilon^{N/2}, \|J'_\varepsilon(u)\|\},$$
$$\langle J'_\varepsilon(u), g_\varepsilon(u)\rangle \geq \min\{\varepsilon^{N/2}, \|J'_\varepsilon(u)\|\}\|J'_\varepsilon(u)\|.$$

Let ζ_ε be a Lipschitz continuous function on $\mathcal{D}^s_{V,\varepsilon}(\mathbb{R}^N)$ such that $0 \leq \zeta_\varepsilon \leq 1$, $\zeta_\varepsilon \equiv 1$ on $\mathcal{X}_\varepsilon^{d\varepsilon^{N/2}} \cap J_\varepsilon^{\mathcal{D}_\varepsilon}$ and $\zeta_\varepsilon \equiv 0$ on $\mathcal{D}^s_{V,\varepsilon}(\mathbb{R}^N)\backslash N_\varepsilon$. Let ξ_ε be a Lipschitz continuous function on $\mathbb{R}$ such that $0 \leq \xi_\varepsilon \leq 1$, $\xi_\varepsilon(l) \equiv 1$ if $|l - \mathcal{D}_\varepsilon\varepsilon^{-N}| \leq \frac{\alpha}{2}$ and $\xi_\varepsilon(l) \equiv 0$ if $|l - \mathcal{D}_\varepsilon\varepsilon^{-N}| \geq \alpha$. Set

$$h_\varepsilon(u) := \begin{cases} -\zeta_\varepsilon(u)\xi_\varepsilon(\varepsilon^{-N}J_\varepsilon(u))g_\varepsilon(u), & \text{if } u \in N_\varepsilon \\ 0, & \text{if } u \in \mathcal{D}^s_{V,\varepsilon}\backslash N_\varepsilon. \end{cases} \tag{2.31}$$

Then there exists a unique solution $\Phi_\varepsilon : \mathcal{D}^s_{V,\varepsilon} \times [0, +\infty) \to \mathcal{D}^s_{V,\varepsilon}$ to the following initial value problem

$$\begin{cases} \frac{d}{d\theta}\Phi_\varepsilon(u, \theta) = h_\varepsilon(\Phi_\varepsilon(u, \theta)), \\ \Phi_\varepsilon(u, 0) = u. \end{cases} \tag{2.32}$$

(See the proof of Lemma 2.3 in [32]). It can be easily check that Φ_ε has the following properties:

$$(1)\ \Phi_\varepsilon(u, \theta) = u \text{ if } \theta = 0 \text{ or } u \in \mathcal{D}^s_{V,\varepsilon}(\mathbb{R}^N)\backslash N_\varepsilon \text{ or } |J_\varepsilon(u) - \mathcal{D}_\varepsilon| \geq \alpha\varepsilon^N.$$

$$(2)\|\frac{d}{d\theta}\Phi_\varepsilon(u, \theta)\| \leq 2\varepsilon^{N/2}. \tag{2.33}$$

(3) $\dfrac{d}{d\theta}J_\varepsilon(\Phi_\varepsilon(u,\theta)) = \langle J'_\varepsilon(\Phi_\varepsilon(u,\theta)), h_\varepsilon(\Phi_\varepsilon(u,\theta)) \le 0.$

Claim 1 For any $\tau \in [0,T]^k$, there exists $\theta_\tau \in [0,+\infty)$ such that

$$\Phi_\varepsilon(\gamma_\varepsilon(\tau),\theta_\tau) \in J_\varepsilon^{\mathcal{D}_\varepsilon - \alpha_0\varepsilon^N}.$$

Proof of Claim 1. Assume by contradiction that there exists $\tau_0 \in [0,T]^k$ such that

$$J_\varepsilon(\Phi_\varepsilon(\gamma_\varepsilon(\tau_0),\theta)) > \mathcal{D}_\varepsilon - \alpha_0\varepsilon^N \tag{2.34}$$

for all $\theta > 0$. Then, by the property (3) in (2.33), we have

$$\mathcal{D}_\varepsilon - \alpha_0\varepsilon^N < J_\varepsilon(\Phi_\varepsilon(\gamma_\varepsilon(\tau_0),\theta)) \le J_\varepsilon(\Phi_\varepsilon(\gamma_\varepsilon(\tau_0),0)) = J_\varepsilon(\gamma_\varepsilon(\tau_0)) \le \mathcal{D}_\varepsilon < \mathcal{D}_\varepsilon + \alpha_0\varepsilon^N, \tag{2.35}$$

which and the choice of α_0 imply that $\xi_\varepsilon(\varepsilon^{-N}J_\varepsilon(\Phi_\varepsilon(\gamma_\varepsilon(\tau_0),\theta))) \equiv 1$.

If $\Phi_\varepsilon(\gamma_\varepsilon(\tau_0),\theta) \in \mathcal{X}_\varepsilon^{d\varepsilon^{N/2}}$ for all $\theta \ge 0$, then by (2.35), we have $\Phi_\varepsilon(\gamma_\varepsilon(\tau_0),\theta) \in \mathcal{X}_\varepsilon^{d\varepsilon^{N/2}} \cap J_\varepsilon^{\mathcal{D}_\varepsilon}$ for all $\theta \ge 0$. Then $\zeta_\varepsilon(\Phi_\varepsilon(\gamma_\varepsilon(\tau_0),\theta)) \equiv 1$ and $|\frac{d}{d\theta}J_\varepsilon(\Phi_\varepsilon(\gamma_\varepsilon(\tau_0),\theta))| \ge \beta^2\varepsilon^N$ for all $\theta \ge 0$. Hence

$$J_\varepsilon(\Phi_\varepsilon(\gamma_\varepsilon(\tau_0),\frac{\alpha}{\beta^2}) \le \mathcal{D}_\varepsilon + \alpha_0\varepsilon^N - \varepsilon^N\int_0^{\frac{\alpha}{\beta^2}}\beta^2 d\theta \le \mathcal{D}_\varepsilon - \alpha_0\varepsilon^N,$$

a contradiction to (2.35).

Assume that $\Phi_\varepsilon(\gamma_\varepsilon(\tau_0),\theta_0) \notin \mathcal{X}_\varepsilon^{d\varepsilon^{N/2}}$ for some $\theta_0 > 0$. Note that (2.34), (2.35) and (2.30) imply that $\gamma_\varepsilon(\tau_0) \in \mathcal{X}_\varepsilon^{\frac{d}{2}\varepsilon^{N/2}}$. Then there exist $0 < \theta_0^1 < \theta_0^2$ such that $\Phi_\varepsilon(\gamma_\varepsilon(\tau_0),\theta_0^1) \in \partial\mathcal{X}_\varepsilon^{\frac{d}{2}\varepsilon^{N/2}}$, $\Phi_\varepsilon(\gamma_\varepsilon(\tau_0),\theta_0^2) \in \partial\mathcal{X}_\varepsilon^{d\varepsilon^{N/2}}$ and $\Phi_\varepsilon(\gamma_\varepsilon(\tau_0),\theta) \in \mathcal{X}_\varepsilon^{d\varepsilon^{N/2}} \backslash \mathcal{X}_\varepsilon^{\frac{d}{2}\varepsilon^{N/2}}$ for all $\theta \in (\theta_0^1,\theta_0^2)$. Then by Proposition 2.10, we have $|\frac{d}{d\theta}J_\varepsilon(\Phi_\varepsilon(\gamma_\varepsilon(\tau_0),\theta)| \ge \sigma^2\varepsilon^N$ for all $\theta \in (\theta_0^1,\theta_0^2)$. By property (2) of (2.33) and mean value theorem, we have

$$\frac{d\varepsilon^{N/2}}{2} \le \|\Phi_\varepsilon(\gamma_\varepsilon(\tau_0),\theta_0^1) - \Phi_\varepsilon(\gamma_\varepsilon(\tau_0),\theta_0^2)\| \le 2\varepsilon^{N/2}|\theta_0^1 - \theta_0^2|,$$

which implies

$$|\theta_0^1 - \theta_0^2| \ge \frac{d}{4}.$$

Hence

$$\begin{aligned} J_\varepsilon(\Phi_\varepsilon(\gamma_\varepsilon(\tau_0),\theta_0^2)) &= J_\varepsilon(\Phi_\varepsilon(\gamma_\varepsilon(\tau_0),\theta_0^1)) + \int_{\theta_0^1}^{\theta_0^2}\frac{d}{d\theta}J_\varepsilon(\Phi_\varepsilon(\gamma_\varepsilon(\tau_0),\theta))d\theta \\ &\le \mathcal{D}_\varepsilon + \alpha_0\varepsilon^N - \varepsilon^N\sigma^2|\theta_0^1 - \theta_0^2| \\ &< \mathcal{D}_\varepsilon + \alpha_0\varepsilon^N - \varepsilon^N\sigma^2\frac{d}{4} \\ &\le \mathcal{D}_\varepsilon + \alpha_0\varepsilon^N - \varepsilon^N\sigma^2\frac{d_0}{4} \\ &\le \mathcal{D}_\varepsilon - \alpha_0\varepsilon^N, \end{aligned} \tag{2.36}$$

which is a contradiction to (2.35). This completes the proof of Claim 1.

By Claim 1, we can define $\theta(\tau) := \inf\{\theta \geq 0 : J_\varepsilon(\Phi_\varepsilon(\gamma_\varepsilon(\tau), \theta)) \leq \mathcal{D}_\varepsilon - \alpha_0\varepsilon^N\}$ and let $\bar{\gamma}_\varepsilon(\tau) := \Phi_\varepsilon(\gamma_\varepsilon(\tau), \theta(\tau))$. We have

Claim 2 $\bar{\gamma}_\varepsilon(\tau) \in \Psi_\varepsilon$.

Proof of Claim 2. Firstly, for any $\tau \in \partial[0, T]^k$, by Proposition 2.6, we have $\gamma_\varepsilon(\tau) \in J_\varepsilon^{\mathcal{D}_\varepsilon - \alpha_0\varepsilon^N}$. Hence $\theta(\tau) = 0$ and $\bar{\gamma}_\varepsilon(\tau) = \gamma_\varepsilon(\tau)$ if $\tau \in \partial[0, T]^k$. If $J_\varepsilon(\gamma_\varepsilon(\gamma_\varepsilon(\tau)) \leq \mathcal{D}_\varepsilon - \alpha_0\varepsilon^N$, then $\vartheta(\tau) = 0$ and so $\bar{\gamma}_\varepsilon(\tau) = \gamma_\varepsilon(\tau) \in \mathcal{X}_\varepsilon^{\nu\varepsilon^{N/2}}$ for large $\nu > 0$. If $J_\varepsilon(\gamma_\varepsilon(\tau)) > \mathcal{D}_\varepsilon - \alpha_0\varepsilon^N$, then by (2.30), $\gamma_\varepsilon(\tau) \in \mathcal{X}^{d\varepsilon^{N/2}/2}$ and by property (3) in (2.33)

$$\mathcal{D}_\varepsilon - \alpha_0\varepsilon^N < J_\varepsilon(\Phi_\varepsilon(\gamma_\varepsilon(\tau), \theta)) \leq \mathcal{D}_\varepsilon < \mathcal{D}_\varepsilon + \alpha_0\varepsilon^N, \quad \text{for all } \theta \in [0, \theta(\tau)).$$

This implies $\xi_\varepsilon(\varepsilon^{-N} J_\varepsilon(\Phi_\varepsilon(\gamma_\varepsilon(\tau_0), \theta))) \equiv 1$ for all $\theta \in [0, \theta(\tau))$. Consequently, if $\bar{\gamma}_\varepsilon(\tau) = \Phi_\varepsilon(\gamma_\varepsilon(\tau), \vartheta(\tau)) \notin \mathcal{X}_\varepsilon^{d\varepsilon^N}$, then by the same argument of (2.36), there exists a $\theta \in (0, \theta(\tau))$ such that

$$J_\varepsilon(\Phi_\varepsilon(\gamma_\varepsilon(\tau), \theta)) < \mathcal{D}_\varepsilon - \alpha_0\varepsilon^N.$$

This contradicts the definition of $\theta(\tau)$. Hence $\bar{\gamma}_\varepsilon(\tau) \in \mathcal{X}_\varepsilon^{d\varepsilon^{N/2}} \subset \mathcal{X}_\varepsilon^{\nu\varepsilon^{N/2}}$.

Secondly, we prove that $\bar{\gamma}_\varepsilon(\tau)$ is continuous. We fix any $\bar{\tau} \in [0, 1]^k$. If $J_\varepsilon(\gamma_\varepsilon(\bar{\tau})) < \mathcal{D}_\varepsilon - \alpha_0\varepsilon^N$, then $\theta(\bar{\tau}) = 0$. Then by the continuity of γ_ε, we conclude that $\bar{\gamma}_\varepsilon(\tau)$ is continuous at $\bar{\tau}$. If $J_\varepsilon(\gamma_\varepsilon(\bar{\tau})) = \mathcal{D}_\varepsilon - \alpha_0\varepsilon^N$, then from the proof of (2.36), we know that $\gamma_\varepsilon(\bar{\tau}) \in \mathcal{X}_\varepsilon^{d\varepsilon^{N/2}}$, and so

$$\|J_\varepsilon'(\gamma_\varepsilon(\bar{\tau}))\| \geq \beta\varepsilon^{N/2} > 0.$$

Thus, from the property (3) in (2.33), we have $J_\varepsilon(\Phi_\varepsilon(\gamma_\varepsilon(\bar{\tau}), \theta(\bar{\tau}) + \omega) < \mathcal{D}_\varepsilon - \alpha_0\varepsilon^N$. By the continuity of γ_ε, we choose $r > 0$ as the constants such that $J_\varepsilon(\Phi_\varepsilon(\gamma_\varepsilon(\tau), \theta(\bar{\tau}))) < \mathcal{D}_\varepsilon - \alpha_0\varepsilon^N$ for all $\tau \in B_r(\bar{\tau})$. Then by the definition of $\theta(\tau)$, we have $\theta(\tau) < \theta(\bar{\tau})$ for all $\tau \in B_r(\bar{\tau}) \cap [0, T]^k$, and then

$$0 \leq \limsup_{\tau \to \bar{\tau}} \theta(\tau) \leq \theta(\bar{\tau}).$$

If $\theta(\bar{\tau}) = 0$, we immediately have

$$\lim_{\tau \to \bar{\tau}} \theta(\tau) = \theta(\bar{\tau}).$$

If $\theta(\tau) > 0$, then for any $0 < \omega < \theta(\bar{\tau})$, similarly we have $J_\varepsilon(\Phi_\varepsilon(\gamma_\varepsilon(\tau), \theta(\bar{\tau}) - \omega)) > \mathcal{D}_\varepsilon - \alpha_0\varepsilon^N$. By the continuity of γ_ε again, we see that

$$\liminf_{\tau \to \bar{\tau}} \theta(\tau) \geq \theta(\bar{\tau}).$$

So $\theta(\cdot)$ is continuous at $\bar{\tau}$. This completes the proof of Claim 2.

Now we have proved that $\bar{\gamma}_\varepsilon(\tau) \in \Psi_\varepsilon$ and $\max_{\tau \in [0,T]^k} \leq \mathcal{D}_\varepsilon - \alpha_0\varepsilon^N$, which contradicts the definition of C_ε. This completes the proof.

□

Lemma 2.12. *Let* $\{u_n\}_{n=1}^\infty$ *be the sequence given by Proposition 2.11. Then* $\{u_n\}$ *has a subsequence which converges to* u_ε *in* $\mathcal{D}_{V,\varepsilon}^s(\mathbb{R}^N)$*. Moreover, there hold* $u_\varepsilon > 0$, $u_\varepsilon \in \mathcal{D}_{V,\varepsilon}^s(\mathbb{R}^N) \cap C^{1,\beta}(\mathbb{R}^N)$ *for some* $\beta \in (0, 1)$ *and* u_ε *is a solution to the penalized problem* (2.3)*(or* (2.4)*).*

Proof. The convergence is from Lemma 2.2. The regularity result follows from Appendix D in [21]. Testing the penalized equation (2.4) with $(u_\varepsilon)_-$ and integrating, we can see that $u_\varepsilon \geq 0$. Suppose to the contrary that there exists $x_0 \in \mathbb{R}^N$ such that $u_\varepsilon(x_0) = 0$, then we have

$$0 = \varepsilon^{2s}(-\Delta)^s u_\varepsilon(x_0) + V(x_0)u_\varepsilon(x_0) < 0,$$

which is a contradiction. Therefore, $u_\varepsilon > 0$.

□

To end this section, we prove that u_ε owns k-peaks.

Lemma 2.13. *Let $\rho > 0$ and u_ε be the solution of* (2.3) *given by Lemma 2.12. Then there exists k families of points $\{x_\varepsilon^i\}$, $i = 1, \ldots, k$, such that*

$$\begin{aligned} &(1)\ \liminf_{\varepsilon\to 0} \|u_\varepsilon\|_{L^\infty(B_{\varepsilon\rho}(x_\varepsilon^i))} > 0,\\ &(2)\ \lim_{\varepsilon\to 0} dist(x_\varepsilon^i, \mathcal{M}_i) = 0,\\ &(3)\ \lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \|u_\varepsilon\|_{L^\infty(U\setminus\cup_{1\leq i\leq k} B_{\varepsilon R}(x_\varepsilon^i))} = 0. \end{aligned}$$

Proof. The proof is trivial by the fact that the (PS) sequence given by Proposition 2.11 satisfies the assumptions of Proposition 2.9. □

3. Back to the original problem

In this section we show that u_ε solves the original problem (1.1). For this purpose, basing on the penalized equation (2.4), all we need to do is to prove that

$$f(u_\varepsilon) \leq \mathcal{P}_\varepsilon(x)u_\varepsilon, \quad x \in \mathbb{R}^N\setminus\Lambda. \tag{3.1}$$

We use comparison principle to prove (3.1), for which we should first linearize the penalized equation (2.4) outside small balls.

Proposition 3.1. *Let $\{x_\varepsilon^i\}$, $i = 1, \ldots, k$ be the k families of points given by Lemma 2.13. Then for $\varepsilon > 0$ small enough and $\delta \in (0, 1)$, there exist $C_\infty > 0$ and $R > 0$ such that*

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u_\varepsilon + (1-\delta)Vu_\varepsilon \leq P_\varepsilon u_\varepsilon, & in\ \mathbb{R}^N\setminus \bigcup_{i=1}^k B_{R\varepsilon}(x_\varepsilon^i),\\ u_\varepsilon \leq C_\infty & in\ \Lambda. \end{cases} \tag{3.2}$$

Proof. That $u_\varepsilon \leq C_\infty$ in Λ is from Lemma 2.13 and the L^∞ estimate in [21, Appendix D]. By the assumption on f, $\inf_U V(x) > 0$ and Lemma 2.13, there exists $R > 0$ such that

$$f(u_\varepsilon) \leq \delta V u_\varepsilon \text{ in } U\setminus\bigcup_{i=1}^k B_{R\varepsilon}(x_\varepsilon^i).$$

Obviously

$$g_\varepsilon(u_\varepsilon) \leq \mathcal{P}_\varepsilon u_\varepsilon \text{ in } \mathbb{R}^N\setminus U.$$

Hence we conclude our result by inserting the previous pointwise bounds into the penalized equation (2.4).

□

Next, we construct a suitable sup-solution to Eq (3.2). Some of the the details are similar to that in Proposition 4.2 of [28]. Let $\tilde{\eta}_\beta(s)$, $s \geq 0$ be a smooth non-increasing function with $\tilde{\eta}_\beta \equiv 1$ on $[0,1]$ and $\tilde{\eta}_\beta \equiv 0$ on $(1+\beta,+\infty)$, where β is a small parameter. Define $\eta_{\beta,R}(|x|) = \tilde{\eta}_\beta(|x|/R)$. Setting $0 < \alpha < N-2s$ and denoting

$$\begin{aligned} f^{\alpha}_{\beta,R}(x) &= \eta_{\beta,R}(x)\frac{1}{R^\alpha} + (1-\eta_{\beta,R}(x))\frac{1}{|x|^\alpha}, \\ f^{\alpha,i}_{\beta,R,\varepsilon}(x) &= f^{\alpha}_{\beta,R}\Big(\frac{x-x^i_\varepsilon}{\varepsilon}\Big), \\ f^{\alpha}_{\beta,R,\varepsilon}(x) &= \sum_{i=1}^{k} f^{\alpha,i}_{\beta,R,\varepsilon}(x). \end{aligned}$$

We have

Proposition 3.2. *Let $\varepsilon > 0$ be small enough. Then for every $x \in \mathbb{R}^N\setminus \bigcup_{i=1}^k B_{R\varepsilon}(x^i_\varepsilon)$, it holds*

$$\varepsilon^{2s}(-\Delta)^s f^{\alpha}_{\beta,R,\varepsilon} + (1-\delta)V(x)f^{\alpha}_{\beta,R,\varepsilon} - \mathcal{P}_\varepsilon(x)f^{\alpha}_{\beta,R,\varepsilon} \geq 0. \tag{3.3}$$

Proof. Fixing any $i \in \{1,\dots,k\}$, a computation shows that

$$\begin{aligned} &\varepsilon^{2s}(-\Delta)^s f^{\alpha,i}_{\beta,R,\varepsilon} + V(x)f^{\alpha,i}_{\beta,R,\varepsilon} - \mathcal{P}_\varepsilon(x)f^{\alpha,i}_{\beta,R,\varepsilon} \\ &= (-\Delta)^s f^{\alpha}_{\beta,R,\varepsilon}\Big(\frac{x-x^i_\varepsilon}{\varepsilon}\Big) + V(x)f^{\alpha}_{\beta,R,\varepsilon}\Big(\frac{x-x^i_\varepsilon}{\varepsilon}\Big) - \mathcal{P}_\varepsilon(x)f^{\alpha}_{\beta,R,\varepsilon}\Big(\frac{x-x^i_\varepsilon}{\varepsilon}\Big) \\ &= \Big((-\Delta)^s f^{\alpha}_{\beta,R,\varepsilon}(y) + V^i_\varepsilon(y)f^{\alpha}_{\beta,R,\varepsilon}(y) - \widehat{\mathcal{P}}^i_\varepsilon(y)f^{\alpha}_{\beta,R,\varepsilon}(y)\Big)\Big|_{y=\frac{x-x^i_\varepsilon}{\varepsilon}}, \end{aligned} \tag{3.4}$$

where $V^i_\varepsilon(\cdot) = V(\varepsilon x \cdot + x^i_\varepsilon)$ and $\widehat{\mathcal{P}}^i_\varepsilon(\cdot) = \mathcal{P}_\varepsilon(\varepsilon \cdot + x^i_\varepsilon)$. But, using the non-increasing property of η_β and the computation of Proposition 4.2 of [28], for any $y \in \mathbb{R}^N\backslash B_R(0)$, when $\varepsilon > 0$ is small enough, we can conclude that

$$(-\Delta)^s f^{\alpha}_{\beta,R,\varepsilon}(y) + V^i_\varepsilon(y)f^{\alpha}_{\beta,R,\varepsilon}(y) - \widehat{\mathcal{P}}^i_\varepsilon(y)f^{\alpha}_{\beta,R,\varepsilon}(y) \geq 0. \tag{3.5}$$

Then for all $x \in \mathbb{R}^N\backslash B_{R\varepsilon}(x^i_\varepsilon)$, it holds

$$\varepsilon^{2s}(-\Delta)^s f^{\alpha,i}_{\beta,R,\varepsilon} + V(x)f^{\alpha,i}_{\beta,R,\varepsilon} - \mathcal{P}_\varepsilon(x)f^{\alpha,i}_{\beta,R,\varepsilon} \geq 0.$$

As a result, we have

$$\begin{aligned} &\varepsilon^{2s}(-\Delta)^s f^{\alpha}_{\beta,R,\varepsilon} + V(x)f^{\alpha}_{\beta,R,\varepsilon} - \mathcal{P}_\varepsilon(x)f^{\alpha}_{\beta,R,\varepsilon} \\ &= \sum_{i=1}^{k}\Big(\varepsilon^{2s}(-\Delta)^s f^{\alpha,i}_{\beta,R,\varepsilon} + V(x)f^{\alpha,i}_{\beta,R,\varepsilon} - \mathcal{P}_\varepsilon(x)f^{\alpha,i}_{\beta,R,\varepsilon}\Big) \geq 0 \end{aligned}$$

for all $x \in \mathbb{R}^N\setminus \bigcup_{i=1}^k B_{R\varepsilon}(x^i_\varepsilon)$. This completes the proof. □

At last, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let

$$\begin{cases} \mathcal{P}_\varepsilon(x) = \frac{\varepsilon^{2s+2\kappa}}{|x|^{2s+\kappa}}\chi_{\mathbb{R}^N\backslash\Lambda}(x), \\ \overline{U}_\varepsilon(x) = CR^\alpha f^{\alpha}_{\beta,R,\varepsilon}(x). \end{cases} \tag{3.6}$$

It is easy to check that $\mathcal{P}_\varepsilon$ satisfies the assumption (2.1).

By Proposition 3.2, choosing the constant $C > 0$ large enough and letting $v_\varepsilon(x) = u_\varepsilon(x) - \overline{U}_\varepsilon(x)$, we have

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s v_\varepsilon(x) + (1-\delta)V(x)v_\varepsilon(x) - \mathcal{P}_\varepsilon(x)v_\varepsilon(x) \le 0, & \text{in } \mathbb{R}^N \setminus \bigcup_{i=1}^k B_{R\varepsilon}(x_\varepsilon^i), \\ \\ v_\varepsilon(x) \le 0 & \text{in } \bigcup_{i=1}^k B_{R\varepsilon}(x_\varepsilon^i). \end{cases}$$

Since $v_\varepsilon^+ \in \mathcal{D}_{V,\varepsilon}^s$(when α is closed to $N-2s$), testing the equation above against with $v_\varepsilon^+(x)$, by the fractional Hardy inequality in (1.7), we find $v_\varepsilon^+(x) = 0,\ x \in \mathbb{R}^N$. Hence $v_\varepsilon(x) \le 0,\ x \in \mathbb{R}^N$. Especially, we have

$$u_\varepsilon(x) \le \overline{U}_\varepsilon(x) = \sum_{i=1}^k f_{\beta,R,\varepsilon}^{\alpha,i}(x) \le \sum_{i=1}^k \frac{C\varepsilon^\alpha}{\varepsilon^\alpha + |x - x_\varepsilon^i|^\alpha}, \quad x \in \mathbb{R}^N.$$

Moreover, letting α be closed to $N-2s$, for all $x \in \mathbb{R}^N \setminus \Lambda$, it holds

$$\frac{f(u_\epsilon)}{u_\epsilon} \le (u_\epsilon)^{\tilde{\kappa}} \le \frac{C\epsilon^{\alpha\tilde{\kappa}}}{|x|^{\alpha\tilde{\kappa}}} \le \frac{\epsilon^{2s+2\kappa}}{|x|^{2s+\kappa}} = \mathcal{P}_\epsilon(x).$$

This gives (3.1). As a result, u_ε solves the original problem.

Remark 3.3. In the local case $s = 1$, we can prove the same result more easily by introducing the same penalized function $\mathcal{P}_\varepsilon$ in this paper. We point out here that we also answer positively to the conjecture proposed by Ambrosetti and Malchiodi in [33] in the nonlocal case.

A. Appendix

In this section we are going to verify Lemma 2.8, (2.23), (2.26), (2.27) and (2.29).

Proposition A.1. *For every $i = 1, \ldots, k$, it holds*

$$\lim_{\varepsilon \to 0} \frac{c_\varepsilon^i}{\varepsilon^N} = c_{\lambda_i}.$$

Proof. The achievement of c_ε^i is easily from the fact that the embedding

$$W^{s,2}(\Omega) \hookrightarrow L^p$$

is compact for $1 \le p < 2_s^*$(see [4] for more details). Thus we only need to prove (2.9).

For every nonnegative $v \in C_c^\infty(\mathbb{R}^N)\setminus\{0\}$ and $x_0 \in \Lambda_i$, let $v_\varepsilon(x) = v(\frac{x - x_0}{\varepsilon})$. Obviously, *supp* $v_\varepsilon \subset \Lambda_i$ and $\gamma(t) = tTv_\varepsilon \in \Gamma_\varepsilon^i$ for ε small enough and T large enough. Therefore,

$$\begin{aligned} c_\varepsilon^i &\le \max_{t\in[0,1]} J_\varepsilon^i(\gamma(t)) \\ &\le \varepsilon^N \max_{t>0} \Big(\frac{t^2}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s}} dxdy \\ &\quad + \frac{t^2}{2} \int_{\mathbb{R}^N} V(\varepsilon x + x_0)|v|^2 dx - \int_{\mathbb{R}^N} F(tv)dx \Big) \end{aligned}$$

and then

$$\limsup_{\varepsilon\to 0}\frac{c_\varepsilon^i}{\varepsilon^N}\le \limsup_{\varepsilon\to 0}\max_{t>0}\Big(\frac{t^2}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}dxdy + \frac{t^2}{2}\int_{\mathbb{R}^N}V(\varepsilon x+x_0)|v|^2dx-\int_{\mathbb{R}^N}F(tv)dx\Big) = \max_{t>0}L_{v(x_0)}(tv).$$

Hence, by the arbitrariness of v and x_0, we have

$$\limsup_{\varepsilon\to 0}\frac{c_\varepsilon^i}{\varepsilon^N}\le c_{\lambda_i}. \tag{A.1}$$

On the other hand, let w_ε be a critical point corresponding to c_ε^i, i.e., $J_\varepsilon^i(w_\varepsilon)=c_\varepsilon^i$ and

$$\varepsilon^{2s}\int_{S_i}\frac{w_\varepsilon(x)-w_\varepsilon(y)}{|x-y|^{N+2s}}dy+V(x)w_\varepsilon(x)=\mathrm{g}_\varepsilon(w_\varepsilon),\quad x\in S_i. \tag{A.2}$$

It follows that

$$\varepsilon^{2s}\int_{S_i}\int_{S_i}\frac{w_\varepsilon(x)-w_\varepsilon(y)}{|x-y|^{N+2s}}w_\varepsilon(x)dydx+\int_{S_i}V(x)|w_\varepsilon(x)|^2=\int_{S_i}\mathrm{g}_\varepsilon(w_\varepsilon)w_\varepsilon.$$

Then by (2.2), it holds

$$\varepsilon^{2s}\int_{S_i}\int_{S_i}\frac{w_\varepsilon(x)-w_\varepsilon(y)}{|x-y|^{N+2s}}w_\varepsilon(x)dydx+\int_{S_i}V(x)|w_\varepsilon(x)|^2 \le C\big(\|w_\varepsilon\|_{L^\infty(\Lambda_i)}^{p-1}+\|w_\varepsilon\|_{L^\infty(\Lambda_i)}^{\tilde{\kappa}}\big)\Big(\varepsilon^{2s}\int_{S_i}\int_{S_i}\frac{w_\varepsilon(x)-w_\varepsilon(y)}{|x-y|^{N+2s}}w_\varepsilon(x)dydx+\int_{S_i}V(x)|w_\varepsilon(x)|^2\Big),$$

from which we conclude that there exists $x_\varepsilon^i\in\overline{\Lambda_i}$ such that for $\rho>0$,

$$\liminf_{\varepsilon\to 0}\|w_\varepsilon\|_{L^\infty(B_{\varepsilon\rho}(x_\varepsilon^i))}>0. \tag{A.3}$$

Going if necessary to a subsequence, we assume that

$$\lim_{\varepsilon\to 0}x_\varepsilon^i\to x^i\in\overline{\Lambda_i}. \tag{A.4}$$

Now, let $\tilde{w}_\varepsilon(x)=w_\varepsilon(x_\varepsilon^i+\varepsilon x)$, then $\tilde{w}_\varepsilon$ satisfies

$$\int_{S_\varepsilon^i}\frac{\tilde{w}_\varepsilon(x)-\tilde{w}_\varepsilon(y)}{|x-y|^{N+2s}}dy+V_\varepsilon(x)\tilde{w}_\varepsilon(x)=\tilde{\mathrm{g}}_\varepsilon(\tilde{w}_\varepsilon)\ \ x\in S_\varepsilon^i, \tag{A.5}$$

where $V_\varepsilon(x)=V(x_\varepsilon^i+\varepsilon x)$, $S_\varepsilon^i=\{x\in\mathbb{R}^N:\varepsilon x+x_\varepsilon^i\in S\}$ and $\tilde{\mathrm{g}}_\varepsilon(\tilde{w}_\varepsilon)=g(\varepsilon x+x_\varepsilon^i,\tilde{w}_\varepsilon)$. Moreover, by (A.1), we have

$$\sup_{\varepsilon>0}\|\tilde{w}_\varepsilon\|_{W^{s,2}(B_R)}<\infty$$

for every $R \in (0, +\infty)$. Thus, by diagonal argument, we conclude that $\tilde{w}_\varepsilon \rightharpoonup \tilde{w}$ weakly in $W^{s,2}(B_R)$ for every $R > 0$. Moreover, it is easy to check by Fatou's Lemma that $\tilde{w} \in H^s(\mathbb{R}^N)$. Then, by (A.4), using Corollary 7.2 in [4] and taking limit in (A.5), we conclude that

$$\int_{\mathbb{R}^N} \frac{\tilde{w}(x) - \tilde{w}(y)}{|x-y|^{N+2s}} dy + V(x^i)\tilde{w} = \chi_{\Lambda^i_*} f(\tilde{w}) \ \ x \in \mathbb{R}^N,$$

where Λ^*_i is the limit of the set $\Lambda^i_\varepsilon = \{x \in \mathbb{R}^N : \varepsilon x + x^i_\varepsilon \in \Lambda_i\}$. But by (A.3) and using the standard bootstrap argument in Appendix D in [21], we have

$$\|\tilde{w}\|_{L^\infty(B_\rho(0))} = \lim_{\varepsilon\to 0} \|\tilde{w}_\varepsilon\|_{L^\infty(B_\rho(0))} \geq \liminf_{\varepsilon\to 0} \|w_\varepsilon\|_{L^\infty(B_\rho(0))} > 0,$$

which combined with the Liouville-type results (see Lemma 3.3 in [27]) implies that $\Lambda^i_* = \mathbb{R}^N$. Hence we have

$$(-\Delta)^s \tilde{w} + V(x^i)\tilde{w} = f(\tilde{w}) \ \text{ in } \mathbb{R}^N.$$

Proceeding as one proves Lemma 3.3 of [28], we have

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \frac{c^i_\varepsilon}{\varepsilon^N} &\geq L_{V(x^i)}(\tilde{w}) + o_R(1) \\
&\quad + \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^N}\Big(\frac{1}{2}\int_{S^i_\varepsilon \setminus B_R} dx \int_{S^i_\varepsilon} \frac{|\tilde{w}_\varepsilon(x) - \tilde{w}_\varepsilon(y)|^2}{|x-y|^{N+2s}} dy \\
&\quad + \frac{1}{2}\int_{S^i_\varepsilon \setminus B_R} V_\varepsilon(x)\tilde{w}^2_\varepsilon(x) dx - \int_{S^i_\varepsilon \setminus B_R} \mathfrak{G}_\varepsilon(\tilde{w}_\varepsilon(x)) dx\Big) \\
&\geq c_{V(x^i)} + o_R(1)
\end{aligned}$$

Therefore,

$$\liminf_{\varepsilon\to 0} \frac{c^i_\varepsilon}{\varepsilon^N} \geq c_{\lambda_i},$$

which and (A.1) complete the proof. □

Lemma A.2. *The estimates* (2.23)*,* (2.26)*,* (2.27) *and* (2.29) *hold.*

Proof. Hereafter, we define $\hat{\eta}_\varepsilon(x) = \eta(2\varepsilon x) = \eta_\varepsilon(2x)$ for all $x \in \mathbb{R}^N$. We first give the proof of (2.26). By the definition of $\bar{\eta}_\varepsilon$, we have

$$\begin{aligned}
2T^2_\varepsilon(\bar{\eta}_\varepsilon)/\varepsilon^N &= \sum_{i=1}^k \varepsilon^{2s-N} \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} (u_\varepsilon(x) - u_\varepsilon(y))\Big(\eta(2(x - p^i_\varepsilon - \varepsilon z_i)) \\
&\qquad - \eta(2(y - p^i_\varepsilon - \varepsilon z_i))\Big) u_\varepsilon(y)|x-y|^{-N-2s} dy \\
&= \sum_{i=1}^k \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \frac{(v^i_\varepsilon(x) - v^i_\varepsilon(y))(\hat{\eta}_\varepsilon(x) - \hat{\eta}_\varepsilon(y)) v^i_\varepsilon(y)}{|x-y|^{N+2s}} dy \\
&:= \sum_{i=1}^k T^{2,i}_\varepsilon(\eta).
\end{aligned}$$

For each $i = 1, \ldots, k$, dividing $\mathbb{R}^N$ into several regions, we have

$$
\begin{aligned}
T_\varepsilon^{2,i}(\eta) &= \int_{B_{\frac{\beta}{\varepsilon}}} dx \int_{B_{\frac{\beta}{\varepsilon}}} \frac{(v_\varepsilon^i(x) - v_\varepsilon^i(y))(\hat{\eta}_\varepsilon(x) - \hat{\eta}_\varepsilon(y))v_\varepsilon^i(y)}{|x-y|^{N+2s}} dy \\
&\quad + \int_{B_{\frac{\beta}{\varepsilon}}} dx \int_{B^c_{\frac{\beta}{\varepsilon}}} \frac{(v_\varepsilon^i(x) - v_\varepsilon^i(y))(\hat{\eta}_\varepsilon(x) - \hat{\eta}_\varepsilon(y))v_\varepsilon^i(y)}{|x-y|^{N+2s}} dy \\
&\quad + \int_{B^c_{\frac{\beta}{\varepsilon}}} dx \int_{B_{\frac{\beta}{\varepsilon}}} \frac{(v_\varepsilon^i(x) - v_\varepsilon^i(y))(\hat{\eta}_\varepsilon(x) - \hat{\eta}_\varepsilon(y))v_\varepsilon^i(y)}{|x-y|^{N+2s}} dy \\
&:= \sum_{j=1}^{3} T_\varepsilon^{2,i,j}(\eta).
\end{aligned}
$$

For $T_\varepsilon^{2,i,1}(\eta)$, by Cauchy inequality, we have

$$
\begin{aligned}
|T_\varepsilon^{2,i,1}(\eta)|^2 &\le C \int_{B_{\frac{\beta}{\varepsilon}}} |v_\varepsilon^i(y)|^2 dy \int_{B_{\frac{\beta}{\varepsilon}}} \frac{|\hat{\eta}_\varepsilon(x) - \hat{\eta}_\varepsilon(y)|^2}{|x-y|^{N+2s}} dx \\
&\le C\varepsilon^2 \int_{B_{\frac{\beta}{\varepsilon}}} |v_\varepsilon^i(z)|^2 dy \int_{B_{\frac{2\beta}{\varepsilon}}} \frac{1}{|z|^{N+2s-2}} dx \\
&= C\varepsilon^{2s}.
\end{aligned}
$$

For $T_\varepsilon^{2,i,2}(\eta)$, by the definition of η, we have

$$
T_\varepsilon^{2,i,2}(\eta) \le \int_{B_{\frac{\beta}{\varepsilon}}} dx \int_{B^c_{\frac{\beta}{\varepsilon}}} \frac{v_\varepsilon^i(x)\hat{\eta}_\varepsilon(x)v_\varepsilon^i(y)}{|x-y|^{N+2s}} dy.
$$

But, using the similar estimate of $T_\varepsilon^{2,i,1}(\eta)$ and fractional Hardy inequality (1.7), we have

$$
\begin{aligned}
&\Big| \int_{B_{\frac{\beta}{\varepsilon}}} dx \int_{B^c_{\frac{\beta}{\varepsilon}}} \frac{v_\varepsilon^i(x)\hat{\eta}_\varepsilon(x)v_\varepsilon^i(y)}{|x-y|^{N+2s}} dy \Big| \\
&\le \int_{B_{\frac{\beta}{\varepsilon}}} dx \int_{B^c_{\frac{3\beta}{\varepsilon}}} \frac{|v_\varepsilon^i(x)|\hat{\eta}_\varepsilon(x)|v_\varepsilon^i(y)|}{|x-y|^{N+2s}} dy \\
&\quad + \int_{B_{\frac{\beta}{\varepsilon}}} dx \int_{B_{\frac{3\beta}{\varepsilon}} \setminus B_{\frac{\beta}{\varepsilon}}} \frac{|v_\varepsilon^i(x)||\hat{\eta}_\varepsilon(x) - \hat{\eta}_\varepsilon(y)||v_\varepsilon^i(y)|}{|x-y|^{N+2s}} dy \Big| \\
&\le \int_{B_{\frac{\beta}{\varepsilon}}} dx \int_{B^c_{\frac{3\beta}{\varepsilon}}} \frac{|v_\varepsilon^i(x)|\hat{\eta}_\varepsilon(x)|v_\varepsilon^i(y)|}{|x-y|^{N+2s}} dy + C\varepsilon^s \\
&\le \Big(\int_{B_{\frac{\beta}{\varepsilon}}} (\hat{\eta}_\varepsilon(x)v_\varepsilon^i(x))^2 dx \int_{B^c_{\frac{3\beta}{\varepsilon}}} \frac{1}{|x-y|^{N+2s}} dy \Big)^{\frac{1}{2}} \\
&\quad \cdot \Big(\int_{B^c_{\frac{3\beta}{\varepsilon}}} \frac{(v_\varepsilon^i(y))^2}{|y|^{2s}} dy \int_{B_{\frac{\beta}{\varepsilon}}} \frac{|y|^{2s}}{|x-y|^{N+2s}} dx \Big)^{\frac{1}{2}} + C\varepsilon^s
\end{aligned}
$$

$$\le C\varepsilon^s.$$

Hence, it holds

$$\limsup_{\varepsilon\to 0} T_\varepsilon^{2,i,2}(\eta) \le 0.$$

Similarly, one has

$$\limsup_{\varepsilon\to 0} T_\varepsilon^{2,i,3}(\eta) \le 0.$$

So

$$\limsup_{\varepsilon\to 0} T_\varepsilon^{2,i}(\eta) \le 0$$

and

$$\limsup_{\varepsilon\to 0} \frac{T_\varepsilon^{2}(\eta)}{\varepsilon^N} \le 0.$$

Secondly, we prove (2.23). By the definition of η, we have

$$\begin{aligned}
|T_\varepsilon^1(\eta)/2|^2 &\le \varepsilon^{4s}\Big(\int_{B_\beta(p_\varepsilon^i+\varepsilon z_i)} (u_\varepsilon(x))^2dx \int_{B_\beta(p_\varepsilon^j+\varepsilon z_j)} \frac{1}{|x-y|^{N+2s}}dy\Big)\\
&\cdot\Big(\int_{B_\beta(p_\varepsilon^j+\varepsilon z_j)} (u_\varepsilon(y))^2dy \int_{B_\beta(p_\varepsilon^j+\varepsilon z_j)} \frac{1}{|x-y|^{N+2s}}dx\Big)\\
&= \varepsilon^{4N+4s}\Big(\int_{B_{\frac{\beta}{\varepsilon}}} (v_\varepsilon^i(x))^2dx \int_{B_{\frac{\beta}{\varepsilon}}} \frac{1}{|\varepsilon x+p_\varepsilon^i+\varepsilon z_i-\varepsilon y-p_\varepsilon^j-\varepsilon z_j|^{N+2s}}dy\Big)\\
&\cdot\Big(\int_{B_{\frac{\beta}{\varepsilon}}} (v_\varepsilon^j(y))^2dy \int_{B_{\frac{\beta}{\varepsilon}}} \frac{1}{|\varepsilon x+p_\varepsilon^i+\varepsilon z_i-\varepsilon y-p_\varepsilon^j-\varepsilon z_j|^{N+2s}}dx\Big)\\
&= \varepsilon^{2N}\Big(\int_{B_{\frac{\beta}{\varepsilon}}} (v_\varepsilon^i(x))^2dx \int_{B_{\frac{\beta}{\varepsilon}}} \frac{1}{|x-y+\frac{p_\varepsilon^i+\varepsilon z_i-p_\varepsilon^j+\varepsilon z_j}{\varepsilon}|^{N+2s}}dy\Big)\\
&\cdot\Big(\int_{B_{\frac{\beta}{\varepsilon}}} (v_\varepsilon^j(y))^2dy \int_{B_{\frac{\beta}{\varepsilon}}} \frac{1}{|x-y+\frac{p_\varepsilon^i+\varepsilon z_i-p_\varepsilon^j-\varepsilon z_j}{\varepsilon}|^{N+2s}}dx\Big)\\
&\le C\varepsilon^{2N+4s}.
\end{aligned}$$

Then we have

$$\frac{T_\varepsilon^1(\eta)}{\varepsilon^N} \le C\varepsilon^s,$$

which gives (2.23).

Thirdly, we give the proof of (2.29). Denoting $A_\varepsilon = \mathbb{R}^N\setminus\bigcup_{i=1}^k B_{2\beta}(p_\varepsilon^i+\varepsilon z_i)$, one can check that

$$\begin{aligned}
\varepsilon^{-2s}T_\varepsilon^3(\breve{\eta}) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\breve{\eta}_\varepsilon(x)u_\varepsilon(x)-\breve{\eta}_\varepsilon(y)u_\varepsilon(y)|^2}{|x-y|^{N+2s}}dxdy\\
&= \int_{A_\varepsilon}dx\int_{A_\varepsilon}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{N+2s}}dxdy + \int_{A_\varepsilon}dx\int_{A_\varepsilon^c}\frac{|u_\varepsilon(x)-\breve{\eta}_\varepsilon(y)u_\varepsilon(y)|^2}{|x-y|^{N+2s}}dxdy\\
&+ \int_{A_\varepsilon^c}dx\int_{A_\varepsilon}\frac{|\breve{\eta}_\varepsilon(x)u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{N+2s}}dxdy + \int_{A_\varepsilon^c}dx\int_{A_\varepsilon^c}\frac{|\breve{\eta}_\varepsilon(x)u_\varepsilon(x)-\breve{\eta}_\varepsilon(y)u_\varepsilon(y)|^2}{|x-y|^{N+2s}}dxdy
\end{aligned}$$

$$
\begin{aligned}
&\le \int_{A_\varepsilon} dx \int_{A_\varepsilon} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{N+2s}}dxdy + C\int_{A_\varepsilon} dx \int_{A_\varepsilon^c} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{N+2s}}dxdy\\
&\quad + C\int_{A_\varepsilon} dx \int_{A_\varepsilon^c} \frac{|(1-\breve{\eta}_\varepsilon(y))u_\varepsilon(y)|^2}{|x-y|^{N+2s}}dxdy + \int_{A_\varepsilon^c} dx \int_{A_\varepsilon^c} \frac{|\breve{\eta}_\varepsilon(x)u_\varepsilon(x)-\breve{\eta}_\varepsilon(y)u_\varepsilon(y)|^2}{|x-y|^{N+2s}}dxdy\\
&\le C\varepsilon^N + C\int_{A_\varepsilon} dx \int_{A_\varepsilon^c} \frac{|(1-\breve{\eta}_\varepsilon(y))u_\varepsilon(y)|^2}{|x-y|^{N+2s}}dxdy\\
&\quad + \int_{A_\varepsilon^c} dx \int_{A_\varepsilon^c} \frac{|\breve{\eta}_\varepsilon(x)u_\varepsilon(x)-\breve{\eta}_\varepsilon(y)u_\varepsilon(y)|^2}{|x-y|^{N+2s}}dxdy\\
&\le C\varepsilon^N.
\end{aligned}
$$

As a result, we get (2.29).

The proof of (2.27) is similar and we omit it. □

Acknowledgments

The authors are grateful to the referee for carefully reading the manuscript and for many valuable comments which largely improved the article. This work was partially supported by NSFC grants (No.12101150; No.11831009) and the Science and Technology Foundation of Guizhou Province ([2021]ZK008).

Conflict of interest

The authors declare there is no conflicts of interest.

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