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*Research article*

# A structure-preserving doubling algorithm for solving a class of quadratic matrix equation with $M$-matrix

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**Abstract:** Consider the problem of finding the maximal nonpositive solvent $\Phi$ of the quadratic matrix equation (QME) $X^2 + BX + C = 0$ with $B$ being a nonsingular $M$-matrix and $C$ an $M$-matrix such that $B^{-1}C \geq 0$. Such QME arises from an overdamped vibrating system. Recently, under the condition that $B - C - I$ is a nonsingular $M$-matrix, Yu et al. (*Appl. Math. Comput.*, 218 (2011): 3303–3310) proved that $\rho(\Phi) \leq 1$ for this QME. In this paper, under the same condition, we slightly improve their result and prove that $\rho(\Phi) < 1$, which is important for the quadratic convergence of the structure-preserving doubling algorithm. Then, a new globally monotonically and quadratically convergent structure-preserving doubling algorithm for solving the QME is developed. Numerical examples are presented to demonstrate the feasibility and effectiveness of our method.

**Keywords:** Quadratic matrix equation; structure-preserving doubling algorithm; $M$-matrix; maximal nonpositive solvent; quadratic convergence

## 1. Introduction

In this paper, we consider the problem of finding the maximal nonpositive solvent of the following quadratic matrix equation (QME)

$$Q_1(X) \equiv \widetilde{A}X^2 + \widetilde{B}X + \widetilde{C} = 0, \tag{1.1}$$

where

| |
|---|
| $\widetilde{A} \in \mathbb{R}^{n\times n}$ is a diagonal matrix with positive diagonal elements,<br>$\widetilde{B} \in \mathbb{R}^{n\times n}$ is a nonsingular $M$-matrix and<br>$\widetilde{C} \in \mathbb{R}^{n\times n}$ is an $M$-matrix such that $\widetilde{B}^{-1}\widetilde{C} \geq 0$. |

Such QME arises from an overdamped vibrating system [1, 2]. By left multiplying $\widetilde{A}^{-1}$ [3], without changing the $M$-matrix structure of it, QME (1.1) can be reduced to the following form

$$Q_2(X) \equiv X^2 + BX + C = 0, \tag{1.2}$$

where $B$ is a nonsingular $M$-matrix and $C$ is an $M$-matrix such that $B^{-1}C \geq 0$. It is known that Eq (1.2) has a maximal nonpositive solvent $\Phi$ under the condition that [3]

$$B - C - I \text{ is a nonsingular } M\text{-matrix.} \tag{1.3}$$

This solvent $\Phi$ is the one of interest.

Various iterative methods have been developed to obtain the maximal nonpositive solvent of QME (1.2) with assumption (1.3), including the Newton's method and Bernoulli-like methods (fixed-point iterative methods) [3], modified Bernoulli-like methods with diagonal update skill [4]. Newton's method is not competitive in terms of CPU time because it is to solve a generalized Sylvester matrix equation in each Newton's iterative step. The fixed-point iterative methods are usually linearly or sublinearly convergent and sometimes can be very slow [3].

There are many researches on iterative methods for other QMEs. For example, the cyclic reduction algorithm and the invariant subspace method for solving the quadratic matrix equation arising from quasi-birth-death problems [5, 6]; the methods for solving the quadratic matrix equation from quadratic eigenvalue problems [7–12]; the fixed-point iteration and the Schur method for solving the quadratic matrix equation arising in noisy Wiener-Hopf problems for Markov chains [13, 14] and others; see [15–18] and the references therein. Our work here is mainly inspired by recent study on highly accurate structure-preserving doubling algorithm for quadratic matrix equation from quasi-birth-and-death process [19]. Structure-preserving doubling algorithms are very efficient iterative methods for solving nonlinear matrix equations. For instance, some structure-preserving doubling algorithms are presented to solve continuous-time algebraic Riccati equations (ARE) [20], periodic discrete-time ARE [21], nonsymmetric ARE [22, 23] and $M$-matrix ARE [24, 25]. For more applications, please refer to [26, 27] and the references therein.

Yu et al. [3] proved $\rho(\Phi) \leq 1$ under (1.3). In this paper, we will slightly improve their result and prove that $\rho(\Phi) < 1$ under the same condition. The property $\rho(\Phi) < 1$ is important since it is desired for the quadratic convergence of structure-preserving doubling algorithms. Based on the new property $\rho(\Phi) < 1$, furthermore, we extend the structure-preserving doubling algorithm for the first standard form (SF1) [27] to solve QME (1.2) and give the quadratically convergent result.

The rest of this paper is organized as follows. In Section 2 we give some notations and state a few basic results on nonnegative and $M$-matrices. The main results of this paper are presented in Section 3. Numerical examples are given in Section 4 to demonstrate the performance of our method. Finally, conclusions are made in Section 5.

## 2. Notations and preliminaries

In this section, we first introduce some necessary notations and terminologies for this paper. $\mathbb{R}^{m\times m}$ is the set of all $m \times m$ real matrices, $\mathbb{R}^n = \mathbb{R}^{n\times 1}$, and $\mathbb{R} = \mathbb{R}^1$. $I_n$ (or simply $I$ if its dimension is clear from the context) is the $n \times n$ identity matrix. For $X \in \mathbb{R}^{m\times n}$, $X_{(i,j)}$ refers to its $(i, j)$th entry. Inequality

$X \leq Y$ means $X_{(i,j)} \leq Y_{(i,j)}$ for all $(i, j)$, and similarly for $X < Y$, $X \geq Y$, and $X > Y$. In particular, $X \geq 0$ means that $X$ is entrywise nonnegative and it is called a nonnegative matrix. $X$ is entrywise nonpositive if $-X$ is entrywise nonnegative. A matrix $A \in \mathbb{R}^{m\times n}$ is *positive*, denoted by $A > 0$, if all its entries are positive. The same understanding goes to vectors. For a square matrix $X$, denote by $\rho(X)$ its spectral radius. A matrix $A \in \mathbb{R}^{n\times n}$ is called a *Z-matrix* if $A_{(i,j)} \leq 0$ for all $i \neq j$. Any $Z$-matrix $A$ can be written as $sI - N$ with $N \geq 0$, and it is called an *M-matrix* if $s \geq \rho(N)$. Specifically, it is a *singular M-matrix* if $s = \rho(N)$, and a *nonsingular M-matrix* if $s > \rho(N)$.

The following results on nonnegative matrices and $M$-matrices can be found in, e.g., [28,29].

**Theorem 2.1.** *Let $A \in \mathbb{R}^{n\times n}$ be a nonnegative matrix. Then the spectral radius, $\rho(A)$, is an eigenvalue of A and there exists a nonnegative right eigenvector $\boldsymbol{x}$ associated with the eigenvalue $\rho(A)$: $A\boldsymbol{x} = \rho(A)\boldsymbol{x}$.*

**Theorem 2.2.** *Let $A \in \mathbb{R}^{n\times n}$ be a Z-matrix. Then the following statements are equivalent:*

(a) *A is a nonsingular M-matrix;*

(b) $A^{-1} \geq 0$*;*

(c) $A\boldsymbol{u} > 0$ *holds for some positive vector* $\boldsymbol{u} \in \mathbb{R}^n$.

**Theorem 2.3.** *Let $A \in \mathbb{R}^{n\times n}$ be an M-matrix. Let $B \in \mathbb{R}^{n\times n}$ be a Z-matrix. If A is nonsingular and $B \geq A$, then B is also a nonsingular M-matrix.*

## 3. The main results

In this section, we give the main results of this paper. Lemma 3.1 below can be found in [3, Theorem 3.1]. The first goal of this paper is to further prove that $\rho(\Phi) < 1$.

**Lemma 3.1.** *Suppose* (1.3)*, then* QME (1.2) *has a maximal nonpositive solvent $\Phi$ with $\rho(\Phi) \leq 1$, also $B + \Phi$ and $B + \Phi - C$ are both nonsingular M-matrices.*

Since $B - C - I$ is a nonsingular $M$-matrix, by Theorems 2.2, there exists a vector $0 < \boldsymbol{u} \in \mathbb{R}^n$ such that

$$\boldsymbol{v} = (B - C - I)\boldsymbol{u} > 0.$$

Unless stated otherwise, throughout the rest of this paper, $\boldsymbol{u}$ and $\boldsymbol{v}$ are reserved for the ones here. The following lemma is inspired by [19, Lemma 3.2], we still give the proof for completeness.

**Lemma 3.2.** *Suppose* (1.3)*, i.e., $B-C-I$ is a nonsingular M-matrix. Then $\rho(X) \neq 1$ for any nonpositive solvent X of Eq* (1.2)*.*

*Proof.* Suppose, to the contrary, that $\rho(X) = 1$ (which is equivalent to $\rho(-X) = 1$), where $X$ is a nonpositive solvent of Eq (1.2). Then according to Theorem 2.1, there exists a nonzero and nonnegative vector $\boldsymbol{z} \in \mathbb{R}^n$ such that $-X\boldsymbol{z} = \boldsymbol{z}$ and thus

$$(X^2 + BX + C)\boldsymbol{z} = 0$$

implies that $(B - C - I)\boldsymbol{z} = 0$, which contradicts with the fact that $B - C - I$ is nonsingular.

Combining Lemmas 3.1 and 3.2, we immediately finish our first goal of this paper. Moreover, we have the following theorem. The theorem is implied by [19, Theorem 3.1] or [30, Theorem 2.3].

**Theorem 3.3.** *Under the assumption* (1.3)*,* QME (1.2) *has a unique maximal nonpositive solvent* $\Phi$*. Moreover, it holds that* $\Phi \le X_0$ *and*

$$-\Phi \boldsymbol{u} \le \boldsymbol{u} - B^{-1}\boldsymbol{v},$$

*where* $X_0 = -B^{-1}C$ *is as defined in Eq* (3.3a)*.*

It can be checked that Theorem 3.3 is applicable to

$$CY^2 + BY + I = 0, \tag{3.1}$$

which is called the *dual equation* of Eq (1.2). In particular, under assumption (1.3), the dual equation Eq (3.1) also has a unique maximal nonpositive solvent, denoted by $\Psi$ hereafter. In conclusion, Theorem 3.4 below gives some of the important results, the proof is similar to that of [19, Theorem 3.4] and thus it is omitted here.

**Theorem 3.4.** *Suppose* (1.3)*. The following statements hold.*

(a) *We have*

$$\begin{aligned} \Phi \le X_0 = -B^{-1}C \le 0, &\quad -\Phi \boldsymbol{u} \le \boldsymbol{u} - B^{-1}\boldsymbol{v}, \\ \Psi \le Y_0 = -B^{-1} \le 0, &\quad -\Psi \boldsymbol{u} \le \boldsymbol{u} - B^{-1}\boldsymbol{v}. \end{aligned}$$

(b) $\rho(\Phi) < 1$ *and* $\rho(\Psi) < 1$.

(c) $I - \Phi\Psi$ *and* $I - \Psi\Phi$ *are nonsingular M-matrices.*

Now we are in position to develop a new structure-preserving doubling algorithm for solving the QME (1.2). Similar to the discussion in the introduction of [19], QME (1.2) is connected with the matrix pencil

$$\mathscr{A}_0 \begin{bmatrix} I \\ X \end{bmatrix} = \mathscr{B}_0 \begin{bmatrix} I \\ X \end{bmatrix} X, \tag{3.2}$$

where

$$\mathscr{A}_0 = \begin{bmatrix} -B^{-1}C & 0 \\ B^{-1}C & I \end{bmatrix} =: \begin{matrix} {\scriptstyle n} \\ {\scriptstyle n} \end{matrix} \overset{\begin{matrix} {\scriptstyle n} & {\scriptstyle n} \end{matrix}}{\begin{bmatrix} E_0 & 0 \\ -X_0 & I \end{bmatrix}}, \tag{3.3a}$$

$$\mathscr{B}_0 = \begin{bmatrix} I & B^{-1} \\ 0 & -B^{-1} \end{bmatrix} =: \begin{matrix} {\scriptstyle n} \\ {\scriptstyle n} \end{matrix} \overset{\begin{matrix} {\scriptstyle n} & {\scriptstyle n} \end{matrix}}{\begin{bmatrix} I & -Y_0 \\ 0 & F_0 \end{bmatrix}}. \tag{3.3b}$$

Now that the matrix pencil $\mathscr{A}_0 - \lambda\mathscr{B}_0$ is in (SF1), it is natural for us to apply the doubling algorithm (see Algorithm 1) for (SF1) [27] to solve Eq (3.2).

The basic idea of the structure-preserving doubling algorithm for (SF1) [23,27] for solving Eq (3.2) is to recursively construct a sequence of matrix pencils $\mathscr{A}_i - \lambda\mathscr{B}_i$ for $i \ge 1$ that have the same block structure as $\mathscr{A}_0 - \lambda\mathscr{B}_0$:

$$\mathscr{A}_i = \begin{matrix} {\scriptstyle n} \\ {\scriptstyle n} \end{matrix} \overset{\begin{matrix} {\scriptstyle n} & {\scriptstyle n} \end{matrix}}{\begin{bmatrix} E_i & 0 \\ -X_i & I \end{bmatrix}}, \quad \mathscr{B}_i = \begin{matrix} {\scriptstyle n} \\ {\scriptstyle n} \end{matrix} \overset{\begin{matrix} {\scriptstyle n} & {\scriptstyle n} \end{matrix}}{\begin{bmatrix} I & -Y_i \\ 0 & F_i \end{bmatrix}} \quad \text{for } i = 1, 2, \ldots \tag{3.5}$$

**Algorithm 1** Doubling Algorithm for (SF1) [27]

**Input:** $X_0,\ Y_0,\ E_0,\ F_0 \in \mathbb{R}^{n\times n}$ determined by Eq (3.3).
**Output:** $X_\infty$ as the limit of $X_i$ if it converges.

1: **for** $i = 0, 1, \dots,$ until convergence **do**
2: compute $E_{i+1},\ F_{i+1},\ X_{i+1},\ Y_{i+1}$ according to

$$E_{i+1} = E_i(I - Y_iX_i)^{-1}E_i, \tag{3.4a}$$
$$F_{i+1} = F_i(I - X_iY_i)^{-1}F_i, \tag{3.4b}$$
$$X_{i+1} = X_i + F_i(I - X_iY_i)^{-1}X_iE_i, \tag{3.4c}$$
$$Y_{i+1} = Y_i + E_i(I - Y_iX_i)^{-1}Y_iF_i. \tag{3.4d}$$

3: **end for**
4: **return** $X_i$ at convergence as the computed solution.

and at the same time

$$\mathscr{A}_i\begin{bmatrix} I \\ X \end{bmatrix} = \mathscr{B}_i\begin{bmatrix} I \\ X \end{bmatrix}M^{2^i} \quad \text{for } i = 0, 1, \dots,$$

where $M = X$.

We observe that as long as $E_k,\ F_k,\ X_k,\ Y_k$ are well-defined (so are $\mathscr{A}_k$ and $\mathscr{B}_k$), we will have

$$\mathscr{A}_k\begin{bmatrix} I \\ \Phi \end{bmatrix} = \mathscr{B}_k\begin{bmatrix} I \\ \Phi \end{bmatrix}\Phi^{2^k}, \quad \mathscr{A}_k\begin{bmatrix} \Psi \\ I \end{bmatrix}\Psi^{2^k} = \mathscr{B}_k\begin{bmatrix} \Psi \\ I \end{bmatrix},$$

where $\mathscr{A}_k$ and $\mathscr{B}_k$ are defined as in Eq (3.5). Or, equivalently,

$$\Phi - X_k = F_k\Phi^{2^k+1}, \quad E_k = (I - Y_k\Phi)\Phi^{2^k}, \tag{3.6a}$$
$$\Psi - Y_k = E_k\Psi^{2^k+1}, \quad F_k = (I - X_k\Psi)\Psi^{2^k}. \tag{3.6b}$$

In the following we will analysis the convergence of Algorithm 1 for solving the QME (1.2) under the assumption (1.3). Theorem 3.5 below is essentially [19, Theorem 6.1] or [23, Theorem 4.1]. The only difference lies in the initial matrices $(E_0, F_0, X_0, Y_0)$.

**Theorem 3.5.** *Under* (1.3)*, the matrix sequences* $\{E_k\}, \{F_k\}, \{X_k\}$ *and* $\{Y_k\}$ *generated by* Algorithm 1 *are well-defined and, moreover, for* $k \geq 1$*,*

(a) $E_k = (I - Y_k\Phi)\Phi^{2^k} \geq 0$*;*

(b) $F_k = (I - X_k\Psi)\Psi^{2^k} \geq 0$*;*

(c) $I - X_kY_k$ *and* $I - Y_kX_k$ *are nonsingular M-matrices;*

(d) $\Phi \leq X_k \leq X_{k-1} \leq 0,\ \Psi \leq Y_k \leq Y_{k-1} \leq 0$*, and*

$$0 \leq X_k - \Phi \leq \Psi^{2^k}(-\Phi)\Phi^{2^k},\ 0 \leq Y_k - \Psi \leq \Phi^{2^k}(-\Psi)\Psi^{2^k}. \tag{3.7}$$

*Proof.* We prove the theorem by mathematical induction.

Since $B$ is a nonsingular $M$-matrix, we immediately conclude that $X_0$, $Y_0$, $E_0$ and $F_0$ are well-defined as in Eq (3.3) and they are all nonpositive. From Theorem 3.4(a), we obtain that $\Phi \le X_0 \le 0$, $\Psi \le Y_0 \le 0$. Therefore,

$$I - X_0Y_0 \ge I - \Phi\Psi, \quad I - Y_0X_0 \ge I - \Psi\Phi.$$

By Theorem 3.4(c), both $I - \Phi\Psi$ and $I - \Psi\Phi$ are nonsingular $M$-matrices; so are $I - X_0Y_0$ and $I - Y_0X_0$ according to Theorem 2.3. Hence, the matrices $E_1$, $X_1$, $F_1$, $Y_1$ generated by the Algorithm 1 are well-defined. Moreover, from Eq (3.4) we have

$$\begin{aligned}
E_1 &= E_0(I - Y_0X_0)^{-1}E_0 \ge 0, \\
F_1 &= F_0(I - X_0Y_0)^{-1}F_0 \ge 0, \\
X_1 &= X_0 + F_0(I - X_0Y_0)^{-1}X_0E_0 \le X_0, \\
Y_1 &= Y_0 + E_0(I - Y_0X_0)^{-1}Y_0F_0 \le Y_0.
\end{aligned}$$

Let $k = 1$ in Eq (3.6), we have

$$\Phi - X_1 = F_1\Phi^3, \quad E_1 = (I - Y_1\Phi)\Phi^2 \ge 0, \tag{3.8a}$$
$$\Psi - Y_1 = E_1\Psi^3, \quad F_1 = (I - X_1\Psi)\Psi^2 \ge 0. \tag{3.8b}$$

Noting that $F_1$, $E_1 \ge 0$ and $\Phi$, $\Psi \le 0$, it follows from Eq (3.8) that $\Phi \le X_1 \le X_0 \le 0$, $\Psi \le Y_1 \le Y_0 \le 0$. By the same reasoning above, we can conclude that $I - X_1Y_1$ and $I - Y_1X_1$ are nonsingular $M$-matrices. Furthermore, it follows from Eq (3.8), $\Phi \le X_1 \le 0$ and $\Psi \le Y_1 \le 0$ that

$$\begin{aligned}
0 \le X_1 - \Phi = F_1(-\Phi)\Phi^2 = (I - X_1\Psi)\Psi^2(-\Phi)\Phi^2 = \Psi^2(-\Phi)\Phi^2 + X_1\Psi^3\Phi^3 \le \Psi^2(-\Phi)\Phi^2, \\
0 \le Y_1 - \Psi = E_1(-\Psi)\Psi^2 = (I - Y_1\Phi)\Phi^2(-\Psi)\Psi^2 = \Phi^2(-\Psi)\Psi^2 + Y_1\Phi^3\Psi^3 \le \Phi^2(-\Psi)\Psi^2.
\end{aligned}$$

This completes the proof of our results for $k = 1$.

Next, suppose that the results hold for all positive integers $k \le \ell$. Hence $E_{\ell+1}$, $X_{\ell+1}$, $F_{\ell+1}$, $Y_{\ell+1}$ are well-defined by Eq (3.4), which, together with the induction hypothesis, guarantee that

$$E_{\ell+1} \ge 0, \; F_{\ell+1} \ge 0, \; X_{\ell+1} \le X_\ell \le 0, \; Y_{\ell+1} \le Y_\ell \le 0. \tag{3.9}$$

On the other hand, Eqs (3.9) and (3.6) for $k = \ell + 1$ say

$$\Phi - X_{\ell+1} = F_{\ell+1}\Phi^{2^{\ell+1}+1} \le 0, \quad E_{\ell+1} = (I - Y_{\ell+1}\Phi)\Phi^{2^{\ell+1}} \ge 0, \tag{3.10a}$$
$$\Psi - Y_{\ell+1} = E_{\ell+1}\Psi^{2^{\ell+1}+1} \le 0, \quad F_{\ell+1} = (I - X_{\ell+1}\Psi)\Psi^{2^{\ell+1}} \ge 0. \tag{3.10b}$$

Thus we have

$$I - X_{\ell+1}Y_{\ell+1} \ge I - \Phi\Psi, \quad I - Y_{\ell+1}X_{\ell+1} \ge I - \Psi\Phi.$$

Following the same line as the proof of the $k = 1$ case, we conclude that $I - X_{\ell+1}Y_{\ell+1}$ and $I - Y_{\ell+1}X_{\ell+1}$ are nonsingular $M$-matrices. Similarly, we deduce from Eq (3.10) that

$$\begin{aligned}
0 \le X_{\ell+1} - \Phi \le \Psi^{2^{\ell+1}}(-\Phi)\Phi^{2^{\ell+1}}, \\
0 \le Y_{\ell+1} - \Psi \le \Phi^{2^{\ell+1}}(-\Psi)\Psi^{2^{\ell+1}}.
\end{aligned}$$

By the induction principle, we have finished the proof.

From Eq (3.7) and Theorem 3.4(b), we can conclude that $X_k$ and $Y_k$ generated by Algorithm 1 converge quadratically to $\Phi$ and $\Psi$, respectively, under assumption (1.3).

## 4. Numerical examples

In this section, we will present numerical results applying Algorithm 1 to solve QME (1.2). We will compare Algorithm 1 (referred to as DA) with two Bernoulli-like methods presented in [3] (referred to, respectively, as BL1 and BL2 as in [4]) and three modified Bernoulli-like methods with diagonal update skill [4] (referred to as BL1-DU, BL2-DU1 and BL2-DU2, respectively). In reporting numerical results, we will record the numbers of iterations (denoted by "Iter"), the elapsed CPU time in seconds (denoted as "CPU") and plot iterative history curves for normalized residual NRes defined by

$$\mathrm{NRes}(X_k) = \frac{\|X_k^2 + BX_k + C\|_\infty}{\|X_k\|_\infty(\|X_k\|_\infty + \|B\|_\infty) + \|C\|_\infty}.$$

All runs terminate if the current iteration satisfies either NRes $< 10^{-12}$ or the number of the prescribed iteration $k_{\max} = 1000$ is exceeded. According to DA, we set $X_0 = -B^{-1}C$ for all methods. All experiments are implemented in MATLAB R2018b with a machine precision $2.22 \times 10^{-16}$ on a PC Windows 10 operating system with an Intel i7-9700 CPU and 8GB RAM.

**Example 1** ( [4]). *Consider the Eq* (1.2) *with*

$$B = \begin{bmatrix} 20 & -10 & & & & \\ -10 & 30 & -10 & & & \\ & -10 & 30 & -10 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -10 & 30 & -10 \\ & & & & -10 & 20 \end{bmatrix}, \quad C = \begin{bmatrix} 15 & -5 & & & & \\ -5 & 15 & -5 & & & \\ & -5 & 15 & -5 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -5 & 15 & -5 \\ & & & & -5 & 15 \end{bmatrix}.$$

**Table 1.** Numerical results for Example 1.

| | $n = 30$ | | | $n = 100$ | | |
|---|---|---|---|---|---|---|
| *Method* | Iter | CPU | NRes | Iter | CPU | NRes |
| DA | 4 | 0.0030 | $1.0292 \times 10^{-16}$ | 4 | 0.0082 | $1.0286 \times 10^{-16}$ |
| BL1 | 10 | 0.0021 | $1.3375 \times 10^{-13}$ | 10 | 0.0052 | $1.3380 \times 10^{-13}$ |
| BL1-DU | 7 | 0.0024 | $5.5076 \times 10^{-13}$ | 7 | 0.0055 | $5.5070 \times 10^{-13}$ |
| BL2 | 12 | 0.0023 | $8.4738 \times 10^{-13}$ | 12 | 0.0050 | $8.4740 \times 10^{-13}$ |
| BL2-DU1 | 11 | 0.0025 | $1.5404 \times 10^{-13}$ | 11 | 0.0054 | $1.5408 \times 10^{-13}$ |
| BL2-DU2 | 9 | 0.0027 | $1.0203 \times 10^{-13}$ | 9 | 0.0049 | $1.0197 \times 10^{-13}$ |

*In Table 1, we record the numerical results for Example 1. We find that* DA *uses the smallest number of iterations and delivers the lowest value of NRes within all the tested methods. For this example, however,* DA *is not the fastest one in terms of the elapsed CPU time. The reason is that each step of the* DA *iteration is expensive than that of the other methods and the iteration number of* DA *can not compensate its additional cost such that* DA *beats the other methods. Figure 1 plots the convergent history for Example 1. Quadratic monotonic convergence of* DA *and monotonic linear convergence of Bernoulli-like methods clearly show.*

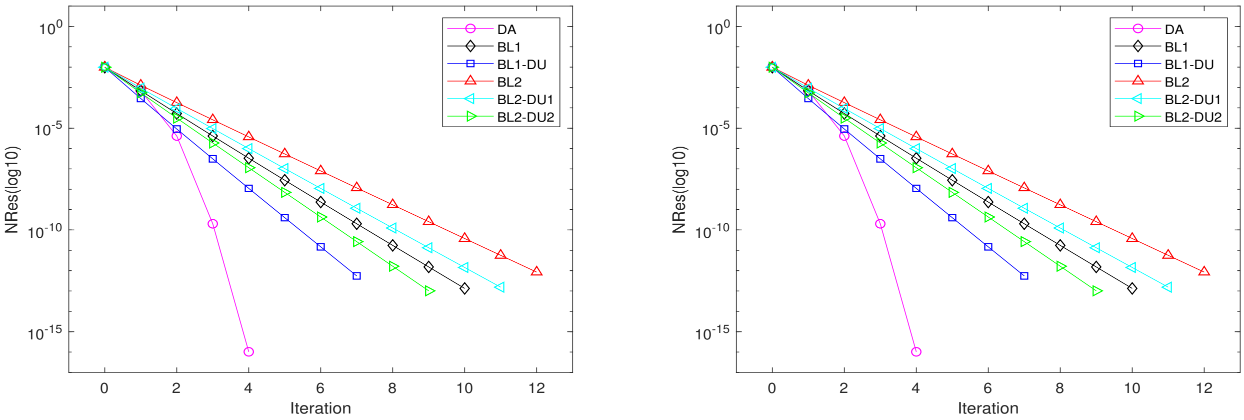


**Figure 1.** Convergent history curves for Example 1. The left one is for $n = 30$ and the right one is for $n = 100$.

**Example 2** ( [4]). *Consider the Eq* (1.2) *with*

$$B = \begin{bmatrix} 4 & -1 & & & & \\ -1 & 4 & -1 & & & \\ & -1 & 4 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 4 & -1 \\ & & & & -1 & 4 \end{bmatrix}, \quad C = I.$$

*Table 2 displays the the numerical results for Example 2. We find that* DA *is the best one for this example in terms of Iter, CPU and NRes. The iteration number of* DA *compensates its additional cost such that* DA *beats the other methods in terms of the elapsed CPU time. Figure 2 shows the convergent history for Example 2. Quadratic monotonic convergence of* DA *and monotonic linear convergence of Bernoulli-like methods again clearly show.*

**Table 2.** Numerical results for Example 2.

| | $n = 30$ | | | $n = 100$ | | |
|---|---|---|---|---|---|---|
| *Method* | Iter | CPU | NRes | Iter | CPU | NRes |
| DA | 7 | 0.0041 | $3.1621 \times 10^{-14}$ | 9 | 0.0136 | $1.9857 \times 10^{-16}$ |
| BL1 | 110 | 0.0046 | $9.8576 \times 10^{-13}$ | 324 | 0.0905 | $9.8002 \times 10^{-13}$ |
| BL1-DU | 77 | 0.0054 | $9.0078 \times 10^{-13}$ | 226 | 0.0668 | $9.8568 \times 10^{-13}$ |
| BL2 | 209 | 0.0070 | $9.0390 \times 10^{-13}$ | 636 | 0.1365 | $9.7354 \times 10^{-13}$ |
| BL2-DU1 | 175 | 0.0069 | $9.7996 \times 10^{-13}$ | 538 | 0.1153 | $9.7411 \times 10^{-13}$ |
| BL2-DU2 | 142 | 0.0059 | $9.4024 \times 10^{-13}$ | 440 | 0.0960 | $9.7441 \times 10^{-13}$ |

## 5. Conclusions

The structure-preserving doubling algorithm for (SF1) [27] is extended to compute the maximal nonpositive solvent of a type of QMEs. It is shown that the approximations generated by the algorithm

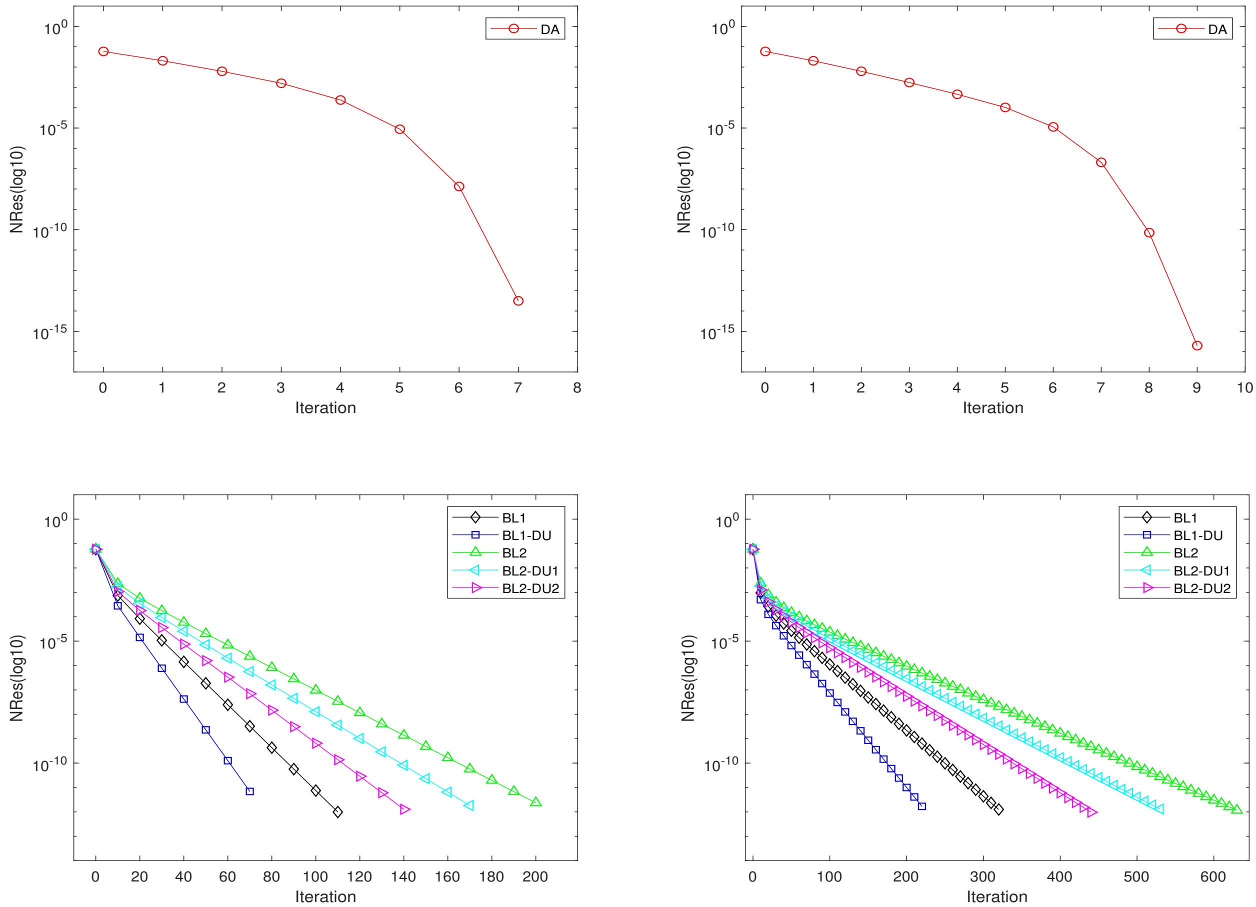


**Figure 2.** Convergent history curves for Example 2. The left two are for $n = 30$ and the right two are for $n = 100$.

are globally monotonically and quadratically convergent. Two numerical examples are presented to demonstrate the feasibility and effectiveness of our method. Our work here can be seen as a new application of the structure-preserving doubling algorithm for (SF1).

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## Conflict of interest

The author declares there is no conflicts of interest.

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