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Research article

Entire positive k-convex solutions to k-Hessian type equations and systems

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Abstract: In this paper, we study the existence of entire positive solutions for the k-Hessian type equation

$$\mathrm{S}_k(D^2u+\alpha I)=p(|x|)f^k(u),\quad x\in\mathbb{R}^n$$

and system

$$\begin{cases}\mathrm{S}_k(D^2u+\alpha I)=p(|x|)f^k(v), & x\in\mathbb{R}^n,\\ \mathrm{S}_k(D^2v+\alpha I)=q(|x|)g^k(u), & x\in\mathbb{R}^n,\end{cases}$$

where D^2u is the Hessian of u and I denotes unit matrix. The arguments are based upon a new monotone iteration scheme.

Keywords: k-Hessian type equation and system; entire positive k-convex solution; monotone iterative; existence

1. Introduction

Consider the existence of entire positive k-convex solutions to the following k-Hessian type equation

$$S_k(D^2u+\alpha I)=p(|x|)f^k(u),\quad x\in\mathbb{R}^n, \tag{E}$$

and system

$$\begin{cases}S_k(D^2u+\alpha I)=p(|x|)f^k(v), & x\in\mathbb{R}^n,\\ S_k(D^2v+\alpha I)=q(|x|)g^k(u), & x\in\mathbb{R}^n,\end{cases} \tag{S}$$

where $k\in\{1,2,\dots,n\}$, $\alpha\geq 0$ is a constant, I is the identity function and p,q are continuous functions on $[0,+\infty)$. Letting $D^2u=(\frac{\partial^2 u}{\partial x_i\partial x_j})$ denote the Hessian of $u\in C^2(\mathbb{R}^n)$ and λ_i $(i\in\{1,2,\dots,n\})$ denote the eigenvalues of D^2u, then

$$S_k(D^2u+\alpha I)=\sum_{\substack{1\le i_1<...<i_k\le n\\ 1\le j_1<...<j_k\le n}}\frac{1}{k!}\delta^{i_1,i_2,...,i_k}_{j_1,j_2,...,j_k}(\lambda_{i_1}+\alpha)(\lambda_{i_2}+\alpha)...(\lambda_{i_k}+\alpha),$$

where $\delta^{i_1,i_2,...,i_k}_{j_1,j_2,...,j_k}$ is the generalized Kronecker symbol, is the k-Hessian type operator. When $\alpha=0$, $S_k(D^2u)$ is the standard k-Hessian operator.

Denote

$$\Gamma_k:=\{\lambda\in\mathbb{R}^n: S_j(\lambda)>0, 1\le j\le k\}.$$

We call a function $u\in C^2(\mathbb{R}^n)$ k-convex in $\mathbb{R}^n$ if $\lambda(D^2u(x)+\alpha I)\in\Gamma_k$ for all $x\in\mathbb{R}^n$.

In particular,

$$S_1(D^2u+\alpha I)=\sum_{i=1}^n\lambda_i=\Delta u;$$
$$S_n(D^2u+\alpha I)=\prod_{i=1}^n\lambda_i=\det(D^2u+\alpha I).$$

The k-Hessian equation is fully nonlinear PDEs for $k\neq 1$ (see Urbas [1] and Wang [2]), and there are many important applications in fluid mechanics, geometric problems and other applied subjects. Many authors have demonstrated increasing interest in k-Hessian equations by different methods, for instance, see ([3–9]) and the references cited therein([10–15]). In particular, problem (E) reduces to the problems studied by Keller [16] and Osserman [17] when $k=1$, $p(|x|)=1$ on $\mathbb{R}^n$ and $f:[0,\infty)\to[0,\infty)$ is continuous and increasing. The authors studied a necessary and sufficient condition

$$\int_1^\infty\frac{dt}{\sqrt{2F(t)}}=\infty,\quad F(t)=\int_0^t f(s)ds$$

for the existence of entire large positive radial solutions to (E). When $k=1$, $f(u)=u^\gamma$ ($\gamma\in(0,1]$) and $p:[0,\infty)\to[0,\infty)$ is continuous, Lair and Wood [18] showed that (E) admits infinitely many entire large positive radial solutions if and only if

$$\int_0^\infty rp(r)dr=\infty.$$

For the case $k=1$, system (S) reduces to the following problem

$$\begin{cases}\Delta u=p(|x|)f(v),\ x\in\mathbb{R}^n,\\ \Delta v=q(|x|)g(u),\ x\in\mathbb{R}^n.\end{cases}\tag{1.1}$$

Lair and Wood [19] analyzed the existence and nonexistence of entire positive radial solutions to Eq (1.1) when $f(v)=v^\beta$, $g(u)=u^\gamma$ ($0<\beta\le\gamma$). For the further results, we can see [20–23] and the reference therein.

When $\alpha=0$, Zhang and Zhou [24] considered the existence of entire positive k-convex solutions to problem (E) and system (S).

For the case $k = n$, Zhang and Liu [25] studied the existence of entire radial large solutions for a Monge-Ampère type equation

$$\det(D^2u) - \alpha\Delta u = a(|x|)f(u),\ x \in \mathbb{R}^n \tag{1.2}$$

and system

$$\begin{cases}\det(D^2u) - \alpha\Delta u = a(|x|)f(v),\ x \in \mathbb{R}^n,\\ \det(D^2v) - \beta\Delta v = b(|x|)g(u),\ x \in \mathbb{R}^n.\end{cases} \tag{1.3}$$

Their results have been improved by Covei [26].

Recently, when $p(|x|) \equiv 1$ on $\mathbb{R}^n$, Dai [27] showed that there exists a subsolution $u \in C^2(\mathbb{R}^n)$ of (E) if and only if

$$\int^{\infty}\Big(\int_0^{\tau} f(t)dt\Big)^{-\frac{1}{k+1}} d\tau = \infty$$

holds.

Motivated by these works mentioned above, in this paper we will obtain some new results on the existence of entire positive k-convex radial solutions for equation (E) and system (S). The arguments are based upon a new monotone iteration scheme.

Let α_0 denote a positive constant. In the following, we always suppose that

$$\alpha_0 > \frac{n}{k}(\frac{nC_{n-1}^{k-1}}{k})^{\frac{1}{k}}\alpha = \frac{n}{k}(C_n^k)^{\frac{1}{k}}\alpha. \tag{1.4}$$

We give the following conditions:

$(f1)$ $f, g :[0,\infty)\to (\alpha_0, \infty)$ are continuous and nondecreasing;

$(f2)$ $p, q :[0,\infty)\to (0, \infty)$ are continuous and nondecreasing.

Define

$$P(\infty) := \lim_{r\to\infty} P(r),\quad P(r) := \int_0^r \Big(\frac{t^{k-n}}{C_0}\int_0^t s^{n-1}p(s)ds\Big)^{\frac{1}{k}} dt,\ r \geq 0; \tag{1.5}$$

$$Q(\infty) := \lim_{r\to\infty} Q(r),\quad Q(r) := \int_0^r \Big(\frac{t^{k-n}}{C_0}\int_0^t s^{n-1}q(s)ds\Big)^{\frac{1}{k}} dt,\ r \geq 0, \tag{1.6}$$

where

$$C_0 = \frac{C_{n-1}^{k-1}}{k}.$$

For an arbitrary $a > 0$, we also define

$$H_{1a}(\infty) := \lim_{r\to\infty} H_{1a}(r),\quad H_{1a}(r) := \int_a^r \frac{d\tau}{f(\tau)},\ r \geq a; \tag{1.7}$$

$$H_{2a}(\infty) := \lim_{r\to\infty} H_{2a}(r),\quad H_{2a}(r) := \int_a^r \frac{d\tau}{f(\tau)+g(\tau)},\ r \geq a, \tag{1.8}$$

and we see that

$$H'_{1a}(r) = \frac{1}{f(r)} > 0,\quad H'_{2a}(r) = \frac{1}{f(r)+g(r)} > 0,\ \forall r > a,$$

and H_{1a}, H_{2a} admit the inverse functions H_{1a}^{-1} and H_{2a}^{-1} on $[0, H_{1a}(\infty))$ and $[0, H_{2a}(\infty))$ respectively.

The main results of this paper can be stated as follows.

Theorem 1.1. *Suppose that $(f1)$ and $(f2)$ hold. If $\alpha = 0$, then Eq (E) admits an entire positive k-convex radial solution $u \in C^2(\mathbb{R}^n)$ satisfying*

$$a + \alpha_0 P(r) \le u \le H_{1a}^{-1}(P(r)),\ \forall r \ge 0.$$

Moreover, if $P(\infty) = \infty$ and $H_{1a}(\infty) = \infty$, then $\lim_{r\to\infty} u(r) = \infty$; if $P(\infty) < H_{1a}(\infty) < \infty$, then u is bounded.
If $\alpha > 0$ and $p(|x|) \ge 1$, then Eq (E) admits an entire positive k-convex radial solution $u \in C^2(\mathbb{R}^n)$ satisfying

$$a + \alpha_0 P(r) - \frac{\alpha}{2} r^2 \le u \le H_{1a}^{-1}(P(r)),\ \forall r \ge 0.$$

If further suppose $H_{1a}(\infty) = \infty$, then $\lim_{r\to\infty} u(r) = \infty$.

Theorem 1.2. *Suppose that $(f1)$ and $(f2)$ hold. If $\alpha = 0$, then system (S) admits an entire positive k-convex radial solution $(u, v) \in C^2(\mathbb{R}^n) \times C^2(\mathbb{R}^n)$ satisfying*

$$\frac{a}{2} + \alpha_0 P(r) \le u \le H_{2a}^{-1}(P(r) + Q(r)),\ \forall r \ge 0;$$
$$\frac{a}{2} + \alpha_0 Q(r) \le v \le H_{2a}^{-1}(P(r) + Q(r)),\ \forall r \ge 0.$$

Moreover, if $P(\infty) = \infty = Q(\infty)$ and $H_{2a}(\infty) = \infty$, then $\lim_{r\to\infty} u(r) = \lim_{r\to\infty} v(r) = \infty$; if $P(\infty) + Q(\infty) < H_{2a}(\infty) < \infty$, then u and v are bounded.
If $\alpha > 0$ and $p(|x|) \ge 1$, $q(|x|) \ge 1$, then system (S) admits an entire positive k-convex radial solution $(u, v) \in C^2(\mathbb{R}^n) \times C^2(\mathbb{R}^n)$ satisfying

$$\frac{a}{2} + \alpha_0 P(r) - \frac{\alpha}{2} r^2 \le u \le H_{2a}^{-1}(P(r) + Q(r)),\ \forall r \ge 0;$$
$$\frac{a}{2} + \alpha_0 Q(r) - \frac{\alpha}{2} r^2 \le v \le H_{2a}^{-1}(P(r) + Q(r)),\ \forall r \ge 0.$$

If further suppose $H_{2a}(\infty) = \infty$, $\lim_{r\to\infty} u(r) = \lim_{r\to\infty} v(r) = \infty$.

2. Preliminary lemmas

For convenience, we give some lemmas for the radial functions before proving the main results.
Let $r = |x| = \sqrt{x_1^2 + ... + x_n^2}$ and $B_R := \{x \in \mathbb{R}^n : |x| < R\}$ for $R \in (0, \infty]$.

Lemma 2.1. *(Lemma 2.1, [25]) Suppose that $\varphi \in C^2[0, R)$ with $\varphi'(0) = 0$. Then, for $u(x) = \varphi(r)$, we have $u(x) \in C^2(B_R)$, and the eigenvalues of $D^2u + \alpha I$ are*

$$\lambda(D^2u + \alpha I) = \begin{cases} (\varphi''(r) + \alpha, \frac{\varphi'(r)}{r} + \alpha, ..., \frac{\varphi'(r)}{r} + \alpha), & r \in (0, R), \\ (\varphi''(0) + \alpha, \varphi''(0) + \alpha, ..., \varphi''(0) + \alpha), & r = 0, \end{cases}$$

and so

$$S_k(D^2u + \alpha I) = \begin{cases} C_{n-1}^{k-1}(\varphi''(r) + \alpha)\frac{(\varphi'(r)+\alpha r)^{k-1}}{r^{k-1}} + C_{n-1}^{k}\frac{(\varphi'(r)+\alpha r)^{k}}{r^{k}}, & r \in (0, R), \\ C_n^k(\varphi''(0) + \alpha)^k, & r = 0. \end{cases}$$

By Lemma 2.1, we can conclude that $u(x) = \varphi(r)$ is a C^2 radial solution of (E) if and only if $\varphi(r)$ satisfies

$$C_{n-1}^{k-1}(\varphi''(r)+\alpha)\frac{(\varphi'(r)+\alpha r)^{k-1}}{r^{k-1}}+C_{n-1}^{k}\frac{(\varphi'(r)+\alpha r)^{k}}{r^{k}}=p(r)f^k(\varphi(r)),\ r\in(0,R). \tag{2.1}$$

Lemma 2.2. *Suppose that $(f1)$ and $(f2)$ hold. For any positive number a, let $\varphi \in C[0,R)\cap C^1(0,R)$ be a solution of the Cauchy problem*

$$\begin{cases}\varphi'(r)=\left(\frac{r^{k-n}}{C_0}\int_0^r s^{n-1}p(s)f^k(\varphi(s))ds\right)^{\frac{1}{k}}-\alpha r,\ r>0,\\ \varphi(0)=a>0.\end{cases} \tag{2.2}$$

Then $\varphi\in C^2[0,R)$, and it satisfies (2.1) with $\varphi'(0)=0$.

Proof. Firstly, we have

$$\begin{aligned}\varphi'(0)&=\lim_{r\to0}\frac{\varphi(r)-\varphi(0)}{r-0}\\&=\lim_{r\to0}\varphi'(r)\\&=\lim_{r\to0}\left(\frac{r^{k-n}}{C_0}\int_0^r s^{n-1}p(s)f^k(\varphi(s))ds\right)^{\frac{1}{k}}-\alpha r=0.\end{aligned}$$

Since

$$\lim_{r\to0}\varphi'(r)=\lim_{r\to0}\left(\frac{r^{k-n}}{C_0}\int_0^r s^{n-1}p(s)f^k(\varphi(s))ds\right)^{\frac{1}{k}}-\alpha r=0=\varphi'(0).$$

This shows that $\varphi(r)\in C^1[0,R)$.

Secondly,

$$\begin{aligned}\varphi''(0)&=\lim_{r\to0}\frac{\varphi'(r)-\varphi'(0)}{r-0}\\&=\lim_{r\to0}\frac{\left(\frac{r^{k-n}}{C_0}\int_0^r s^{n-1}p(s)f^k(\varphi(s))ds\right)^{\frac{1}{k}}-\alpha r}{r}\\&=\lim_{r\to0}\left(\frac{\frac{k-n}{C_0}r^{k-n-1}\int_0^r s^{n-1}p(s)f^k(\varphi(s))ds+\frac{r^{k-1}}{C_0}p(r)f^k(\varphi(r))}{r^k}\right)^{\frac{1}{k}}-\alpha\\&=\left(\frac{1}{nC_0}p(0)\right)^{\frac{1}{k}}f(\varphi(0))-\alpha.\end{aligned}$$

It is easy to know that $\varphi(r)\in C^2(0,R)$ for $r\in(0,R)$. By calculating,

$$\begin{aligned}\lim_{r\to0}\varphi''(r)&=\lim_{r\to0}\frac{k-n}{k}r^{-\frac{n}{k}}(\int_0^r\frac{1}{C_0}p(s)f^k(\varphi(s))s^{n-1}ds)^{\frac{1}{k}}\\&\quad+\lim_{r\to0}\frac{1}{kC_0}r^{\frac{k-n}{k}}(\int_0^r\frac{1}{C_0}p(s)f^k(\varphi(s))s^{n-1}ds)^{\frac{1}{k}-1}r^{n-1}p(r)f^k(\varphi(r))-\alpha\end{aligned}$$

$$= \frac{k-n}{k}\Big(\frac{1}{nC_0}p(0)\Big)^{\frac{1}{k}} f(\varphi(0)) + \frac{n}{k}\Big(\frac{1}{nC_0}p(0)\Big)^{\frac{1}{k}} f(\varphi(0)) - \alpha$$
$$= \Big(\frac{1}{nC_0}p(0)\Big)^{\frac{1}{k}} f(\varphi(0)) - \alpha.$$

Hence, $\varphi(r) \in C^2[0, R)$. And by direct calculation, we can prove that $\varphi(r)$ satisfies (2.1).

Remark 2.3. *When $p(r) \equiv 1$, Lemma 2.2 is consistent with Lemma 2.3 in [25].*

Lemma 2.4. *(Lemma 2.2, [25]) Suppose that $(f1), (f2)$ hold and $\varphi(r) \in C^2[0, R)$ satifies (2.1) with $\varphi'(0) = 0$. Then $\varphi'(r) \geq 0$ and $\varphi''(r) + \alpha > 0$.*

Proof. From (2.1), we have

$$C_{n-1}^{k-1}(r^{n-k}(\varphi'(r) + \alpha r)^k)' = kr^{n-1}p(r)f^k(\varphi(r)).$$

Noticing that $\varphi'(0) = 0$ and intergrating from 0 to r, combining with (1.7), we have

$$\varphi'(r) = \Big(\frac{r^{k-n}}{C_0}\int_0^r s^{n-1}p(s)f^k(\varphi(s))ds\Big)^{\frac{1}{k}} - \alpha r.$$

If $\alpha = 0$, then we can easily prove that $\varphi'(r) > 0$; if $\alpha > 0$ and $p(|x|) > 1$, then we have

$$\begin{aligned}\varphi'(r) &\geq \alpha_0\Big(\frac{r^{k-n}}{C_0}\int_0^r s^{n-1}ds\Big)^{\frac{1}{k}} - \alpha r\\ &> (nC_0)^{-\frac{1}{k}}\frac{n}{k}(nC_0)^{\frac{1}{k}}\alpha r - \alpha r\\ &= \alpha(\frac{n}{k} - 1)r \geq 0.\end{aligned}$$

On the other hand, by calculating, for $0 < s < r$, we have

$$\begin{aligned}\varphi''(r) + \alpha &= \frac{k-n}{k}r^{-\frac{n}{k}}(\int_0^r \frac{1}{C_0}p(s)f^k(\varphi(s))s^{n-1}ds)^{\frac{1}{k}}\\ &+ \frac{1}{kC_0}r^{\frac{k-n}{k}}(\int_0^r \frac{1}{C_0}p(s)f^k(\varphi(s))s^{n-1}ds)^{\frac{1}{k}-1}r^{n-1}p(r)f^k(\varphi(r))\\ &= r^{-\frac{n}{k}}(\int_0^r \frac{1}{C_0}p(s)f^k(\varphi(s))s^{n-1}ds)^{\frac{1}{k}-1}\Big[\frac{k-n}{k}\int_0^r \frac{1}{C_0}p(s)f^k(\varphi(s))s^{n-1}ds\\ &+ \frac{1}{kC_0}r^n p(r)f^k(\varphi(r))\Big]\\ &\geq r^{-\frac{n}{k}}(\int_0^r \frac{1}{C_0}p(s)f^k(\varphi(s))s^{n-1}ds)^{\frac{1}{k}-1}\Big[\frac{k-n}{k}\frac{1}{C_0}p(r)f^k(\varphi(r))\frac{r^n}{n}\\ &+ \frac{1}{kC_0}r^n p(r)f^k(\varphi(r))\Big]\\ &= r^{-\frac{n}{k}}(\int_0^r \frac{1}{C_0}p(s)f^k(\varphi(s))s^{n-1}ds)^{\frac{1}{k}-1}\Big[\frac{1}{nC_0}r^n p(r)f^k(\varphi(r))\Big]\\ &> 0.\end{aligned} \tag{2.3}$$

This gives the proof of Lemma 2.4.

Remark 2.5. *When $p(r) \equiv 1$, Lemma 2.4 is consistent with Lemma 2.2 in [25].*

3. Proof of the main results

In this section, we prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Firstly, we consider the equations

$$C_{n-1}^{k-1}(u''(r)+\alpha)\frac{(u'(r)+\alpha r)^{k-1}}{r^{k-1}}+C_{n-1}^{k}\frac{(u'(r)+\alpha r)^{k}}{r^{k}}=p(r)f^k(u(r)),\ r>0, \tag{3.1}$$

$$u'(r)=\Big(\frac{r^{k-n}}{C_0}\int_0^r s^{n-1}p(s)f^k(u(s))ds\Big)^{\frac{1}{k}}-\alpha r,\ r>0,\ u(0)=a, \tag{3.2}$$

and

$$u(r)=a+\int_0^r\Big(\frac{t^{k-n}}{C_0}\int_0^t s^{n-1}p(s)f^k(u(s))ds\Big)^{\frac{1}{k}}dt-\frac{\alpha}{2}r^2,\ r\geq 0. \tag{3.3}$$

Apparently, solutions in $C[0,\infty)$ to (3.3) are solutions in $C[0,\infty)\cap C^1(0,\infty)$ to (3.2).

Let $\{u_m\}_{m\geq 1}$ be the sequences of positive continuous functions defined on $[0,\infty)$ by

$$u_0(r)=a,\ u_m(r)=a+\int_0^r\Big(\frac{t^{k-n}}{C_0}\int_0^t s^{n-1}p(s)f^k(u_{m-1}(s))ds\Big)^{\frac{1}{k}}dt-\frac{\alpha}{2}r^2,\ r\geq 0.$$

Obviously, for all $r\geq 0$ and $m\in\mathbb{N}$, we have

$$\begin{aligned} u_m(r)&=a+\int_0^r\Big(\frac{t^{k-n}}{C_0}\int_0^t s^{n-1}p(s)f^k(u_{m-1}(s))ds\Big)^{\frac{1}{k}}dt-\frac{\alpha}{2}r^2\\ &\geq a+\alpha_0\int_0^r\Big(\frac{t^{k-n}}{C_0}\int_0^t s^{n-1}p(s)ds\Big)^{\frac{1}{k}}dt-\frac{\alpha}{2}r^2\\ &\geq a+\alpha_0 P(r)-\frac{\alpha}{2}r^2. \end{aligned}$$

Therefore, $u_m(r)\geq a$, and $u_0(r)<u_1(r)$. Since $(f1)$ holds, we have $u_1(r)<u_2(r)$ for $r\geq 0$. According to the above reasons, we obtain that the sequences $\{u_m\}$ is increasing on $[0,\infty)$. Also, we obtain by $(f1)$ and $(f2)$ that for each $r>0$

$$\begin{aligned} u_m'(r)&=\Big(\frac{r^{k-n}}{C_0}\int_0^r s^{n-1}p(s)f^k(u_{m-1}(s))ds\Big)^{\frac{1}{k}}-\alpha r\\ &\leq f(u_m(r))\Big(\frac{r^{k-n}}{C_0}\int_0^r s^{n-1}p(s)ds\Big)^{\frac{1}{k}}-\alpha r\\ &\leq f(u_m(r))P'(r). \end{aligned}$$

Therefore,

$$\int_a^{u_m(r)}\frac{1}{f(\tau)}d\tau\leq P(r),\ r>0.$$

This shows that

$$H_{1a}(u_m(r))\leq P(r),\ \forall r\geq 0, \tag{3.4}$$

and

$$u_m(r) \le H_{1a}^{-1}(P(r)),\ \forall r \ge 0. \tag{3.5}$$

It follows that the sequences $\{u_m\}$, $\{u_m'\}$ are bounded on $[0, R_0]$ for an arbitrary $R_0 > 0$. By Arzelà-Ascoli theorem, $\{u_m\}$ has subsequences converging uniformly to u on $[0, R_0]$. Since $\{u_m\}$ is increasing on $[0, \infty)$, we see that $\{u_m\}$ itself converges uniformly to u on $[0, R_0]$. By arbitrariness of R_0 and Lemma 2.2, we get that u is an entire positive k-convex radial solution to (E), and u satisfies

$$a + \alpha_0 P(r) - \frac{\alpha}{2}r^2 < u(r) \le H_{1a}^{-1}(P(r)),\ \ \forall r \ge 0. \tag{3.6}$$

If $\alpha = 0$, by (3.6), it is easy to obtain that if $P(\infty) = \infty$ and $H_{1a}(\infty) = \infty$, then $\lim_{r\to\infty} u(r) = \infty$; if $P(\infty) < H_{1a}(\infty) < \infty$, then u is bounded. If $\alpha > 0$, combining the fact that $p(|x|) \ge 1$, $\alpha_0 > \frac{n}{k}(C_n^k)^{\frac{1}{k}}\alpha$ and $H_{1a}(\infty) = \infty$, it is obvious that $\lim_{r\to\infty} u(r) = \infty$. This finishes the proof of Theorem 1.1.

Remark 3.1. Theorem 1.1 generalizes Theorem 1.1 with $\alpha > 0$ in [24]. In the case $\alpha > 0$, since $P(\infty) = \infty$ for the positivity of u, it is difficult to ensure if there is bounded positive entire solution of (E).

Proof of Theorem 1.2. Consider the following systems

$$\begin{cases} C_{n-1}^{k-1}(u''(r)+\alpha)\frac{(u'(r)+\alpha r)^{k-1}}{r^{k-1}} + C_{n-1}^{k}\frac{(u'(r)+\alpha r)^{k}}{r^{k}} = p(r)f^k(v(r)),\ r > 0, \\ C_{n-1}^{k-1}(v''(r)+\alpha)\frac{(v'(r)+\alpha r)^{k-1}}{r^{k-1}} + C_{n-1}^{k}\frac{(v'(r)+\alpha r)^{k}}{r^{k}} = q(r)g^k(u(r)),\ r > 0, \end{cases}$$

and

$$\begin{cases} u(r) = \frac{a}{2} + \int_0^r \left(\frac{t^{k-n}}{C_0}\int_0^t s^{n-1}p(s)f^k(v(s))ds\right)^{\frac{1}{k}}dt - \frac{\alpha}{2}r^2,\ r \ge 0, \\ v(r) = \frac{a}{2} + \int_0^r \left(\frac{t^{k-n}}{C_0}\int_0^t s^{n-1}q(s)g^k(u(s))ds\right)^{\frac{1}{k}}dt - \frac{\alpha}{2}r^2,\ r \ge 0. \end{cases}$$

Let $\{u_m\}_{m\ge1}$ and $\{v_m\}_{m\ge1}$ be the sequences of positive continuous functions defined on $[0, \infty)$ by

$$\begin{cases} v_0 = \frac{a}{2}, \\ u_m(r) = \frac{a}{2} + \int_0^r \left(\frac{t^{k-n}}{C_0}\int_0^t s^{n-1}p(s)f^k(v_{m-1}(s))ds\right)^{\frac{1}{k}}dt - \frac{\alpha}{2}r^2,\ r \ge 0, \\ v_m(r) = \frac{a}{2} + \int_0^r \left(\frac{t^{k-n}}{C_0}\int_0^t s^{n-1}q(s)g^k(u_m(s))ds\right)^{\frac{1}{k}}dt - \frac{\alpha}{2}r^2,\ r \ge 0. \end{cases}$$

Similarly, for all $r \ge 0$ and $m \in \mathbb{N}$, when $m \ge 1$, we have

$$u_m(r) > \frac{a}{2} + \alpha_0 P(r) - \frac{\alpha}{2}r^2;$$
$$v_m(r) > \frac{a}{2} + \alpha_0 Q(r) - \frac{\alpha}{2}r^2.$$

Therefore, $u_m(r) \ge \frac{a}{2}$, $v_m(r) \ge \frac{a}{2}$ and $v_0(r) < v_1(r)$. Since f, g are continuous and nondecreasing, we have $u_1(r) < u_2(r)$, $\forall r \ge 0$, and $v_1(r) < v_2(r)$, $\forall r \ge 0$. According to the above reasons, we obtain that the sequences $\{u_m\}$ and $\{v_m\}$ are increasing on $[0, \infty)$.

Moreover, for $r > 0$, by $(f1)$ and $(f2)$, one can prove that

$$u_m'(r) \le (f(v_m(r) + u_m(r)) + g(v_m(r) + u_m(r)))P'(r);$$

$$v'_m(r) \le (f(v_m(r) + u_m(r)) + g(v_m(r) + u_m(r)))Q'(r),$$

and

$$u'_m(r) + v'_m(r) \le [f(v_m(r) + u_m(r)) + g(v_m(r) + u_m(r))](P'(r) + Q'(r)).$$

Therefore,

$$\int_a^{u_m(r)+v_m(r)} \frac{1}{f(\tau) + g(\tau)} d\tau \le P(r) + Q(r), \quad r > 0,$$

which shows that

$$H_{2a}(u_m(r) + v_m(r)) \le P(r) + Q(r), \quad \forall r \ge 0,$$

and

$$u_m(r) + v_m(r) \le H_{2a}^{-1}(P(r) + Q(r)), \quad \forall r \ge 0.$$

It so follows that the sequences $\{u_m\}$, $\{u'_m\}$ and $\{v_m\}$, $\{v'_m\}$ are bounded on $[0, R_0]$ for an arbitrary $R_0 > 0$. By Arzelà-Ascoli theorem, $\{u_m\}$ and $\{v_m\}$ have subsequences converging uniformly to u and v respectively on $[0, R_0]$. Since $\{u_m\}$, $\{v_m\}$ are increasing on $[0, \infty)$, we see that $\{u_m\}$ itself converges uniformly to u on $[0, R_0]$, so is $\{v_m\}$. By arbitrariness of R_0 and Lemma 2.2, we get that (u, v) is an entire positive k-convex radial solution to (S).

The rest proof is similar to that of Theorem 1.1. So we omit it here.

4. Conclusions

In this paper, we use a new monotone iteration scheme to obtain some new existence results of entire positive solutions for a k-Hessian type equation and system.

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Conflict of interest

The authors declare there is no conflict of interest.

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