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*Research article*

# General decay for a system of viscoelastic wave equation with past history, distributed delay and Balakrishnan-Taylor damping terms

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**Abstract:** The subject of this research is a coupled system of nonlinear viscoelastic wave equations with distributed delay components, infinite memory and Balakrishnan-Taylor damping. Assume the kernels $g_i : \mathbf{R}_+ \to \mathbf{R}_+$ holds true the below

$$g_i'(t) \leq -\zeta_i(t)G_i(g_i(t)), \quad \forall t \in \mathbf{R}_+, \quad \text{for} \quad i = 1, 2,$$

in which $\zeta_i$ and $G_i$ are functions. We demonstrate the stability of the system under this highly generic assumptions on the behaviour of $g_i$ at infinity and by dropping the boundedness assumptions in the historical data.

**Keywords:** wave equation; infinite memory; distributed delay term; viscoelastic term; relaxation function

## 1. Introduction

In this research, we mainly focused on wave equation to study and examine the coupled system. In this system, we assumed a bounded domain $\Omega \in \mathbf{R}^N$ where $\partial\Omega$ indicates sufficiently smooth boundary of $\Omega \in \mathbf{R}^N$ and take the positive constants $\xi_0, \xi_1, \sigma, \beta_1, \beta_3$ where $m \geq 1$ for $N = 1, 2$, and $1 < m \leq \frac{N+2}{N-2}$

for $N \geq 3$. The coupled system with these terms is given by

$$\begin{cases} v_{tt} - \Big(\xi_0 + \xi_1 \|\nabla v\|_2^2 + \delta(\nabla v, \nabla v_t)_{L^2(\Omega)}\Big)\Delta v(t) + \displaystyle\int_0^\infty g_1(s)\Delta v(t-s)ds \\ \quad + \beta_1 |v_t(t)|^{m-2} v_t(t) + \displaystyle\int_{\tau_1}^{\tau_2} |\beta_2(r)||v_t(t-r)|^{m-2} v_t(t-r)dr + f_1(v,w) = 0. \\ w_{tt} - \Big(\xi_0 + \xi_1 \|\nabla w\|_2^2 + \delta(\nabla w, \nabla w_t)_{L^2(\Omega)}\Big)\Delta w(t) + \displaystyle\int_0^\infty g_2(s)\Delta w(t-s)ds \\ \quad + \beta_3 |w_t(t)|^{m-2} w_t(t) + \displaystyle\int_{\tau_1}^{\tau_2} |\beta_4(r)||w_t(t-r)|^{m-2} w_t(t-r)dr + f_2(v,w) = 0. \\ v(z,-t) = v_0(z), \;\; v_t(z,0) = v_1(z), w(z,-t) = w_0(z), \;\; w_t(z,0) = w_1(z), \;\; in \;\; \Omega \\ v_t(z,-t) = j_0(z,t), \;\; w_t(z,-t) = \varrho_0(z,t), \;\; in \;\; \Omega \times (0,\tau_2) \\ v(z,t) = w(z,t) = 0, \;\; in \;\; \partial\Omega \times (0,\infty) \end{cases} \tag{1.1}$$

in which $G = \Omega \times (\tau_1, \tau_2) \times (0, \infty)$ and $\tau_1 < \tau_2$ are taken to be non-negative constants in a manner that $\beta_2, \beta_4 : [\tau_1, \tau_2] \to \mathbb{R}$ indicates distributive time delay while $g_i$, $i = 1, 2$ are positive.

The viscoelastic damping term, whose kernel is the function $g$, is a physical term used to describe the link between the strain and stress histories in a beam that was inspired by the Boltzmann theory. There are several publications that discuss this subject and produce a lot of fresh and original findings [1–5], particularly the hypotheses regarding the initial condition [6–12] and the kernel. See [13–17]. As it concerns to the plate equation and the span problem, Balakrishnan and Taylor introduced a novel damping model in [18] that they dubbed the Balakrishnan-Taylor damping. Here are a few studies that specifically addressed the research of this dampening for further information [18–23].

Several applications and real-world issues are frequently affected by the delay, which transforms numerous systems into interesting research topics. Numerous writers have recently studied the stability of the evolution systems with time delays, particularly the effect of distributed delay. See [24–26].

In [1], the authors presented the stability result of the system over a considerably broader class of kernels in the absence of delay and Balakrishnan-Taylor damping $\xi_0 = 1, \xi_1 = \delta = \beta_i = 0, i = 1, \ldots, 4$.

Based on everything said above, one specific problem may be solved by combining these damping terms (distributed delay terms, Balakrishnan-Taylor damping and infinite memory), especially when the past history and the distributed delay

$$\int_{\tau_1}^{\tau_2} |\beta_i(r)||u_t(t-r)|^{m-2} u_t(t-r)dr, \quad i = 2, 4$$

are added. We shall attempt to throw light on it since we think it represents a fresh topic that merits investigation and analysis in contrast to the ones mentioned before. Our study is structured into multiple sections: in the second section, we establish the assumptions, notions, and lemmas we require; in the final section, we substantiate our major finding.

## 2. Fundamental knowledge

In this section of the paper, we will introduce some basic results related to the theory for the analysis of our problem. Let us take the below:

(G1) $h_i : \mathbf{R}_+ \to \mathbf{R}_+$ are a non-increasing $C^1$ functions fulfills the following

$$g_i(0) > 0, \; , \;\; \xi_0 - \int_0^\infty h_i(s)ds = l_i > 0, \;\; i = 1, 2, \tag{2.1}$$

and

$$g_0 = \int_0^\infty h_1(s)ds, \;\; \widehat{g}_0 = \int_0^\infty g_2(s)ds,$$

(G2) One can find a function $C^1$ functions $G_i : \mathbf{R}_+ \to \mathbf{R}_+$ holds true $G_i(0) = G_i'(0) = 0$.

The functions $G_i(t)$ are strictly increasing and convex of class $C^2(\mathbf{R}_+)$ on $(0, \varrho]$, $r \leq g_i(0)$ or linear in a manner that

$$g_i'(t) \leq -\zeta_i(t) G_i(g_i(t)), \; \forall t \geq 0, \;\; \text{for} \;\; i = 1, 2, \tag{2.2}$$

in which $\zeta_i(t)$ are a $C^1$ functions fulfilling the below

$$\zeta_i(t) > 0, \; \zeta_i'(t) \leq 0, \;\; \forall t \geq 0. \tag{2.3}$$

(G3) $\beta_2, \beta_4 : [\tau_1, \tau_2] \to \mathbf{R}$ are a bounded function fulfilling the below

$$\begin{aligned} \int_{\tau_1}^{\tau_2} |\beta_2(r)| dr < \beta_1, \\ \int_{\tau_1}^{\tau_2} |\beta_4(r)| dr < \beta_3. \end{aligned} \tag{2.4}$$

(G4) $f_i : \mathbf{R}^2 \to \mathbf{R}$ are $C^1$ functions with $f_i(0, 0) = 0$, and one can find a function $F$ in a way that

$$\begin{aligned} & f_1(c, e) = \frac{dF}{dc}(c, e), \quad f_2(c, e) = \frac{dF}{de}(c, e), \\ & F \geq 0, \quad\;\; a f_1(c, e) + e f_2(c, e) = F(c, e) \geq 0, \end{aligned} \tag{2.5}$$

and

$$\frac{df_i}{dc}(c, e) + \frac{df_i}{de}(c, e) \leq d(1 + c^{p_i - 1} + e^{p_i - 1}). \;\; \forall (c, e) \in \mathbf{R}^2. \tag{2.6}$$

Take the below

$$(g \circ \phi)(t) := \int_\Omega \int_0^\infty h(r) |\phi(t) - \phi(t - r)|^2 dr dz,$$

and

$$\begin{aligned} M_1(t) &:= \Big(\xi_0 + \xi_1 \|\nabla v\|_2^2 + \delta(\nabla v(t), \nabla v_t(t))_{L^2(\Omega)}\Big), \\ M_2(t) &:= \Big(\xi_0 + \xi_1 \|\nabla w\|_2^2 + \delta(\nabla w(t), \nabla w_t(t))_{L^2(\Omega)}\Big). \end{aligned}$$

**Lemma 2.1.** *(Sobolev-Poincare inequality [27]). Assume that $2 \leq q < \infty$ for $n = 1, 2$ and $2 \leq q < \frac{2n}{n-2}$ for $n \geq 3$. Then, one can find $c_* = c(\Omega, q) > 0$ in a manner that*

$$\|v\|_q \leq c_* \|\nabla v\|_2, \;\; \forall v \in G_0^1(\Omega).$$

Moreover, choose the below as in [26]:

$$x(z,\rho,r,t) = v_t(z,t-r\rho),$$
$$y(z,\rho,r,t) = w_t(z,t-r\rho)$$

with

$$\begin{cases} rx_t(z,\rho,r,t) + x_\rho(z,\rho,r,t) = 0, \quad sy_t(z,\rho,r,t) + y_\rho(z,\rho,r,t) = 0 \\ x(z,0,r,t) = v_t(z,t), \quad y(z,0,r,t) = w_t(z,t). \end{cases} \tag{2.7}$$

Take the auxiliary variable (see [28])

$$\eta^t(z,s) = v(z,t) - v(z,t-s), s \geq 0,$$
$$\vartheta^t(z,s) = w(z,t) - w(z,t-s), s \geq 0.$$

Then

$$\begin{aligned} \eta_t^t(z,s) + \eta_s^t(z,s) &= v_t(z,t), \\ \vartheta_t^t(z,s) + \vartheta_s^t(z,s) &= w_t(z,t). \end{aligned} \tag{2.8}$$

Rewrite the problem (1.1) as follows

$$\begin{cases} v_{tt} - \Big(l_1 + \xi_1\|\nabla v\|_2^2 + \delta(\nabla v, \nabla v_t)_{L^2(\Omega)}\Big)\Delta v(t) + \displaystyle\int_0^\infty g_1(s)\Delta\eta^t(s)ds \\ \quad +\beta_1|v_t(t)|^{m-2}v_t(t) + \displaystyle\int_{\tau_1}^{\tau_2} |\beta_2(s)||x(z,1,r,t)|^{m-2}x(z,1,r,t)dr + f_1(v,w) = 0, \\ w_{tt} - \Big(l_2 + \xi_1\|\nabla w\|_2^2 + \delta(\nabla w, \nabla w_t)_{L^2(\Omega)}\Big)\Delta w(t) + \displaystyle\int_0^\infty g_2(s)\Delta\vartheta^t(s)ds \\ \quad +\beta_3|w_t(t)|^{m-2}w_t(t) + \displaystyle\int_{\tau_1}^{\tau_2} |\beta_4(r)||y(z,1,r,t)|^{m-2}y(z,1,r,t)dr + f_2(v,w) = 0, \\ rx_t(z,\rho,r,t) + x_\rho(z,\rho,r,t) = 0, \\ ry_t(z,\rho,r,t) + y_\rho(z,\rho,r,t) = 0, \\ \eta_t^t(z,s) + \eta_s^t(z,s) = v_t(z,t) \\ \vartheta_t^t(z,s) + \vartheta_s^t(z,s) = w_t(z,t), \end{cases} \tag{2.9}$$

where

$$(z,\rho,r,t) \in \Omega \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty).$$

with

$$\begin{cases} v(z,-t) = v_0(z),\ v_t(z,0) = v_1(z),\ w(z,-t) = w_0(z),\ w_t(z,0) = w_1(z), \ \ in\ \ \Omega \\ x(z,\rho,r,0) = j_0(z,\rho r), \ \ y(z,\rho,r,0) = \varrho_0(z,\rho r), \ \ in\ \ \Omega \times (0,1) \times (0,\tau_2) \\ v(z,t) = \eta^t(z,s) = 0, \ \ z \in \ \partial\Omega, \ \ t,s \in (0,\infty), \\ \eta^t(z,0) = 0, \ \forall t \geq 0, \ \ \eta^0(z,s) = \eta_0(s) = 0, \ \forall s \geq 0, \\ w(z,t) = \vartheta^t(z,s) = 0, \ \ z \in \ \partial\Omega, \ \ t,s \in (0,\infty), \\ \vartheta^t(z,0) = 0, \ \forall t \geq 0, \ \ \vartheta^0(z,s) = \vartheta_0(s) = 0, \ \forall s \geq 0. \end{cases} \tag{2.10}$$

In the upcoming Lemma, the energy functional will be introduced.

**Lemma 2.2.** *Let the energy functional is symbolized by E, then it is given by*

$$\begin{aligned} E(t) &= \frac{1}{2}(\|v_t\|_2^2 + \|w_t\|_2^2) + \frac{\xi_1}{4}(\|\nabla v(t)\|_2^4 + \|\nabla w(t)\|_2^4) + \int_\Omega F(v,w)dz \\ &+\frac{1}{2}\Big(l_1\|\nabla v(t)\|_2^2 + l_2\|\nabla w(t)\|_2^2\Big) + \frac{1}{2}((g_1 \circ \nabla v)(t) + (g_2 \circ \nabla w)(t)) \\ &+\frac{m-1}{m}\int_0^1\int_{\tau_1}^{\tau_2} s\Big(|\beta_2(r)|\|x(z,\rho,r,t)\|_m^m + |\beta_4(r)|\|y(z,\rho,r,t)\|_m^m\Big)drd\rho. \end{aligned} \tag{2.11}$$

*The above fulfills the below*

$$\begin{aligned} E'(t) &\leq -\gamma_0\Big(\|v_t(t)\|_m^m + \|w_t(t)\|_m^m\Big) + \frac{1}{2}\Big((g_1' \circ \nabla v)(t) + (g_2' \circ \nabla w)(t)\Big) \\ &-\frac{\delta}{4}\Big\{\Big(\frac{d}{dt}\Big\{\|\nabla v(t)\|_2^2\Big\}\Big)^2 + \Big(\frac{d}{dt}\Big\{\|\nabla w(t)\|_2^2\Big\}\Big)^2\Big\} \leq 0, \end{aligned} \tag{2.12}$$

*in which* $\gamma_0 = \min\{\beta_1 - \int_{\tau_1}^{\tau_2}|\beta_2(r)|dr,\ \beta_3 - \int_{\tau_1}^{\tau_2}|\beta_4(r)|dr\}$.

*Proof.* To prove the result, we take the inner product of (2.9) with $v_t, w_t$ and after that integrating over $\Omega$, the following is obtained

$$\begin{aligned} &(v_{tt}(t), v_t(t))_{L^2(\Omega)} - (M_3(t)\Delta v(t), v_t(t))_{L^2(\Omega)} \\ &+(\int_0^\infty h_1(s)\Delta\eta^t(s)ds, v_t(t))_{L^2(\Omega)} + \beta_1(|v_t|^{m-2}v_t, v_t)_{L^2(\Omega)} \\ &+\int_{\tau_1}^{\tau_2} |\beta_2(s)|(|x(z,1,r,t)|^{m-2}x(z,1,r,t), v_t(t))_{L^2(\Omega)}dr \\ &+(w_{tt}(t), w_t(t))_{L^2(\Omega)} - (M_4(t)\Delta w(t), w_t(t))_{L^2(\Omega)} \\ &+(\int_0^\infty h_2(s)\Delta\vartheta^t(s)ds, w_t(t))_{L^2(\Omega)} + \beta_3(|w_t|^{m-2}w_t, w_t)_{L^2(\Omega)} \\ &+\int_{\tau_1}^{\tau_2} |\beta_4(s)|(|y(z,1,r,t)|^{m-2}y(z,1,r,t), w_t(t))_{L^2(\Omega)}dr \\ &+(f_1(v,w), v_t(t))_{L^2(\Omega)} + (f_2(v,w), w_t(t))_{L^2(\Omega)} = 0. \end{aligned} \tag{2.13}$$

in which

$$\begin{aligned} M_3(t) &:= \Big(l_1 + \xi_1\|\nabla v\|_2^2 + \delta(\nabla v(t), \nabla v_t(t))_{L^2(\Omega)}\Big), \\ M_4(t) &:= \Big(l_2 + \xi_1\|\nabla w\|_2^2 + \delta(\nabla w(t), \nabla w_t(t))_{L^2(\Omega)}\Big). \end{aligned}$$

Using mathematical skills, the following is obtained

$$(v_{tt}(t), v_t(t))_{L^2(\Omega)} = \frac{1}{2}\frac{d}{dt}\Big(\|v_t(t)\|_2^2\Big), \tag{2.14}$$

further simplification leads us to the following

$$
\begin{aligned}
& -(M_3(t)\Delta v(t), v_t(t))_{L^2(\Omega)} \\
= & -(\Big(l_1 + \xi_1\|\nabla v\|_2^2 + \delta(\nabla v(t), \nabla v_t(t))_{L^2(\Omega)}\Big)\Delta v(t), v_t(t))_{L^2(\Omega)} \\
= & \Big(l_1 + \xi_1\|\nabla v\|_2^2 + \delta(\nabla v(t), \nabla v_t(t))_{L^2(\Omega)}\Big)\int_\Omega \nabla v(t).\nabla v_t(t)dz \\
= & \Big(l_1 + \xi_1\|\nabla v\|_2^2 + \delta(\nabla v(t), \nabla v_t(t))_{L^2(\Omega)}\Big)\frac{d}{dt}\Big\{\int_\Omega |\nabla v(t)|^2 dz\Big\} \\
= & \frac{d}{dt}\Big\{\frac{1}{2}\Big(l_1 + \frac{\xi_1}{2}\|\nabla v\|_2^2\Big)\|\nabla v(t)\|_2^2\Big\} + \frac{\delta}{4}\frac{d}{dt}\Big\{\|\nabla v(t)\|_2^2\Big\}^2.
\end{aligned}
\tag{2.15}
$$

The following is obtained after calculation

$$
\begin{aligned}
(\int_0^\infty g_1(s)\Delta\eta^t(s)ds, v_t(t))_{L^2(\Omega)} &= \int_\Omega \nabla v_t \int_0^\infty g_1(s)\nabla\eta^t(s)dsdz \\
&= \int_0^\infty g_1(s)\int_\Omega \nabla v_t \nabla\eta^t(s)dzds \\
&= \int_0^\infty g_1(s)\int_\Omega (\nabla\eta_t^t + \nabla\eta_s^t)\nabla\eta^t(s)dzds \\
&= \int_0^\infty g_1(s)\int_\Omega \nabla\eta_t^t\nabla\eta^t(s)dzds \\
&\quad + \int_\Omega\int_0^\infty g_1(s)\nabla\eta_s^t\nabla\eta^t(\nabla)d\nabla dz \\
&= \frac{1}{2}\frac{d}{dt}(g_1 \circ \nabla v)(t) - \frac{1}{2}(g_1' \circ \nabla v)(t).
\end{aligned}
\tag{2.16}
$$

In the same way, we have

$$
\begin{aligned}
(w_{tt}(t), w_t(t))_{L^2(\Omega)} &= \frac{1}{2}\frac{d}{dt}\Big(\|w_t(t)\|_2^2\Big), \\
-(M_4(t)\Delta w(t), w_t(t))_{L^2(\Omega)} &= \frac{d}{dt}\Big\{\frac{1}{2}\Big(l_2 + \frac{\xi_1}{2}\|\nabla w\|_2^2\Big)\|\nabla w(t)\|_2^2\Big\} \\
&\quad + \frac{\delta}{4}\frac{d}{dt}\Big\{\|\nabla w(t)\|_2^2\Big\}^2, \qquad (2.17) \\
(\int_0^\infty g_2(s)\Delta\vartheta^t(s)ds, w_t(t))_{L^2(\Omega)} &= \frac{1}{2}\frac{d}{dt}(g_2 \circ \nabla w)(t) - \frac{1}{2}(g_2' \circ \nabla w)(t).
\end{aligned}
$$

Now, multiplying the equation (2.9) by $-x|\beta_2(r)|$, $-y|\beta_4(r)|$, and integrating over $\Omega\times(0,1)\times(\tau_1,\tau_2)$ and utilizing (2.7), the below is obtained

$$
\begin{aligned}
& \frac{d}{dt} \quad \frac{m-1}{m} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} r|\beta_2(r)|.|x(z,\rho,r,t)|^m drd\rho dz \\
= \quad & -(m-1) \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} |\beta_2(r)|.|y|^{m-1} x_\rho drd\rho dz \\
= \quad & -\frac{m-1}{m} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} |\beta_2(r)| \frac{d}{d\rho} |x(z,\rho,r,t)|^m drd\rho dz \\
= \quad & \frac{m-1}{m} \int_\Omega \int_{\tau_1}^{\tau_2} |\beta_2(r)| \Big( |x(z,0,r,t)|^m - |x(z,1,r,t)|^m \Big) drdz \\
= \quad & \frac{m-1}{m} \Big( \int_{\tau_1}^{\tau_2} |\beta_2(r)| dr \Big) \int_\Omega |v_t(t)|^m dz \\
& -\frac{m-1}{m} \int_\Omega \int_{\tau_1}^{\tau_2} |\beta_2(r)|.|x\,(z,1,r,t)\,|^m drdz \\
= \quad & \frac{m-1}{m} \Big( \int_{\tau_1}^{\tau_2} |\beta_2(r)| dr \Big) \|v_t(t)\|_m^m \\
& -\frac{m-1}{m} \int_{\tau_1}^{\tau_2} |\beta_2(r)| \|x\,(z,1,r,t)\,\|_m^m dr. \qquad (2.18)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
& \frac{d}{dt}\frac{m-1}{m} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} r|\beta_4(r)|.|y(z,\rho,r,t)|^m drd\rho dz \qquad (2.19) \\
= \quad & \frac{m-1}{m} \Big( \int_{\tau_1}^{\tau_2} |\beta_4(r)| dr \Big) \|w_t(t)\|_m^m - \frac{m-1}{m} \int_{\tau_1}^{\tau_2} |\beta_4(r)| \|y\,(z,1,r,t)\,\|_m^m dr.
\end{aligned}
$$

Here, we utilize the inequalities of Young as

$$
\begin{aligned}
& \int_{\tau_1}^{\tau_2} |\beta_2(r)| \Big( |x(z,1,r,t)|^{m-2} x(z,1,r,t), v_t(t) \Big)_{L^2(\Omega)} ds \qquad (2.20) \\
\leq \quad & \frac{1}{m} \Big( \int_{\tau_1}^{\tau_2} |\beta_2(r)| dr \Big) \|v_t(t)\|_m^m + \frac{m-1}{m} \int_{\tau_1}^{\tau_2} |\beta_2(r)| \|x\,(z,1,r,t)\,\|_m^m dr,
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_{\tau_1}^{\tau_2} |\beta_4(r)| \Big( |y(z,1,r,t)|^{m-2} y(z,1,r,t), w_t(t) \Big)_{L^2(\Omega)} dr \qquad (2.21) \\
\leq \quad & \frac{1}{m} \Big( \int_{\tau_1}^{\tau_2} |\beta_4(r)| dr \Big) \|w_t(t)\|_m^m + \frac{m-1}{m} \int_{\tau_1}^{\tau_2} |\beta_4(r)| \|y\,(z,1,r,t)\,\|_m^m dr.
\end{aligned}
$$

Finally, we have

$$
(f_1(v,w), v_t(t))_{L^2(\Omega)} + (f_2(v,w), w_t(t))_{L^2(\Omega)} = \frac{d}{dt} \int_\Omega F(v,w) dz. \qquad (2.22)
$$

Thus, after replacement of (2.14)–(2.22) into (2.13), we determined (2.11) and (2.12). As a result, we obtained that $E$ is a non-increasing function by (2.2)–(2.5), which is required.

**Theorem 2.3.** *Take the function* $\mathbf{U} = (v, v_t, w, w_t, x, y, \eta^t, \vartheta^t)^T$ *and assume that (2.1)–(2.5) holds true. Then, for any* $\mathbf{U}_0 \in \mathcal{H}$*, then one can find a unique solution* $\mathbf{U}$ *of problems (2.9) and (2.10) in a manner that*

$$U \in C(\mathbf{R}_+, \mathcal{G}).$$

*If* $\mathbf{U}_0 \in \mathcal{G}_1$*, then U fulfills the following*

$$U \in C^1(\mathbf{R}_+, \mathcal{G}) \cap C(\mathbf{R}_+, \mathcal{G}_1),$$

*in which*

$$\begin{aligned} \mathcal{G} &= (G_0^1(\Omega) \times L^2(\Omega))^2 \times (L^2(\Omega, (0,1), (\tau_1, \tau_2)))^2 \times (L_{g_1} \times L_{g_2}). \\ \mathcal{G}_1 &= \Big\{ U \in \mathcal{G}/v, w \in G^2 \cap G_0^1, v_t, w_t \in G_0^1(\Omega), x, y, x_\rho, y_\rho \in L^2(\Omega, (0,1), (\tau_1, \tau_2)), \\ &\quad (\eta^t, \vartheta^t) \in L_{g_1} \times L_{g_2}, \eta^t(z, 0) = \vartheta^t(z, 0) = 0, x(z, 0, r, t) = v_t, \\ &\quad y(z, 0, r, t) = w_t \Big\}. \end{aligned}$$

## 3. Analysis of stability

Here, the stability of the systems (2.9) and (2.10) will be established and investigated. For which the following lemma is needed

**Lemma 3.1.** *Let us suppose that (2.1) and (2.2) fulfills.*

$$\int_\Omega \left( \int_0^\infty g_i(s)(v(t) - v(t-s))ds \right)^2 dz \leq C_{\kappa,i}(h_i \circ v)(t), \quad i = 1, 2. \tag{3.1}$$

*where*

$$\begin{aligned} C_{\kappa_i} &:= \int_0^\infty \frac{g_i^2(s)}{\kappa g_i(s) - g_i'(s)} ds \\ h_i(t) &:= \kappa g_i(t) - g_i'(t), \quad i = 1, 2. \end{aligned}$$

*Proof.*

$$\begin{aligned} & \int_\Omega \left( \int_0^\infty g_i(s)(v(t) - v(t-s))ds \right)^2 dz \\ = & \int_\Omega \left( \int_{-\infty}^t g_i(t-s)(v(t) - v(t-s))ds \right)^2 dz \\ = & \int_\Omega \left( \int_{-\infty}^t \frac{g_i(t-s)}{\sqrt{\kappa g_i(t-s) - g_i'(t-s)}} \sqrt{\kappa g_i(t-s) - g_i'(t-s)} \right. \\ & \left. (v(t) - v(s))ds \right)^2 dz \end{aligned} \tag{3.2}$$

which is obtained through Young's inequality (Eq 3.1).

**Lemma 3.2.** *(Jensens inequality). Let* $f : \Omega \to [c, e]$ *and* $h : \Omega \to \mathbf{R}$ *are integrable functions in a manner that for any* $z \in \Omega$, $h(z) > 0$ *and* $\int_\Omega h(z)dz = k > 0$. *Furthermore, assume a convex function* $G$ *such that* $G : [c, e] \to \mathbf{R}$. *Then*

$$G\Big(\frac{1}{k}\int_\Omega f(z)h(z)dz\Big) < \frac{1}{k}\int_\Omega G(f(z))h(z)dz. \tag{3.3}$$

**Lemma 3.3.** *It is mentioned in [12] that one can find a positive constant* $\beta, \widehat{\beta}$ *in a manner that*

$$\begin{aligned} I_1(t) &= \int_\Omega \int_t^\infty g_1(s)|\nabla\eta^t(\delta)|^2 ds dz \le \beta\mu(t), \\ I_2(t) &= \int_\Omega \int_t^\infty g_2(s)|\nabla\vartheta^t(\delta)|^2 ds dz \le \widehat{\beta}\widehat{\mu}(t), \end{aligned} \tag{3.4}$$

*in which*

$$\begin{aligned} \mu(t) &= \int_0^\infty g_1(t+s)\Big(1 + \int_\Omega \nabla v_0^2(z, s)dz\Big)ds, \\ \widehat{\mu}(t) &= \int_0^\infty g_2(t+s)\Big(1 + \int_\Omega \nabla w_0^2(z, s)dz\Big)ds. \end{aligned}$$

*Proof.* As the function $E(t)$ is decreasing and utilizing (2.11), we have the following

$$\begin{aligned} \int_\Omega |\nabla\eta^t(s)|^2 dz &= \int_\Omega (\nabla v(z,t) - v(z, t-s)^2 dz \\ &\le 2\int_\Omega \nabla v^2(z,t)dz + 2\int_\Omega \nabla v^2(z, t-s)dz \\ &\le 2\sup_{s>0}\int_\Omega \nabla v^2(z,s)dz + 2\int_\Omega \nabla v^2(z, t-x)dz \\ &\le \frac{4E(0)}{l_1} + 2\int_\Omega \nabla v^2(z, t-s)dz, \end{aligned} \tag{3.5}$$

for any $t, s \ge 0$. Further, we have

$$\begin{aligned} I_1(t) &\le \frac{4E(0)}{l_1}\int_t^\infty g_1(s)ds + 2\int_t^\infty g_1(s)\int_\Omega \nabla v^2(z, t-s)dzds \\ &\le \frac{4E(0)}{l_1}\int_0^\infty g_1(t+s)ds + 2\int_0^\infty g_1(t+s)\int_\Omega \nabla v_0^2(z,s)dzds \\ &\le \beta\mu(t), \end{aligned} \tag{3.6}$$

in which $\beta = \max\{\frac{4E(0)}{l_1}, 2\}$ and $\mu(t) = \int_0^\infty g_1(t+s)(1 + \int_\Omega \nabla u_0^2(z,s)dz)ds$.

In the same way, we can deduce that

$$\begin{aligned} I_2(t) &\le \frac{4E(0)}{l_2}\int_0^\infty g_2(t+s)ds + 2\int_0^\infty g_2(t+s)\int_\Omega \nabla w_0^2(z,s)dzds \\ &\le \widehat{\beta}\widehat{\mu}(t), \end{aligned} \tag{3.7}$$

in which $\widehat{\beta} = \max\{\frac{4E(0)}{l_2}, 2\}$ and $\widehat{\mu}(t) = \int_0^\infty g_2(t+s)(1 + \int_\Omega \nabla w_0^2(z,s)dz)ds$.

In the upcoming part, we set the following

$$\Psi(t) \;:=\; \int_\Omega \Big(v(t)v_t(t) + w(t)w_t(t)\Big)dz + \frac{\delta}{4}\Big(\|\nabla v(t)\|_2^4 + \|\nabla w(t)\|_2^4\Big), \tag{3.8}$$

and

$$\begin{aligned}\Phi(t) \;:=\; & -\int_\Omega v_t \int_0^\infty g_1(s)(v(t) - v(t-s))dsdz \\ & -\int_\Omega w_t \int_0^\infty g_2(s)(w(t) - w(t-s))dsdz,\end{aligned} \tag{3.9}$$

and

$$\Theta(t) \;:=\; \int_0^1 \int_{\tau_1}^{\tau_2} re^{-\rho r}\Big(|\beta_2(r)|.\|x(z,\rho,r,t)\|_m^m + |\beta_4(r)|.\|y(z,\rho,r,t)\|_m^m\Big)drd\rho. \tag{3.10}$$

**Lemma 3.4.** *In (3.8), the functional* $\Psi(t)$ *fulfills the following*

$$\begin{aligned}\Psi'(t) \;\leq\; & \|v_t\|_2^2 + \|w_t\|_2^2 - (l - \varepsilon(c_1 + c_2) - \sigma_1)\Big(\|\nabla v\|_2^2 + \|\nabla w\|_2^2\Big) \\ & -\xi_1\Big(\|\nabla v\|_2^4 + \|\nabla w\|_2^4\Big) + c(\varepsilon)\Big(\|v_t\|_m^m + \|w_t\|_m^m\Big) \\ & +c(\sigma_1)\Big(C_{\kappa,1}(g_1 \circ \nabla v)(t) + C_{\kappa,2}(h_2 \circ \nabla w)(t)\Big) - \int_\Omega F(v,w)dz \\ & +c(\varepsilon)\int_{\tau_1}^{\tau_2}\Big(|\beta_2(r)|\|x(z,1,r,t)\|_m^m + |\beta_4(r)|\|y(z,1,r,t)\|_m^m\Big)dr.\end{aligned} \tag{3.11}$$

*for any* $\varepsilon, \sigma_1 > 0$ *with* $l = \min\{l_1, l_2\}$.

*Proof.* To prove the result, differentiate (3.8) first and then apply (2.9), we have the following

$$\begin{aligned}\Psi'(t) \;=\; & \|v_t\|_2^2 + \int_\Omega v_{tt}vdz + \delta\|\nabla v\|_2^2\int_\Omega \nabla v_t \nabla vdz \\ & +\|w_t\|_2^2 + \int_\Omega w_{tt}wdz + \delta\|\nabla w\|_2^2\int_\Omega \nabla w_t\nabla wdz \\ =\; & \|v_t\|_2^2 + \|w_t\|_2^2 - \xi_0(\|\nabla v\|_2^2 + \|\nabla w\|_2^2) - \xi_1(\|\nabla v\|_2^4 + \|\nabla w\|_2^4) \\ & \underbrace{-\beta_1\int_\Omega |v_t|^{m-2}v_tvdz}_{I_{11}} \underbrace{-\beta_3\int_\Omega |w_t|^{m-2}w_twdz}_{I_{12}} \\ & +\underbrace{\int_\Omega \nabla v(t)\int_0^\infty g_1(s)\nabla v(t-s)dsdz}_{I_{21}} \\ & +\underbrace{\int_\Omega \nabla w(t)\int_0^\infty g_2(s)\nabla w(t-s)dsdz}_{I_{22}}\end{aligned}$$

$$
\begin{aligned}
&-\underbrace{\int_{\Omega}\int_{\tau_1}^{\tau_2}|\beta_2(r)||x(z,1,r,t)|^{m-2}x(z,1,r,t)v\,dr\,dz}_{I_{31}}\\
&-\underbrace{\int_{\Omega}\int_{\tau_1}^{\tau_2}|\beta_4(r)||y(z,1,r,t)|^{m-2}y(z,1,r,t)w\,dr\,dz}_{I_{32}}\\
&-\underbrace{\int_{\Omega}(vf_1(v,w)+wf_2(v,w))dz}_{I_4}. \qquad (3.12)
\end{aligned}
$$

We estimate the last 6 terms of the RHS of (3.12). The following is obtained by applying Young's, Sobolev-Poincare and Hölder's inequalities on (2.1) and (2.11), we have

$$
\begin{aligned}
I_{11} &\leq \varepsilon\beta_1^m\|v\|_m^m + c(\varepsilon)\|v_t\|_m^m\\
&\leq \varepsilon\beta_1^m c_p^m\|\nabla v\|_2^m + c(\varepsilon)\|v_t\|_m^m\\
&\leq \varepsilon\beta_1^m c_p^m\Big(\frac{E(0)}{l_1}\Big)^{(m-2)/2}\|\nabla v\|_2^2 + c(\varepsilon)\|v_t\|_m^m\\
&\leq \varepsilon c_{11}\|\nabla v\|_2^2 + c(\varepsilon)\|v_t\|_m^m. \qquad (3.13)
\end{aligned}
$$

In addition to this, for any $\sigma_1 > 0$, by Lemma 3.1, we have the below

$$
\begin{aligned}
I_{21} &\leq (\int_0^\infty g_1(s)ds)\|\nabla v\|_2^2 - \int_\Omega \nabla v(t)\int_0^\infty g_1(s)(\nabla v(t)-\nabla v(t-s))ds\,dz\\
&\leq (\xi_0 - l_1 + \sigma_1)\|\nabla v\|_2^2 + \frac{c}{\sigma_1}C_{\kappa,1}(h_1\circ\nabla v)(t). \qquad (3.14)
\end{aligned}
$$

Taking same steps to $I_{12}$, the below is obtained

$$
I_{31} \leq \varepsilon c_{21}\|\nabla v\|_2^2 + c(\varepsilon)\int_{\tau_1}^{\tau_2}|\beta_2(r)|.\|x(z,1,r,t)\|_m^m dr. \qquad (3.15)
$$

Same steps for $I_{11}, I_{21}$ and $I_{31}$, we have

$$
\begin{aligned}
I_{12} &\leq \varepsilon c_{12}\|\nabla w\|_2^2 + c(\varepsilon)\|w_t\|_m^m\\
I_{22} &\leq (\xi_0 - l_2 + \sigma_1)\|\nabla w\|_2^2 + \frac{c}{\sigma_1}C_{\kappa,2}(h_2\circ\nabla w)(t),\\
I_{32} &\leq \varepsilon c_{22}\|\nabla w\|_2^2 + c(\varepsilon)\int_{\tau_1}^{\tau_2}|\beta_4(r)|.\|y(z,1,r,t)\|_m^m dr. \qquad (3.16)
\end{aligned}
$$

Combining (3.13)–(3.21), (3.12) and (2.5), the required (3.11) is obtained.

**Lemma 3.5.** *For any $\sigma, \sigma_2, \sigma_3 > 0$, the functional $\Phi(t)$ introduced in (3.9) holds true*

$$
\begin{aligned}
\Phi'(t) &\leq -\Big(l_0-\sigma_3\Big)\Big(\|v_t\|_2^2+\|w_t\|_2^2\Big) + \xi_1\sigma\Big(\|\nabla v\|_2^4+\|\nabla w\|_2^4\Big)\\
&\quad +\sigma\Big(\xi_0+\widehat{l_0}^{\,2}+c\widehat{l}\Big)\|\nabla v\|_2^2 + \sigma\Big(\xi_0+\widehat{h}_0^2+cl_2\Big)\|\nabla w\|_2^2
\end{aligned}
$$

$$
\begin{aligned}
&+\sigma_2 2\delta E(0)\Big(\frac{1}{l_1}\Big(\frac{1}{2}\frac{d}{dt}\|\nabla v\|_2^2\Big)^2+\frac{1}{l_2}\Big(\frac{1}{2}\frac{d}{dt}\|\nabla w\|_2^2\Big)^2\Big)\\
&+c(\sigma,\sigma_2,\sigma_3)\Big(C_{\kappa,1}(h_1\circ\nabla v)(t)+C_{\kappa,2}(h_2\circ\nabla w)(t)\Big)\\
&+c(\sigma)\Big(\|v_t\|_m^m+\int_{\tau_1}^{\tau_2}|\beta_2(r)|\|x(z,1,r,t)\|_m^m dr\Big)\\
&+c(\sigma)\Big(\|w_t\|_m^m+\int_{\tau_1}^{\tau_2}|\beta_4(r)|\|y(z,1,r,t)\|_m^m dr\Big). \qquad (3.17)
\end{aligned}
$$

*where* $\widehat{l}=\max\{l_1,l_2\}$, $l_0=\min\{g_0,\widehat{g_0}\}$ *and* $\widehat{l_0}=\max\{g_0,\widehat{g_0}\}$.

*Proof.* To prove the result, simplification of (3.9) and (2.9) through mathematical skills leads us to the following

$$
\begin{aligned}
\Phi'(t) &= -\int_\Omega v_{tt}\int_0^\infty g_1(s)(v(t)-v(t-s))dsdz\\
&\quad -\int_\Omega v_t\frac{\partial}{\partial t}\Big(\int_0^\infty g_1(s)(v(t)-v(t-s))ds\Big)dz\\
&\quad -\int_\Omega w_{tt}\int_0^\infty g_2(s)(w(t)-w(t-s))dsdz\\
&\quad -\int_\Omega w_t\frac{\partial}{\partial t}\Big(\int_0^\infty g_2(s)(w(t)-w(t-s))ds\Big)dz\\
&= \underbrace{(\xi_0+\xi_1\|\nabla v\|_2^2)\int_\Omega\nabla v\int_0^\infty g_1(s)(\nabla v(t)-\nabla v(t-s))dsdz}_{J_{11}}\\
&\quad \underbrace{+(\xi_0+\xi_1\|\nabla w\|_2^2)\int_\Omega\nabla w\int_0^\infty g_2(s)(\nabla w(t)-\nabla w(t-s))dsdz}_{J_{12}}\\
&\quad \underbrace{+\delta\int_\Omega\nabla v\nabla v_t dz.\int_\Omega\nabla v\int_0^\infty g_1(s)(\nabla v(t)-\nabla v(t-s))dsdz}_{J_{21}}\\
&\quad \underbrace{+\delta\int_\Omega\nabla w\nabla w_t dz.\int_\Omega\nabla w\int_0^\infty g_2(s)(\nabla w(t)-\nabla w(t-s))dsdz}_{J_{22}}\\
&\quad \underbrace{-\int_\Omega\Big(\int_0^\infty g_1(s)\nabla v(t-s)ds\Big).\Big(\int_0^\infty g_1(s)(\nabla v(t)-\nabla v(t-s))ds\Big)dz}_{J_{31}}\\
&\quad \underbrace{-\int_\Omega\Big(\int_0^\infty g_2(s)\nabla w(t-s)ds\Big).\Big(\int_0^\infty g_2(s)(\nabla w(t)-\nabla w(t-s))ds\Big)dz}_{J_{32}}\\
&\quad \underbrace{-\beta_1\int_\Omega|v_t|^{m-2}v_t\Big(\int_0^\infty g_1(s)(\nabla v(t)-\nabla v(t-s))ds\Big)dz}_{J_{41}}
\end{aligned}
$$

$$
\underbrace{-\beta_3\int_\Omega |w_t|^{m-2}w_t\Big(\int_0^\infty g_2(s)(\nabla w(t)-\nabla w(t-s))ds\Big)dx}_{J_{42}}
$$
$$
\underbrace{-\int_\Omega\int_{\tau_1}^{\tau_2}|\beta_2(r)||x(z,1,r,t)|^{m-2}x(z,1,r,t)\times\int_0^\infty g_1(s)(\nabla v(t)-\nabla v(t-s))ds\Big)dsdz}_{J_{51}}
$$
$$
\underbrace{-\int_\Omega\int_{\tau_1}^{\tau_2}|\beta_4(r)||y(z,1,r,t)|^{m-2}y(z,1,r,t)\times\int_0^\infty g_2(s)(\nabla w(t)-\nabla w(t-s))ds\Big)dsdz}_{J_{51}}
$$
$$
\underbrace{-\int_\Omega v_t\frac{\partial}{\partial t}\Big(\int_0^\infty g(s)(v(t)-v(t-s))ds\Big)dz}_{J_{61}}
$$
$$
\underbrace{-\int_\Omega w_t\frac{\partial}{\partial t}\Big(\int_0^\infty g_2(s)(w(t)-w(t-s))ds\Big)dz}_{J_{62}}
$$
$$
\underbrace{-\int_\Omega f_1(v,w).\Big(\int_0^\infty g_1(s)(v(t)-v(t-s))ds\Big)dz}_{J_{71}}
$$
$$
\underbrace{-\int_\Omega f_2(v,w).\Big(\int_0^\infty g_2(s)(w(t)-w(t-s))ds\Big)dz}_{J_{72}}. \tag{3.18}
$$

Here, we will find our the approximation of the terms of the RHS of (3.18). Using the well-known Young's, Sobolev-Poincare and Hölder's inequalities on (2.1), (2.11) and Lemma 3.1, we proceed as follows

$$
\begin{aligned}
|J_{11}| &\le (\xi_0+\xi_1\|\nabla v\|_2^2)\Big(\sigma\|\nabla v\|_2^2+\frac{1}{4\sigma}C_{\kappa,1}(h_1\circ\nabla v)(t)\Big)\\
&\le \sigma\xi_0\|\nabla v\|_2^2+\sigma\xi_1\|\nabla v\|_2^4+\Big(\frac{\xi_0}{4\sigma}+\frac{\xi_1 E(0)}{4l_1\xi}\Big)C_{\kappa,1}(h_1\circ\nabla v)(t),
\end{aligned} \tag{3.19}
$$

and

$$
\begin{aligned}
J_{21} &\le \sigma_2\delta\Big(\int_\Omega \nabla v\nabla v_t dz\Big)^2\|\nabla v\|_2^2+\frac{\delta}{4\sigma_2}C_{\kappa,1}(h_1\circ\nabla v)(t)\\
&\le \sigma_2\frac{2\delta E(0)}{l_1}\Big(\frac{1}{2}\frac{d}{dt}\|\nabla v\|_2^2\Big)^2+\frac{\delta}{4\sigma_2}C_{\kappa,1}(h_1\circ\nabla v)(t),
\end{aligned} \tag{3.20}
$$

$$
\begin{aligned}
|J_{31}| &\leq \int_\Omega \Big(\int_0^\infty g_1(s)\nabla v(t)ds\Big)\Big(\int_0^\infty g_1(s)(\nabla v(t-s)-\nabla v(t))ds\Big)dz \\
&\quad -\int_\Omega \Big(\int_0^\infty g_1(s)(\nabla v(t)-\nabla v(t-s))ds\Big)^2 dz \\
&\leq \delta g_0^2\|\nabla v\|_2^2 + \Big(1+\frac{1}{4\delta}\Big)C_{\kappa,1}(h_1\circ\nabla v)(t), \qquad (3.21)
\end{aligned}
$$

$$
\begin{aligned}
|J_{41}| &\leq c(\sigma)\|\nabla v_t\|_m^m + \sigma\beta_1^m \int_\Omega \Big(\int_0^\infty g_1(s)(v(t)-v(t-s))ds\Big)^m dz \\
&\leq c(\sigma)\|\nabla v_t\|_m^m + \sigma\Big(\beta_1^m c_p^m [\frac{4g_0E(0)}{l_1}]^{(m-2)}\Big)C_{\kappa,1}(h_1\circ\nabla v)(t) \\
&\leq c(\sigma)\|\nabla v_t\|_m^m + \sigma c_3 C_{\kappa,1}(h_1\circ\nabla v)(t). \qquad (3.22)
\end{aligned}
$$

In the same, we obtained the following

$$
J_{51} \leq c(\sigma)\|x(z,1,r,t)\|_m^m + \sigma c_4 C_{\kappa,1}(h_1\circ\nabla v)(t), \qquad (3.23)
$$

and to find the approximation of $J_{61}$, we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\Big(\int_0^\infty g_1(s)(v(t)-v(t-s))ds\Big) &= \frac{\partial}{\partial t}\Big(\int_{-\infty}^t g_1(t-s)(v(t)-v(s))ds\Big) \\
&= \int_{-\infty}^t g_1'(t-s)(v(t)-v(s))ds \\
&\quad +\Big(\int_{-\infty}^t g_1(t-s)ds\Big)v_t(t) \\
&= \int_0^\infty g_1'(s)(v(t)-v(t-s))ds + g_0 v_t(t),
\end{aligned}
$$

the (2.2) implies that

$$
J_{61} \leq -(g_0-\sigma_3)\|v_t\|_2^2 + \frac{c}{\sigma_3}C_{\kappa,1}(h_1\circ\nabla v)(t). \qquad (3.24)
$$

In the same steps, the estimation of $J_{i2}$, $i = 1, .., 6$ are obtained and

$$
\begin{aligned}
J_{71} &\leq c\sigma l_1\|\nabla v\|_2^2 + c(\sigma)C_{\kappa,1}(h_1\circ\nabla v)(t) \\
J_{72} &\leq c\sigma l_2\|\nabla w\|_2^2 + c(\sigma)C_{\kappa,2}(h_2\circ\nabla v)(t). \qquad (3.25)
\end{aligned}
$$

Here, put (3.19)–(3.25) into (3.18), the required result is obtained.

**Lemma 3.6.** *The functional $\Theta(t)$ introduced in (3.10) fulfills the below*

$$
\begin{aligned}
\Theta'(t) \quad \leq \quad & -\gamma_1 \int_0^1 \int_{\tau_1}^{\tau_2} r\Big(|\beta_2(r)|.\|x(z,\rho,r,t)\|_m^m + |\beta_4(r)|.\|y(z,\rho,r,t)\|_m^m\Big) dr d\rho \\
& -\gamma_1 \int_{\tau_1}^{\tau_2} \Big(|\beta_2(s)|.\|x\,(z,1,r,t)\,\|_m^m + |\beta_4(r)|.\|y\,(z,1,r,t)\,\|_m^m\Big) dr \\
& +\beta_5\Big(\|v_t(t)\|_m^m + \|w_t(t)\|_m^m\Big). \qquad (3.26)
\end{aligned}
$$

*in which* $\beta_5 = \max\{\beta_1, \beta_3\}$.

*Proof.* To prove the result, using $\Theta(t)$, and (2.9), we obtained the following

$$
\begin{aligned}
\Theta'(t) \quad = \quad & -m \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} e^{-r\rho} |\beta_2(r)|.|x|^{m-1} x_\rho\,(z,\rho,r,t)\, dr d\rho dz \\
& -m \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} e^{-r\rho} |\beta_4(r)|.|y|^{m-1} y_\rho\,(z,\rho,r,t)\, dr d\rho dz \\
= \quad & -\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} r e^{-r\rho} |\beta_2(r)|.|x(z,\rho,r,t)|^m dr d\rho dz \\
& -\int_\Omega \int_{\tau_1}^{\tau_2} |\beta_2(r)| \Big[e^{-r}|x\,(z,1,r,t)\,|^m - |x\,(z,0,r,t)\,|^m\Big] dr dz \\
& -\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} r e^{-r\rho} |\beta_4(r)|.|y(z,\rho,r,t)|^m dr d\rho dz \\
& -\int_\Omega \int_{\tau_1}^{\tau_2} |\beta_4(r)| \Big[e^{-r}|y\,(z,1,r,t)\,|^m - |y\,(z,0,r,t)\,|^m\Big] dr dz
\end{aligned}
$$

Utilizing $x(z,0,r,t) = v_t(z,t), y(z,0,r,t) = w_t(z,t)$, and $e^{-r} \leq e^{-r\rho} \leq 1$, for any $0 < \rho < 1$, moreover, select $\gamma_1 = e^{-\tau_2}$, we have

$$
\begin{aligned}
\Theta'(t) \quad \leq \quad & -\gamma_1 \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} r\Big(|\beta_2(r)|.|z(z,\rho,r,t)|^m + |\beta_4(r)|.|y(z,\rho,r,t)|^m\Big) dr d\rho dz \\
& -\gamma_1 \int_\Omega \int_{\tau_1}^{\tau_2} \Big(|\beta_2(r)||x(z,1,r,t)|^m + |\beta_4(r)||y(z,1,r,t)|^m\Big) dr dz \\
& + \int_{\tau_1}^{\tau_2} |\beta_2(r)| dr \int_\Omega |v_t|^m(t) dz + \int_{\tau_1}^{\tau_2} |\beta_4(r)| dr \int_\Omega |w_t|^m(t) dz,
\end{aligned}
$$

applying (2.4), the required proof is obtained.

In the next step, we below functional are introduced

$$
\begin{aligned}
\mathcal{A}_1(t) \quad := \quad & \int_\Omega \int_0^t \varphi_1(t-s) \nabla v(s)^2 ds dz, \\
\mathcal{A}_2(t) \quad := \quad & \int_\Omega \int_0^t \varphi_2(t-s) \nabla w(s)^2 ds dz, \qquad (3.27)
\end{aligned}
$$

in which $\varphi_1(t) = \int_t^\infty g_1(s) ds, \varphi_2(t) = \int_t^\infty g_2(s) ds$.

**Lemma 3.7.** *Let us suppose that (2.1) and (2.2) satisfied. Then, the functional* $F_1 = \mathcal{A}_1 + \mathcal{A}_2$ *and fulfills the following*

$$\begin{aligned} F_1'(t) \quad \leq \quad & -\frac{1}{2}\Big((g_1 \circ \nabla v)(t) + (g_2 \circ \nabla w)(t)\Big) \\ & +3g_0 \int_\Omega \nabla v^2 dz + 3\widehat{g_0} \int_\Omega \nabla w^2 dz \\ & +\frac{1}{2} \int_\Omega \int_t^\infty g_1(s)(\nabla v(t) - \nabla v(t-s))^2 ds dz \\ & +\frac{1}{2} \int_\Omega \int_t^\infty g_2(s)(\nabla w(t) - \nabla w(t-s))^2 ds dz. \end{aligned} \tag{3.28}$$

*Proof.* We can easily prove this lemma with the help of Lemma 3.7 in [13] and Lemma 3.4 in [15].

Now, we have sufficient mathematical tools to prove the below mentioned Theorem.

**Theorem 3.8.** *Take (2.1)–(2.5), then one can find positive constants* $\varsigma_i, i = 1, 2, 3$ *and positive function* $\varsigma_4(t)$ *in a way that the energy functional mentioned in (2.11) fulfills*

$$E(t) \leq \varsigma_1 D_2^{-1}\left(\frac{\varsigma_2 + \varsigma_3 \int_0^t \widehat{\zeta}(v) D_4(\varsigma_4(v)\mu_0(v))dv}{\int_0^t \zeta_0(v)dv}\right), \tag{3.29}$$

*in which*

$$D_2(t) = tD'(\varepsilon_0 t), \;\; D_3(t) = tD'^{-1}(t), \;\; D_4(t) = \overline{D}_3^*(t), \tag{3.30}$$

*and*

$$\mu_0 = \max\{\mu, \widehat{\mu}\}, \;\; \widehat{\zeta} = \max\{\zeta_1, \zeta_2\}, \;\; \zeta_0 = \min\{\zeta_1, \zeta_2\},$$

*which are increasing and convex in* $(0, \varrho]$.

*Proof.* For the proof, we define the below functional

$$\mathcal{G}(t) \;\; := \;\; NE(t) + N_1\Psi(t) + N_2\Phi(t) + N_3\Theta(t), \tag{3.31}$$

we determined the positive constants $N, N_i, i = 1, 2, 3$. Simplifying (3.36) and utilizing 2.12, the Lemmas 3.4–3.6, we have

$$\begin{aligned} \mathcal{G}'(t) \;\; := \;\; & NE'(t) + N_1\Psi'(t) + N_2\Phi'(t) + N_3\Theta'(t) \\ \leq \;\; & -\Big\{N_2(l_0 - \sigma_3) - N_1\Big\}\Big(\|v_t\|_2^2 + \|w_t\|_2^2\Big) \\ & -\Big\{N_3\xi_1 - N_2\xi_1\sigma\Big\}\Big(\|\nabla v\|_2^4 + \|\nabla w\|_2^4\Big) \\ & -\Big\{N_1(l - \varepsilon(c_1 + c_2) - \sigma_1) - N_2\sigma(\xi 0 + \widehat{l_0}^2 + c\widehat{l})\Big\}\Big(\|\nabla v\|_2^2 + \|\nabla w\|_2^2\Big) \\ & -\Big\{\frac{N\delta}{4} - N_2\sigma_2\frac{2\delta E(0)}{l}\Big\}\Big[\Big(\frac{1}{2}\frac{d}{dt}\|\nabla v\|_2^2\Big)^2 + \Big(\frac{1}{2}\frac{d}{dt}\|\nabla w\|_2^2\Big)^2\Big] \\ & +\Big\{N_1c(\sigma_1) + N_2c(\sigma, \sigma_2, \sigma_3)\Big\}\Big(C_{\kappa,1}(h_1 \circ \nabla v)(t) + C_{\kappa,2}(h_2 \circ \nabla w)(t)\Big) \end{aligned}$$

$$
\begin{aligned}
&+\frac{N}{2}\Big((g_1'\circ\nabla v)(t)+(g_2'\circ\nabla w)(t)\Big)\\
&-\Big\{\gamma_0 N-N_1c(\varepsilon)-N_2c(\sigma)-N_3\beta_5\Big\}\Big(\|v_t\|_m^m+\|w_t\|_m^m\Big)\\
&-\Big(\gamma_1 N_3-N_1c(\varepsilon)-N_2c(\sigma)\Big)\int_{\tau_1}^{\tau_2}|\beta_2(r)|\|x(z,1,r,t)\|_m^m ds\Big)\\
&-N_3\gamma_1\int_0^1\int_{\tau_1}^{\tau_2} r|\beta_2(r)|.\|x(z,\rho,r,t)\|_m^m dr d\rho\\
&-\Big(\gamma_1 N_3-N_1c(\varepsilon)-N_2c(\sigma)\Big)\int_{\tau_1}^{\tau_2}|\beta_4(r)|\|y(z,1,r,t)\|_m^m dr\Big)\\
&-N_3\gamma_1\int_0^1\int_{\tau_1}^{\tau_2} r|\beta_4(r)|.\|y(z,\rho,r,t)\|_m^m dr d\rho-N_1\int_\Omega F(v,w)dz. \qquad (3.32)
\end{aligned}
$$

We select the various constants at this point such that the values included in parenthesis are positive in this stage. Here, putting

$$\sigma_3=\frac{l_0}{2},\ \ \varepsilon=\frac{l}{4(c_1+c_2)},\ \ \sigma_1=\frac{l}{4},\ \ \sigma_2=\frac{lN}{16E(0)N_2},\ \ N_1=\frac{l_0}{4}N_2.$$

Thus, we arrive at

$$
\begin{aligned}
\mathcal{H}'(t)\ \leq\ & -\frac{l_0}{4}N_2\Big(\|w_t\|_2^2+\|w_t\|_2^2\Big)-\zeta_1N_2\Big(\frac{l_0}{4}-\delta\Big)\Big(\|\nabla w\|_2^4+\|\nabla u\|_2^4\Big)\\
&-N_2\Big(\frac{ll_0}{8}-\delta(\zeta_0+\widehat{h_0}^2+\widehat{cl})\Big)\Big(\|\nabla w\|_2^2+\|\nabla u\|_2^2\Big)\\
&-\frac{N\delta}{8}\Big[\Big(\frac{1}{2}\frac{d}{dt}\|\nabla v\|_2^2\Big)^2+\Big(\frac{1}{2}\frac{d}{dt}\|\nabla w\|_2^2\Big)^2\Big]\\
&+N_2c(\sigma,\sigma_1,\sigma_2,\sigma_3)\Big(C_{\kappa,1}(h_1\circ\nabla v)(t)+C_{\kappa,2}(h_2\circ\nabla w)(t)\Big)\\
&+\frac{N}{2}\Big((g_1'\circ\nabla v)(t)+(g_2'\circ\nabla v)(t)\Big)-N_1\int_\Omega F(v,w)dz\\
&-\Big(\gamma_0N-N_2c(\sigma,\varepsilon)-N_3\beta_5\Big)\Big(\|v_t\|_m^m+\|w_t\|_m^m\Big)\\
&-\Big(\gamma_1N_3-N_2c(\sigma,\varepsilon)\Big)\int_{\tau_1}^{\tau_2}|\beta_2(r)|\|x(z,1,r,t)\|_m^m ds\Big)\\
&-N_3\gamma_1\int_0^1\int_{\tau_1}^{\tau_2} r|\beta_2(r)|.\|x(z,\rho,r,t)\|_m^m dr d\rho\\
&-\Big(\gamma_1N_3-N_2c(\sigma,\varepsilon)\Big)\int_{\tau_1}^{\tau_2}|\beta_4(r)|\|y(z,1,r,t)\|_m^m dr\Big)\\
&-N_3\gamma_1\int_0^1\int_{\tau_1}^{\tau_2} r|\beta_4(r)|.\|y(z,\rho,r,t)\|_m^m dr d\rho. \qquad (3.33)
\end{aligned}
$$

In the upcoming, we select $\sigma$ in a manner that

$$\sigma<\min\Big\{\frac{l_0}{4},\frac{ll_0}{8(\xi_0+\widehat{g_0}^2+\widehat{cl})}\Big\}.$$

After that, we take $N_2$ in a way that

$$N_2\Big(\frac{ll_0}{8} - \sigma(\xi_0 + \widehat{g_0}^2 + \widehat{cl})\Big) > 4l_0,$$

and take $N_3$ large enough in a way that

$$\gamma_1 N_3 - N_2 c(\sigma, \varepsilon) > 0.$$

As a result, for positive constants $d_i, i = 1,2,3,4,5$, (3.33) can be written as

$$\begin{aligned}
\mathcal{H}'(t) \leq\ & -d_1(\|v_t\|_2^2 + \|w_t\|_2^2) - d_2(\|\nabla v\|_2^4 + \|\nabla w\|_2^4) - 4l_0(\|\nabla v\|_2^2 + \|\nabla w\|_2^2) \\
& -\frac{N\delta}{8}\Big[\Big(\frac{1}{2}\frac{d}{dt}\|\nabla v\|_2^2\Big)^2 + \Big(\frac{1}{2}\frac{d}{dt}\|\nabla w\|_2^2\Big)^2\Big] \\
& -\Big(\frac{N}{2} - d_3 C_\kappa\Big)\Big((h_1 \circ \nabla v)(t) + (h_2 \circ \nabla w)(t)\Big) \\
& +\frac{N\kappa}{2}\Big((g_1 \circ \nabla v)(t) + (g_2 \circ \nabla w)(t)\Big) \qquad (3.34)\\
& -(\gamma_0 N - c)\Big(\|v_t\|_m^m + \|w_t\|_m^m\Big) - d_5\int_\Omega F(v,w)dz \\
& -d_4\int_0^1\int_{\tau_1}^{\tau_2} s\Big(|\beta_2(r)|.\|x(z,\rho,r,t)\|_m^m + |\beta_4(r)|.\|y(z,\rho,r,t)\|_m^m\Big)drd\rho,
\end{aligned}$$

in which $C_\kappa = \max\{C_{\kappa,1}, C_{\kappa,2}\}$.

We know that $\frac{\kappa g_i^2(s)}{\kappa g_i(s) - g_i(s)} \leq g_i(s)$, then from from Lebesgue Dominated Convergence, we have the below

$$\lim_{\kappa\to0^+} \kappa C_{\kappa,i} = \lim_{\kappa\to0^+}\int_0^\infty \frac{\kappa g_i^2(s)}{\kappa g_i(s) - g_i(s)}ds = 0, \quad i = 1,2 \qquad (3.35)$$

which leads to

$$\lim_{\kappa\to0^+} \kappa C_\kappa = 0.$$

As a result of this, one can find $0 < \kappa_0 < 1$ in a manner that if $\kappa < \kappa_0$, then

$$\kappa C_\kappa \leq \frac{1}{d_3}. \qquad (3.36)$$

From (3.8)–(3.10) through mathematical skills, we have the following

$$\begin{aligned}
|\mathcal{H}(t) - NE(t)| \leq\ & \frac{N_1}{2}\Big(\|v_t(t)\|_2^2 + \|w_t(t)\|_2^2 + c_p\|\nabla w(t)\|_2^2 + c_p\|\nabla w(t)\|_2^2\Big) \\
& +\delta\frac{N_1}{4}\Big(\|\nabla v(t)\|_2^4 + \|\nabla w(t)\|_2^4\Big) + \frac{N_2}{2}\Big(\|v_t(t)\|_2^2 + \|w_t(t)\|_2^2\Big) \\
& +\frac{N_2}{2}c_p\Big(C_{\kappa,1}(g_1 \circ \nabla v)(t) + C_{\kappa,2}(g_2 \circ \nabla w)(t)\Big) \qquad (3.37)\\
& +N_3\int_0^1\int_{\tau_1}^{\tau_2} re^{-\rho r}\Big(|\beta_2(r)|.\|x(z,\rho,r,t)\|_m^m + |\beta_4(r)|.\|y(z,\rho,r,t)\|_m^m\Big)drd\rho.
\end{aligned}$$

By the fact $e^{-\rho r} < 1$ and (2.2), we have the below

$$|\mathcal{H}(t) - NE(t)| \leq C(N_1, N_2, N_3)E(t) = C_1E(t). \tag{3.38}$$

that is

$$(N - C_1)\, E\,(t) \leq \mathcal{H}\,(t) \leq (N + C_1)\, E\,(t)\,. \tag{3.39}$$

Here, set $\kappa = \frac{1}{2N}$ and take $N$ large enough in a manner that

$$N - C_1 > 0, \;, \;\; \gamma_0 N - c > 0, \;\; \frac{1}{2}N - \frac{1}{2\kappa_0} > 0, \;\; \kappa = \frac{1}{2N} < \kappa_0,$$

we find

$$\mathcal{H}'\,(t) \leq -k_2E(t) + \frac{1}{4}((g_1 \circ \nabla v)(t) + (g_2 \circ \nabla w)(t)) \tag{3.40}$$

for some $k_2 > 0$, and

$$c_5E\,(t) \leq \mathcal{H}\,(t) \leq c_6E\,(t)\,, \forall t \geq 0 \tag{3.41}$$

for some $c_5, c_6 > 0$, we have

$$\mathcal{H}(t) \sim E(t).$$

After that, the below cases are considered:

**Case 3.9.** $G_i, i = 1, 2$ *are linear. Multiplying (3.40) by* $\zeta_0(t) = \min\{\zeta_1(t), \zeta_2(t)\}$*, we find*

$$\begin{aligned} \zeta_0(t)\mathcal{H}'\,(t) &\leq -k_2\zeta_0(t)E(t) + \frac{1}{4}\zeta_0(t)((g_1 \circ \nabla v)(t) + (g_2 \circ \nabla w)(t)) \\ &\leq -k_2\zeta_0(t)E(t) + \frac{1}{4}\zeta_1(t)(g_1 \circ \nabla v)(t) + \frac{1}{4}\zeta_2(t)(g_2 \circ \nabla w)(t). \end{aligned} \tag{3.42}$$

*The last two terms in (3.42), we have*

$$\begin{aligned} \frac{\zeta_1(t)}{4}(g_1 \circ \nabla v)(t) &= \frac{\zeta_1(t)}{4}\int_\Omega\int_0^\infty g_1(\delta)|\nabla\eta^t(s)|^2dsdz \\ &= \underbrace{\frac{\zeta_1(t)}{4}\int_\Omega\int_0^t g_1(s)|\nabla\eta^t(s)|^2dsdz}_{I_1} \\ &\quad + \underbrace{\frac{\zeta_1(t)}{4}\int_\Omega\int_t^\infty g_1(s)|\nabla\eta^t(s)|^2dsdz}_{I_2} \end{aligned} \tag{3.43}$$

*To estimate* $I_1$*, using (2.11),*

$$\begin{aligned} I_1 &\leq \frac{1}{4}\int_\Omega\int_0^t \zeta_1(s)g_1(s)|\nabla\eta^t(s)|^2dsdz \\ &= -\frac{1}{4}\int_\Omega\int_0^t g_1'(s)|\nabla\eta^t(s)|^2dsdz \\ &\leq -\frac{1}{2l_1}E'(t), \end{aligned} \tag{3.44}$$

*and by (3.4), we get*

$$I_2 \leq \frac{\beta}{4}\zeta_1(t)\mu(t). \tag{3.45}$$

*In the same way, we obtained*

$$\frac{\zeta_2(t)}{4}(g_2 \circ \nabla w)(t) \leq -\frac{1}{2l_2}E'(t) + \frac{\widehat{\beta}}{4}\zeta_2(t)\widehat{\mu}(t). \tag{3.46}$$

*As a result of this, we get*

$$\zeta_0(t)\mathcal{H}'(t) \leq -k_2\zeta_0(t)E(t) - \frac{1}{\widetilde{l}}E'(t) + 2\beta_0 w(t), \tag{3.47}$$

*where* $\beta_0 = \max\{\frac{\beta}{4}, \frac{\widehat{\beta}}{4}\}$ *and* $w(t) = \widehat{\zeta}(t)\mu_0(t)$.

*Applying* $\zeta_i'(t) \leq 0$, *we get*

$$\mathcal{H}_1'(t) \leq -k_2\zeta_0(t)E(t) + 2\beta_0 w(t), \tag{3.48}$$

*with*

$$\mathcal{H}_1(t) = \zeta_0(t)\mathcal{H}(t) + \frac{1}{\widetilde{l}}E(t) \sim E(t),$$

*we have*

$$k_4E(t) \leq \mathcal{H}_1(t) \leq k_5E(t), \tag{3.49}$$

*then, the following is obtained from (3.48)*

$$\begin{aligned} k_2E(T)\int_0^T \zeta_0(t)dt &\leq k_2\int_0^T \zeta_0(t)E(t)dt \\ &\leq \mathcal{H}_1(0) - \mathcal{H}_1(T) + 2\beta_0\int_0^T w(t)dt \\ &\leq \mathcal{H}_1(0) + 2\beta_0\int_0^T \widehat{\zeta}(t)\mu_0(t)dt. \end{aligned}$$

*Further analysis implies that*

$$E(T) \leq \frac{1}{k_2}\Big(\frac{\mathcal{G}_1(0) + 2\beta_0\int_0^T \widehat{\xi}(t)\mu_0(t)dt}{\int_0^T \xi_0(t)dt}\Big),$$

*From the linearity of D, the linearity of the functions* $D_2, D_2'$ *and* $D_4$ *can easily be determined. This implies that*

$$E(T) \leq \lambda_1 D_2^{-1}\Big(\frac{\frac{\mathcal{H}_1(0)}{k_2} + \frac{2\beta_0}{k_2}\int_0^T \widehat{\zeta}(t)\mu_0(t)dt}{\int_0^T \zeta_0(t)dt}\Big), \tag{3.50}$$

*which gives (3.29) with* $\varsigma_1 = \lambda_1$, $\varsigma_2 = \frac{\mathcal{H}_1(0)}{k_2}$, $\varsigma_3 = \frac{2\beta_0}{\lambda_2 k_2}$ , *and* $\varsigma_4(t) = Id(t) = t$. *Hence, the required proof is completed.*

**Case 3.10.** *Let $H_i, i = 1, 2$ are nonlinear. Then, with the help of (3.28) and (3.40). Assume the positive functional*

$$\mathcal{H}_2(t) = \mathcal{H}(t) + F_1(t)$$

*then for all $t \geq 0$ and for some $k_3 > 0$, the following holds true*

$$\begin{aligned} \mathcal{H}_2'(t) &\leq -k_3 E(t) + \frac{1}{2}\int_\Omega \int_t^\infty g_1(s)(\nabla v(t) - \nabla v(t-s))^2 ds dz \\ &+ \frac{1}{2}\int_\Omega \int_t^\infty g_2(s)(\nabla w(t) - \nabla w(t-s))^2 ds dz, \end{aligned} \tag{3.51}$$

*with the help of (3.4), we have*

$$\begin{aligned} k_3 \int_0^t E(x)dx &\leq \mathcal{H}_2(0) - \mathcal{H}_2(t) + \beta_0 \int_0^t \mu_0(\varsigma)d\varsigma \\ &\leq \mathcal{H}_2(0) + \beta_0 \int_0^t \mu_0(\varsigma)d\varsigma. \end{aligned} \tag{3.52}$$

*Therefore*

$$\int_0^t E(x)dx \leq k_6 \mu_1(t), \tag{3.53}$$

*where $k_6 = \max\{\frac{\mathcal{H}_2(0)}{k_3}, \frac{\beta_0}{k_3}\}$ and $\mu_1(t) = 1 + \int_0^t \mu_0(\varsigma)d\varsigma$.*

**Corollary 3.11.** *The following is obtained from (2.11) and (3.53):*

$$\begin{aligned} &\int_0^t \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 dz ds \\ &+ \int_0^t \int_\Omega |\nabla w(t) - \nabla w(t-s)|^2 dz ds \\ \leq\ & 2\int_0^t \int_\Omega \nabla v^2(t) - \nabla v^2(t-s) dz ds \\ &+ 2\int_0^t \int_\Omega \nabla w^2(t) - \nabla w^2(t-s) dz ds \\ \leq\ & \frac{4}{l_0}\int_0^t E(t) - E(t-s) ds \\ \leq\ & \frac{8}{l_0}\int_0^t E(x)dx \leq \frac{8k_6}{l_0}\mu_1(t). \end{aligned} \tag{3.54}$$

*Now, we define $\phi_i(t), i = 1, 2$ by*

$$\begin{aligned} \phi_1(t) &:= \mathcal{B}(t)\int_0^t \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 dz ds, \\ \phi_2(t) &:= \mathcal{B}(t)\int_0^t \int_\Omega |\nabla w(t) - \nabla w(t-s)|^2 dz ds \end{aligned} \tag{3.55}$$

*where* $\mathcal{B}(t) = \frac{\mathcal{B}_0}{\mu_1(t)}$ *and* $0 < \mathcal{B}_0 < \min\{1, \frac{l}{8k_6}\}$.

*Then, by (3.53), we have*

$$\phi_i(t) < 1, \quad \forall t > 0, \quad i = 1, 2 \tag{3.56}$$

*Further, we suppose that* $\phi_i(t) > 0, \ \forall t > 0, \ i = 1, 2.$ *In addition to this, we define another functional* $\Gamma_1, \Gamma_2$ *by*

$$\begin{aligned} \Gamma_1(t) &:= -\int_0^t g_1'(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 dz ds, \\ \Gamma_2(t) &:= -\int_0^t g_2'(s) \int_\Omega |\nabla w(t) - \nabla w(t-s)|^2 dz ds \end{aligned} \tag{3.57}$$

*Here, obviously* $\Gamma_i(t) \leq -cE'(t), \ i = 1, 2$. *As* $G_i(0) = 0, \ i = 1, 2$ *and* $G_i(t)$ *are convex strictly on* $(0, \varrho]$, *then*

$$G_i(\lambda z) \leq \lambda G_i(z), \quad 0 < \lambda < 1, \ z \in (0, \varrho], \quad i = 1, 2. \tag{3.58}$$

*Applying (2.3) and (3.56), we get*

$$\begin{aligned} \Gamma_1(t) &= \frac{-1}{\mathcal{B}(t)\phi_1(t)} \int_0^t \phi_1(t)(g_1'(s)) \int_\Omega \mathcal{B}(t)|\nabla v(t) - \nabla v(t-s)|^2 dz ds \\ &\geq \frac{1}{\mathcal{B}(t)\phi_1(t)} \int_0^t \phi_1(t)\zeta_1(s)G_1(g_1(s)) \int_\Omega \mathcal{B}(t)|\nabla v(t) - \nabla v(t-s)|^2 dz ds \\ &\geq \frac{\zeta_1(t)}{\mathcal{B}(t)\phi_1(t)} \int_0^t G_1(\phi_1(t)g_1(s)) \int_\Omega \mathcal{B}(t)|\nabla v(t) - \nabla v(t-s)|^2 dz ds \\ &\geq \frac{\zeta_1(t)}{\mathcal{B}(t)} G_1\Big(\frac{1}{\phi_1(t)} \int_0^t \phi_1(t)g_1(s) \int_\Omega \mathcal{B}(t)|\nabla v(t) - \nabla v(t-s)|^2 dz ds\Big) \\ &= \frac{\zeta_1(t)}{\mathcal{B}(t)} G_1\Big(\mathcal{B}(t) \int_0^t g_1(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 dz ds\Big) \\ &= \frac{\zeta_1(t)}{\mathcal{B}(t)} \overline{G_1}\Big(\mathcal{B}(t) \int_0^t g_1(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 dz ds\Big). \end{aligned} \tag{3.59}$$

$$\Gamma_2(t) \geq \frac{\zeta_2(t)}{\mathcal{B}(t)} \overline{G_2}\Big(\mathcal{B}(t) \int_0^t g_2(s) \int_\Omega |\nabla w(t) - \nabla w(t-s)|^2 dz ds\Big). \tag{3.60}$$

*Taking the same steps,* $\overline{G_i}, i = 1, 2$ *are* $C^2$*-extension of* $G_i$ *that are convex strictly and increasing strictly on* $\mathbf{R}_+$. *From (3.59), we have the following*

$$\begin{aligned} \int_0^t g_1(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 dz ds &\leq \frac{1}{\mathcal{B}(t)} \overline{G_1}^{-1}\Big(\frac{\mathcal{B}(t)\Gamma_1(t)}{\zeta_1(t)}\Big) \\ \int_0^t g_2(s) \int_\Omega |\nabla w(t) - \nabla w(t-s)|^2 dz ds &\leq \frac{1}{\mathcal{B}(t)} \overline{G_2}^{-1}\Big(\frac{\mathcal{B}(t)\Gamma_2(t)}{\zeta_2(t)}\Big). \end{aligned} \tag{3.61}$$

*Putting (3.61) and (3.4) into (3.40), we have*

$$\mathcal{H}'(t) \leq -k_2E(t) + \frac{c}{\mathcal{B}(t)}\overline{G_1}^{-1}\Big(\frac{\mathcal{B}(t)\Gamma_1(t)}{\zeta_1(t)}\Big) + \frac{c}{\mathcal{B}(t)}\overline{G_2}^{-1}\Big(\frac{\mathcal{B}(t)\Gamma_2(t)}{\zeta_2(t)}\Big) + k_6\mu_0(t) \quad (3.62)$$

*Here, introduce $\mathcal{K}_1(t)$ for $\varepsilon_0 < r$ by*

$$\mathcal{K}_1(t) = D'\left(\varepsilon_0\frac{\mathcal{B}(t)E(t)}{E(0)}\right)\mathcal{H}(t) + E(t), \quad (3.63)$$

*in which $D' = \min\{G_1, G_2\}$ and is equivalent to $E(t)$. Because of this $E'(t) \leq 0, \overline{G_i}' > 0$, and $\overline{G_i}'' > 0, i = 1, 2$. Also applying (3.62), we obtained that*

$$\begin{aligned}
\mathcal{K}_1'(t) &= \varepsilon_0\Big(\frac{E'(t)\mathcal{B}(t)}{E(0)} + \frac{E(t)\mathcal{B}'(t)}{E(0)}\Big)D''\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right)\mathcal{H}(t) \\
&\quad + D'\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right)\mathcal{H}'(t) + E'(t) \\
&\leq -k_2E(t)D'\left(\varepsilon_0\frac{\mathcal{B}(t)E(t)}{E(0)}\right) + k_6\mu_0(t)D'\left(\varepsilon_0\frac{\mathcal{B}(t)E(t)}{E(0)}\right) \\
&\quad + \frac{c}{\mathcal{B}(t)}\overline{G_1}^{-1}\left(\frac{\mathcal{B}(t)\Gamma_1(t)}{\zeta_1(t)}\right)D'\left(\varepsilon_0\frac{\mathcal{B}(t)E(t)}{E(0)}\right) \\
&\quad + \frac{c}{\mathcal{B}(t)}\overline{G_2}^{-1}\left(\frac{\mathcal{B}(t)\Gamma_2(t)}{\zeta_2(t)}\right)D'\left(\varepsilon_0\frac{\mathcal{B}(t)E(t)}{E(0)}\right) + E'(t) \quad (3.64)
\end{aligned}$$

*According to [29], we introduce the conjugate function of $\overline{G_i}$ by $\overline{G_i}^*$, which fulfills*

$$AB \leq \overline{G_i}^*(A) + \overline{G_i}(B), \quad i = 1, 2 \quad (3.65)$$

*For $A = D'(\varepsilon_0(E(t)\mathcal{B}(t))/(E(0))))$ and $B_i = \overline{G_i}^{-1}((\mathcal{B}(t)\Gamma_i(t))/(\zeta_i(t))), \quad i = 1, 2$ and applying (3.64), we have*

$$\begin{aligned}
\mathcal{K}_1'(t) &\leq -k_2E(t)D'\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right) + k_6\mu_0(t)D'\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right) \\
&\quad + \frac{c}{\mathcal{B}(t)}\overline{G_1}^*\left(D'\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right)\right) + \frac{c}{\mathcal{B}(t)}\frac{\mathcal{B}(t)\Gamma_1(t)}{\zeta_1(t)} \\
&\quad + \frac{c}{\mathcal{B}(t)}\overline{G_2}^*\left(D'\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right)\right) + \frac{c}{\mathcal{B}(t)}\frac{\mathcal{B}(t)\Gamma_2(t)}{\zeta_2(t)} + E'(t) \\
&\leq -k_2E(t)D'\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right) + k_6\mu_0(t)D'\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right) \\
&\quad + \frac{c}{\mathcal{B}(t)}D'\Big(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\Big)(\overline{G_1}')^{-1}\left[D'\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right)\right] \quad (3.66) \\
&\quad + \frac{c}{\mathcal{B}(t)}D'\Big(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\Big)(\overline{G_2}')^{-1}\left[D'\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right)\right] \\
&\quad + \frac{c\Gamma_1(t)}{\zeta 1(t)} + \frac{c\Gamma_2(t)}{\zeta_2(t)}.
\end{aligned}$$

*Here, we multiply (3.66) by $\zeta_0(t)$ and get*

$$\begin{aligned}
\zeta_0(t)\mathcal{K}_1'(t) &\leq -k_2\zeta_0(t)E(t)D'\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right) + k_6\zeta_0(t)\mu_0(t)D'\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right) \\
&\quad + \frac{2c\zeta_0(t)}{\mathcal{B}(t)}\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}D'\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right) + c\Gamma_1(t) + c\Gamma_2(t) \\
&\leq -k_2\zeta_0(t)E(t)D'\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right) + k_6\zeta_0(t)\mu_0(t)D'\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right) \\
&\quad + \frac{2c\zeta_0(t)}{\mathcal{B}(t)}\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}D'\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right) - cE'(t) \qquad (3.67)
\end{aligned}$$

*where we utilized the following $\varepsilon_0(\mathcal{B}(t)E(t)/E(0)) < r$, $D' = \min\{G_1, G_2\}$ and $\Gamma_i < -cE'(t), i = 1, 2$, and define the functional $\mathcal{K}_2(t)$ as*

$$\mathcal{K}_2(t) = \zeta_0(t)\mathcal{K}_1(t) + cE(t) \qquad (3.68)$$

*Effortlessly, one can prove that $\mathcal{K}_2(t) \sim E(t)$, i.e., one can find two positive constants $m_1$ and $m_2$ in a manner that*

$$m_1\mathcal{K}_2(t) \leq E(t) \leq m_2\mathcal{K}_2(t), \qquad (3.69)$$

*then, we have*

$$\begin{aligned}
\mathcal{K}_2'(t) &\leq -\beta_6\zeta_0(t)\frac{E(t)}{E(0)}D'\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right) + k_6\zeta_0(t)\mu_0(t)D'\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right) \\
&= -\beta_6\frac{\zeta_0(t)}{\mathcal{B}(t)}D_2\left(\frac{E(t)\mathcal{B}(t)}{E(0)}\right) + k_6\zeta_0(t)\mu_0(t)D'\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right), \qquad (3.70)
\end{aligned}$$

*where $\beta_6 = (k_2E(0) - 2c\varepsilon_0)$ and $D_2(t) = tD'(\varepsilon_0 t)$.*

*Choosing $\varepsilon_0$ so small such that $\beta_6 > 0$, since $D_2'(t) = D'(\varepsilon_0 t) + \varepsilon_0 tD''(\varepsilon_0 t)$. As $D_2'(t), D_2(t) > 0$ on* (0, 1] *and $G_i$ on* $(0, \varrho]$ *are strictly increasing. Applying Young's inequality (3.65) on the last term in (3.70)*
*with $A = D'\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right)$ and $B = \frac{k_6}{\delta}\mu(t)$, we find*

$$\begin{aligned}
k_6\mu_0(t)D'\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right) &= \frac{\sigma}{\mathcal{B}(t)}\Big(\frac{k_6}{\sigma}\mathcal{B}(t)\mu_0(t)\Big)\Big(D'\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right)\Big) \\
&< \frac{\sigma}{\mathcal{B}(t)}D_3^*\Big(\frac{k_6}{\sigma}\mathcal{B}(t)\mu_0(t)\Big) + \frac{\sigma}{\mathcal{B}(t)}D_3\Big(D'\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right)\Big) \\
&< \frac{\sigma}{\mathcal{B}(t)}D_4\Big(\frac{k_6}{\sigma}\mathcal{B}(t)\mu_0(t)\Big) \\
&\quad + \frac{\sigma}{\mathcal{B}(t)}\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right)D'\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right) \\
&< \frac{\sigma}{\mathcal{B}(t)}D_4\Big(\frac{k_6}{\sigma}\mathcal{B}(t)\mu_0(t)\Big) + \frac{\sigma\varepsilon_0}{\mathcal{B}(t)}D_2\left(\varepsilon_0\frac{E(t)\mathcal{B}(t)}{E(0)}\right). \qquad (3.71)
\end{aligned}$$

*Here, choose $\sigma$ small enough in a manner that $\beta_6 - \sigma\varepsilon_0 > 0$ and combining (3.70) and (3.71), we have*

$$\mathcal{K}_2'(t) \leq -\beta_7 \frac{\zeta_0(t)}{\mathcal{B}(t)} D_2\left(\frac{E(t)\mathcal{B}(t)}{E(0)}\right) + \frac{\sigma\zeta_0(t)}{\mathcal{B}(t)} D_4\Big(\frac{k_6}{\delta}\mathcal{B}(t)\mu_0(t)\Big). \tag{3.72}$$

*where $\beta_7 = \beta_6 - \sigma\varepsilon_0 > 0$, $D_3(t) = tD'^{-1}(t)$ and $D_4(t) = \overline{D}_3^*(t)$.*

*In light of fact $E' < 0$ and $\mathcal{B}' < 0$, then $D_2(\frac{E(t)\mathcal{B}(t)}{E(0)})$ is decreasing. As a consequences of this, for $0 \leq t \leq T$, we have*

$$D_2\Big(\frac{E(T)\mathcal{B}(T)}{E(0)}\Big) < D_2\Big(\frac{E(t)\mathcal{B}(t)}{E(0)}\Big). \tag{3.73}$$

*In the next step, combine (3.72) with (3.73) and multiply by $\mathcal{B}(t)$, the following is obtained*

$$\mathcal{B}(t)\mathcal{K}_2'(t) + \beta_7\zeta_0(t)D_2\left(\frac{E(T)\mathcal{B}(T)}{E(0)}\right) < \sigma\zeta_0(t)D_4\Big(\frac{k_6}{\sigma}\mathcal{B}(t)\mu_0(t)\Big). \tag{3.74}$$

*Since $\mathcal{B}' < 0$, then for any $0 < t < T$*

$$\begin{aligned}(\mathcal{B}\mathcal{K}_2)'(t) + \beta_7\zeta_0(t)D_2\left(\frac{E(T)\mathcal{B}(T)}{E(0)}\right) &< \sigma\zeta_0(t)D_4\Big(\frac{k_6}{\sigma}\mathcal{B}(t)\mu_0(t)\Big)\\ &< \sigma\widehat{\zeta}(t)D_4\Big(\frac{k_6}{\sigma}\mathcal{B}(t)\mu_0(t)\Big).\end{aligned} \tag{3.75}$$

*Simplify (3.75) over $[0, T]$ and apply $\mathcal{B}(0) = 1$, the following is obtained*

$$D_2\left(\frac{E(T)\mathcal{B}(T)}{E(0)}\right)\int_0^T \zeta_0(t)dt < \frac{\mathcal{K}_2(0)}{\beta_7} + \frac{\sigma}{\beta_7}\int_0^T \widehat{\zeta}(t)D_4\Big(\frac{k_6}{\sigma}\mathcal{B}(t)\mu_0(t)\Big)dt. \tag{3.76}$$

*Consequently, we have*

$$D_2\left(\frac{E(T)\mathcal{B}(T)}{E(0)}\right) < \frac{\frac{\mathcal{K}_2(0)}{\beta_7} + \frac{\sigma}{\beta_7}\int_0^T \widehat{\zeta}(t)D_4(\frac{k_6}{\sigma}\mathcal{B}(t)\mu_0(t))dt}{\int_0^T \zeta_0(t)dt}. \tag{3.77}$$

*As a results of this, we obtain*

$$\left(\frac{E(T)\mathcal{B}(T)}{E(0)}\right) < D_2^{-1}\Big(\frac{\frac{\mathcal{K}_2(0)}{\beta_7} + \frac{\sigma}{\beta_7}\int_0^T \widehat{\zeta}(t)D_4(\frac{k_6}{\sigma}\mathcal{B}(t)\mu_0(t))dt}{\int_0^T \zeta_0(t)dt}\Big). \tag{3.78}$$

*As a result of this, we get*

$$E(T) < \frac{E(0)}{\mathcal{B}(T)}D_2^{-1}\Big(\frac{\frac{\mathcal{K}_2(0)}{\beta_7} + \frac{\sigma}{\beta_7}\int_0^T \widehat{\zeta}(t)D_4(\frac{k_6}{\sigma}\mathcal{B}(t)\mu_0(t))dt}{\int_0^T \zeta_0(t)dt}\Big). \tag{3.79}$$

*where, we have (3.29) with $\varsigma_1 = \frac{E(0)}{\mathcal{B}(T)}$, $\varsigma_2 = \frac{\mathcal{K}_2(0)}{\beta_7}$, $\varsigma_3 = \frac{\sigma}{\beta_7}$ , and $\varsigma_4(t) = \frac{k_6}{\sigma}\mathcal{B}(t)$.*

*Hence, the required result is obtained 3.8.*

## 4. Conclusions

The purpose of this work was to study when the coupled system of nonlinear viscoelastic wave equations with distributed delay components, infinite memory and Balakrishnan-Taylor damping. Assume the kernels $g_i : \mathbf{R}_+ \to \mathbf{R}_+$ holds true the below

$$g_i'(t) \leq -\zeta_i(t)G_i(g_i(t)), \ \ \forall t \in \mathbf{R}_+, \ \text{ for } \ i = 1, 2,$$

in which $\zeta_i$ and $G_i$ are functions. We prove the stability of the system under this highly generic assumptions on the behaviour of $g_i$ at infinity and by dropping the boundedness assumptions in the historical data. This type of problem is frequently found in some mathematical models in applied sciences. Especially in the theory of viscoelasticity. What interests us in this current work is the combination of these terms of damping, which dictates the emergence of these terms in the problem. In the next work, we will try to using the same method with same problem. But in added of other dampings.

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## Conflict of interest

The authors declare there is no conflicts of interest.

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