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Research article

Determination of the 3D Navier-Stokes equations with damping

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Abstract: This paper is concerned the determination of trajectories for the three-dimensional Navier-Stokes equations with nonlinear damping subject to periodic boundary condition. By using the energy estimate of Galerkin approximated equation, the finite number of determining modes and asymptotic determined functionals have been shown via the Grashof numbers for the non-autonomous and autonomous damped Navier-Stokes fluid flow respectively.

Keywords: determination; Navier-Stokes equation; damping

1. Introduction

The three-dimensional Navier-Stokes equations with damping describe the flow when there exists resistance in the fluid motion. The damping is related to various physical phenomena, such as air drag, friction effects or relative motion, caused by internal friction of fluid and the limitation of flow channel interface leading to friction and collision between fluid particles and walls.

This paper is concerned with the asymptotic behavior of the 3D Navier-Stokes equations with nonlinear damping for a viscous incompressible fluid on the torus $\Omega = \prod_{i=1}^{3}(0, L_i) \in \mathbb{R}^3$ and $t \in \mathbb{R}^+$ in the space-periodic case:

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \alpha|u|^{\beta-1}u + \nabla p = f \ \text{ in } \ \Omega\times\mathbb{R}^+, \\ \nabla\cdot u = 0 \ \text{ in } \ \Omega\times\mathbb{R}^+, \\ u(x+L_ie_i, t) = u(x,t), \quad i = 1,2,3, \\ u(x, t=0) = u_0(x), \end{cases} \tag{1.1}$$

where the kinematic viscosity $\nu > 0$ and the external force $f = f(x,t)$ are given in appropriate Sobolev space and the constant $\alpha > 0$ is a characteristic parameter of the elasticity for fluid flow, $\beta \geq 1$ is a fixed positive parameter which describes the increasing radio. The above system formed by the unknown three-component velocity field $u = (u_1, u_2, u_3)$ and the scalar pressure p represents the conservation law of momentum and mass.

When the parameter α disappears, system (1.1) reduces to the 3D classic incompressible Navier-Stokes equations, whose well-posedness (see [1–7]) and dynamic systems (see [8, 9]) have been extensively investigated based on non-uniqueness of weak solution and global existence of strong solution in bounded or periodic domain. Now for the more general case $\alpha > 0$, Cai and Jiu have proved that the Cauchy problem of the 3D Navier-Stokes equations with nonlinear damping term has global weak solution for $\beta \geq 1$, global strong solution for $\beta \geq \frac{7}{2}$ and the uniqueness for $\frac{7}{2} \leq \beta \leq 5$ in [10]. Since the nonlinear damping term $\alpha|u|^{\beta-1}u$ leads to more regularity than the classical Navier-Stokes equations, the research on infinite dimensional dynamic systems for (1.1) are progressively improved, such as [11–14].

Based on the development of 2D/3D Navier-Stokes equations, there have many related literatures paying attention to the determination and reduction of incompressible flow flows, for instance in [15, 16], the authors give the upper bound of determining modes for the 2D Navier-Stokes equations in the periodic case.

Inspired by [10, 15–18], we consider the determination of trajectories for the damped Navier-Stokes model (1.1). The main results and features can be summarized as follows.

(I) As shown in [10], the nonlinear damping term $\alpha|u|^{\beta-1}u$ resulting in more regularity, the system (1.1) possesses a global strong solution satisfying

$$u \in L^{\infty}(0, T; \dot{V}_{per}) \cap L^2(0, T; H^2(\Omega)^3) \cap L^{\infty}(0, T; L^{\beta+1}(\Omega)^3), \tag{1.2}$$

which guarantees the research on asymptotic behavior can be achieved. However, more assumption on u and f are needed as

$$\int_{\Omega} u dx = \int_{\Omega} f dx = 0, \ \ \forall \ \ t \in \mathbb{R}^+, \tag{1.3}$$

which implies Poincaré's inequality holds.

(II) For the non-autonomous case, by using the estimates on Galerkin's approximated equation, the Mazur inequality and continuous embedding $\dot{V}_{per} \subset L^{\frac{3(\beta-1)}{2}}(\Omega)^3$, the finite determining modes m has been presented via the restriction on generalized Grashof number G_r as $m \geq CG_r^3$ for $\frac{7}{2} \leq \beta \leq 5$.

(III) For the autonomous case, the problem is called asymptotic determination if there exists finite Fourier functionals $\mathcal{F} = \{F_i\}$ with $i = 1, 2, \cdots, n$, such that the trajectories inside global attractor can be determined. Based on the existence of global attractor in [14], we can proved that the autonomous system (1.1) is asymptotic determining if $n > C(\frac{L}{\nu\lambda_1})^{\frac{3}{2}}$ is large enough, where L is defined in Section 3.4.

The structure of this paper is organized as follows. In Section 2, the preliminaries and functional setting are stated. The main results are shown in Section 3, the further research and some comments are also presented in this part.

2. Preliminaries and functional settings

2.1. Functional spaces

Following the notation in [4, 6, 7, 16], denote

$$\begin{aligned}\dot{H}_{per} &= \{u \in L^2_{per}(\Omega)^3 \Big| \int_\Omega u dx = 0, \nabla \cdot u = 0\} \\ &= \{u = \sum_{k\in\mathbb{Z}^3\backslash\{0\}} \hat{u}_k e^{2\pi i \frac{k}{L}\cdot x} \Big| \hat{u}_{-k} = \hat{u}_k, \frac{k}{L}\cdot \hat{u}_k = 0, \sum_{k\in\mathbb{Z}^3\backslash\{0\}} |\hat{u}_k|^2 < \infty\}\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}\dot{V}_{per} &= \{u \in H^1_{per}(\Omega)^3 \Big| \int_\Omega u dx = 0, \nabla \cdot u = 0\} \\ &= \{u = \sum_{k\in\mathbb{Z}^3\backslash\{0\}} \hat{u}_k e^{2\pi i \frac{k}{L}\cdot x} \Big| \hat{u}_{-k} = \hat{u}_k, \frac{k}{L}\cdot \hat{u}_k = 0, \sum_{k\in\mathbb{Z}^3\backslash\{0\}} \Big|\frac{k}{L}\Big|^2 |\hat{u}_k|^2 < \infty\},\end{aligned} \tag{2.2}$$

where $\hat{u}_k$ denotes the k·th Fourier coefficient, $k = (k_1, k_2, k_3) \in \mathbb{Z}^3\backslash\{0\}$ and $L = (L_1, L_2, L_3)$.

It is easy to check that $\dot{H}_{per}$ and $\dot{V}_{per}$ are Hilbert spaces with the inner products

$$(u, v)_{\dot{H}_{per}} = \int_\Omega u(x)\cdot v(x) dx, \quad (u, v)_{\dot{V}_{per}} = \sum_{i,j=1}^{3} \int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} dx \tag{2.3}$$

and the norms

$$\|u\|_{\dot{H}_{per}} = \Big(\sum_{k\in\mathbb{Z}^3\backslash\{0\}} |\hat{u}_k|^2\Big)^{\frac{1}{2}}, \quad \|u\|_{\dot{V}_{per}} = \Big(\sum_{k\in\mathbb{Z}^3\backslash\{0\}} \Big|\frac{k}{L}\Big|^2 |\hat{u}_k|^2\Big)^{\frac{1}{2}} \tag{2.4}$$

respectively.

2.2. *The Helmholtz-Weyl decomposition*

In this part, the Helmholtz-Leray projector P_L on the Lebesgue space $\dot{L}^2_{per}(\Omega)$ is defined via the Helmholtz-Weyl decomposition

$$\dot{L}^2_{per}(\Omega)^3 = \dot{H}_{per} \oplus G(\Omega), \tag{2.5}$$

where the curl field $\dot{H}_{per}$ is defined as (2.1) whose element satisfies the weakly divergence free condition

$$< u, \nabla\phi >= 0 \ \text{ for every } \ u \in \dot{H}_{per}, \ \ \phi \in C_0^\infty(\Omega), \tag{2.6}$$

and the gradient field $G(\Omega)$ on the torus Ω is defined by

$$G(\Omega) = \{u \in \dot{L}^2_{per}(\Omega)^3 | u = \nabla g, \ \ g \in H^1(\Omega)\}. \tag{2.7}$$

The decomposition (2.5) means, every function $u \in \dot{L}^2_{per}(\Omega)^3$ can be decomposed uniquely as

$$u = h + \nabla g, \tag{2.8}$$

where the function h belongs to $\dot{H}_{per}$ and the scalar function g belongs to $H^1(\Omega)$ (see [6]).

Now, the Helmholtz-Leray projector P_L on the torus Ω is defined as

$$P_L : \dot{L}^2_{per}(\Omega)^3 \to \dot{H}_{per}, \tag{2.9}$$

where $u \in \dot{L}^2_{per}(\Omega)^3$ and $h \in \dot{H}_{per}$, i.e., $P_L u = h$ for $u \in \dot{L}^2_{per}(\Omega)^3$.

2.3. The Stokes operator

The Stokes operator A is defined as

$$Au = P_L(-\Delta u) = -P_L \Delta u, \quad \text{for any} \;\; u \in H^2(\Omega)^3, \tag{2.10}$$

with noting that $A = -\Delta$ in the three-dimensional periodic case (see [4]). Since A is a positive operator, we can deduce that the eigenvalues λ_i of the operator A are positive and satisfy

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i < \infty \;\; \text{and} \;\; \lim_{i \to \infty} \lambda_i = \infty. \tag{2.11}$$

Hence, for u belongs to $\dot{H}_{per}$, the Poincaré inequality

$$\|u\|^2_{\dot{H}_{per}} \leq \frac{1}{\lambda_1} \|u\|^2_{\dot{V}_{per}}, \quad \text{for all} \;\; u \in \dot{V}_{per} \tag{2.12}$$

holds.

2.4. The bilinear operator and trilinear operator

The bilinear and trilinear operators are defined as

$$B(u, v) = P_L(u \cdot \nabla)v \;\; \text{and} \;\; b(u, v, w) = (P_L(u \cdot \nabla)v, w)_{\dot{H}_{per}}, \tag{2.13}$$

which satisfy

$$b(u, v, w) = -b(u, w, v) \;\; \text{and} \;\; b(u, v, v) = 0, \quad \text{for any} \;\; u, v, w \in \dot{V}_{per}. \tag{2.14}$$

2.5. Some lemmas

Lemma 2.1. *(The Ladyzhenskaya inequality) For u defined on the tours $\Omega \subset \mathbb{R}^3$, the following estimates*

$$\|u(x)\|_{L^4_{per}(\Omega)^3} \leq \|u\|^{\frac{1}{4}}_{\dot{H}_{per}} \|u\|^{\frac{3}{4}}_{\dot{V}_{per}}, \tag{2.15}$$

$$\|u(x)\|_{L^\infty_{per}(\Omega)^3} \leq \|u\|^{\frac{1}{2}}_{\dot{V}_{per}} \|Au\|^{\frac{1}{2}}_{\dot{H}_{per}}. \tag{2.16}$$

hold.

Proof. See, e.g., [6] for more detail.

Lemma 2.2. *(The Mazur inequality) Let $p \geq 0$ is an arbitrary non-negative constant, one can deduce the following inequality for any x, $y \in \mathbb{R}$*

$$2^{-p}|x - y|^{p+1} \leq |x|x|^p - y|y|^p| \leq (p + 1)|x - y|(|x|^p + |y|^p). \tag{2.17}$$

Proof. See, e.g., Kuang [19].

Lemma 2.3. *(The generalized Gronwall inequality) Let $\eta = \eta(t)$ and $\phi = \phi(t)$ be locally integrable real-valued functions on $[0, \infty)$ that satisfy the following conditions for some $T > 0$:*

$$\begin{aligned} &\liminf_{t\to\infty} \tfrac{1}{T}\int_t^{t+T} \eta(\tau)d\tau > 0, \\ &\limsup_{t\to\infty} \tfrac{1}{T}\int_t^{t+T} \eta^-(\tau)d\tau < \infty, \\ &\lim_{t\to\infty} \tfrac{1}{T}\int_t^{t+T} \phi^+(\tau)d\tau = 0, \end{aligned} \tag{2.18}$$

where $\eta^-(t) = max\{-\eta(t), 0\}$ and $\phi^+(t) = max\{\phi(t), 0\}$. Suppose that $\gamma = \gamma(t)$ is an absolutely continuous nonnegative function on $[0, \infty)$ that satisfies the following inequality almost everywhere on $[0, \infty)$:

$$\frac{d\gamma}{dt} + \eta\gamma \leq \phi. \tag{2.19}$$

Then $\gamma \to 0$, as $t \to 0$.

Proof. See, the detailed proof in [4, 16].

3. Main results: determination for autonomous and non-autonomous problem (1.1)

3.1. The Fourier modes

Recalling the Galerkin decomposition $u = \sum_{i=1}^{\infty} \hat{u}_i\omega_i$ associated with the eigenfunctions ω_i of the Stokes operator, we denote the first m Fourier modes $P_m u$ and residual modes $Q_m u$ as follows

$$P_m u(x,t) = \sum_{i=1}^{m} \hat{u}_i\omega_i, \quad Q_m u(x,t) = \sum_{i=m+1}^{\infty} \hat{u}_i\omega_i, \tag{3.1}$$

which satisfy

$$(P_m u(x,t), Q_m u(x,t))_{\dot{H}_{per}} = 0, \tag{3.2}$$

according to orthogonal properties of eigenfunctions $\omega_i (i = 1, 2, \cdots, \infty)$. Moreover, the Poincaré-Wirtinger inequality with respect to the function $Q_m u$ with zero space average under the periodic case

$$\|Q_m u(x,t)\|^2_{\dot{H}_{per}} \leq \tfrac{1}{\lambda_{m+1}}\|Q_m u(x,t)\|^2_{\dot{V}_{per}} \tag{3.3}$$

and the inverse Poincaré-Wirtinger inequality for $P_m u$

$$\|P_m u(x,t)\|^2_{\dot{V}_{per}} \leq \lambda_m\|P_m u(x,t)\|^2_{\dot{H}_{per}} \tag{3.4}$$

are true.

3.2. The Grashof number

The Grashof number G_r is a dimensionless number for fluid dynamics whose approximate is the ratio of the buoyant to viscous forces acting on a fluid. Following [4, 16], we define the Grashof number with regard to the first eigenvalues of the Stokes operator λ_1, fluid viscosity ν and external force term f.

Definition 3.1. *(The Grashof number) In the three-dimensional space, we define the Grashof number by*

$$G_r = \frac{F}{\nu^2 \lambda_1}, \tag{3.5}$$

where $F^2 = \limsup_{t\to\infty} \frac{1}{T} \int_t^{t+T} \|f(s)\|^2_{\dot{H}_{per}} ds$ *with* $F > 0$.

3.3. Determining modes for non-autonomous case

Considering two solenoidal vector fields $u(x,t)$, $v(x,t)$ and two scalar functions $p(x,t)$, $q(x,t)$ respectively satisfying 3D Navier-Stokes equations with the damping term sharing the same periodic boundary condition

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \alpha|u|^{\beta-1}u + \nabla p = f \ \text{ in } \ \Omega\times\mathbb{R}^+, \\ \nabla\cdot u = 0 \ \text{ in } \ \Omega\times\mathbb{R}^+ \end{cases} \tag{3.6}$$

and

$$\begin{cases} \partial_t v - \nu\Delta v + (v\cdot\nabla)v + \alpha|v|^{\beta-1}v + \nabla q = g \ \text{ in } \ \Omega\times\mathbb{R}^+, \\ \nabla\cdot v = 0 \ \text{ in } \ \Omega\times\mathbb{R}^+, \end{cases} \tag{3.7}$$

where $f = f(x,t)$ and $g = g(x,t)$ are the corresponding external force terms for the above two systems, then we can derive the appropriate evolution equations for u and v by using the Helmholtz-Weyl decomposition P_L as

$$\tfrac{du}{dt} + \nu Au + B(u,u) + \alpha P_L|u|^{\beta-1}u = P_L f \ \text{ in } \ \Omega\times\mathbb{R}^+ \tag{3.8}$$

and

$$\tfrac{dv}{dt} + \nu Au + B(v,v) + \alpha P_L|v|^{\beta-1}v = P_L g \ \text{ in } \ \Omega\times\mathbb{R}^+ \tag{3.9}$$

respectively.

Theorem 3.2. *Suppose that* $\beta \geq 1$, $T > 0$, *the initial value* $u_0 \in \dot{H}_{per}$ *and the external force term* $f \in \dot{H}_{per}$. *Then there exists a global weak solution of the initial boundary value problem (1.1) such that*

$$u \in L^\infty(0,T;\dot{H}_{per}) \cap L^2(0,T;\dot{V}_{per}) \cap L^{\beta+1}(0,T;L^{\beta+1}(\Omega)^3). \tag{3.10}$$

Moreover, suppose that $\frac{7}{2} \leq \beta \leq 5$, $u_0 \in \dot{V}_{per} \cap L^{\beta+1}(\Omega)^3$, *there exists a unique global strong solution to system (1.1) satisfying*

$$u \in L^\infty(0,T;\dot{V}_{per}) \cap L^2(0,T;H^2(\Omega)^3) \cap L^\infty(0,T;L^{\beta+1}(\Omega)^3). \tag{3.11}$$

Proof. Here by using the Galerkin approximated approach and localized technique to achieve a priori estimate, then by virtue of compact argument and limiting process, we can obtain the desired results, we skip the detail in this part. The proof is similar as the existence of global weak and strong solution in $\mathbb{R}^n$ can be seen in Cai and Jiu [10], and the bounded domain in Song and Hou [14], except some minor revision for our problem with periodic boundary.

Theorem 3.3. *Assume u is the strong solution in Theorem 3.2, then for any $t, T > 0$, u satisfies the following estimate*

$$\int_t^{t+T} \|Au\|_{\dot{H}_{per}}^2 d\tau \le \frac{C}{\nu^2} \int_t^{t+T} \|f\|_{\dot{H}_{per}}^2 d\tau, \tag{3.12}$$

when T is large enough.

Proof. Multiplying (3.8) by Au and integrate over Ω, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u\|_{\dot{V}_{per}}^2 + \nu\|Au\|_{\dot{H}_{per}}^2 + \alpha \int_\Omega |u|^{\beta-1}|\nabla u|^2 dx + \frac{\alpha(\beta-1)}{4} \int_\Omega |u|^{\beta-3}\Big|\nabla|u|^2\Big|^2 dx \\ &\quad = (f, Au)_{\dot{H}_{per}} - b(u,u,Au). \end{aligned} \tag{3.13}$$

The Hölder inequality, the Young and Gagliardo-Nirenberg inequalities result in

$$\begin{aligned} |b(u,u,Au)| &\le C\Big(\int_\Omega |u\cdot\nabla u|^2 dx\Big)^{\frac{1}{2}}\|Au\|_{\dot{H}_{per}} \\ &\le \frac{\nu}{8}\|Au\|_{\dot{H}_{per}}^2 + \frac{C}{\nu}\int_\Omega |u\cdot\nabla u|^2 dx \\ &\le \frac{\nu}{8}\|Au\|_{\dot{H}_{per}}^2 + \frac{C}{\nu}\|u\|_{L^{\beta+1}(\Omega)^3}^2 \|\nabla u\|^2_{L^{\frac{2(\beta+1)}{\beta-1}}(\Omega)^3} \\ &\le \frac{\nu}{8}\|Au\|_{\dot{H}_{per}}^2 + \frac{C}{\nu}\|u\|_{L^{\beta+1}(\Omega)^3}^2 \|Au\|_{\dot{H}_{per}}^{\frac{2(11-\beta)}{\beta+7}} \|u\|_{L^{\beta+1}(\Omega)^3}^{\frac{4(\beta-2)}{\beta+7}} \\ &\le \frac{\nu}{8}\|Au\|_{\dot{H}_{per}}^2 + \frac{C}{\nu}\|u\|_{L^{\beta+1}(\Omega)^3}^{\frac{6(\beta+1)}{\beta+7}} \|Au\|_{\dot{H}_{per}}^{\frac{2(11-\beta)}{\beta+7}} \\ &\le \frac{\nu}{4}\|Au\|_{\dot{H}_{per}}^2 + \frac{C}{\nu^3}\|u\|_{L^{\beta+1}(\Omega)^3}^{2(\beta+1)}, \end{aligned} \tag{3.14}$$

and

$$|(f, Au)_{\dot{H}_{per}}| \le \frac{C}{\nu}\|f\|_{\dot{H}_{per}}^2 + \frac{\nu}{4}\|Au\|_{\dot{H}_{per}}^2 \tag{3.15}$$

for $\frac{7}{2} \le \beta \le 5$.

Substituting (3.14) and (3.15) into (3.13) for any t, $t_0 \in \mathbb{R}^+$, we derive

$$\nu \int_{t_0}^t \|Au\|_{\dot{H}_{per}}^2 d\tau \le \frac{C}{\nu}\int_{t_0}^t \|f\|_{\dot{H}_{per}}^2 d\tau + \frac{C}{\nu^3}\int_{t_0}^t \|u\|_{L^{\beta+1}(\Omega)^3}^{2(\beta+1)} d\tau + \|u(t_0)\|_{\dot{V}_{per}}^2. \tag{3.16}$$

Then, the uniform boundedness of $\|u(t)\|_{\dot{V}_{per}}$ and $\|u(t)\|_{L^{\beta+1}(\Omega)^3}$ results in the following estimate

$$\int_{t_0}^t \|Au\|_{\dot{H}_{per}}^2 d\tau \le \frac{C}{\nu^2}\int_{t_0}^t \|f\|_{\dot{H}_{per}}^2 d\tau \tag{3.17}$$

for sufficiently large constant C which is bigger then the one in (3.16), which leads to (3.12), the proof is complete.

Theorem 3.4. *Assume that $\frac{7}{2} \le \beta \le 5$, u is the global strong solution in Theorem 3.2. Then the first m modes are determining of (1.1), i.e.,*

$$\|P_m u - P_m v\|_{\dot{H}_{per}} \to 0, \quad as \quad t \to \infty, \tag{3.18}$$

implies

$$\|u - v\|_{\dot{H}_{per}} \to 0, \quad as \quad t \to \infty \tag{3.19}$$

provided that $m \in \mathbb{R}^+$ satisfies

$$m \ge CG_r^3, \tag{3.20}$$

where C is the constant only depending on λ_1, and G_r is the Grashof number (see Definition 3.1).

Proof. Let $w = u - v$. Then w satisfies

$$\begin{aligned}&(\tfrac{dw}{dt}, \omega_i) + \nu(Aw, \omega_i) + b(w, u, \omega_i) + b(v, w, \omega_i) + \alpha(|u|^{\beta-1}u - |v|^{\beta-1}v, \omega_i)_{\dot{H}_{per}} \\ &\quad = (f - g, \omega_i)_{\dot{H}_{per}}\end{aligned} \tag{3.21}$$

in distribution sense.

Since $Q_m u = \sum_{i=m+1}^{\infty} \hat{u}_i(t)\omega_i$, $P_m u$ and $Q_m u$ are orthogonal, it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|Q_m w\|^2_{\dot{H}_{per}} + \nu\|Q_m w\|^2_{\dot{V}_{per}} + b(w, u, Q_m w) + b(v, w, Q_m w) \\ &\quad + \alpha(|u|^{\beta-1}u - |v|^{\beta-1}v, Q_m w)_{\dot{H}_{per}} = (f(x,t) - g(x,t), Q_m w)_{\dot{H}_{per}}.\end{aligned} \tag{3.22}$$

Based on the monotonicity of the nonlinear term, we can get

$$\begin{aligned}&\alpha(|u|^{\beta-1}u - |v|^{\beta-1}v, Q_m w)_{\dot{H}_{per}} \\ &= \alpha(|u|^{\beta-1}u - |v|^{\beta-1}v, w)_{\dot{H}_{per}} - \alpha(|u|^{\beta-1}u - |v|^{\beta-1}v, P_m w)_{\dot{H}_{per}} \\ &\geq -\alpha(|u|^{\beta-1}u - |v|^{\beta-1}v, P_m w)_{\dot{H}_{per}},\end{aligned} \tag{3.23}$$

which leads to the following inequality

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|Q_m w\|^2_{\dot{H}_{per}} + \nu\|Q_m w\|^2_{\dot{V}_{per}} \\ &\leq |b(w, u, Q_m w)| + |b(v, w, Q_m w)| + |\alpha(|u|^{\beta-1}u - |v|^{\beta-1}v, P_m w)_{\dot{H}_{per}}| \\ &\quad + |(f(x,t) - g(x,t), Q_m w)_{\dot{H}_{per}}|.\end{aligned} \tag{3.24}$$

Next, the estimates for every term in (3.24) will be proceed for applying the generalized Gronwall inequality (Lemma 2.3) with $\xi = \|Q_m w\|_{\dot{H}_{per}}$.

Noting that $u = P_m u + Q_m u$ and the properties of trilinear operators (2.14), we write

$$|b(w, u, Q_m w)| \leq |b(P_m w, Q_m w, u)| + |b(Q_m w, u, Q_m w)| =: b_1 + b_2 \tag{3.25}$$

and

$$|b(v, w, Q_m w)| = |b(v, P_m w, Q_m w)| = |b(v, Q_m w, P_m w)| =: b_3. \tag{3.26}$$

The Hölder and Young inequalities for $b_i (i = 1, 2, 3)$, and Lemma 2.1 result in

$$\begin{aligned}b_1 &\leq C\|P_m w\|_{L^4_{per}(\Omega)^3}\|Q_m w\|_{\dot{V}_{per}}\|u\|_{L^4_{per}(\Omega)^3} \\ &\leq C\|P_m w\|^{\frac{1}{4}}_{\dot{H}_{per}}\|P_m w\|^{\frac{3}{4}}_{\dot{V}_{per}}\|Q_m w\|_{\dot{V}_{per}}\|u\|^{\frac{1}{4}}_{\dot{H}_{per}}\|u\|^{\frac{3}{4}}_{\dot{V}_{per}},\end{aligned} \tag{3.27}$$

$$\begin{aligned}b_2 &\leq C\|Q_m w\|_{\dot{H}_{per}}\|Q_m w\|_{\dot{V}_{per}}\|u\|_{L^\infty(\Omega)^3} \\ &\leq \frac{\nu}{2}\|Q_m w\|^2_{\dot{V}_{per}} + \frac{C}{2\nu}\|Q_m w\|^2_{\dot{H}_{per}}\|Au\|^2_{\dot{H}_{per}}\end{aligned}$$

and

$$
\begin{aligned}
b_3 &\le C\|v\|_{L^4_{per}(\Omega)^3}\|Q_m w\|_{\dot{V}_{per}}\|P_m w\|_{L^4_{per}(\Omega)^3} \\
&\le C\|v\|^{\frac{1}{4}}_{\dot{H}_{per}}\|v\|^{\frac{3}{4}}_{\dot{V}_{per}}\|Q_m w\|_{\dot{V}_{per}}\|P_m w\|^{\frac{1}{4}}_{\dot{H}_{per}}\|P_m w\|^{\frac{3}{4}}_{\dot{V}_{per}},
\end{aligned} \tag{3.28}
$$

where C represents different variable-independent constants.

Noting that $\frac{7}{2} \le \beta \le 5$, we have the following embedding

$$
\dot{V}_{per} \subset L^{\frac{3(\beta-1)}{2}}(\Omega)^3 \tag{3.29}
$$

by the Sobolev theorem. For the damping term in (3.24), applying (3.29), Mazur's inequality (Lemma 2.2), Hölder's inequality, Young's inequality and $\dot{V}_{per} \subset L^6(\Omega)^3$, we obtain

$$
\begin{aligned}
&|\alpha(|u|^{\beta-1}u - |v|^{\beta-1}v, P_m w)_{\dot{H}_{per}}| \\
&\le \alpha\beta \int_\Omega |w|((|u|^{\beta-1} + |v|^{\beta-1})|P_m w| dx \\
&\le \alpha\beta\|w\|_{L^6(\Omega)^3}\Big(\|u\|^{\beta-1}_{L^{\frac{3(\beta-1)}{2}}(\Omega)^3} + \|v\|^{\beta-1}_{L^{\frac{3(\beta-1)}{2}}(\Omega)^3}\Big)\|P_m w\|_{L^6(\Omega)^3} \\
&\le C\alpha\beta\|w\|_{\dot{V}_{per}}\Big(\|u\|^{\beta-1}_{\dot{V}_{per}} + \|v\|^{\beta-1}_{\dot{V}_{per}}\Big)\|P_m w\|_{\dot{V}_{per}}.
\end{aligned} \tag{3.30}
$$

For the remaining external force term in (3.24), the Cauchy-Schwarz inequality results in

$$
|(f(t) - g(t), Q_m w)_{\dot{H}_{per}}| \le \|f(t) - g(t)\|_{L^2(\Omega)}\|Q_m w\|_{\dot{H}_{per}}. \tag{3.31}
$$

Combining (3.22), (3.27), (3.28), (3.30) and (3.31), we conclude

$$
\begin{aligned}
&\frac{d}{dt}\|Q_m w\|^2_{\dot{H}_{per}} + \nu\|Q_m w\|^2_{\dot{V}_{per}} - \frac{C}{\nu}\|Q_m w\|^2_{\dot{H}_{per}}\|Au\|^2_{\dot{H}_{per}} \\
\le\ & C\|P_m w\|^{\frac{1}{4}}_{\dot{H}_{per}}\|P_m w\|^{\frac{3}{4}}_{\dot{V}_{per}}\|Q_m w\|_{\dot{V}_{per}}\|u\|^{\frac{1}{4}}_{\dot{H}_{per}}\|u\|^{\frac{3}{4}}_{\dot{V}_{per}} \\
&+C\|v\|^{\frac{1}{4}}_{\dot{H}_{per}}\|v\|^{\frac{3}{4}}_{\dot{V}_{per}}\|Q_m w\|_{\dot{V}_{per}}\|P_m w\|^{\frac{1}{4}}_{\dot{H}_{per}}\|P_m w\|^{\frac{3}{4}}_{\dot{V}_{per}} \\
&+C\alpha\beta\|w\|_{\dot{V}_{per}}\Big(\|u\|^{\beta-1}_{\dot{V}_{per}} + \|v\|^{\beta-1}_{\dot{V}_{per}}\Big)\|P_m w\|_{\dot{V}_{per}} \\
&+\|f(t) - g(t)\|_{\dot{H}_{per}}\|Q_m w\|_{\dot{H}_{per}},
\end{aligned} \tag{3.32}
$$

which can be rewritten in the form

$$
\frac{d\gamma(t)}{dt} + \eta(t)\gamma(t) \le \phi(t) \tag{3.33}
$$

from (3.3) and (3.4) and the notations

$$
\begin{aligned}
\gamma(t) &= \|Q_m w\|^2_{\dot{H}_{per}}, \\
\eta(t) &= \nu\lambda_{m+1} - \frac{C}{\nu}\|Au\|^2_{\dot{H}_{per}}, \\
\phi(t) &= C\|P_m w\|^{\frac{1}{4}}_{\dot{H}_{per}}\|P_m w\|^{\frac{3}{4}}_{\dot{V}_{per}}\|Q_m w\|_{\dot{V}_{per}}\|u\|^{\frac{1}{4}}_{\dot{H}_{per}}\|u\|^{\frac{3}{4}}_{\dot{V}_{per}} \\
&\quad +C\|P_m w\|_{\dot{H}_{per}}\Big(\lambda_m^{\frac{3}{4}}\|v\|^{\frac{1}{4}}_{\dot{H}_{per}}\|v\|^{\frac{3}{4}}_{\dot{V}_{per}}\|Q_m w\|_{\dot{V}_{per}} \\
&\quad +\alpha\beta\lambda_m\|w\|_{\dot{V}_{per}}(\|u\|^{\beta-1}_{\dot{V}_{per}} + \|v\|^{\beta-1}_{\dot{V}_{per}})\Big) \\
&\quad +\|f(t) - g(t)\|_{\dot{H}_{per}}\|Q_m w\|_{\dot{H}_{per}}.
\end{aligned} \tag{3.34}
$$

The uniform boundedness of u, v, w in $\dot{H}_{per}$, $\dot{V}_{per}$, $L^{\beta+1}(\Omega)^3$ from Theorem 3.2 together with convergences $\|P_m w\|_{\dot{H}_{per}}$ and $\|f(t)-g(t)\|_{L^2(\Omega)}$ yield

$$\phi(t) \to 0 \ \text{ as } \ t \to \infty \tag{3.35}$$

provided that (2.18) is true. Hence, (2.18) is verified by

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} \eta(\tau)d\tau \\
&= \nu\lambda_{m+1} - \limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\frac{C}{\nu}\|Au\|^2_{\dot{H}_{per}}d\tau \\
&\geq \nu\lambda_{m+1} - \frac{CF^2}{\nu^3} \\
&> 0
\end{aligned} \tag{3.36}$$

via the estimate

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|Au\|^2_{\dot{H}_{per}}d\tau \leq \frac{CF^2}{\nu^2} \tag{3.37}$$

from Theorem 3.3.

According to $\lambda_m = C\lambda_1 m^{\frac{2}{3}}$, and the definition of Grashof number G_r (see Definition 3.1), we conclude that

$$m > CG_r^3. \tag{3.38}$$

The proof is complete.

Remark 3.1. *In Theorem 3.4, we give the determining modes for weak solution of system (1.1) when the strong solution exists. However, if there is only the existence of weak solution, the determining modes for system (1.1) is still open.*

3.4. Asymptotic determining functionals for autonomous case

In this part, we consider the 3D damped Navier-Stokes equations with autonomous force $f(x)$ as

$$\begin{cases}
\partial_t u - \nu\Delta u + (u\cdot\nabla)u + \alpha|u|^{\beta-1}u + \nabla p = f(x) \ \text{ in } \ \Omega\times\mathbb{R}^+, \\
\nabla\cdot u = 0 \ \text{ in } \ \Omega\times\mathbb{R}^+, \\
u(x+L_i e_i, t) = u(x,t), \quad i = 1,2,3, \\
u(x, t=0) = u_0.
\end{cases} \tag{3.39}$$

Similar as the non-autonomous case above, the equivalent abstract form of (3.39) can be given by

$$\tfrac{du}{dt} + \nu Au + B(u,u) + \alpha P_L|u|^{\beta-1}u = P_L f(x) \ \text{ in } \ \Omega\times\mathbb{R}^+, \tag{3.40}$$

where P_L still denotes the Helmholtz-Leray projector. Based on the well-posedness of (3.40), Li et al. study the existence of a finite dimensional global attractor as following.

Theorem 3.5. *Assume that $f \in \dot{H}_{per}$ and $u_0 \in \dot{V}_{per}$, Then the semigroup $\{L_t\}_{t\geq 0}$ generated by problem (3.39) possesses a $\dot{V}_{per}$-global attractor $\mathcal{A}$.*

Proof. See, e.g., Li et al. [12], which studied the damped Navier-Stokes equations in the non-slip boundary condition. We can similarly get the dynamical system to the problem (3.39) on the periodic boundary condition.

Consider the system $\mathcal{F} = \{F_1, F_2, \cdots, F_n\}$ of linear functionals generated by the corresponding Fourier modes $F_n(u) =: (u, \omega_n)$, where ω_n denotes the eigenfunctions of the Stokes operator. In what follows, we will prove the asymptotic determination for (3.39) with autonomous external force $f(x)$ according to Theorem 3.5, namely, the system $\mathcal{F}$ is determining if n is large enough.

Theorem 3.6. *Let $\frac{7}{2} \leq \beta \leq 5$ and n satisfy the inequality*

$$n > C(\frac{L}{\nu\lambda_1})^{\frac{3}{2}}, \tag{3.41}$$

where $L = L(\nu, \alpha, \beta, \|u\|_{L^\infty(0,T;\dot{V}_{per})}, \|u\|_{L^2(0,T;H^2(\Omega)^3)}, \|u\|_{L^\infty(0,T;L^{\beta+1}(\Omega)^3)})$ is a dimensionless constant, λ_1 denotes the primary eigenvalue. Then the system $\mathcal{F}$ of the first n Fourier modes is asymptotically determining for the dynamical system generated by (3.39).

Proof. Let $u^*(t)$ and $v^*(t)$ be two trajectories inside the finite dimensional global attractor $\mathcal{A}$ in Theorem 3.5. Denote $w^*(t) = u^*(t) - v^*(t)$, then it is easy to check that $w^*(t)$ satisfies

$$\begin{aligned} & \frac{dw^*}{dt} + \nu A w^* \\ = \ & B(v^*, v^*) - B(u^*, u^*) + \alpha P_L |v^*|^{\beta-1} v^* - \alpha P_L |u^*|^{\beta-1} u^* \ \text{ in } \ \Omega \times \mathbb{R}^+. \end{aligned} \tag{3.42}$$

Next, multiplying (3.42) by Aw^* and integrating over Ω, noting that P_L is symmetric and (2.14), we obtain

$$\begin{aligned} & \tfrac{1}{2}\tfrac{d}{dt}\|w^*\|^2_{\dot{V}_{per}} + \nu\|Aw^*\|^2_{\dot{H}_{per}} \\ & \quad \leq |b(w^*, u^*, Aw^*)| + |b(v^*, w^*, Aw^*)| + \alpha|(|u^*|^{\beta-1}u^* - |v^*|^{\beta-1}v^*, Aw^*)|. \end{aligned} \tag{3.43}$$

The Hölder inequality, Young's inequality and Lemma 2.1 yield the estimates

$$\begin{aligned} |b(w^*, u^*, Aw^*)| & \leq C\|w^*\|_{L^\infty_{per}(\Omega)^3}\|u^*\|_{\dot{V}_{per}}\|Aw^*\|_{\dot{H}_{per}} \\ & \leq C\|w^*\|^{\frac{1}{2}}_{\dot{V}_{per}}\|u^*\|_{\dot{V}_{per}}\|Aw^*\|^{\frac{3}{2}}_{\dot{H}_{per}} \\ & \leq \tfrac{\nu}{6}\|Aw^*\|^2_{\dot{H}_{per}} + C_1(\nu)\|w^*\|^2_{\dot{V}_{per}}\|u^*\|^4_{\dot{V}_{per}} \end{aligned} \tag{3.44}$$

and

$$\begin{aligned} |b(v^*, w^*, Aw^*)| & \leq C\|v^*\|_{L^\infty_{per}(\Omega)^3}\|w^*\|_{\dot{V}_{per}}\|Aw^*\|_{\dot{H}_{per}} \\ & \leq C\|v^*\|^{\frac{1}{2}}_{\dot{V}_{per}}\|Av^*\|^{\frac{1}{2}}_{\dot{H}_{per}}\|w^*\|_{\dot{V}_{per}}\|Aw^*\|_{\dot{H}_{per}} \\ & \leq \tfrac{\nu}{6}\|Aw^*\|^2_{\dot{H}_{per}} + C_2(\nu)\|v^*\|_{\dot{V}_{per}}\|Av^*\|_{\dot{H}_{per}}\|w^*\|^2_{\dot{V}_{per}}, \end{aligned} \tag{3.45}$$

where $C_1(\nu) = \frac{729C}{32\nu^3}$ and $C_2(\nu) = \frac{3C}{2\nu}$ depend on the dimensionless constant ν only.

For the damping term in (3.43), by using Hölder's and Young's inequalities, we obtain

$$\begin{aligned} &\alpha|(|u^*|^{\beta-1}u^* - |v^*|^{\beta-1}v^*, Aw^*)| \\ &\leq C(\int_\Omega |\alpha|u^*|^{\beta-1}u^* - \alpha|v^*|^{\beta-1}v^*|^2 dx)^{\frac{1}{2}} \|Aw^*\|_{\dot{H}_{per}} \\ &\leq \frac{\nu}{6}\|Aw^*\|^2_{\dot{H}_{per}} + C_2(\nu)\int_\Omega |\alpha|u^*|^{\beta-1}u^* - \alpha|v^*|^{\beta-1}v^*|^2 dx. \end{aligned} \tag{3.46}$$

Hence, Mazur's inequality, $\dot{V}_{per} \subset \dot{L}^6_{per}(\Omega)$ and Hölder's inequality result in

$$\begin{aligned} &\int_\Omega \left|\alpha|u^*|^{\beta-1}u^* - \alpha|v^*|^{\beta-1}v^*\right|^2 dx \\ &\leq C\int_\Omega (|u^*|^{\beta-1}|w^*| + \left||u^*|^{\beta-1} - |v^*|^{\beta-1}\right||v^*|)^2 dx \\ &\leq C\int_\Omega (|u^*|^{2(\beta-1)}|w^*|^2 dx + C\int_\Omega (|u^*|^{\beta-2} + |v^*|^{\beta-2})^2|v^*|^2|w^*|^2 dx \\ &\leq C\|u^*\|^{2(\beta-1)}_{L^{3(\beta-1)}(\Omega)^3}\|w^*\|^2_{L^6(\Omega)^3} + C(\|u^*\|^{2(\beta-2)}_{L^{6(\beta-2)}(\Omega)^3} + \|v^*\|^{2(\beta-2)}_{L^{6(\beta-2)}(\Omega)^3}) \\ &\quad \|v^*\|^2_{L^6(\Omega)^3}\|w^*\|^2_{L^6(\Omega)^3} \\ &\leq C\|u^*\|^{2(\beta-1)}_{L^{3(\beta-1)}(\Omega)^3}\|w^*\|^2_{\dot{V}_{per}} + C(\|u^*\|^{2(\beta-2)}_{L^{6(\beta-2)}(\Omega)^3} + \|v^*\|^{2(\beta-2)}_{L^{6(\beta-2)}(\Omega)^3}) \\ &\quad \|v^*\||^2_{\dot{V}_{per}}\|w^*\||^2_{\dot{V}_{per}}. \end{aligned} \tag{3.47}$$

Substituting (3.44)–(3.47) into (3.43), we conclude

$$\begin{aligned} &\frac{d}{dt}\|w^*\|^2_{\dot{V}_{per}} + \nu\|Aw^*\|^2_{\dot{H}_{per}} \\ &\leq 2\|w^*\|^2_{\dot{V}_{per}}\Big(C_1\|u^*\|^4_{\dot{V}_{per}} + C_2\|v^*\|_{\dot{V}_{per}}\|Av^*\|_{\dot{H}_{per}} + C_3\|u^*\|^{2(\beta-1)}_{L^{3(\beta-1)}(\Omega)^3} \\ &\quad + C_3(\|u^*\|^{2(\beta-2)}_{L^{6(\beta-2)}(\Omega)^3} + \|v^*\|^{2(\beta-2)}_{L^{6(\beta-2)}(\Omega)^3})\|v^*\||^2_{\dot{V}_{per}}\Big), \end{aligned} \tag{3.48}$$

where $C_3 = C_3(\nu, \alpha, \beta)$ is a dimensionless constant.

Owing to $\frac{7}{2} < \beta < 5$, we have

$$\begin{aligned} &\int_0^t \|u^*\|^{2(\beta-1)}_{L^{3(\beta-1)}(\Omega)^3} d\tau \leq C\|u^*\|^{\frac{2(\beta+1)^2}{\beta+7}}_{L^\infty(0,t;L^{\beta+1}(\Omega)^3)}\|u^*\|^{\frac{8(\beta-2)}{\beta+7}}_{L^2(0,T;H^2(\Omega)^3)} t^{\frac{3(5-\beta)}{\beta+7}}, \\ &\int_0^t \|u^*\|^{2(\beta-2)}_{L^{6(\beta-2)}(\Omega)^3} d\tau \leq C\|u^*\|^{\frac{2(\beta-1)^3}{\beta+7}}_{L^\infty(0,t;L^{\beta+1}(\Omega)^3)}\|u^*\|^{\frac{2(5\beta-13)}{\beta+7}}_{L^2(0,T;H^2(\Omega)^3)} t^{\frac{4(5-\beta)}{\beta+7}} \end{aligned} \tag{3.49}$$

are all bounded for any $0 < t < \infty$.

Denote

$$\begin{aligned} L = 2\Big(&C_1\|u^*\|^4_{\dot{V}_{per}} + C_2\|v^*\|_{\dot{V}_{per}}\|Av^*\|_{\dot{H}_{per}} + C_3\|u^*\|^{2(\beta-1)}_{L^{3(\beta-1)}(\Omega)^3} \\ &+ C_3(\|u^*\|^{2(\beta-2)}_{L^{6(\beta-2)}(\Omega)^3} + \|v^*\|^{2(\beta-2)}_{L^{6(\beta-2)}(\Omega)^3})\|v^*\||^2_{\dot{V}_{per}}\Big), \end{aligned} \tag{3.50}$$

which is a finite dimensionless constant because of (3.11). Then there exists an n large enough such that $\nu\lambda_{n+1} > L$ holds, the estimate (3.48) can be reduced to

$$\frac{d}{dt}\|w^*\|^2_{\dot{V}_{per}} + (\nu\lambda_{n+1} - L)\|w^*\|^2_{\dot{V}_{per}} \leq 0, \tag{3.51}$$

which yields

$$\|w^*(t)\|^2_{\dot{V}_{per}} \leq e^{-C'(t-s)}\|w^*(s)\|^2_{\dot{V}_{per}}, \quad s \leq t \tag{3.52}$$

for some positive constant C'.

Since u^* and v^* belong to the global attractor $\mathcal{A}$, we have $\|w^*(t)\|_{\dot{V}_{per}} \to 0$ as $t \to \infty$. Hence, the system $\mathcal{F}$ is asymptotically determining for the dynamical system of (3.39) when $n > C(\frac{L}{\nu\lambda_1})^{\frac{3}{2}}$ because of $\lambda_{n+1} = C\lambda_1 n^{\frac{2}{3}}$. Therefore, the proof is completed.

3.5. Further research

In this section, we discuss some open interrelated issues that might be studied in the prospective investigations and research.

(I) The determining modes in the periodic case is presented in this paper, which can be proved similarly in the whole space case $\mathbb{R}^3$(where the boundary conditions can be considered as $|u| \to 0$ as $|x| \to \infty$) by utilizing the relevant conclusions in [10], but the situation turns out to be quite different in the Dirichlet condition, due to the fact

$$A = -P_L\Delta \neq -\Delta, \tag{3.53}$$

which implies $b(u, u, Au) \neq 0$ in the Dirichlet boundary condition. So the main ideas and difficulties of the subject is summarized as follows

$$\begin{cases} \partial_t u - \nu\Delta u + (u \cdot \nabla)u + \alpha|u|^{\beta-1}u + \nabla p = f \ \text{ in } \ \widetilde{\Omega} \times \mathbb{R}^+, \\ \nabla \cdot u = 0 \ \text{ in } \ \widetilde{\Omega} \times \mathbb{R}^+, \\ u(x,t) = 0 \ \text{ on } \ \partial\widetilde{\Omega}, \\ u(x, t = 0) = u_0(x), \end{cases} \tag{3.54}$$

where $\widetilde{\Omega}$ denotes an open bounded region in three dimensions. The determination and reduction for our problem defined on bounded domain is our objective in future.

(II) For the generalized fluid flow, Ladyzhenskaya proposed a revised version of Ladyzhenskaya-type Navier-Stokes model in the 1960s, where the surrounding flow problem was considered

$$\begin{cases} \partial_t u - \nabla \cdot [(\nu_0 + \nu_1\|Du\|_{L^2}^{q-2})Du] + (u \cdot \nabla)u + \nabla p = f, \\ \nabla \cdot u = 0, \end{cases} \tag{3.55}$$

where $Du = \frac{1}{2}(\nabla u + \nabla u^T)$, Guo and Zhu studied the partial regularity of the distribution solution of the initial boundary value problem in the three-dimensional case of the model (see [20]). In order to overcome the difficulties of the model, Lions proposed a new class of polished Navier-Stokes equations (see [5]) with the establishment of the monotonicity method

$$\begin{cases} \partial_t u - \nu_0\Delta u - \nu_1 \sum\limits_{i=1}^{n} \frac{\partial}{\partial x_i}(|\nabla u|^{q-1}\frac{\partial u}{\partial x_i}) + (u \cdot \nabla)u + \nabla p = f, \\ \nabla \cdot u = 0. \end{cases} \tag{3.56}$$

Exploiting the technique of this work, it is actually possible to study the above models (3.55) and (3.56) in the whole space $\mathbb{R}^3$, torus $\mathbb{T}^3$, and bounded smooth region Ω respectively, the main difficulty for dealing with their determining modes lies in the unknown spectral relationship of the corresponding operator.

(III) A further meaningful research is concerned with finite dimensional reduction of the Navier-Stokes equations with damping (1.1). If we can get the Lipschitz property

$$\|h(u) - h(v)\|_{(\dot{H}_{per})'} \leq L\|u - v\|_{\dot{H}_{per}}, \quad \forall u, v \in \dot{H}_{per}, \tag{3.57}$$

where $h(u)$ denotes $(u \cdot \nabla)u + \alpha|u|^{\beta-1}u$, and

$$L < \lambda_{N+1}, \tag{3.58}$$

then the first N Fourier modes is asymptotically determining for the dynamic system of (1.1) (see [21]), which leads to the reduction of (1.1) and even the existence of inertial manifold.

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Conflict of interest

The authors declare there is no conflict of interest.

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