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Research article

A high-order numerical scheme for right Caputo fractional differential equations with uniform accuracy

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Abstract: In this paper, we mainly study the high-order numerical scheme of right Caputo time fractional differential equations with uniform accuracy. Firstly, we construct the high-order finite difference method for the right Caputo fractional ordinary differential equations (FODEs) based on piecewise quadratic interpolation. The local truncation error of right Caputo FODEs is given, and the stability analysis of the right Caputo FODEs is proved in detail. Secondly, the time fractional partial differential equations (FPDEs) with right Caputo fractional derivative is studied by coupling the time-dependent high-order finite difference method and the spatial central second-order difference scheme. Finally, three numerical examples are used to verify that the convergence order of high-order numerical scheme is $3-\lambda$ in time with uniform accuracy.

Keywords: right Caputo fractional; high order numerical scheme; optimal convergence order; local truncation error; stability analysis

1. Introduction

It is known that fractional calculus is widely used, especially in engineering and science, and it is mainly described and realized through fractional differential equations (FDEs). For example: finance, biology, automatic control theory, mechanics, information theory and other fields have real applications. The Caputo-type fractional derivatives are divided into right Caputo derivative and left Caputo derivative. According to the research [1], by means of piecewise quadratic interpolation, a FODE high-order numerical scheme with consistent precision is given, and the unconditionally stable and consistent sharp numerical order is strict. A general high order method was used to solve FODE based on block-by-block method in [2]. In [3], by means of the $L2$ formula, a high-order compact finite difference method with respect to time FPDE is constructed. The high-order finite-difference methods with respect to time FPDE is introduced, moreover the first proof the stability of $3-\alpha$ was presented in [4]. The finite-difference method introduced in [5] is a meshless generalized type and is applied to deal with 3D variable-order time FPDE. In [6], the Caputo fractional derivative is approximated by a fractional numerical differentiation method and uses a high-order numerical approach to deal with time-dependent FPDEs. In [7], in order to obtain the fully discrete form of the spatial fractional diffusion wave equation, the time-dependent second-order difference method and the spatial finite element method are proposed. According to the study [8], the time fractional diffusion equation in regard to the Mittag-Leffler matrix function is solved by the line method. In [9], a high-order numerical scheme is introduced to infinitely approximate the Riemann-Liouville fractional derivative. In [10], some numerical methods are proposed to solve fractional calculus, such as piecewise interpolation and Simpson's method. The paper [11] presented a fractional Taylor-type formula for the right Caputo fractional derivative, meanwhile the derivative is characterized by a fractional differential formula and a fractional integral remainder. In [12], the right Caputo fractional derivative is obtained by making use of the singularity-preserving spectrum configuration approach of fractional differential equations, the convergence analysis to obtain and the best error estimate is obtained. In [13], based on the derived operation matrix, using the Gauss-Lobatto quadrature formula is not only effective for studying fractional optimal control problems, but also for fractional order variational problems. At the same time, it is also verified that the Lagrange multiplier method is still valid for them. Discuss two second-order numerical differential equations of Caputo derivative operator, moreover, proving its unconditional convergence and stability with the help of discrete energy approach in [14]. In [15], some numerical approaches to fractional derivatives are constructed, which are Caputo-type derivatives with finite real-valued lower bounds and Riemann-Liouville-type derivatives with infinite lower bounds. In [16], based on cubic Lagrangian interpolation polynomials, proposed a high-order numerical method to approximate Caputo fractional derivatives. In [17], in order to determine the spatially related source terms in the opposite problem of the time-fractional diffusion equation, the time finite difference method and the spatially local discontinuous Galerkin method are used to construct a numerical scheme. A high-order form of the Caputo-type fractional convection-diffusion equation is constructed under the Dirichlet boundary condition in [18]. In [19], the numerical solution for the coupled nonlinear time-dependent FPDE with Caputo derivatives was given by the implicit trapezoidal method. In [20], the high-order scheme of Caputo fractional derivative approximation was developed and applied to solve the time-dependent FPDEs. With smooth conditions on the solution, piecewise polynomials were popularly used to solve fractional differential equations; refer to the literature [21–25] for details. For no-smooth solutions, one should introduce the non-uniform meshes for the fractional differential equations, and readers can refer to [26–28] for details.

In the current literature, we learn that there are limited high-order numerical schemes for studying right Caputo FDEs with uniform accuracy. Therefore, based on the idea of [1], this paper constructs a high-order numerical scheme with uniform precision convergence for right Caputo FDEs.

The main content of this paper is organized as follows: We adopt a high-order numerical scheme to solve the right Caputo FODE, and Section 2 analyzes the local truncation error and stability of this scheme in detail. In Section 3, we study time-dependent FPDE with right Caputo derivative, implementing discrete-time FPDE by finite-difference methods in time, and second-order central-difference methods in space. In Section 4, some numerical examples are given to verify efficient high-order numerical methods and support our theoretical findings. Finally, some conclusions from this work are drawn in Section 5.

2. The high-order numerical scheme of the right Caputo FODE

2.1. Construction of the high-order numerical scheme

Consider the following FODE

$$ {}_tD_b^{\lambda} m(t) = g(t, m(t)), \quad a \le t \le b, \quad 0 < \lambda < 1, \tag{2.1} $$

where the initial condition is $m(b) = m_0$, and m_0 is a known constant, ${}_tD_b^{\lambda} m(t)$ in (2.1) defined as the right Caputo fractional derivative of order λ which is given by

$$ {}_tD_b^{\lambda} m(t) = \frac{-1}{\Gamma(1-\lambda)} \int_t^b (\rho - t)^{-\lambda} m'(\rho) d\rho, \tag{2.2} $$

here $\Gamma(\cdot)$ means Gamma function in [29].

Now, we will construct a high-order scheme of (2.1) and divide the interval $[a, b]$ into W uniform sub-intervals . Suppose $a = t_0 < \ldots < t_q < \ldots < t_W = b$, $t_q = q\eta$, $q = 0, 1, 2, \ldots, W$, and $\eta = \frac{b-a}{W}$ is the step size, m_q is the numerical solution of (2.1) at t_q, and $g_q = g(t_q, m_q)$.

In order to discretize the right Caputo fractional derivative of (2.2), one can firstly determine the values of $m(t)$ at t_{W-2}, t_{W-1}, t_W, and employ quadratic Lagrange interpolation to $m(t)$ on $[t_{W-2},\ t_W]$ as follows

$$ m(t) \approx \gamma_{[t_{W-2}, t_W]} m(t) = \sum_{i=0}^{2} \kappa_{i,W-2}(t) m_{W-2+i}, \tag{2.3} $$

where the quadratic Lagrangian interpolation basis function at points t_{W-2}, t_{W-1} and t_W are $\kappa_{i,W-2}, i = 0, 1, 2$, which are defined as follows

$$ \kappa_{0,W-2}(t) = \frac{(t - t_{W-1})(t - t_W)}{2\eta^2}, \kappa_{1,W-2}(t) = \frac{(t - t_{W-2})(t - t_W)}{-\eta^2}, \kappa_{2,W-2}(t) = \frac{(t - t_{W-2})(t - t_{W-1})}{2\eta^2}. \tag{2.4} $$

For $q = W-1, W-2$, substituting (2.3) into (2.2), we can obtain

$$ \begin{aligned} {}_tD_b^{\lambda} m(t_{W-1}) &= \frac{-1}{\Gamma(1-\lambda)} \int_{t_{W-1}}^{t_W} (\rho - t_{W-1})^{-\lambda} m'(\rho) d\rho \\ &\approx \frac{-1}{\Gamma(1-\lambda)} \int_{t_{W-1}}^{t_W} (\rho - t_{W-1})^{-\lambda} [\gamma_{[t_{W-2}, t_W]} m(\rho)]' d\rho \\ &= E_{W-1}^{0,0} m_{W-2} + E_{W-1}^{1,0} m_{W-1} + E_{W-1}^{2,0} m_W, \qquad (2.5) \\ {}_tD_b^{\lambda} m(t_{W-2}) &= \frac{-1}{\Gamma(1-\lambda)} \int_{t_{W-2}}^{t_W} (\rho - t_{W-2})^{-\lambda} m'(\rho) d\rho \\ &\approx \frac{-1}{\Gamma(1-\lambda)} \int_{t_{W-2}}^{t_W} (\rho - t_{W-2})^{-\lambda} [\gamma_{[t_{W-2}, t_W]} m(\rho)]' d\rho \\ &= E_{W-2}^{0,0} m_{W-2} + E_{W-2}^{1,0} m_{W-1} + E_{W-2}^{2,0} m_W, \qquad (2.6) \end{aligned} $$

where the expression of $E_{W-1}^{i,0}$, $E_{W-2}^{i,0}$ are as follows

$$ E_{W-1}^{i,0} = \frac{-1}{\Gamma(1-\lambda)} \int_{t_{W-1}}^{t_W} (\rho - t_{W-1})^{-\lambda} \kappa'_{i,W-2}(\rho) d\rho, i = 0, 1, 2, \tag{2.7} $$

$$E^{i,0}_{W-2} = \frac{-1}{\Gamma(1-\lambda)}\int_{t_{W-2}}^{t_W}(\rho - t_{W-2})^{-\lambda}\kappa'_{i,W-2}(\rho)d\rho, i = 0, 1, 2. \tag{2.8}$$

When $q \le W-3$, firstly in the interval of $[t_{l-1}, t_l]$, the approximation solution of $m(t)$ at points t_{l-1}, t_l, t_{l+1} is

$$m(t) \approx \gamma_{[t_{l-1},t_l]}m(t) = \sum_{i=0}^{2}\omega_{i,l}(t)m_{l-1+i}, \tag{2.9}$$

where

$$\omega_{0,l}(t) = \frac{(t-t_l)(t-t_{l+1})}{2\eta^2}, \omega_{1,l}(t) = \frac{(t-t_{l-1})(t-t_{l+1})}{-\eta^2}, \omega_{2,l}(t) = \frac{(t-t_{l-1})(t-t_l)}{2\eta^2}. \tag{2.10}$$

Substituting (2.3) and (2.9) into (2.2), we have

$$\begin{aligned}
{}_tD^\lambda_b m(t_q) &= \frac{-1}{\Gamma(1-\lambda)}\int_{t_q}^{t_W}(\rho - t_q)^{-\lambda}m'(\rho)d\rho \\
&= \frac{-1}{\Gamma(1-\lambda)}\Big[\int_{t_{W-1}}^{t_W}(\rho - t_q)^{-\lambda}m'(\rho)d\rho + \sum_{l=q+1}^{W-1}\int_{t_{l-1}}^{t_l}(\rho - t_q)^{-\lambda}m'(\rho)d\rho\Big] \\
&\approx \frac{-1}{\Gamma(1-\lambda)}\Big\{\int_{t_{W-1}}^{t_W}(\rho - t_q)^{-\lambda}[\gamma_{[t_{W-2},t_W]}m(\rho)]'d\rho \\
&\quad + \sum_{l=q+1}^{W-1}\int_{t_{l-1}}^{t_l}(\rho - t_q)^{-\lambda}[\gamma_{[t_{l-1},t_l]}m(\rho)]'d\rho\Big\} \\
&= E^{0,0}_q m_{W-2} + E^{1,0}_q m_{W-1} + E^{2,0}_q m_W \\
&\quad + \sum_{l=q+1}^{W-1}(E^{0,l}_q m_{l-1} + E^{1,l}_q m_l + E^{2,l}_q m_{l+1}),
\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}
E^{i,0}_q &= \frac{-1}{\Gamma(1-\lambda)}\int_{t_{W-1}}^{t_W}(\rho - t_q)^{-\lambda}\kappa'_{i,W-2}(\rho)d\rho, i = 0, 1, 2, \\
E^{i,l}_q &= \frac{-1}{\Gamma(1-\lambda)}\int_{t_{l-1}}^{t_l}(\rho - t_q)^{-\lambda}\omega'_{i,l}(\rho)d\rho, i = 0, 1, 2;\; l = W-1, \ldots, q+1,
\end{aligned} \tag{2.12}$$

and $\kappa_{i,W-2}(t)$, $\omega_{i,l}(t)$ are in (2.4) and (2.10), respectively.

In summary, the linear combination of m_l approximates the right Caputo derivative ${}_tD^\lambda_b m(t_q)$. After calculation, it is found that each $E^{i,l}_q$ is proportional to $\frac{\eta^{-\lambda}}{\Gamma(3-\lambda)}$. After sorting (2.5), (2.6) and (2.11), the discrete Caputo derivative ${}_\eta L^\lambda_b m_q$ be obtained as follows

$${}_\eta L^\lambda_b m_q = \begin{cases}
\frac{\eta^{-\lambda}}{\Gamma(3-\lambda)}(\bar{E}_0 m_{W-2} + \bar{E}_1 m_{W-1} + \bar{E}_2 m_W), \quad q = W-1, \\
\frac{\eta^{-\lambda}}{\Gamma(3-\lambda)}(E_0 m_{W-2} + E_1 m_{W-1} + E_2 m_W), \quad q = W-2, \\
\frac{\eta^{-\lambda}}{\Gamma(3-\lambda)}\Big[\bar{F}_q m_{W-2} + \bar{G}_q m_{W-1} + \bar{H}_q m_W \\
\quad + \sum_{l=q+1}^{W-1}(F_l m_{l-1} + G_l m_l + H_l m_{l+1})\Big], \quad q \le W-3,
\end{cases} \tag{2.13}$$

where the values of all the coefficients "E, F, G, H" have been carefully calculated as follows

$$\begin{aligned}
&\bar{E}_0 = \frac{\lambda}{2}, \bar{E}_1 = 2-2\lambda, \bar{E}_2 = \frac{3\lambda-4}{2}, E_0 = \frac{\lambda+2}{2^\lambda}, E_1 = -\frac{4\lambda}{2^\lambda}, E_2 = \frac{3\lambda-2}{2^\lambda};\\
&\bar{F}_q = -\frac{2-\lambda}{2}[(W-q)^{1-\lambda} + (W-q-1)^{1-\lambda}] + (W-q)^{2-\lambda} - (W-q-1)^{2-\lambda};\\
&\bar{G}_q = 2(2-\lambda)(W-q)^{1-\lambda} - 2(W-q)^{2-\lambda} + 2(W-q-1)^{2-\lambda};\\
&\bar{H}_q = -\frac{3(2-\lambda)}{2}(W-q)^{1-\lambda} + \frac{2-\lambda}{2}(W-q-1)^{1-\lambda} + (W-q)^{2-\lambda} - (W-q-1)^{2-\lambda};\\
&F_l = -\frac{3(2-\lambda)}{2}(l-q-1)^{1-\lambda} + \frac{2-\lambda}{2}(l-q)^{1-\lambda} + (l-q)^{2-\lambda} - (l-q-1)^{2-\lambda};\\
&G_l = 2(2-\lambda)(l-q-1)^{1-\lambda} - 2(l-q)^{2-\lambda} + 2(l-q-1)^{2-\lambda};\\
&H_l = -\frac{2-\lambda}{2}[(l-q)^{1-\lambda} + (l-q-1)^{1-\lambda}] + (l-q)^{2-\lambda} - (l-q-1)^{2-\lambda}.
\end{aligned} \tag{2.14}$$

With the observe of the above coefficients, we will find that all the coefficients only depend on the constant of λ. With the help of (2.13), we have

$$ {}_tD_b^\lambda m(t_q) \approx {}_\eta L_b^\lambda m_q, q = W-1, \ldots, 1, 0.$$

Therefore, this sufficiently shows that the high-order numerical scheme corresponding to the above Eq (2.1) is

$$ {}_\eta L_b^\lambda m_q = g(t_q, m_q), \quad q = W-1, \ldots, 1, 0. \tag{2.15}$$

2.2. Error estimation

In order to analysis the local error of scheme (2.15), we bring in the following operator

$$ {}_\eta L_b^\lambda m(t_q) = \begin{cases} \frac{\eta^{-\lambda}}{\Gamma(3-\lambda)}[\bar{E}_0 m(t_{W-2}) + \bar{E}_1 m(t_{W-1}) + \bar{E}_2 m(t_W)], \quad q = W-1,\\ \frac{\eta^{-\lambda}}{\Gamma(3-\lambda)}[E_0 m(t_{W-2}) + E_1 m(t_{W-1}) + E_2 m(t_W)], \quad q = W-2,\\ \frac{\eta^{-\lambda}}{\Gamma(3-\lambda)}\Big[\bar{F}_q m(t_{W-2}) + \bar{G}_q m(t_{W-1}) + \bar{H}_q m(t_W)\\ \quad + \sum_{l=q+1}^{W-1}(F_l m(t_{l-1}) + G_l m(t_l) + H_l m(t_{l+1}))\Big], \quad q \le W-3, \end{cases} \tag{2.16}$$

where the values of all the coefficients "E, F, G, H" are defined by (2.14), which is replace m_q with $m(t_q)$ in (2.13).

Theorem 2.1. *Suppose $m(t) \in C^3[a, b], 0 < \lambda < 1$, let*

$$R^q(\eta) = {}_tD_b^\lambda m(t_q) - {}_\eta L_b^\lambda m(t_q), q = W-1, \ldots, 1, 0, \tag{2.17}$$

we have

$$|R^q(\eta)| \le C_m \eta^{3-\lambda}, \tag{2.18}$$

where C_m is a positive constant independent of η.

Proof. The following error estimates are estimated with the help of Taylor's theorem,

$$m(t)-\gamma_{[t_{W-2},t_W]}m(t) = \frac{m^{(3)}\left(\xi_W(\rho)\right)}{6}\left(\rho-t_{W-2}\right)\left(\rho-t_{W-1}\right)\left(\rho-t_W\right), \tag{2.19}$$

$$m(t)-\gamma_{[t_{l-1},t_l]}m(t) = \frac{m^{(3)}\left(\xi_l(\rho)\right)}{6}\left(\rho-t_{l-1}\right)\left(\rho-t_l\right)\left(\rho-t_{l+1}\right), \tag{2.20}$$

where $\xi_W(\rho)\in[t_{W-2},t_W]$, $\xi_l(\rho)\in[t_{l-1},t_{l+1}]$.

When $q=W-1$, according to (2.19), we get the following

$$\begin{aligned}
&|R^{W-1}(\eta)| = |{}_tD_b^{\lambda}m(t_{W-1})-{}_{\eta}L_b^{\lambda}m(t_{W-1})| \\
=\;& \frac{1}{\Gamma(1-\lambda)}\Big|\int_{t_{W-1}}^{t_W}(\rho-t_{W-1})^{-\lambda}d\big[m(\rho)-\gamma_{[t_{W-2},t_W]}m(\rho)\big]\Big| \\
=\;& \frac{\lambda}{\Gamma(1-\lambda)}\Big|\int_{t_{W-1}}^{t_W}\big[m(\rho)-\gamma_{[t_{W-2},t_W]}m(\rho)\big](\rho-t_{W-1})^{-\lambda-1}d\rho\Big| \\
=\;& \frac{\lambda}{\Gamma(1-\lambda)}\Big|\int_{t_{W-1}}^{t_W}\frac{m^{(3)}(\xi_W(\rho))}{6}(\rho-t_{W-2})(\rho-t_W)(\rho-t_{W-1})^{-\lambda}d\rho\Big| \\
\le\;& \frac{\lambda}{\Gamma(1-\lambda)}\int_{t_{W-1}}^{t_W}\Big|\frac{m^{(3)}(\xi_W(\rho))}{6}(\rho-t_{W-2})(\rho-t_W)(\rho-t_{W-1})^{-\lambda}\Big|d\rho \\
\le\;& \frac{\lambda}{6\Gamma(1-\lambda)}\max_{t\in[a,b]}|m^{(3)}(t)|\int_{t_{W-1}}^{t_W}(\rho-t_{W-2})(t_W-\rho)(\rho-t_{W-1})^{-\lambda}d\rho \\
=\;& \frac{\lambda}{6\Gamma(1-\lambda)}\max_{t\in[a,b]}|m^{(3)}(t)|\Big[\frac{1}{1-\lambda}\int_{t_{W-1}}^{t_W}(\rho-t_{W-1})^{1-\lambda}(2\rho-t_W-t_{W-2})d\rho\Big] \\
=\;& \frac{\lambda}{6\Gamma(2-\lambda)}\max_{t\in[a,b]}|m^{(3)}(t)|\Big[\frac{1}{2-\lambda}(t_W-t_{W-1})^{2-\lambda}(t_W-t_{W-2}) \\
&\quad -\frac{2}{2-\lambda}\int_{t_{W-1}}^{t_W}(\rho-t_{W-1})^{2-\lambda}d\rho\Big] \\
=\;& \frac{\lambda}{6\Gamma(2-\lambda)}\max_{t\in[a,b]}|m^{(3)}(t)|\Big[\frac{2}{2-\lambda}\eta^{3-\lambda}-\frac{2}{2-\lambda}\int_{t_{W-1}}^{t_W}(\rho-t_{W-1})^{2-\lambda}d\rho\Big] \\
=\;& \frac{\lambda}{3\Gamma(3-\lambda)}\max_{t\in[a,b]}|m^{(3)}(t)|\Big(1-\frac{1}{3-\lambda}\Big)\eta^{3-\lambda} \\
=\;& \frac{\lambda(2-\lambda)}{3\Gamma(4-\lambda)}\max_{t\in[a,b]}|m^{(3)}(t)|\eta^{3-\lambda}\le C_m\eta^{3-\lambda},
\end{aligned} \tag{2.21}$$

where C_m is a positive constant independent of η.

When $q=W-2$, by (2.19), we have

$$
\begin{aligned}
&|R^{W-2}(\eta)| = |{}_tD_b^{\lambda}m(t_{W-2}) - {}_{\eta}L_b^{\lambda}m(t_{W-2})| \\
= \ & \Big|\frac{-1}{\Gamma(1-\lambda)}\int_{t_{W-2}}^{t_W}(\rho - t_{W-2})^{-\lambda}\{m'(\rho) - [\gamma_{[t_{W-2},t_W]}m(\rho)]'\}d\rho\Big| \\
= \ & \frac{1}{\Gamma(1-\lambda)}\Big|\int_{t_{W-2}}^{t_W}(\rho - t_{W-2})^{-\lambda}d\big[m(\rho) - \gamma_{[t_{W-2},t_W]}m(\rho)\big]\Big| \\
= \ & \frac{\lambda}{\Gamma(1-\lambda)}\Big|\int_{t_{W-2}}^{t_W}\big[m(\rho) - \gamma_{[t_{W-2},t_W]}m(\rho)\big](\rho - t_{W-2})^{-\lambda-1}d\rho\Big| \\
= \ & \frac{\lambda}{\Gamma(1-\lambda)}\Big|\int_{t_{W-2}}^{t_W}\frac{m^{(3)}(\xi_W(\rho))}{6}(\rho - t_{W-1})(\rho - t_W)(\rho - t_{W-2})^{-\lambda}d\rho\Big| \\
\leq \ & \frac{\lambda}{6\Gamma(1-\lambda)}\max_{t\in[a,b]}|m^{(3)}(t)|\int_{t_{W-2}}^{t_W}\Big|(\rho - t_{W-1})(\rho - t_W)(\rho - t_{W-2})^{-\lambda}\Big|d\rho \\
\leq \ & \frac{\lambda}{6\Gamma(1-\lambda)}\max_{t\in[a,b]}|m^{(3)}(t)|\int_{t_{W-2}}^{t_W}\eta\cdot 2\eta\cdot(\rho - t_{W-2})^{-\lambda}d\rho \\
= \ & \frac{\lambda}{6\Gamma(1-\lambda)}\max_{t\in[a,b]}|m^{(3)}(t)|2\eta^2\int_{t_{W-2}}^{t_W}(\rho - t_{W-2})^{-\lambda}d\rho \\
= \ & \frac{\lambda}{6\Gamma(1-\lambda)}\max_{t\in[a,b]}|m^{(3)}(t)|2\eta^2\frac{1}{1-\lambda}(2\eta)^{1-\lambda} \leq C_m\eta^{3-\lambda}. \qquad (2.22)
\end{aligned}
$$

When $q \leq W-3$, according to (2.20), we obtain that

$$
\begin{aligned}
|R^q(\eta)| \ &= \ |{}_tD_b^{\lambda}m(t_q) - {}_{\eta}L_b^{\lambda}m(t_q)| \\
&= \ \Big|\frac{-1}{\Gamma(1-\lambda)}\int_{t_q}^{t_W}(\rho - t_q)^{-\lambda}[m(\rho) - \gamma_{[t_q,t_W]}m(\rho)]'d\rho\Big| \\
&= \ \frac{1}{\Gamma(1-\lambda)}\Big|\int_{t_{W-1}}^{t_W}(\rho - t_q)^{-\lambda}d\big[m(\rho) - \gamma_{[t_{W-2},t_W]}m(\rho)\big] \\
&\quad + \sum_{l=q+2}^{W-1}\int_{t_{l-1}}^{t_l}(\rho - t_q)^{-\lambda}d\big[m(\rho) - \gamma_{[t_{l-1},t_l]}m(\rho)\big] \\
&\quad + \int_{t_q}^{t_{q+1}}(\rho - t_q)^{-\lambda}d\big[m(\rho) - \gamma_{[t_{q-1},t_q]}m(\rho)\big]\Big| \\
&= \ \frac{\lambda}{\Gamma(1-\lambda)}\Big|\int_{t_{W-1}}^{t_W}\big[m(\rho) - \gamma_{[t_{W-2},t_W]}m(\rho)\big](\rho - t_q)^{-\lambda-1}d\rho \\
&\quad + \sum_{l=q+2}^{W-1}\int_{t_{l-1}}^{t_l}\big[m(\rho) - \gamma_{[t_{l-1},t_l]}m(\rho)\big](\rho - t_q)^{-\lambda-1}d\rho \\
&\quad + \int_{t_q}^{t_{q+1}}\big[m(\rho) - \gamma_{[t_{q-1},t_q]}m(\rho)\big](\rho - t_q)^{-\lambda-1}d\rho\Big| \\
&= \ \frac{\lambda}{\Gamma(1-\lambda)}|S_1 + S_2 + S_3|. \qquad (2.23)
\end{aligned}
$$

Next, we estimate each item at the right hand side of (2.23). By using (2.19), S_1 is calculated as following

$$\begin{aligned}
|S_1| &= \Big|\int_{t_{W-1}}^{t_W} [m(\rho) - \gamma_{[t_{W-2},t_W]} m(\rho)](\rho - t_q)^{-\lambda-1} d\rho\Big| \\
&= \Big|\int_{t_{W-1}}^{t_W} \frac{m^{(3)}(\xi_W(\rho))}{6}(\rho - t_{W-1})(\rho - t_W)(\rho - t_{W-2})(\rho - t_q)^{-\lambda-1} d\rho\Big| \\
&\leq \frac{1}{6}\max_{t\in[a,b]}|m^{(3)}(t)| \int_{t_{W-1}}^{t_W} \Big|(\rho - t_{W-1})(\rho - t_W)(\rho - t_{W-2})(\rho - t_q)^{-\lambda-1}\Big| d\rho \\
&\leq \frac{1}{6}\max_{t\in[a,b]}|m^{(3)}(t)| \int_{t_{W-1}}^{t_W} \eta\cdot\eta\cdot 2\eta\cdot(t_W - t_q)^{-\lambda-1} d\rho \\
&= \frac{1}{6}\max_{t\in[a,b]}|m^{(3)}(t)| 2\eta^3\cdot\eta\cdot\eta^{-\lambda-1}(W-q)^{-\lambda-1} \leq \frac{1}{3}\max_{t\in[a,b]}|m^{(3)}(t)|\eta^{3-\lambda}.
\end{aligned}$$

S_2 and S_3 are calculated as follows

$$\begin{aligned}
|S_2| &= \Big|\sum_{l=q+2}^{W-1}\int_{t_{l-1}}^{t_l} [m(\rho) - \gamma_{[t_{l-1},t_l]} m(\rho)](\rho - t_q)^{-\lambda-1} d\rho\Big| \\
&= \Big|\sum_{l=q+2}^{W-1}\int_{t_{l-1}}^{t_l} \frac{m^{(3)}(\xi_l(\rho))}{6}(\rho - t_{l-1})(\rho - t_l)(\rho - t_{l+1})(\rho - t_q)^{-\lambda-1} d\rho\Big| \\
&\leq \frac{1}{6}\max_{t\in[a,b]}|m^{(3)}(t)| \sum_{l=q+2}^{W-1}\int_{t_{l-1}}^{t_l} \Big|(\rho - t_{l-1})(\rho - t_l)(\rho - t_{l+1})(\rho - t_q)^{-\lambda-1}\Big| d\rho \\
&\leq \frac{\eta^3}{3}\max_{t\in[a,b]}|m^{(3)}(t)| \int_{t_{q+1}}^{t_{W-1}} (\rho - t_q)^{-\lambda-1} d\rho \\
&= \frac{1}{3\lambda}\max_{t\in[a,b]}|m^{(3)}(t)|\eta^{3-\lambda}[1 - (W-1-q)^{-\lambda}] \\
&\leq \frac{1}{3\lambda}\max_{t\in[a,b]}|m^{(3)}(t)|\eta^{3-\lambda}, \qquad (2.24) \\
|S_3| &= \Big|\int_{t_q}^{t_{q+1}} [m(\rho) - \gamma_{[t_{q-1},t_q]} m(\rho)](\rho - t_q)^{-\lambda-1} d\rho\Big| \\
&= \Big|\int_{t_q}^{t_{q+1}} \frac{m^{(3)}(\xi_q(\rho))}{6}(\rho - t_{q-1})(\rho - t_{q+1})(\rho - t_q)^{-\lambda} d\rho\Big| \\
&\leq \frac{1}{6}\max_{t\in[a,b]}|m^{(3)}(t)| \int_{t_q}^{t_{q+1}} \Big|(\rho - t_{q-1})(\rho - t_{q+1})(\rho - t_q)^{-\lambda}\Big| d\rho \\
&= \frac{1}{6}\max_{t\in[a,b]}|m^{(3)}(t)| \int_{t_q}^{t_{q+1}} (\rho - t_{q-1})(t_{q+1} - \rho)(\rho - t_q)^{-\lambda} d\rho \\
&= \frac{1}{6(1-\lambda)}\max_{t\in[a,b]}|m^{(3)}(t)| \int_{t_q}^{t_{q+1}} (\rho - t_{q-1})(t_{q+1} - \rho) d(\rho - t_q)^{1-\lambda} \\
&= \frac{1}{6(1-\lambda)}\max_{t\in[a,b]}|m^{(3)}(t)| \int_{t_q}^{t_{q+1}} (\rho - t_q)^{1-\lambda}(2\rho - t_{q-1} - t_{q+1}) d\rho
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{6(1-\lambda)}\max_{t\in[a,b]}|m^{(3)}(t)|\Big[\frac{1}{2-\lambda}(t_{q+1}-t_q)^{2-\lambda}(t_{q+1}-t_{q-1})\\
&\quad -\frac{2}{2-\lambda}\int_{t_q}^{t_{q+1}}(\rho-t_q)^{2-\lambda}d\rho\Big]\\
&= \frac{2}{6(1-\lambda)(2-\lambda)}\max_{t\in[a,b]}|m^{(3)}(t)|[\eta^{3-\lambda}-\frac{1}{3-\lambda}(t_{q+1}-t_q)^{3-\lambda}]\\
&= \frac{1}{3(1-\lambda)(3-\lambda)}\max_{t\in[a,b]}|m^{(3)}(t)|\eta^{3-\lambda}. \qquad (2.25)
\end{aligned}
$$

Bringing (2.24) and (2.25) into (2.23), we can get

$$
|R^q(\eta)| = \frac{\lambda}{\Gamma(1-\lambda)}|S_1+S_2+S_3| \le C_m\eta^{3-\lambda}. \qquad (2.26)
$$

Combining (2.21), (2.22) with (2.26), Theorem 2.1 has been proved.

2.3. Stability analysis

We will now analyze the stability of the scheme (2.15), analogous to integer order differential equations, considering the problem

$$
g(t,m(t)) = -\theta m(t), \qquad (2.27)
$$

where $\theta > 0$ is a real number, first introducing the symbol,

$$
\eta_0 = \Gamma(3-\lambda)\eta^\lambda, \alpha_0 = F_{q+1} = \frac{4-\lambda}{2}. \qquad (2.28)
$$

Now rewrite the scheme (2.15) as follows, for $q \le W-3$,

$$
\begin{aligned}
m_q + \alpha_0^{-1}\eta_0\theta m_q &= -\alpha_0^{-1}[(G_{q+1}+F_{q+2})m_{q+1} + (H_{q+1}+G_{q+2}+F_{q+3})m_{q+2}\\
&+ \sum_{s=q+3}^{W-3}(H_{s-1}+G_s+F_{s+1})m_s + (\bar{F}_q + H_{W-3}+G_{W-2}+F_{W-1})m_{W-2}\\
&+(H_{W-2}+G_{W-1}+\bar{G}_q)m_{W-1} + (H_{W-1}+\bar{H}_q)m_W].
\end{aligned} \qquad (2.29)
$$

When $q = W-1, W-2$, we have

$$
\begin{aligned}
m_{W-1} + \eta_0\frac{1}{2(1-\lambda)}\theta m_{W-1} &= -\frac{1}{2(1-\lambda)}(\bar{E}_0 m_{W-2} + \bar{E}_2 m_W),\\
m_{W-2} + \eta_0\frac{2^\lambda}{\lambda+2}\theta m_{W-2} &= -\frac{2^\lambda}{\lambda+2}(E_1 m_{W-1} + E_2 m_W).
\end{aligned} \qquad (2.30)
$$

Equations (2.29) and (2.30) are solved together, and to further simplify the representation, the coefficients are introduced, when $q \le W-3$,

$$
\begin{aligned}
&d^q_{q+1} = -(G_{q+1}+F_{q+2})\alpha_0^{-1},\ d^q_{q+2} = -(H_{q+1}+G_{q+2}+F_{q+3})\alpha_0^{-1},\\
&d^q_s = -(H_{s-1}+G_s+F_{s+1})\alpha_0^{-1},\quad s = q+3, q+4, \ldots, W-3,\\
&d^q_{W-2} = -(\bar{F}_q + H_{W-3}+G_{W-2}+F_{W-1})\alpha_0^{-1},\\
&d^q_{W-1} = -(H_{W-2}+G_{W-1}+\bar{G}_q)\alpha_0^{-1},\\
&d^q_W = -(H_{W-1}+\bar{H}_q)\alpha_0^{-1}.
\end{aligned} \qquad (2.31)
$$

Therefore, (2.29) is re-expressed as follows

$$m_q + \alpha_0^{-1}\eta_0\theta m_q = \sum_{s=q+1}^{W} d_s^q m_s, \quad q \le W-3. \tag{2.32}$$

Before analyzing the stability of (2.32), two auxiliary lemmas are given.

Lemma 2.2. *For all* $0 < \lambda < 1, q \le W-3$*, the coefficients in the problem (2.32) satisfy:*

1) $\sum_{s=q+1}^{W} d_s^q = 1$;
2) $d_s^q > 0, \ s = W, W-1, \ldots, q+4, q+3$;
3) $0 < d_{q+1}^q < \frac{4}{3}$;
4) *The symbol for* d_{q+2}^q *can not be determined;*
5) $\frac{1}{4}(d_{q+1}^q)^2 + d_{q+2}^q > 0$.

Proof. For a detailed proof, see Appendix A.

In Lemma 2.2, we find that the value of d_{q+2}^q can be positive or negative. In order to prove that the high order scheme is absolutely stability for $\lambda \in (0,1)$, let

$$\tau = \frac{1}{2}d_{q+1}^q, \tag{2.33}$$

then the Eq (2.32) can be re-expressed as follows

$$\begin{aligned}
&m_q - \tau m_{q+1} + \alpha_0^{-1}\eta_0\theta m_q \\
&= \tau(m_{q+1} - \tau m_{q+2}) + (\tau^2 + d_{q+2}^q)(m_{q+2} - \tau m_{q+3}) \\
&\quad + (\tau^3 + \tau d_{q+2}^q + d_{q+3}^q)(m_{q+3} - \tau m_{q+4}) \\
&\quad + \ldots + (\tau^{W-q-2} + \tau^{W-q-4} d_{q+2}^q + \ldots + \tau d_{W-3}^q + d_{W-2}^q)(m_{W-2} - \tau m_{W-1}) \\
&\quad + (\tau^{W-1-q} + \tau^{W-3-q} d_{q+2}^q + \ldots + \tau d_{W-2}^q + d_{W-1}^q)(m_{W-1} - \tau m_W) \\
&\quad + (\tau^{W-q} + \tau^{W-2-q} d_{q+2}^q + \ldots + \tau d_{W-1}^q + d_W^q) m_W.
\end{aligned}$$

Now we denote

$$\bar{d}_s^q = \tau^{s-q} + \sum_{k=q+2}^{s} \tau^{s-k} d_k^q, \ s = W, W-1, \ldots, q+2, \tag{2.34}$$

$$\bar{m}_s = m_s - \tau m_{s+1}, \ s = W-1, W-2, \ldots, q. \tag{2.35}$$

The equivalent form of (2.29) is

$$\bar{m}_q + \alpha_0^{-1}\eta_0\theta m_q = \tau \bar{m}_{q+1} + \sum_{s=q+2}^{W-1} \bar{d}_s^q \bar{m}_s + \bar{d}_W^q m_W. \tag{2.36}$$

Lemma 2.3. *For all* $0 < \lambda < 1, q \leq W-3$*, the coefficients of (2.36) satisfy*

1) $0 < \tau < \frac{2}{3}$;
2) $\bar{d}_s^q > 0, s = W, W-1, \ldots, q+2$;
3) $\tau + \sum_{s=q+2}^{W-1} \bar{d}_s^q + \bar{d}_W^q \leq 1$.

Proof. 1) According to 3) in Lemma 2.2 and (2.33), it is obviously provable.

2) When $s = q+2$,

$$\bar{d}_{q+2}^q = \tau^2 + d_{q+2}^q = \frac{1}{4}(d_{q+1}^q)^2 + d_{q+2}^q.$$

According to 5) in Lemma 2.2, we have $\bar{d}_{q+2}^q > 0$, thus

$$\bar{d}_s^q = \bar{d}_{s+1}^q \tau + d_s^q, s = W, W-1, \ldots, q+3,$$

and by 2) in Lemma 2.2, $\tau > 0$, we have

$$\bar{d}_s^q > 0, s = W, W-1, \ldots, q+2.$$

3) Assume $Q_q = \tau + \sum_{s=q+2}^{W-1} \bar{d}_s^q + \bar{d}_W^q$, for (2.34), we have

$$\begin{aligned} Q_q &= \sum_{s=1}^{W-q} \tau^s + d_{q+2}^q \sum_{s=0}^{W-q-2} \tau^s + \ldots + d_{W-1}^q \sum_{s=0}^{1} \tau^s + d_W^q \\ &= \tau \frac{1-\tau^{W-q}}{1-\tau} + d_{q+2}^q \frac{1-\tau^{W-q-1}}{1-\tau} + \ldots + d_{W-2}^q \frac{1-\tau^3}{1-\tau} + d_{W-1}^q \frac{1-\tau^2}{1-\tau} + d_W^q. \end{aligned}$$

Further available,

$$(1-\tau)Q_q = \tau(1-\tau^{W-q}) + d_{q+2}^q(1-\tau^{W-q-1}) + \ldots + d_{W-1}^q(1-\tau^2) + d_W^q(1-\tau).$$

According to 1), 2), 5) in Lemma 2.2 and (2.33), we get

$$\begin{aligned} (1-\tau)Q_q &\leq \tau(1-\tau^{W-q}) + d_{q+2}^q(1-\tau^{W-q-1}) + \sum_{s=q+3}^{W} d_s^q \\ &= (\tau + \sum_{s=q+2}^{W} d_s^q) - \tau^{W-q-1}(\tau^2 + d_{q+2}^q) \\ &= (1-\tau) - \tau^{W-q-1}(\tau^2 + d_{q+2}^q) < (1-\tau). \end{aligned}$$

Therefore, $Q_q = \tau + \sum_{s=q+2}^{W-1} \bar{d}_s^q + \bar{d}_W^q \leq 1$. The proof of Lemma 2.3 is then completed.

Lemma 2.4. *For all* $0 < \lambda < 1$*, there is*

$$\bar{m}_q^2 + \alpha_0^{-1}\eta_0\theta m_q^2 \leq m_W^2, q = W-1, \ldots, 1, 0. \tag{2.37}$$

Proof. For a detailed proof, see Appendix B.

Theorem 2.5. *When g is defined by (2.27), the high-order numerical scheme to (2.1) is stable, and have*

$$|m_q| \leq 3|m_W|, q = W-1, \ldots, 1, 0.$$

Proof. According to Lemma 2.4, we get

$$|\bar{m}_q| \leq |m_W|, q = W-1, \ldots, 1, 0. \tag{2.38}$$

From (2.35) and $m_s = \bar{m}_s + \tau m_{s+1} = \ldots = \bar{m}_s + \tau \bar{m}_{s+1} + \ldots + \tau^{W-1-s}\bar{m}_{W-1} + \tau^{W-s} m_W$, bying using (2.38) and 1) in Lemma 2.3, we have

$$\begin{aligned} |m_q| &= |\bar{m}_q + \tau \bar{m}_{q+1} + \ldots + \tau^{W-1-q}\bar{m}_{W-1} + \tau^{W-q} m_W| \\ &\leq |\bar{m}_q| + \tau|\bar{m}_{q+1}| + \ldots + \tau^{W-1-q}|\bar{m}_{W-1}| + \tau^{W-q}|m_W| \\ &\leq (1 + \tau + \ldots + \tau^{W-1-q} + \tau^{W-q})|m_W| \leq \frac{1}{1-\tau}|m_W| \leq 3|m_W|. \end{aligned}$$

To sum up, Theorem 2.5 certificate is completed.

3. High-order numerical schemes of FPDEs

3.1. High-order numerical schemes of 1D time FPDEs

Consider the following time FPDEs

$$\begin{cases} {}_tD_b^{\lambda} m(y,t) - \dfrac{\partial^2 m(y,t)}{\partial y^2} = g(y,t), \quad a \leq t \leq b, \quad 0 < \lambda < 1, \quad y \in [c,d], \\ m(y,b) = m_0(y), \qquad \forall y \in [c,d], \\ m(c,t) = m(d,t) = 0, \quad \forall t \in [a,b], \end{cases} \tag{3.1}$$

where λ represents the order of a fractional derivative in regard to time t, $m_0(y)$ is a known function, and the relevant definition of ${}_tD_b^{\lambda} m(y,t)$ is shown below

$$ {}_tD_b^{\lambda} m(y,t) = \frac{-1}{\Gamma(1-\lambda)} \int_t^b (\rho - t)^{-\lambda} \frac{\partial m(y,\rho)}{\partial \rho} d\rho, \quad 0 < \lambda < 1. \tag{3.2}$$

To construct a time-dependent high-order numerical scheme, $[a,b]$ is divided into W equal parts. Marking $\eta = \frac{b-a}{W}, t_q = q\eta, q = 0,1,2,\ldots,W$, the interval $[c,d]$ is divided into M equal parts, set $\Delta y = \frac{d-c}{M}, y_i = c + i\Delta y, i = 0,1,\ldots,M$. And let the numerical solution of (3.1) at (y_i, t_q) as $m_{i,q}$. Next, construct the discrete scheme of (3.1) at time t, the derivation process is similar to the derivation of (2.13), then we have

$$ {}_\eta L_b^{\lambda} m(y,t_q) = \begin{cases} \dfrac{\eta^{-\lambda}}{\Gamma(3-\lambda)}[\bar{E}_0 m(y,t_{W-2}) + \bar{E}_1 m(y,t_{W-1}) + \bar{E}_2 m(y,t_W)], \quad q = W-1, \\ \dfrac{\eta^{-\lambda}}{\Gamma(3-\lambda)}[E_0 m(y,t_{W-2}) + E_1 m(y,t_{W-1}) + E_2 m(y,t_W)], \quad q = W-2, \\ \dfrac{\eta^{-\lambda}}{\Gamma(3-\lambda)}\Big\{\bar{F}_q m(y,t_{W-2}) + \bar{G}_q m(y,t_{W-1}) + \bar{H}_q m(y,t_W) \\ \quad + \displaystyle\sum_{l=q+1}^{W-1} [F_l m(y,t_{l-1}) + G_l m(y,t_l) + H_l m(y,t_{l+1})]\Big\}, \quad q \leq W-3, \end{cases} \tag{3.3}$$

where the values of the coefficients "E, F, G, H" are defined by (2.14).

Therefore, the semi-discrete scheme of (3.1) is

$$\begin{aligned}
&\frac{\partial^2 m(y,t_{W-1})}{\partial y^2}+g(y,t_{W-1})=\frac{\eta^{-\lambda}}{\Gamma(3-\lambda)}[\bar{E}_0 m(y,t_{W-2})\\
&\qquad+\bar{E}_1 m(y,t_{W-1})+\bar{E}_2 m(y,t_W)],\quad q=W-1,\\
&\frac{\partial^2 m(y,t_{W-2})}{\partial y^2}+g(y,t_{W-2})=\frac{\eta^{-\lambda}}{\Gamma(3-\lambda)}[E_0 m(y,t_{W-2})\\
&\qquad+E_1 m(y,t_{W-1})+E_2 m(y,t_W)],\quad q=W-2,\\
&\frac{\partial^2 m(y,t_q)}{\partial y^2}+g(y,t_q)=\frac{\eta^{-\lambda}}{\Gamma(3-\lambda)}\Big\{\bar{F}_q m(y,t_{W-2})+\bar{G}_q m(y,t_{W-1})+\bar{H}_q m(y,t_W)\\
&\qquad+\sum_{l=q+1}^{W-1}[F_l m(y,t_{l-1})+G_l m(y,t_l)+H_l m(y,t_{l+1})]\Big\},\quad q\le W-3.
\end{aligned}\tag{3.4}$$

First analyze the truncation error of the above scheme, similar to Theorem 2.1, we have the following lemma.

Lemma 3.1. *Suppose that $m(y,t)$ is the solution of (3.1), and it has fourth-order continuous partial derivative in regard to time variable t, then*

$$\bar{R}(y,t_q)=\left|{}_tD_b^{\lambda}m(y,t_q)-{}_{\eta}L_b^{\lambda}m(y,t_q)\right|\le C_m\eta^{3-\lambda},\ q=W-1,\dots,1,0,\tag{3.5}$$

where C_m is a constant independent of η.

On the discretization of $\dfrac{\partial^2 m(y,t)}{\partial y^2}$, for the fixed t_q, use the idea of the central second-order difference scheme to discretize, and the method is as follows

$$\frac{\partial^2 m(y_i,t_q)}{\partial y^2}\approx\frac{m(y_{i-1},t_q)-2m(y_i,t_q)+m(y_{i+1},t_q)}{\Delta y^2},\quad q=W-1,\dots,1,0.\tag{3.6}$$

Lemma 3.2. *For a fixed time t, use the second-order central difference method to discretize $\frac{\partial^2 m(y,t)}{\partial y^2}$, the scheme (3.6) is known that it has the second-order accuracy under suitable conditions.*

Proof. Its detail proof can be found in reference [30].

We make use of the finite difference scheme (3.4) in the discretization of time and (3.6) in the discretization of space, and the high-order numerical scheme of the FPDE (3.1) is as follows

$$\begin{aligned}
&\frac{m_{i-1,W-1}-2m_{i,W-1}+m_{i+1,W-1}}{\Delta y^2}+g_{i,W-1}=\frac{\eta^{-\lambda}}{\Gamma(3-\lambda)}(\bar{E}_0 m_{i,W-2}+\bar{E}_1 m_{i,W-1}+\bar{E}_2 m_{i,W}),\\
&\frac{m_{i-1,W-2}-2m_{i,W-2}+m_{i+1,W-2}}{\Delta y^2}+g_{i,W-2}=\frac{\eta^{-\lambda}}{\Gamma(3-\lambda)}(E_0 m_{i,W-2}+E_1 m_{i,W-1}+E_2 m_{i,W}),\\
&\frac{m_{i-1,q}-2m_{i,q}+m_{i+1,q}}{\Delta y^2}+g_{i,q}=\frac{\eta^{-\lambda}}{\Gamma(3-\lambda)}\Big[\bar{F}_q m_{i,W-2}+\bar{G}_q m_{i,W-1}
\end{aligned}\tag{3.7}$$

$$+\bar{H}_q m_{i,W} + \sum_{l=q+1}^{W-1} (F_l m_{i,l-1} + G_l m_{i,l} + H_l m_{i,l+1})\Big],$$

where $m_{i,q}$ represents the numerical solution of $m(y_i, t_q)$, $g_{i,q}$ represents $g(y_i, t_q)$, and $i = 1, 2, \ldots, M-1$. The error of the discrete scheme is given below, here, we first bring in two defined the operators

$$\begin{aligned}
&\mathcal{L}(m(y_i, t_q)) = {}_tD_b^\lambda m(y_i, t_q) - \frac{\partial^2 m(y_i, t_q)}{\partial y^2}, \quad \forall y \in \bar{I}, \forall t \in [a, b], \\
&\mathcal{L}^{\Delta y,\eta}(m(y_i, t_q)) = {}_\eta L_b^\lambda m(y_i, t_q) - \frac{m(y_{i-1}, t_q) - 2m(y_i, t_q) + m(y_{i+1}, t_q)}{\Delta y^2}.
\end{aligned} \tag{3.8}$$

Theorem 3.3. *Assume $m(y, t)$ is the solution of the Eq (3.1) and regarding the time variable t and space variable y both have 4th-order continuous partial derivatives, so*

$$|\mathcal{L}(m(y_i, t_q)) - \mathcal{L}^{\Delta y,\eta} m(y_i, t_q)| \le C_m(\eta^{3-\lambda} + \Delta y^2), \tag{3.9}$$

where C_m is a constant independent of η and Δy.

Proof. According to Lemmas 3.1, 3.2 and (3.8), we have already proved above, we are able to immediately gain

$$\begin{aligned}
&|\mathcal{L}(m(y_i, t_q)) - \mathcal{L}^{\Delta y,\eta} m(y_i, t_q)| \\
&= \left| {}_tD_b^\lambda m(y_i, t_q) - \frac{\partial^2 m(y_i, t_q)}{\partial y^2} - {}_\eta L_b^\lambda m(y_i, t_q) + \frac{m(y_{i-1}, t_q) - 2m(y_i, t_q) + m(y_{i+1}, t_q)}{\Delta y^2} \right| \\
&\le \left| {}_tD_b^\lambda m(y_i, t_q) - {}_\eta L_b^\lambda m(y_i, t_q) \right| + \left| \frac{m(y_{i-1}, t_q) - 2m(y_i, t_q) + m(y_{i+1}, t_q)}{\Delta y^2} - \frac{\partial^2 m(y_i, t_q)}{\partial y^2} \right| \\
&\le C_m(\eta^{3-\lambda} + \Delta y^2).
\end{aligned} \tag{3.10}$$

Theorem 3.3 is then proved.

3.2. High-order numerical schemes of 2D FPDEs

Consider the following FPDEs

$$\begin{cases}
{}_tD_b^\lambda m(y, z, t) - \dfrac{\partial^2 m(y, z, t)}{\partial y^2} - \dfrac{\partial^2 m(y, z, t)}{\partial z^2} = g(y, z, t), t \in (a, b), (y, z) \in [c, d]^2, \\
m(y, z, b) = m_0(y, z), \qquad \forall (y, z) \in [c, d]^2, \\
m(c, z, t) = m(d, z, t) = m(y, c, t) = m(y, d, t) = 0, \forall t \in [a, b], \forall y, z \in [c, d],
\end{cases} \tag{3.11}$$

where λ represents the order of the fractional derivative in regard to time t, $m_0(y, z)$ is a known function, and the relevant definition of ${}_tD_b^\lambda m(y, z, t)$ is shown below

$${}_tD_b^\lambda m(y, z, t) = \frac{-1}{\Gamma(1-\lambda)} \int_t^b (\rho - t)^{-\lambda} \frac{\partial m(y, z, \rho)}{\partial \rho} d\rho, \quad 0 < \lambda < 1. \tag{3.12}$$

Similar to (3.7), let $\Delta y = \Delta z = \frac{d-c}{M}, z_l = c + l\Delta z, l = 0, 1, \ldots, M$, we have

$$\delta_y^2 m_{i,l,W-1} + \delta_z^2 m_{i,l,W-1} + g_{i,l,W-1} = \frac{\eta^{-\lambda}}{\Gamma(3-\lambda)} (\bar{E}_0 m_{i,l,W-2} + \bar{E}_1 m_{i,l,W-1} + \bar{E}_2 m_{i,l,W}),$$

$$\delta_y^2 m_{i,l,W-2} + \delta_z^2 m_{i,l,W-2} + g_{i,l,W-2} = \frac{\eta^{-\lambda}}{\Gamma(3-\lambda)}(E_0 m_{i,l,W-2} + E_1 m_{i,l,W-1} + E_2 m_{i,l,W}),$$

$$\delta_y^2 m_{i,l,q} + \delta_z^2 m_{i,l,q} + g_{i,l,q} = \frac{\eta^{-\lambda}}{\Gamma(3-\lambda)}\Big[\bar{F}_l m_{i,l,W-2} + \bar{G}_l m_{i,l,W-1} \tag{3.13}$$

$$+\bar{H}_l m_{i,l,W} + \sum_{s=l+1}^{W-1}(F_s m_{i,l-1,q} + G_s m_{i,l,q} + H_s m_{i,l+1,q})\Big],$$

where $m_{i,l,q}$ represents the numerical solution of $m(y_i, z_l, t_q)$, $g_{i,l,q}$ represents $g(y_i, z_l, t_q)$,

$$\delta_y^2 m_{i,l,q} = \frac{m_{i-1,l,q} - 2m_{i,l,q} + m_{i+1,l,q}}{\Delta y^2}, \quad \delta_z^2 m_{i,l,q} = \frac{m_{i,l-1,q} - 2m_{i,l,q} + m_{i,l+1,q}}{\Delta z^2}, \tag{3.14}$$

where $i, l = 1, 2, \ldots, M-1$. The error of the scheme is given below, we first define the two operators

$$\begin{aligned} \mathcal{L}(m(y_i, z_l, t_q)) &= {}_tD_b^\lambda m(y_i, z_l, t_q) - \frac{\partial^2 m(y_i, z_l, t_q)}{\partial y^2} - \frac{\partial^2 m(y_i, z_l, t_q)}{\partial z^2}, \\ \mathcal{L}^{\Delta y, \Delta z, \eta}(m(y_i, z_l, t_q)) &= {}_\eta L_b^\lambda m(y_i, z_l, t_q) - \delta_y^2 m(y_i, z_l, t_q) - \delta_z^2 m(y_i, z_l, t_q), \end{aligned} \tag{3.15}$$

where δ_y^2, δ_z^2 are defined as following for $i, l = 1, 2, \ldots, M-1$,

$$\begin{aligned} \delta_y^2 m(y_{i-1}, z_l, t_q) &= \frac{m(y_i, z_l, t_q) - 2m(y_i, z_l, t_q) + m(y_{i+1}, z_l, t_q)}{\Delta y^2}, \\ \delta_z^2 m(y_i, z_l, t_q) &= \frac{m(y_i, z_{l-1}, t_q) - 2m(y_i, z_l, t_q) + m(y_i, z_{l+1}, t_q)}{\Delta z^2}. \end{aligned} \tag{3.16}$$

The proof is given below that similar to Theorem 3.3.

Theorem 3.4. *Assume $m(y, z, t)$ is the solution of the Eq (3.11) and has 4th-order continuous partial derivatives with respect to t, y, z, then*

$$|\mathcal{L}(m(y_i, z_l, t_q)) - \mathcal{L}^{\Delta y, \Delta z, \eta} m(y_i, z_l, t_q)| \le C_m(\eta^{3-\lambda} + \Delta y^2 + \Delta z^2), \tag{3.17}$$

where C_m is a constant independent of $\eta, \Delta y, \Delta z$.

4. Numerical results

In this section, three computational examples will be used to verify our conclusions and methods in the previous section.

Example 4.1. Case (1). the exact solution is smooth. For the problem (2.1), let the initial value $m_0 = 0$, and

$$g(t, m) = \frac{24}{\Gamma(5-\lambda)}(1-t)^{4-\lambda} + (1-t)^4 - m(t), \tag{4.1}$$

$$g(t, m) = \frac{24}{\Gamma(5-\lambda)}(1-t)^{4-\lambda} + (1-t)^8 - m^2(t). \tag{4.2}$$

It can be verified that the exact solutions of the above two cases are all $m(t) = (1-t)^4$. We take $a = 0$, $b = 1$, the step size is $\eta = \frac{1}{W}$, $W = 2^{\bar{n}}$, where $\bar{n} = 4, \ldots, 10$. The following two cases for (4.1) and (4.2) use several different values of λ, and we will calculate the convergence order with the help of $\ln(e_{2\eta}/e_{\eta})/\ln 2$, where $e_{\eta} = \max_{q} |m(t_q) - m_q|$.

Firstly, when the function g is defined by (4.1) which g is a linear case of m, it can be seen from Table 1 that when λ takes $0.3, 0.5$ and 0.7, their convergence orders tend to be $2.7, 2.5$ and 2.3, respectively. In Table 2, we take $\lambda \to 0$ or 1, that is $\lambda = 0.01$ and 0.99, we find convergence orders close to 2.99 and 2.01, respectively. Therefore, from Tables 1 and 2, when λ take different values, even when it tends to 0 or 1, the convergence rate is still $3 - \lambda, \lambda \in (0, 1)$, and this is basically consistent with our theoretical expected results.

Secondly, when g is defined by (4.2) which g is the nonlinear case of m, it can be seen from Table 3 that when λ takes $0.2, 0.5$, and 0.8, the convergence order tends to $2.8, 2.5$, and 2.2, respectively. Similar to the above Table 2, $\lambda = 0.01$ and 0.99 are also taken in Table 4, and the convergence order tends to 2.99 and 2.01, respectively. In short, the results in Tables 3 and 4 still verify that our order of convergence is $3 - \lambda$. Therefore, in case (1) of the Example 4.1, when $0 < \lambda < 1$, by taking different values of λ, we find that the obtained convergence orders are all $3 - \lambda$, which again confirms the theoretical prediction in Theorem 2.1.

Table 1. Maximum errors and convergence rates for the right hand side (4.1) with $\lambda = 0.3$, 0.5 and 0.7.

η	$\lambda = 0.3$	Rate	$\lambda = 0.5$	Rate	$\lambda = 0.7$	Rate
$\frac{1}{16}$	4.0865E-04	-	1.2178E-03	-	3.0797E-03	-
$\frac{1}{32}$	6.8093E-05	2.5853	2.2836E-04	2.4148	6.5760E-04	2.2275
$\frac{1}{64}$	1.1060E-05	2.6222	4.1759E-05	2.4512	1.3692E-04	2.2639
$\frac{1}{128}$	1.7693E-06	2.6441	7.5362E-06	2.4702	2.8174E-05	2.2809
$\frac{1}{256}$	2.8029E-07	2.6582	1.3498E-06	2.4811	5.7624E-06	2.2896
$\frac{1}{512}$	4.4100E-08	2.6681	2.4068E-07	2.4876	1.1748E-06	2.2942
$\frac{1}{1024}$	6.8703E-09	2.6823	4.2780E-08	2.4921	2.3914E-07	2.2965

Table 2. Maximum errors and convergence rates for the right hand side (4.1) with $\lambda = 0.01$, 0.99.

η	$\lambda = 0.01$	Rate	$\lambda = 0.99$	Rate
$\frac{1}{16}$	7.5976E-05	-	1.0084E-02	-
$\frac{1}{32}$	4.6806E-06	4.0208	2.6311E-03	1.9384
$\frac{1}{64}$	2.8833E-07	4.0209	6.6795E-04	1.9778
$\frac{1}{128}$	1.7759E-08	4.0211	1.6760E-04	1.9947
$\frac{1}{256}$	2.4453E-09	2.8604	4.1826E-05	2.0025
$\frac{1}{512}$	3.3004E-10	2.8893	1.0411E-05	2.0063
$\frac{1}{1024}$	4.2900E-11	2.9436	2.5881E-06	2.0082

Table 3. Maximum errors and convergence rates for the right hand side (4.2) with $\lambda = 0.2$, 0.5 and 0.8.

η	$\lambda = 0.2$	Rate	$\lambda = 0.5$	Rate	$\lambda = 0.8$	Rate
$\frac{1}{16}$	1.8078E-04	-	1.1076E-03	-	4.5826E-03	-
$\frac{1}{32}$	2.9134E-05	2.6334	2.1050E-04	2.3956	1.0574E-03	2.1157
$\frac{1}{64}$	4.5379E-06	2.6826	3.8742E-05	2.4418	2.3634E-04	2.1616
$\frac{1}{128}$	6.9296E-07	2.7112	7.0178E-06	2.4648	5.2111E-05	2.1812
$\frac{1}{256}$	1.0446E-07	2.7298	1.2597E-06	2.4779	1.1417E-05	2.1904
$\frac{1}{512}$	1.5603E-08	2.7430	2.2491E-07	2.4857	2.4934E-06	2.1950
$\frac{1}{1024}$	2.3102E-09	2.7557	4.0018E-08	2.4906	5.4365E-07	2.1974

Table 4. Maximum errors and convergence rates for the right hand side (4.2) with $\lambda = 0.01$ and 0.99.

η	$\lambda = 0.01$	Rate	$\lambda = 0.99$	Rate
$\frac{1}{16}$	5.3448E-06	-	1.0113E-02	-
$\frac{1}{32}$	7.9246E-07	2.7537	2.6799E-03	1.9160
$\frac{1}{64}$	1.1367E-07	2.8015	6.8385E-04	1.9704
$\frac{1}{128}$	1.5986E-08	2.8230	1.7191E-04	1.9921
$\frac{1}{256}$	2.2170E-09	2.8501	4.2933E-05	2.0015
$\frac{1}{512}$	3.0407E-10	2.8661	1.0690E-05	2.0058
$\frac{1}{1024}$	3.9733E-11	2.9360	2.6578E-06	2.0079

Case (2). The exact solution is non-smooth. Consider the problem (2.1), and $m_0 = 0$ with an exact analytical solution:

$$m(t) = (1 - t)^{\lambda}. \tag{4.3}$$

It can be checked that the corresponding right hand side

$$g(t, m) = \Gamma(1 + \lambda) + t^{\lambda} - m(t). \tag{4.4}$$

The choices of the fractional order are now also taken as $\lambda = 0.3, 0.5$ and 0.7. Other settings are the same as previous case (1) in the Example 4.1.

The results are given in Table 5, from which we can obtain that when $0 < \lambda < 1$, the convergence order is close to λ. The reason lies that the exact solution function has singularity at $b = 1$ and is non-smooth on $[0, 1]$.

Table 5. Maximum errors and convergence rates for the right hand side (4.4) with $\lambda = 0.3$, 0.5 and 0.7.

η	$\lambda = 0.3$	Rate	$\lambda = 0.5$	Rate	$\lambda = 0.7$	Rate
$\frac{1}{16}$	9.0233E-02	-	1.2294E-01	0.1750	1.0782E-01	0.5092
$\frac{1}{32}$	9.2055E-02	-0.0288	9.9332E-02	0.3076	7.1437E-02	0.5940
$\frac{1}{64}$	9.1231E-02	0.0129	7.6437E-02	0.3779	4.5892E-02	0.6384
$\frac{1}{128}$	8.4961E-02	0.1027	5.7148E-02	0.4195	2.8976E-02	0.6633
$\frac{1}{256}$	7.6174E-02	0.1575	4.1959E-02	0.4457	1.8112E-02	0.6779
$\frac{1}{512}$	6.6599E-02	0.1937	3.0444E-02	0.4628	1.1253E-02	0.6865
$\frac{1}{1024}$	5.7213E-02	0.2191	2.1915E-02	0.4742	6.9671E-03	0.6917

Case (3). We consider an exact analytical solution and the corresponding right hand side are (4.3) and (4.4), respectively.

In this case, we choose graded mesh is $t_i = 1 - (1 - i/W)^\beta$ with a grading parameter $\beta > 1$ based on the idea of [27] for $i = 0, 1, 2, \ldots, W$. The choices of the fractional order are now also taken as $\lambda = 0.3, 0.5, 0.7$ and $W = 8, 16, 32, 64, 128, 256, 512$ and $\beta = 3$.

The results are given in Table 6, from which we can obtain that when $0 < \lambda < 1$, the convergence order is close to $min(\beta\lambda, 3 - \lambda)$ which is result of the Lemma 8 in [27].

Table 6. Maximum errors and convergence rates for the right hand side (4.4) with $\lambda = 0.3$, 0.5, 0.7 and $\beta = 3$.

W	$\lambda = 0.3$	Rate	$\lambda = 0.5$	Rate	$\lambda = 0.7$	Rate
8	2.1258E-01	-	1.1954E-01	-	8.9002E-02	-
16	1.1304E-01	0.9111	4.4942E-02	1.4114	2.2005E-02	2.0160
32	6.0616E-02	0.8990	1.6266E-02	1.4661	5.2992E-03	2.0539
64	3.2545E-02	0.8972	5.8004E-03	1.4876	1.2454E-03	2.0891
128	1.7466E-02	0.8978	2.0570E-03	1.4956	2.9101E-04	2.0974
256	9.3689E-03	0.8986	7.2805E-04	1.4984	6.7909E-05	2.0994
512	5.0234E-03	0.8992	2.5750E-04	1.4994	1.5841E-05	2.0998

Example 4.2. To demonstrate the validity of the algorithm and observe the approximation of the numerical solution to the exact solution, we solve the Eq (3.1) by means of the scheme (3.7), and the corresponding right-hand member as following

$$g(y,t) = \Big[\frac{24}{\Gamma(5-\lambda)}(1-t)^{4-\lambda} + 4\pi^2(1-t)^4\Big]\sin 2\pi y.$$

The analytic solution was verified as $m(y,t) = (1-t)^4 \sin 2\pi y$.

We take $a = 0, b = 1$, the interval of the space be [0, 1], and set $e_{\eta,\Delta y} = \max_{i,q} |m_{i,q} - m(y_i, t_q)|$. We start by looking at spatial accuracy. To prevent the effect of time-dependent error on spatial accuracy, we need to fix the time step to be small enough. Let $W = 10,000$ in Table 7, by comparing the error of λ, Δy under different values and the order of convergence. When $0 < \lambda < 1$, the spatial accuracy is second-order convergence, this result is accord with the theoretical analysis obtained in Theorem 3.3.

Table 7. Maximum error and spatial convergence rates with $\lambda = 0.4$, 0.6 and 0.8.

Δy	$\lambda = 0.4$	Rate	$\lambda = 0.6$	Rate	$\lambda = 0.8$	Rate
$\frac{1}{4}$	2.2127E-01	-	2.1751E-01	-	2.1283E-01	-
$\frac{1}{8}$	5.0604E-02	2.1285	4.9863E-02	2.1250	4.8940E-02	2.1206
$\frac{1}{16}$	1.2380E-02	2.0312	1.2205E-02	2.0305	1.1988E-02	2.0295
$\frac{1}{32}$	3.0784E-03	2.0078	3.0354E-03	2.0076	2.9817E-03	2.0073
$\frac{1}{64}$	7.6857E-04	2.0019	7.5786E-04	2.0019	7.4449E-04	2.0018
$\frac{1}{128}$	1.9208E-04	2.0005	1.8940E-04	2.0005	1.8606E-04	2.0005
$\frac{1}{256}$	4.8016E-05	2.0001	4.7347E-05	2.0001	4.6512E-05	2.0001
$\frac{1}{512}$	1.2004E-05	2.0000	1.1837E-05	2.0000	1.1628E-05	2.0000

Next, we check the time-dependent convergence rate. In Table 8, we also list the value of $e_{\eta,\Delta y}$ and the corresponding order, when λ, η takes a series of different values, where $\Delta y = O(\eta^{\frac{3-\lambda}{2}})$ is taken. When λ takes $0.4, 0.6$, and 0.8, the convergence order tends to $2.6, 2.4$, and 2.2, respectively, this can show that the convergence order in the time is $3 - \lambda$.

Table 8. Maximum error and time convergence rates with $\lambda = 0.4$, 0.6 and 0.8.

η	$\lambda = 0.4$	Rate	$\lambda = 0.6$	Rate	$\lambda = 0.8$	Rate
$\frac{1}{4}$	2.3612E-02	-	3.3816E-02	-	3.7639E-02	-
$\frac{1}{8}$	3.7052E-03	2.6719	5.9787E-03	2.4998	8.9774E-03	2.0679
$\frac{1}{16}$	6.1432E-04	2.5925	1.1055E-03	2.4351	2.0248E-03	2.1485
$\frac{1}{32}$	1.0199E-04	2.5905	2.1213E-04	2.3817	4.4334E-04	2.1913
$\frac{1}{64}$	1.7019E-05	2.5833	4.0298E-05	2.3962	9.5825E-05	2.2100
$\frac{1}{128}$	2.8131E-06	2.5969	7.6348E-06	2.4001	2.0869E-05	2.1990
$\frac{1}{256}$	4.6507E-07	2.5966	1.4495E-06	2.3971	4.5424E-06	2.1999
$\frac{1}{512}$	7.6865E-08	2.5971	2.7470E-07	2.3996	9.9072E-07	2.1969

Example 4.3. In order to further test the feasibility of the algorithm, we use the scheme (3.13) to solve the Eq (3.11), the maximum error is $e_{\eta,\Delta y,\Delta z} = \max\limits_{i,l,q} |m_{i,l,q} - m(y_i, z_l, t_q)|$, and take the right side of the equation as follows

$$g(y, z, t) = \Big[\frac{24}{\Gamma(5-\lambda)}(1-t)^{4-\lambda} + 8\pi^2(1-t)^4\Big] \sin 2\pi y \sin 2\pi z,$$

and its exact solution is $m(y, z, t) = (1-t)^4 \sin 2\pi y \sin 2\pi z$.

To study the convergence rate of the space, in Table 9, let $W = 6000$, the domain of space be $[0, 1] \times [0, 1]$. By observing the error and convergence order with λ and $\Delta y = \Delta z$ choosing different values, we find that its spatial accuracy is second-order convergence.

Table 9. Maximum error and spatial convergence rates with $\lambda = 0.3$, 0.5 and 0.8.

$\Delta y = \Delta z$	$\lambda = 0.3$	Rate	$\lambda = 0.5$	Rate	$\lambda = 0.8$	Rate
$\frac{1}{8}$	3.9558E-02	-	3.9298E-02	-	3.8777E-02	-
$\frac{1}{16}$	1.1142E-02	1.8279	1.1071E-02	1.8277	1.0928E-02	1.8271
$\frac{1}{32}$	2.9611E-03	1.9118	2.9424E-03	1.9117	2.9048E-03	1.9116
$\frac{1}{64}$	7.6352E-04	1.9554	7.5870E-04	1.9554	7.4902E-04	1.9553
$\frac{1}{128}$	1.9387E-04	1.9776	1.9265E-04	1.9776	1.9019E-04	1.9776
$\frac{1}{256}$	4.8847E-05	1.9888	4.8539E-05	1.9888	4.7920E-05	1.9887
$\frac{1}{512}$	1.2259E-05	1.9944	1.2182E-05	1.9944	1.2027E-05	1.9943

In Table 10, where $\Delta y = \Delta z = O(\eta^{\frac{3-\lambda}{2}})$ is taken, studying the convergence order in time. When λ takes $0.3, 0.5$, and 0.8, the convergence order tends to $2.7, 2.5$, and 2.2, respectively, the numerical results reveal the theoretical analysis is accord with the numerical results, the convergence order is $3 - \lambda$.

Table 10. Maximum error and time convergence rates with $\lambda = 0.3$, 0.5 and 0.8.

η	$\lambda = 0.3$	Rate	$\lambda = 0.5$	Rate	$\lambda = 0.8$	Rate
$\frac{1}{8}$	2.6952E-03	-	4.5754E-03	-	7.8296E-03	-
$\frac{1}{16}$	4.5757E-04	2.5583	7.9379E-04	2.5271	1.8635E-03	2.0709
$\frac{1}{32}$	7.0351E-05	2.7014	1.4346E-04	2.4681	4.1626E-04	2.1625
$\frac{1}{64}$	1.1002E-05	2.6768	2.5513E-05	2.4913	9.0761E-05	2.1973
$\frac{1}{128}$	1.6959E-06	2.6976	4.5175E-06	2.4976	1.9852E-05	2.1928

5. Conclusions

In this paper, the high-order numerical scheme of the right Caputo FODE is constructed, and the local truncation error and stability are analyzed in detail based on the idea and methods of [1]. Secondly, the high-order scheme is used to solve the time FPDEs. Thirdly, three numerical examples are used to verify the validity of our conclusions that the optimal convergence rate of time is $3 - \lambda, \lambda \in (0, 1)$ with uniform accuracy. Due to the length limitation of the paper, we only give the local error estimate for FPDEs. The convergence analysis can be directly obtained by using the method of stability of FODE in this paper and the ideas of [4]. In the future, We will study higher order numerical schemes with low smoothness based on the good idea of [26–28] by using the non-uniform mesh and we expect that the constructed efficient high-order scheme can be applied to the fractional order optimal control problem and topology optimization nonlocal problem of composites plate based on the idea of [31–33].

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Conflict of interest

The authors declare there is no conflicts of interest.

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Appendix

A. The proof of Lemma 2.2

Proof. 1) can also be checked by a direct calculation using the definition of d_s^q and summing them for all $s = q+1, \ldots, W$. For example, for $q = W-3$, by using (2.14) and (2.28), we have

$$\begin{aligned} d_W^{W-3} + d_{W-1}^{W-3} + d_{W-2}^{W-3} = & -[(\bar{H}_{W-3} + H_{W-1}) + (\bar{G}_{W-3} + H_{W-2} + G_{W-1}) \\ & +(\bar{F}_{W-3} + G_{W-2} + F_{W-1})]\alpha_0^{-1} = 1. \end{aligned}$$

The case of other values of q can be verified similarly.

2) Because

$$\begin{aligned} -H_{s-1} - G_s - F_{s+1} = & -\frac{3}{2}(2-\lambda)(s-q-1)^{1-\lambda} + \frac{1}{2}(2-\lambda)(s-q-2)^{1-\lambda} \\ & -\frac{1}{2}(2-\lambda)(s-q+1)^{1-\lambda} + \frac{3}{2}(2-\lambda)(s-q)^{1-\lambda} \\ & -3(s-q-1)^{2-\lambda} + (s-q-2)^{2-\lambda} + 3(s-q)^{2-\lambda} - (s-q+1)^{2-\lambda}, \end{aligned}$$

when $q \le W-3$, using Taylor's formula, we have

$$
\begin{aligned}
&-H_{s-1}-G_s-F_{s+1} \qquad \text{(A.1)}\\
&=\frac{1}{2}(2-\lambda)(s-q-2)^{1-\lambda}\Big[1-3(1+\frac{1}{s-q-2})^{1-\lambda}+3(1+\frac{2}{s-q-2})^{1-\lambda}\\
&\quad-(1+\frac{3}{s-q-2})^{1-\lambda}\Big]+(s-q-2)^{2-\theta}\Big[1-3(1+\frac{1}{s-q-2})^{2-\lambda}\\
&\quad+3(1+\frac{2}{s-q-2})^{2-\lambda}-(1+\frac{3}{s-q-2})^{2-\lambda}\Big]\\
&=\frac{1}{2}(2-\lambda)(s-q-2)^{1-\lambda}\Big\{1-3\Big[1+\frac{(1-\lambda)}{1!}(\frac{1}{s-q-2})+\dots\Big]\\
&\quad+3\Big[1+\frac{1-\lambda}{1!}(\frac{2}{s-q-2})+\frac{(1-\lambda)(-\lambda)}{2!}(\frac{2}{s-q-2})^2+\dots\Big]\\
&\quad-\Big[1+\frac{1-\lambda}{1!}(\frac{3}{s-q-2})+\frac{(1-\lambda)(-\lambda)}{2!}(\frac{3}{s-q-2})^2+\dots\Big]\Big\}\\
&\quad+(s-q-2)^{2-\lambda}\{1-3\Big[1+\frac{2-\lambda}{1!}(\frac{1}{s-q-2})\\
&\quad+\frac{(2-\lambda)(1-\lambda)}{2!}(\frac{1}{s-q-2})^2+\dots\Big].\\
&\quad+3\Big[1+\frac{2-\lambda}{1!}(\frac{2}{s-q-2})+\frac{(2-\lambda)(1-\lambda)}{2!}(\frac{2}{s-q-2})^2+\dots\Big]\\
&\quad-\Big[1+\frac{2-\lambda}{1!}(\frac{3}{s-q-2})+\frac{(2-\lambda)(1-\lambda)}{2!}(\frac{3}{s-q-2})^2+\dots\Big]\Big\}\\
&=\frac{1}{2}(2-\lambda)(1-\lambda)(-\lambda)(s-q-2)^{-\lambda-2}\sum_{x=0}^{+\infty}a_x\\
&\quad-(2-\lambda)(1-\lambda)(-\lambda)(s-q-2)^{-\lambda-1},
\end{aligned}
$$

where

$$a_x=\prod_{r=0}^{x}(-\lambda-1-r)(\frac{1}{s-q-2})^x\frac{-3x-18+(3x+24)2^{3+x}-(x+10)3^{3+x}}{(x+4)!},$$

it is calculated that when $r\geq 6$, a_x is a convergent alternating series with positive first term, so

$$0<\sum_{x=0}^{+\infty}a_x<4(\lambda+1),$$

when $q=W-1,W-2$, using a method similar to (A.1), it can be shown that, $d^q_{W-1}>0, d^q_{W-1}>0$,

In summary,

$$d^q_s=\frac{-H_{s-1}-G_s-F_{s+1}}{\alpha_0}>0, s=q+3,q+4,\dots,W.$$

3) Calculated from (2.31)

$$d^q_{q+1}=\frac{2^{-\lambda}(2\lambda-12)-3\lambda+12}{4-\lambda},$$

thus

$$d_{q+1}^{q} - \frac{4}{3} = \frac{20 - 5\lambda + (3\lambda - 18)2^{1-\lambda}}{3(4-\lambda)},$$

let $\varphi(\lambda) = 20 - 5\lambda + (3\lambda - 18)2^{1-\lambda}$, by calculation we get

$$\varphi'(\lambda) = -(3\lambda - 18)\cdot 2^{1-\lambda}\ln 2 - 5 + 3\cdot 2^{1-\lambda},$$
$$\varphi''(\lambda) = (3\lambda - 18)2^{1-\lambda}(\ln 2)^2 - 6\ln 2\cdot 2^{1-\lambda} < 0,$$

therefore $\varphi'(\lambda)$ is monotonically decreasing and $0 < \varphi'(1) < \varphi'(\lambda) < \varphi'(0)$, so $\varphi(\lambda)$ is monotonically increasing and $\varphi(0) < \varphi(\lambda) < \varphi(1) = 0$, that is, there is

$$d_{q+1}^{q} - \frac{4}{3} < 0.$$

Let $\tilde{\phi}(\lambda) = 2^{-\lambda}(2\lambda - 12) - 3\lambda + 12$, a tedious but routine calculation gives $\tilde{\phi}(\lambda) > 0$, Therefore $d_{q+1}^{q} > 0$, in short, $0 < d_{q+1}^{q} < \frac{4}{3}$.

4) Because

$$\begin{aligned} d_{q+2}^{q} &= -(H_{q+1} + G_{q+2} + F_{q+3})\alpha_0^{-1} \\ &= \frac{1}{4-\lambda}[2^{1-\lambda}(18 - 3\lambda) - 3^{1-\lambda}(8 - \lambda) + 3\lambda - 12] \doteq \frac{1}{4-\lambda}\phi_1(\lambda), \end{aligned}$$

where $4 - \lambda > 0, \lambda \in (0,1)$, so the sign of d_{q+2}^{q} is determined by $\phi_1(\lambda)$. It can be known by calculation, $\phi_1'(0) > 0, \phi_1'(1) < 0$, $\phi_1(\lambda)$ increase at first and then decrease, and $\phi_1(0) = 0, \phi_1(1) = -1 < 0$. Therefore, when $\lambda \in (0,1)$, d_{q+2}^{q} have positive and negative values. Therefore, The symbol for d_{q+2}^{q} can not be determined.

5) Because

$$\frac{1}{4}(d_{q+1}^{q})^2 + d_{q+2}^{q} = \frac{1}{4(4-\lambda)^2}\varphi_1(\lambda),$$

where $\varphi_1(\lambda) = (3\lambda - 12)(4 - \lambda) + 3(\lambda^2 - 10\lambda + 24)2^{2-\lambda} + 4(12\lambda - \lambda^2 - 32)3^{1-\lambda} + (6 - \lambda)^2 4^{1-\lambda}$. According to Lagrange's mean value theorem, $4^{1-\lambda} > \frac{4}{3+\lambda}\cdot 3^{1-\lambda}$, we get

$$\begin{aligned} \phi_1(\lambda) &> (3\lambda - 12)(4 - \lambda) + 3(\lambda^2 - 10\lambda + 24)2^{2-\lambda} + 4(12\lambda - \lambda^2 - 32)3^{1-\lambda} \\ &\quad +(6 - \lambda)^2\cdot\frac{4}{3+\lambda}\cdot 3^{1-\lambda} \\ &= c_1 + c_2\cdot 2^{1-\lambda} + c_3\cdot 2^{1-\lambda}(1 + \tfrac{1}{2})^{1-\lambda} \\ &= c_1 + 2^{1-\lambda}\{c_2 + c_3[1 + \frac{1-\lambda}{1!}\frac{1}{2} + \frac{(1-\lambda)(-\lambda)}{2!}(\frac{1}{2})^2] \\ &\quad +c_3\cdot[\frac{(1-\lambda)(-\lambda)(-\lambda-1)}{3!}(\frac{1}{2})^3 + \ldots]\} \\ &= c_1 + 2^{1-\lambda}\{c_2 + c_3c_4 + c_3(\lambda - \lambda^2)\sum_{n=0}^{+\infty} f_n\}, \end{aligned}$$

where

$$c_1 = (3\lambda - 12)(4-\lambda), \quad c_3 = \frac{4}{3+\lambda}[(6-\lambda)^2 + (3+\lambda)(12\lambda - \lambda^2 - 32)],$$
$$c_2 = 6(\lambda^2 - 10\lambda + 24), \quad c_4 = 1 + \frac{1-\lambda}{1!}\frac{1}{2} + \frac{(1-\lambda)(-\lambda)}{2!}(\frac{1}{2})^2,$$
$$f_n = \Pi_{r=0}^{n}(-\lambda - 1 - r)\frac{-1}{(n+3)!}(\frac{1}{2})^{n+3},$$
$$f_0 = \frac{(\lambda+1)}{3}(\frac{1}{2})^4 > 0, \quad \left|\frac{f_{n+1}}{f_n}\right| = \frac{\lambda + 2 + n}{2n+8} < 1,$$

so $\sum_{n=0}^{+\infty} f_n$ is a convergent alternating series, that is, $0 < \sum_{n=0}^{+\infty} f_n < f_0$. Thus

$$\begin{aligned}\phi_1(\lambda) &> c_1 + 2^{1-\lambda}\{c_2 + c_3c_4 + c_3(\lambda - \lambda^2)\sum_{n=0}^{+\infty} f_n\} \geq c_1 + 2^{1-\lambda}\{c_2 + c_3c_4 + c_3(\lambda-\lambda^2)f_0\} \\ &= c_1 + 2^{1-\lambda}\{c_2 + c_3[c_4 + (\lambda - \lambda^2)\frac{\lambda+1}{3!}(\frac{1}{2})^3]\} \doteq c_1 + 2^{1-\lambda}\{c_2 + c_3c_5\} \doteq \varphi_2(\lambda),\end{aligned}$$

where

$$c_5 = c_4 + (1-\lambda)\lambda\frac{\lambda+1}{3!}(\frac{1}{2})^3 = 1 + \frac{6\lambda^2 - \lambda^3 - 29\lambda + 24}{48}.$$

It can be obtained by calculation

$$\begin{aligned}\varphi_2(\lambda) = c_1 + 2^{1-\lambda}\{c_2 + c_3c_5\} &= \frac{1}{36+12\lambda}\{-36(\lambda^3 - 5\lambda^2) + 288(\lambda - 6) + 2^{1-\lambda}(\lambda^6 - 16\lambda^5 + 97\lambda^4 \\ &\quad -278\lambda^3) + 2^{3-\lambda}(22\lambda^2 + 183\lambda + 216)\} \doteq \frac{1}{36+12\lambda}\varphi_3(\lambda).\end{aligned}$$

Because of $\lambda \in (0,1)$, so $\lambda^3 < \lambda^2, \lambda^6 > 0$, then

$$\begin{aligned}\varphi_3(\lambda) &> \{-36(\lambda^2 - 5\lambda^2) + 288(\lambda-6) + 2^{1-\lambda}(0 - 16\lambda^5 + 97\lambda^4 - 278\lambda^3) + 2^{3-\lambda}(22\lambda^2 + 183\lambda + 216)\} \\ &= 144(\lambda^2 + 2\lambda - 12) + 2^{1-\lambda}[-16\lambda^5 + 97\lambda^4 - 278\lambda^3 + 4(22\lambda^2 + 183\lambda + 216)] \\ &= 144(\lambda^2 + 2\lambda - 12) + 2^{1-\lambda}\cdot\varphi_4(\lambda) \doteq \bar{\varphi}_3(\lambda).\end{aligned}$$

By Calculating, we get $\bar{\varphi}_3''(\lambda) < 0$, so $\bar{\varphi}_3'(\lambda)$ is monotonically decreasing, so $\bar{\varphi}_3'(1) < \bar{\varphi}_3'(\lambda) < \bar{\varphi}_3'(0)$, a similar method can be used to find the first derivative of $\bar{\varphi}_3'(\lambda)$, because $\bar{\varphi}_3'(0) > 0, \bar{\varphi}_3'(1) < 0$, thus $\bar{\varphi}_3'(\lambda)$ changes from positive to negative, so $\bar{\varphi}_3(\lambda)$ first increases and then decreases, and $\bar{\varphi}_3(0) = 0, \bar{\varphi}_3(1) = 191 > 0$, so $\bar{\varphi}_3(\lambda) > 0, \varphi_1(\lambda) > \varphi_2(\lambda) > \bar{\varphi}_3(\lambda) > 0$. To sum up,

$$\frac{1}{4}(d_{q+1}^q)^2 + d_{q+2}^q > 0, \lambda \in (0,1).$$

The proof of Lemma 2.2 is completed.

B. The proof of Lemma 2.4

Proof. Bringing (2.27) into scheme (2.13), when $q = W-1, W-2$,

$$\begin{aligned}{}_\eta D_b^\lambda m_{W-1} &= \frac{\eta^{-\lambda}}{\Gamma(3-\lambda)}(\bar{E}_0 m_{W-2} + \bar{E}_1 m_{W-1} + \bar{E}_2 m_W) = -\theta m_{W-1}, \\ {}_\eta D_b^\lambda m_{W-2} &= \frac{\eta^{-\lambda}}{\Gamma(3-\lambda)}(E_0 m_{W-2} + E_1 m_{W-1} + E_2 m_W) = -\theta m_{W-2},\end{aligned} \tag{B.1}$$

and $\beta_0 = \eta^\lambda\theta$ the relevant value of "E" is detailed in (2.14). It can be obtained by calculation,

$$m_{W-1} = \frac{Y_1}{Y_2}m_W, \quad m_{W-2} = \frac{Y_3}{Y_2}m_W,$$

where

$$\begin{aligned}
&Y_1 = I - \Gamma(3-\lambda)\beta_0\bar{E}_2, \quad Y_3 = I - \Gamma(3-\lambda)\beta_0 E_2;\\
&Y_2 = \beta_0^2 + \Gamma(3-\lambda)\left(\bar{E}_1 + E_0\right)\beta_0 + I, \quad I = \frac{4-2\lambda}{2^\lambda[\Gamma(3-\lambda)]^2};\\
&\bar{E}_1 + E_0 = 2 - 2\lambda + \frac{\lambda+2}{2^\lambda}.
\end{aligned}$$

So, when $q = W-1$,

$$\begin{aligned}
&\bar{m}_{W-1}^2 + \alpha_0^{-1}\eta_0\theta m_{W-1}^2 = (m_{W-1} - \tau m_W)^2 + \theta\eta_0\alpha_0^{-1}m_{W-1}^2\\
&= \left(\frac{Y_1}{Y_2}m_W - \tau m_W\right)^2 + \alpha_0^{-1}\eta_0\theta\left(\frac{Y_1}{Y_2}m_W\right)^2\\
&= \frac{a_0 + a_1\Gamma(3-\lambda)\beta_0 + a_2\Gamma(3-\lambda)^2\beta_0^2 + a_3\Gamma(3-\lambda)^3\beta_0^3 + a_4\Gamma(3-\lambda)^4\beta_0^4}{e_0 + e_1\Gamma(3-\lambda)\beta_0 + e_2\Gamma(3-\lambda)^2\beta_0^2 + e_3\Gamma(3-\lambda)^3\beta_0^3 + e_4\Gamma(3-\lambda)^4\beta_0^4}m_W^2,
\end{aligned}$$

After detailed calculation, it can be obtained,

$$\begin{aligned}
a_0 &= 4(\lambda^4 - 12\lambda^3 + 52\lambda^2 - 96\lambda + 64) + 2^{4-2\lambda}\left(\lambda^4 - 16\lambda^3 + 88\lambda^2 - 192\lambda + 144\right)\\
&\quad - 2^{4-\lambda}\left(\lambda^4 - 14\lambda^3 + 68\lambda^2 - 136\lambda + 96\right) > 0,\\
a_1 &= 2^{2+\lambda}\left(3\lambda^4 - 32\lambda^3 + 116\lambda^2 - 160\lambda + 64\right) - 2^{4-2\lambda}\left(\lambda^4 - 12\lambda^3 + 32\lambda^2 + 48\lambda - 144\right)\\
&\quad + 2^{4-\lambda}\left(4\lambda^4 - 50\lambda^3 + 188\lambda^2 - 184\lambda - 48\right) - 4\left(13\lambda^4 - 144\lambda^3 + 508\lambda^2 - 576\lambda + 64\right) > 0,\\
a_2 &= \left(9\lambda^4 - 84\lambda^3 + 244\lambda^2 - 224\lambda + 64\right)4^\lambda - 2^{1+\lambda}\left(21\lambda^4 - 170\lambda^3 + 296\lambda^2 + 608\lambda - 896\right)\\
&\quad + \left(\lambda^4 - 8\lambda^3 - 8\lambda^2 + 96\lambda + 144\right)4^{1-\lambda} - 2^{2-\lambda}\left(7\lambda^4 - 58\lambda^3 - 4\lambda^2 + 648\lambda - 288\right)\\
&\quad + 61\lambda^4 - 4\left(137\lambda^3 - 221\lambda^2 - 488\lambda + 480\right) > 0,\\
a_3 &= -4^{1+\lambda}\left(9\lambda^3 - 69\lambda^2 + 152\lambda - 80\right) + 2^{3-\lambda}\left(\lambda^3 - 10\lambda^2 + 12\lambda + 72\right)\\
&\quad + 2^{1+\lambda}\left(27\lambda^3 - 250\lambda^2 + 592\lambda - 96\right) - 4\left(10\lambda^3 - 100\lambda^2 + 216\lambda + 144\right) > 0,\\
a_4 &= 9\left(\lambda^2 - 8\lambda + 16\right)4^\lambda - 6\left(\lambda^2 - 10\lambda + 24\right)2^{1+\lambda} + 4\left(\lambda^2 - 12\lambda + 36\right) > 0,
\end{aligned}$$

and

$$\begin{aligned}e_0-a_0&=12\left(\lambda^4-12\lambda^3+52\lambda^2-96\lambda+64\right)-4^{2-\lambda}\left(\lambda^4-16\lambda^3+88\lambda^2-192\lambda+144\right)\\&\quad+2^{4-\lambda}\left(\lambda^4-14\lambda^3+68\lambda^2-136\lambda+96\right)>0,\\e_1-a_1&=2^{2+\lambda}\left(5\lambda^4-56\lambda^3+220\lambda^2-352\lambda+192\right)+4^{1-\lambda}\left(\lambda^4-12\lambda^3+32\lambda^2+48\lambda-144\right)\\&\quad-2^{4-\lambda}\left(4\lambda^4-50\lambda^3+188\lambda^2-184\lambda-48\right)+4\left(9\lambda^4-112\lambda^3+460\lambda^2-704\lambda+320\right)>0,\\e_2-a_2&=4^{\lambda}\left(7\lambda^4-76\lambda^3+284\lambda^2-416\lambda+192\right)+2^{1+\lambda}\left(13\lambda^4-122\lambda^3+328\lambda^2-208\lambda+64\right)\\&\quad-4^{1-\lambda}\left(\lambda^4-8\lambda^3-8\lambda^2+96\lambda+144\right)+2^{2-\lambda}\left(7\lambda^4-58\lambda^3-4\lambda^2+648\lambda-288\right)\\&\quad-57\lambda^4+4\left(133\lambda^3-233\lambda^2-456\lambda+544\right)>0,\\e_3-a_3&=4^{1+\lambda}\left(5\lambda^3-33\lambda^2+56\lambda-16\right)-2^{3-\lambda}\left(\lambda^3-10\lambda^2+12\lambda+72\right)\\&\quad-2^{1+\lambda}\left(23\lambda^3-226\lambda^2+592\lambda-224\right)+4\left(10\lambda^3-100\lambda^2+216\lambda+144\right)>0,\\e_4-a_4&=-5\left(\lambda^2-8\lambda+16\right)4^{\lambda}+3\left(\lambda^2-10\lambda+24\right)2^{2+\lambda}-4\left(\lambda^2-12\lambda+36\right)>0.\end{aligned}$$

With the above calculation, for all $\lambda\in(0,1)$, when $q=W-1$, we have

$$a_q\geq 0, e_q\geq 0, a_q\leq e_q, q=1,2,3,4.$$

Therefore, we obtain

$$\bar{m}_{W-1}^2+\alpha_0^{-1}\eta_0\theta m_{W-1}^2\leq m_W^2. \tag{B.2}$$

When $q=W-2$, it can be proved that

$$\bar{m}_{W-2}^2+\alpha_0^{-1}\eta_0\theta m_{W-2}^2\leq m_W^2 \tag{B.3}$$

holds by a similar method as $q=W-1$.

The following proves that when $q=W-3$, bring in (2.32) there is

$$m_{W-3}+\alpha_0^{-1}\theta\eta_0 m_{W-3}=d_{W-2}^{W-3}m_{W-2}+d_{W-1}^{W-3}m_{W-1}+d_W^{W-3}m_W.$$

Due to $\tau=\frac{1}{2}[3-\frac{2^{1-\lambda}(6-\lambda)}{4-\lambda}]$, $d_{W-2}^{W-3}=3-\frac{3^{1-\lambda}(\lambda+4)}{4-\lambda}$, and $\tau\neq\frac{1}{2}d_{W-2}^{W-3}$, according to (2.36) we get

$$\bar{m}_{W-3}+\alpha_0^{-1}\eta_0\theta m_{W-3}=\bar{d}_{W-2}^{W-3}\bar{m}_{W-2}+\bar{d}_{W-1}^{W-3}\bar{m}_{W-1}+\bar{d}_W^{W-3}m_W, \tag{B.4}$$

where $\bar{d}_{W-2}^{W-3}=d_{W-2}^{W-3}-\tau, \bar{d}_{W-1}^{W-3}=\tau\bar{d}_{W-2}^{W-3}+d_{W-1}^{W-3}, \bar{d}_W^{W-3}=\tau\bar{d}_{W-1}^{W-3}+d_W^{W-3}$.

Next, we will prove $\bar{d}_{W-2}^{W-3}\geq 0, \bar{d}_{W-1}^{W-3}\geq 0, \bar{d}_W^{W-3}\geq 0$. By carefully calculation,

$$\bar{d}_{W-2}^{W-3}=d_{W-2}^{W-3}-\tau=\frac{6-\frac{3}{2}\lambda-3^{-\lambda}(12+3\lambda)+2^{-\lambda}(6-\lambda)}{4-\lambda}>0.$$

Because $d_{W-1}^{W-3}=\dfrac{1}{4-\lambda}3^{1-\lambda}(4\lambda+4)-3$, then

$$\bar{d}_{W-1}^{W-3}=-\frac{3}{4}+(\frac{4\lambda}{8-2\lambda}-\frac{1}{2})\cdot 3^{1-\lambda}+\frac{24+2\lambda-\lambda^2}{(4-\lambda)^2}\cdot 3^{1-\lambda}\cdot 2^{-\lambda}-(1+\frac{2}{4-\lambda})^2 4^{-\lambda}.$$

By a tedious but routine calculation gives $\bar{d}_{W-1}^{W-3} > 0$.
Because $d_W^{W-3} = \frac{2}{4-\lambda}(\frac{4-\lambda}{2} - \frac{\lambda}{2} \cdot 3^{2-\lambda}) > 0$, so $\bar{d}_W^{W-3} = \tau\bar{d}_{W-1}^{W-3} + d_W^{W-3} > 0$.
Next calculate $\bar{d}_{W-2}^{W-3} + \bar{d}_{W-1}^{W-3} + \bar{d}_W^{W-3} \leq 1$. By carefully calculate, we have

$$
\begin{aligned}
&\bar{d}_{W-2}^{W-3} + \bar{d}_{W-1}^{W-3} + \bar{d}_W^{W-3} = d_{W-2}^{W-3} - \tau + \tau\bar{d}_{W-2}^{W-3} + d_{W-1}^{W-3} + \tau\bar{d}_{W-1}^{W-3} + d_W^{W-3} \\
&= d_{W-2}^{W-3} - \tau + \tau\left(d_{W-2}^{W-3} - \tau\right) + d_{W-1}^{W-3} + \tau^2(d_{W-2}^{W-3} - \tau) + \tau d_{W-1}^{W-3} + d_W^{W-3} \\
&= (d_{W-2}^{W-3} - \tau)\left(1 + \tau + \tau^2\right) + d_{W-1}^{W-3}(1 + \tau) + d_W^{W-3} \\
&= (d_{W-2}^{W-3} - \tau)\frac{1-\tau^3}{1-\tau} + d_{W-1}^{W-3}\frac{1-\tau^2}{1-\tau} + \frac{1-\tau}{1-\tau}d_W^{W-3} \doteq Q_{W-3}.
\end{aligned}
$$

According to 1) in Lemma 2.2, we get

$$
\begin{aligned}
&(1-\tau)Q_{W-3} = (d_{W-2}^{W-3} - \tau)(1-\tau^3) + d_{W-1}^{W-3}(1-\tau^2) + d_W^{W-3}(1-\tau) \\
&= (d_{W-2}^{W-3} + d_{W-1}^{W-3} + d_W^{W-3} - \tau) - (d_{W-2}^{W-3} - \tau)\tau^3 - \tau^2 d_{W-1}^{W-3} - \tau d_{W-3}^{W} \\
&\leq (1-\tau) - \tau^2[\tau(d_{W-2}^{W-3} - \tau) + d_{W-1}^{W-3}] = (1-\tau) - \tau^2\bar{d}_{W-1}^{W-3} \leq 1 - \tau.
\end{aligned}
$$

In summary,

$$
\bar{d}_{W-2}^{W-3} + \bar{d}_{W-1}^{W-3} + \bar{d}_W^{W-3} \leq 1. \tag{B.5}
$$

When $q = W-3$, multiply $2\bar{m}_{W-3}$ on both sides of (B.4) to have

$$
\begin{aligned}
&2\bar{m}_{W-3}(\bar{m}_{W-3} + \alpha_0^{-1}\eta_0\theta m_{W-3}) \\
&= 2\bar{m}_{W-3}(\bar{d}_{W-2}^{W-3}\bar{m}_{W-2} + \bar{d}_{W-1}^{W-3}\bar{m}_{W-1} + \bar{d}_W^{W-3}m_W) \\
&\leq \bar{d}_{W-2}^{W-3}\bar{m}_{W-2}^2 + \bar{d}_{W-1}^{W-3}\bar{m}_{W-1}^2 + \bar{d}_W^{W-3}m_W^2 + (\bar{d}_{W-2}^{W-3} + \bar{d}_{W-1}^{W-3} + \bar{d}_W^{W-3})\bar{m}_{W-3}^2.
\end{aligned}
$$

For the left side of the above equation

$$
\begin{aligned}
&2\bar{m}_{W-3}(\bar{m}_{W-3} + \alpha_0^{-1}\eta_0\theta m_{W-3}) = 2\bar{m}_{W-3}^2 + 2\bar{m}_{W-3}\alpha_0^{-1}\eta_0\theta m_{W-3} \\
&= 2\bar{m}_{W-3}^2 + \alpha_0^{-1}\eta_0\theta(m_{W-3}^2 + \bar{m}_{W-3}^2 - \tau^2 m_{W-2}^2).
\end{aligned}
$$

By using (B.5), we have

$$
\begin{aligned}
&2\bar{m}_{W-3}^2 + \alpha_0^{-1}\eta_0\theta(m_{W-3}^2 + \bar{m}_{W-3}^2 - \tau^2 m_{W-2}^2) \\
&\leq \bar{d}_{W-2}^{W-3}\bar{m}_{W-2}^2 + \bar{d}_{W-1}^{W-3}\bar{m}_{W-1}^2 + \bar{d}_W^{W-3}m_W^2 + \bar{m}_{W-3}^2.
\end{aligned} \tag{B.6}
$$

By carefully calculation, we have

$$
\bar{d}_{W-2}^{W-3} - \tau = -\frac{3^{1-\lambda}}{4-\lambda}[4 + \lambda - (4 - \frac{2}{3}\lambda)(\frac{3}{2})^\lambda] < 0, \tag{B.7}
$$

It is easy to check as follows

$$
\tau + \bar{d}_{W-1}^{W-3} + \bar{d}_W^{W-3} \leq 1. \tag{B.8}
$$

From conditions (B.2), (B.3), (B.8) and 3) in Lemma 2.3, (B.6) becomes following as

$$\begin{aligned}\bar{m}_{W-3}^2+\alpha_0^{-1}\eta_0\theta m_{W-3}^2 &\le \tau\left(\bar{m}_{W-2}^2+\alpha_0^{-1}\eta_0\theta m_{W-2}^2\right)+\bar{d}_{W-1}^{W-3}\left(\bar{m}_{W-1}^2+\alpha_0^{-1}\eta_0\theta m_{W-1}^2\right)+\bar{d}_W^{W-3}m_W^2 \\ &\le\left(\tau+\bar{d}_{W-1}^{W-3}+\bar{d}_W^{W-3}\right)m_W^2\le m_W^2.\end{aligned} \tag{B.9}$$

Therefore, when $q = W-3$, (2.37) holds.
When $q = W-4$, there is

$$\bar{m}_{W-4}^2+\alpha_0^{-1}\eta_0\theta m_{W-4}=\tau\bar{m}_{W-3}+\bar{d}_{W-2}^{W-4}\bar{m}_{W-2}+\bar{d}_{W-1}^{W-4}\bar{m}_{W-1}+\bar{d}_W^{W-4}m_W. \tag{B.10}$$

According to Lemma 2.3, we have $\tau+\sum_{s=W-2}^{W-1}\bar{d}_s^q+\bar{d}_W^q\le 1, \bar{d}_s^{W-4}>0, s=W-2,W-1,W$, multiply both sides of (B.10) by $2\bar{m}_{W-4}$ at the same time, we have

$$\begin{aligned}&2\bar{m}_{W-4}^2+\alpha_0^{-1}\eta_0\theta(m_{W-4}^2+\bar{m}_{W-4}^2-\tau^2m_{W-3}^2)\\ \le&\tau\bar{m}_{W-3}^2+\bar{d}_{W-2}^{W-4}\bar{m}_{W-2}^2+\bar{d}_{W-1}^{W-4}\bar{m}_{W-1}^2+\bar{d}_W^{W-4}m_W^2+(\tau+\bar{d}_{W-2}^{W-4}+\bar{d}_{W-1}^{W-4}+\bar{d}_W^{W-4})\bar{m}_{W-4}^2,\end{aligned}$$

Due to $0<\tau<\frac{2}{3}$, using (B.2), (B.3) and (B.9), then

$$\begin{aligned}\bar{m}_{W-4}^2+\alpha_0^{-1}\eta_0\theta m_{W-4} &\le \tau(\bar{m}_{W-3}^2+\alpha_0^{-1}\eta_0\theta m_{W-3}^2)+\bar{d}_{W-2}^{W-4}(\bar{m}_{W-2}^2+\eta_0\alpha_0^{-1}\theta m_{W-2}^2)\\ &\quad+\bar{d}_{W-1}^{W-4}\left(\bar{m}_{W-1}^2+\eta_0\alpha_0^{-1}\theta m_{W-1}^2\right)+\bar{d}_W^{W-4}m_W^2\\ &\le\left(\tau+\bar{d}_{W-2}^{W-4}+\bar{d}_{W-1}^{W-4}+\bar{d}_W^{W-4}\right)m_W^2\le m_W^2.\end{aligned}$$

When $q\le W-5$, using a similar method above, multiply $2\bar{m}_q$ on both sides of (2.36), and after sorting, we can get,

$$2\bar{m}_q^2+\alpha_0^{-1}\eta_0\theta(m_q^2+\bar{m}_q^2-\tau^2m_{q+1}^2)\le\tau\bar{m}_{q+1}^2+\sum_{s=q+2}^{W-1}\bar{d}_s^q\bar{m}_s^2+\bar{d}_W^qm_W^2+(\tau+\sum_{s=q+2}^{W-1}\bar{d}_s^q+\bar{d}_W^q)\bar{m}_q^2,$$

by mathematical induction, Lemma 2.3, we obtain,

$$\begin{aligned}\bar{m}_q^2+\alpha_0^{-1}\eta_0\theta m_q^2 &\le \tau(\bar{m}_{q+1}^2+\alpha_0^{-1}\eta_0\theta m_{q+1}^2)+\sum_{s=q+2}^{W-1}\bar{d}_s^q(\bar{m}_s^2+\alpha_0^{-1}\eta_0\theta m_s^2)+\bar{d}_W^qm_W^2\\ &\le(\tau+\sum_{s=q+2}^{W-1}\bar{d}_s^q+\bar{d}_W^q)m_W^2\le m_W^2.\end{aligned}$$

In summary, the proof of Lemma 2.4 is completed.

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