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Research article

Numerical analysis of a fourth-order linearized difference method for nonlinear time-space fractional Ginzburg-Landau equation

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Abstract: An efficient difference method is constructed for solving one-dimensional nonlinear time-space fractional Ginzburg-Landau equation. The discrete method is developed by adopting the $L2\text{-}1_\sigma$ scheme to handle Caputo fractional derivative, while a fourth-order difference method is invoked for space discretization. The well-posedness and a priori bound of the numerical solution are rigorously studied, and we prove that the difference scheme is unconditionally convergent in pointwise sense with the rate of $O(\tau^2 + h^4)$, where τ and h are the time and space steps respectively. In addition, the proposed method is extended to solve two-dimensional problem, and corresponding theoretical analysis is established. Several numerical tests are also provided to validate our theoretical analysis.

Keywords: Ginzburg-Landau equation; fractional derivative; $L2\text{-}1_\sigma$ scheme; difference method; convergence

1. Introduction

Progress in the past decades has indicated that fractional partial differential equations can be used to model many physical phenomena with non-locality and long-range interaction [1–7]. The fractional Ginzburg-Landau equation is regarded as an important model to describe many dynamical processes [8], and various theoretical researches have been carried out, such as global well-posedness [9, 10] and long-time asymptotic behavior [11]. Besides, based on the Banach fixed point theorem, Shen et al. [12] analyzed the well-posedness of mild solution of time-space fractional Ginzburg-Landau equation driven by Gaussian noise. Xu et al. [13] studied the well-posedness of time-space fractional Ginzburg-Landau equation with fractional Brownian motion.

In this paper, we investigate finite difference method for the nonlinear time-space fractional

Ginzburg-Landau equation (TSFGLE) with the extended Dirichlet boundary condition [14, 15]

$$\frac{\partial^\alpha u}{\partial t^\alpha} - (\nu + i\xi)\frac{\partial^\beta u}{\partial |x|^\beta} + (\kappa + i\gamma)|u|^2 u - \zeta u = 0, \quad t \in (0, T], \tag{1.1}$$

$$u(x, 0) = \psi(x), \quad x \in \Omega = (a, b), \tag{1.2}$$

$$u(x, t) = 0, \quad x \in \mathbb{R} \setminus \Omega, \ t \in [0, T], \tag{1.3}$$

where $i = \sqrt{-1}$, parameters $0 < \alpha \leq 1$, $1 < \beta \leq 2$, $\nu > 0$, $\kappa > 0$, ξ, γ and ζ are known real numbers, and $\psi(x)$ denotes a sufficiently smooth initial value function. The Caputo fractional derivative [3] is given as

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} := \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\eta)^\alpha} \frac{\partial u(x,\eta)}{\partial \eta} d\eta, & 0 < \alpha < 1, \\ \partial_t u(x, t), & \alpha = 1. \end{cases} \tag{1.4}$$

The Riesz fractional operator $\frac{\partial^\beta u}{\partial |x|^\beta}$ is defined by

$$\frac{\partial^\beta u(x, t)}{\partial |x|^\beta} := -\frac{1}{2\cos(\pi\beta/2)}\left({}_aD_x^\beta u(x, t) + {}_xD_b^\beta u(x, t)\right), \quad 1 < \beta \leq 2, \tag{1.5}$$

where the left and right Riemann-Liouville fractional derivatives are given as

$${}_aD_x^\beta u(x, t) := \frac{1}{\Gamma(2-\beta)} \frac{\partial^2}{\partial x^2} \int_a^x \frac{u(\eta, t) d\eta}{(x - \eta)^{\beta - 1}}, \quad {}_xD_b^\beta u(x, t) =: \frac{1}{\Gamma(2-\beta)} \frac{\partial^2}{\partial x^2} \int_x^b \frac{u(\eta, t) d\eta}{(\eta - x)^{\beta - 1}}.$$

Since the explicit expression of solution of TSFGLE is difficult to obtain, it is very urgent to develop numerical methods for quantitative analysis. We note that when $\alpha = 1$, model (1.1) becomes the space fractional Ginzburg-Landau equation, many numerical studies have been conducted, such as finite difference method [16–22], spectral method [23, 24] and finite element method [25]. When $\alpha = 1$ and $\beta = 2$, Eq (1.1) reduces to the standard nonlinear complex Ginzburg-Landau equation [26, 27]. In particular, when $\nu = 0$ and $\kappa = 0$, model (1.1) collapses to the nonlinear fractional Schrödinger equation, and many efforts have been devoted to solving this equation, please see Refs. [28–31] for more details.

However, there is limited work on numerical research for the TSFGLE (1.1)–(1.3). Recently, Zaky et al. [32] constructed a nonlinear method for solving TSFGLE based on $L2$-1_σ scheme and Legendre spectral discretization, and the error estimate in L^2-norm is established. Since this method is fully implicit, a nonlinear equation needs to be solved iteratively at each time layer. Subsequently, the authors [33] further studied the Alikhanov Legendre-Galerkin spectral method for the coupled TSFGLEs. To the best of our knowledge, there is not linearized numerical method for model (1.1), and the optimal convergence analysis in the pointwise sense has not been considered. The main novelty of this work is that a high-order linearized difference method is proposed for solving the TSFGLE based on $L2$-1_σ scheme for time discretization and compact difference method for space discretization, and optimal error estimate in the pointwise sense is established rigorously without imposing any restriction on the time-space grid ratio in one-dimensional case. Theoretical results show that our method has second-order accuracy in time and fourth-order accuracy in space. The discrete scheme is efficient in the sense that only a linear system needs to be solved at each time step.

In addition, the proposed method is extended to solve the two-dimensional problem, and the corresponding convergence theorem is also analyzed.

The remainder of this article is organized as follows. A linearized high-order difference scheme is constructed in Section 2. The unique solvability, boundedness property and error estimate of the proposed scheme are rigorously analyzed in Section 3. The numerical scheme and corresponding convergence analysis for the two-dimensional problem are studied in Section 4. Some numerical tests are presented in Section 5. Finally, we give some conclusions in Section 6.

2. The fully discrete linearized difference scheme

This section is devoted to constructing a linearized difference scheme for solving TSFGLE (1.1)–(1.3). We divide the rectangle $[a,b]\times[0,T]$ into a uniform partition by the mesh $\Omega=\Omega_h\times\Omega_\tau$, where $\Omega_h=\{x_j \mid x_j=a+jh,\ h=\frac{b-a}{M}, 0\le j\le M\}$ and $\Omega_\tau=\{t_n \mid t_n=n\tau, 0\le n\le N, \tau=\frac{T}{N}\}$.

2.1. $L2$-1_σ formula for the Caputo fractional derivative

Many efforts have been devoted to designing difference approximations of Caputo fractional derivative, including the well-known $L1$ formula [34], $L1$-2 formula [35] and $L2$-1_σ method [36]. Moreover, Li' group constructed and analyzed a series of higher-order numerical methods [37, 38]. In order to construct a high-order linearized difference scheme for TSFGLE and establish convergence theorem, in this work the $L2$-1_σ formula will be invoked to discretize the Caputo fractional derivative.

Lemma 2.1. *([36]) For $0<\alpha\le 1, \sigma=1-\frac{\alpha}{2}$ and $t_{n+\sigma}=(n+\sigma)\tau$, assume $v\in C^3[0,T]$, then it holds that*

$$\left|{}_0^CD_t^\alpha v(t_{n+\sigma})-\tau^{-\alpha}\sum_{s=0}^{n}c_s^{(n,\sigma)}\Big(v(t_{n+1-s})-v(t_{n-s})\Big)\right|=O(\tau^{3-\alpha}),\tag{2.1}$$

where $c_0^{(0,\sigma)}=a_0$ for $n=0$, and

$$c_s^{(n,\sigma)}=\begin{cases}a_0+b_1, & s=0,\\ a_s+b_{s+1}-b_s, & 1\le s\le n-1,\\ a_n-b_n, & s=n,\end{cases}$$

for $n\ge 1$, in which $a_0=\frac{\sigma^{1-\alpha}}{\Gamma(2-\alpha)}$, $a_s=\frac{1}{\Gamma(2-\alpha)}[(s+\sigma)^{1-\alpha}-(s-1+\sigma)^{1-\alpha}]$ for $s\ge 1$, and $b_s=\frac{1}{\Gamma(3-\alpha)}[(s+\sigma)^{2-\alpha}-(s-1+\sigma)^{2-\alpha}]-\frac{1}{2\Gamma(2-\alpha)}[(s+\sigma)^{1-\alpha}+(s-1+\sigma)^{1-\alpha}]$ for $s\ge 1$.

2.2. Compact difference method for spatial discretization

In general, high-order difference scheme performs better than low-order method in terms of computational accuracy, and high-order difference approximations for Riesz fractional derivative have been analyzed, see literatures [39–43] for more details. In this paper, we use a fourth-order difference scheme to approximate the Riesz fractional derivative. For a fixed h, denote

$$\delta_h^\beta v(x_j):=h^{-\beta}\sum_{s=1}^{M-1}g_{j-s}^{(\beta)}v(x_s),\ \ (1<\beta\le 2),\tag{2.2}$$

where

$$g_s^{(\beta)} = \frac{(-1)^s\Gamma(\beta+1)}{\Gamma(\beta/2-s+1)(\Gamma(\beta/2+s+1)}.$$

According to Ref. [44], the fractional central difference scheme (2.2) is second-order convergent for discretizing Riesz fractional derivative as $h \to 0$, that is

$$\delta_h^\beta v(x_j) := -\frac{\partial^\beta v(x_j)}{\partial|x|^\beta} + O(h^2), \quad (1 < \beta \le 2). \tag{2.3}$$

Based on the second-order fractional central difference scheme (2.2), a fourth-order compact difference scheme was constructed in Ref. [39].

Lemma 2.2. *[39] Assume $v \in L_1(R)$ and $v \in \Psi^{4+\beta}(R) := \left\{v \,\middle|\, \int_{-\infty}^{+\infty}(1+|\eta|)^{4+\beta}|\widehat{v}(\eta)|d\eta < +\infty\right\}$. Denote*

$$\Delta_h^\beta v(x_j) = \frac{\beta}{24}\delta_h^\beta v(x_j-h) - (1+\frac{\beta}{12})\delta_h^\beta v(x_j) + \frac{\beta}{24}\delta_h^\beta v(x_j+h) = \sum_{s=1}^{M-1}\breve{g}_{j-s}^{(\beta)} v(x_s), \tag{2.4}$$

where $\breve{g}_s^{(\beta)} = \frac{\beta}{24}g_{s-1}^{(\beta)} - (1+\frac{\beta}{12})g_s^{(\beta)} + \frac{\beta}{24}g_{s+1}^{(\beta)}$, then for $1 \le j \le M-1$, we have

$$\Delta_h^\beta v(x_j) = -\frac{\partial^\beta v(x_j)}{\partial|x|^\beta} + O(h^4).$$

2.3. The linearized discrete scheme

For simplicity, we denote u_j^n as the numerical approximation of the exact solution $u(x_j, t_n)$, $\nabla_t^\alpha u_j^{n+\sigma} = \tau^{-\alpha}\sum_{s=0}^{n} c_s^{(n,\sigma)}(u_j^{n+1-s} - u_j^{n-s})$, $\bar{u}_j^{n+\sigma} = \sigma u_j^{n+1} + (1-\sigma)u_j^n$ and $\tilde{u}_j^{n+\sigma} = (1+\sigma)u_j^n - \sigma u_j^{n-1}$. Considering model (1.1) at the grid point $(x_j, t_{n+\sigma})$, applying the $L2$-1_σ formula for the approximation of Caputo fractional derivative and the fourth-order difference scheme (2.4) for discretizing Riesz fractional derivative, a linearized high-order discrete method for $1 \le n \le N-1$ is constructed as

$$\nabla_t^\alpha u_j^{n+\sigma} + (\nu + i\xi)\Delta_h^\beta \bar{u}_j^{n+\sigma} + (\kappa + i\gamma)|\tilde{u}_j^{n+\sigma}|^2\bar{u}_j^{n+\sigma} - \zeta\bar{u}_j^{n+\sigma} = 0, \ 1 \le j \le M-1. \tag{2.5}$$

Here we mention that an extrapolation technique with second-order accuracy has been used to handle the nonlinear term in (1.1). Besides, the first level approximation u_j^1 is chosen as $u_j^1 = \hat{u}_j^{m_\alpha}$, which is computed by the following iterative method

$$\frac{c_0^{(0,\sigma)}(\hat{u}_j^m - u_j^0)}{\tau^\alpha} + (\nu + i\xi)\Delta_h^\beta\hat{u}_j^{\sigma_m} + (\kappa + i\gamma)|\hat{u}_j^{\sigma_{m-1}}|^2\hat{u}_j^{\sigma_m} - \zeta\hat{u}_j^{\sigma_m} = 0, \quad m = 1, 2, \cdots, m_\alpha, \tag{2.6}$$

where $\hat{u}_j^{\sigma_m} = \sigma\hat{u}_j^m + (1-\sigma)u_j^0$ and $\hat{u}_j^{\sigma_0} = u_j^0$, $m_\alpha = \lceil\frac{2}{\alpha}\rceil$. The initial value u_j^0 is taken as

$$u_j^0 = \psi(x_j), \quad 1 \le j \le M-1. \tag{2.7}$$

2.4. Some useful lemmas

Let $\mathcal{V}_h = \{v \mid v = (v_1, v_2, \cdots, v_{M-1})\}$. For any $v, w \in \mathcal{V}_h$, the discrete inner product and corresponding l^2-norm as well as the maximum norm are defined respectively as

$$(v, w) := h \sum_{j=1}^{M-1} v_j \overline{w}_j, \quad \|v\| := \sqrt{(v, v)}, \quad \|v\|_\infty := \max_{1\le i\le M-1} |v_i|, \tag{2.8}$$

where $\overline{w}$ denotes the conjugate of w.

For a given constant $\epsilon \in [0, 1]$, we give the definition of the fractional Sobolev semi-norm $|v|_{H^\epsilon}$ and the norm $\|v\|_{H^\epsilon}$ as

$$|v|^2_{H^\epsilon} := \int_{-\pi/h}^{\pi/h} |\eta|^{2\epsilon} |\widehat{v}(\eta)|^2 d\eta, \ \ \|v\|^2_{H^\epsilon} := \int_{-\pi/h}^{\pi/h} (1 + |\eta|^{2\epsilon}) |\widehat{v}(\eta)|^2 d\eta,$$

where $\widehat{v}(\eta)$ denotes the semi-discrete Fourier transform of v. By Parseval's theorem, $(v, w) = \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \widehat{v}(\eta)\overline{\widehat{w}(\eta)} d\eta$, it holds that $\|v\|^2_{H^\epsilon} = \|v\|^2 + |v|^2_{H^\epsilon}$.

Lemma 2.3. *[45] (Discrete Sobolev embedding inequality). For $u \in H^\epsilon$ with $\frac{1}{2} < \epsilon \le 1$, it holds that*

$$\|v\|_\infty \le C_\epsilon \|v\|_{H^\epsilon}, \tag{2.9}$$

where $C_\epsilon = C(\epsilon)$ is a positive constant independent of h.

Lemma 2.4. *[46] For any $1 < \beta \le 2$, it holds that*

$$(\Delta_h^\beta v, v) = \|\Lambda^\beta v\|^2, \tag{2.10}$$

where Λ^β denotes a linear operator. Furthermore, there exists a constant $C_\beta = \frac{\beta}{6} + 1$ such that

$$\Big(\frac{2}{\pi}\Big)^\beta |v|^2_{H^{\beta/2}} \le (\Delta_h^\beta v, v) \le C_\beta |v|^2_{H^{\beta/2}}. \tag{2.11}$$

Noting that conclusions $\|v\| \le C(\delta_h^\beta v, v)$ in Ref. [47] and $(\delta_h^\beta v, v) \le |v|^2_{H^{\beta/2}}$ in Ref. [48], combined with Lemma 2.4, we have the following inequality.

Lemma 2.5. *For $1 < \beta \le 2$, there exists a positive constant $\hat{C}_\beta$ independent of h satisfying that*

$$\|v\| \le \hat{C}_\beta \|\Lambda^\beta v\|. \tag{2.12}$$

3. Theoretical analysis

Denote $u^n = (u_1^n, u_2^n, \cdots, u_{M-1}^n)^T$, $\bar{u}^{n+\sigma} = (\bar{u}_1^{n+\sigma}, \bar{u}_2^{n+\sigma}, \cdots, \bar{u}_{M-1}^{n+\sigma})^T$, $u^n \cdot v^n = (u_1^n v_1^n, u_2^n v_2^n, \cdots, u_{M-1}^n v_{M-1}^n)^T$, and other notations $\tilde{u}^{n+\sigma}$, $\hat{u}^m$ and $\hat{u}^{\sigma_m}$ can be defined similarly.

3.1. Unique solvability

Lemma 3.1. *([36]) The coefficients $c_s^{(n,\sigma)}$ in L2-1_σ approximation of Lemma 2.1 satisfy*

$$c_0^{(n,\sigma)} > c_1^{(n,\sigma)} > \cdots > c_s^{(n,\sigma)} > c_{s+1}^{(n,\sigma)} \cdots > c_n^{(n,\sigma)} > 0,$$
$$(2\sigma - 1)c_0^{(n,\sigma)} - \sigma c_1^{(n,\sigma)} > 0.$$

Theorem 3.1. *The difference method* (2.5) *and* (2.6) *with initial condition* (2.7) *is uniquely solvable.*

Proof. It is equivalent to proving that the resulting homogeneous system of the proposed scheme has only a trivial solution at each time layer. The homogeneous equation of the iterative scheme (2.6) can be expressed as

$$c_0^{(0,\sigma)}\hat{u}_j^m + (\nu + i\xi)\sigma\tau^\alpha \Delta_h^\beta \hat{u}_j^m + (\kappa + i\gamma)\sigma\tau^\alpha |\hat{u}_j^{\sigma_{m-1}}|^2 \hat{u}_j^m - \zeta\sigma\tau^\alpha \hat{u}_j^m = 0, \quad m = 1, 2, \cdots, m_\alpha. \tag{3.1}$$

Taking the inner product of both sides with $\hat{u}^m$, then considering the real part gives

$$(c_0^{(0,\sigma)} - \zeta\sigma\tau^\alpha)\|\hat{u}^m\|^2 + \nu\sigma\tau^\alpha \|\Lambda^\beta \hat{u}^m\|^2 + \kappa\sigma\tau^\alpha \|\hat{u}^{\sigma_{m-1}} \cdot \hat{u}^m\|^2 = 0, \quad m = 1, 2, \cdots, m_\alpha, \tag{3.2}$$

where Lemma 2.4 has been used in deriving (3.2). According to Lemma 3.1 we know that $c_0^{(0,\sigma)} > 0$. If $\zeta \le 0$, it is obvious that $\|\hat{u}^m\| = 0$; if $\zeta > 0$, we can also deduce that $\|\hat{u}^m\| = 0$ when $\tau \le \sqrt[\alpha]{\frac{c_0^{(0,\sigma)}}{\zeta\sigma}}$, and it further implies that $\hat{u}_j^m = 0,\ j = 1, 2, \cdots, M - 1$. Thus the iterative process (2.6) is uniquely solvable when $\tau \le \sqrt[\alpha]{\frac{c_0^{(0,\sigma)}}{\zeta\sigma}}$.

Now we turn to study the homogeneous system of the discrete scheme (2.5), i.e.,

$$c_0^{(n,\sigma)} u_j^{n+1} + (\nu + i\xi)\sigma\tau^\alpha \Delta_h^\beta u_j^{n+1} + (\kappa + i\gamma)\sigma\tau^\alpha |\tilde{u}_j^{n+\sigma}|^2 u_j^{n+1} - \zeta\sigma\tau^\alpha u_j^{n+1} = 0,\ 1 \le n \le N - 1. \tag{3.3}$$

Taking the discrete inner product of (3.3) with u^{n+1}, then considering the real part yields

$$(c_0^{(n,\sigma)} - \zeta\sigma\tau^\alpha)\|u^{n+1}\|^2 + \nu\sigma\tau^\alpha \|\Lambda^\beta u^{n+1}\|^2 + \kappa\sigma\tau^\alpha \|\tilde{u}^{n+\sigma} \cdot u^{n+1}\|^2 = 0,\ \ 1 \le n \le N - 1. \tag{3.4}$$

Similarly, we can conclude from (3.4) that there exists a unique solution for difference scheme (2.5) when $\tau \le \sqrt[\alpha]{\frac{c_0^{(n,\sigma)}}{\zeta\sigma}}$. Therefore, we have completed the proof of Theorem 3.1.

3.2. Boundedness

Lemma 3.2. *[36, 49] For a given sequence of complex functions $\{\omega^n\}_{n=0}^N$, we have*

$$2Re\Big(\nabla_t^\alpha \omega^{n+\sigma}, \bar{\omega}^{n+\sigma}\Big) > \frac{1}{\tau^\alpha} \sum_{s=1}^{n+1} c_{n+1-s}^{(n,\sigma)} (\|\omega^s\|^2 - \|\omega^{s-1}\|^2). \tag{3.5}$$

Lemma 3.3. *([50]) (Discrete fractional Grönwall inequality) Suppose that $\{w^n\}_{n=0}^N$ is a nonnegative sequence and $p^n > 0$, and for $0 \le n \le N - 1$ the following inequality is satisfied*

$$\frac{1}{\tau^\alpha} \sum_{s=1}^{n+1} c_{n+1-s}^{(n,\sigma)} \Big((w^s)^2 - (w^{s-1})^2\Big) \le \theta(\bar{w}^{n+\sigma})^2 + \bar{w}^{n+\sigma} p^n,$$

then it holds that when $\tau \leq \frac{1}{\sqrt[\alpha]{11\Gamma(2-\alpha)\theta/2}}$,

$$w^n \leq 2E_\alpha(\frac{11\theta t_n^\alpha}{2})\Big(w^0 + \frac{11\Gamma(1-\alpha)}{4}\max_{1\leq s\leq n}\{t_s^\alpha p^s\}\Big),$$

here $E_\alpha(\cdot)$ *is the Mittag-Leffler function.*

Theorem 3.2. *The solutions of the fully discrete scheme* (2.5)–(2.7) *are bounded in discrete* L^2*-norm, and it holds for* $0 \leq n \leq N$ *that*

$$\|u^n\| \leq \begin{cases} \|u^0\|, & if\ \zeta \leq 0, \\ Q\|u^0\|, & if\ \zeta > 0, \end{cases} \tag{3.6}$$

where $Q = 2E_\alpha(11\zeta T^\alpha)$.

Proof. Computing the inner product of (2.5) with $\bar{u}^{n+\sigma}$, then considering the real part gives

$$Re(\nabla_t^\alpha u^{n+\sigma}, \bar{u}^{n+\sigma}) + \nu\|\Lambda^\beta \bar{u}^{n+\sigma}\|^2 + \kappa\|\tilde{u}^{n+\sigma}\cdot\bar{u}^{n+\sigma}\|^2 - \zeta\|\bar{u}^{n+\sigma}\|^2 = 0,\ 1 \leq n \leq N-1. \tag{3.7}$$

Taking the discrete inner product of (2.6) for $m = m_\alpha$ with $\hat{u}^{\sigma_m}$, then taking the real part of the resulting system and noticing that $u^1 = \hat{u}^{m_\alpha}$, thus we have

$$Re(\nabla_t^\alpha u^\sigma, \bar{u}^\sigma) + \nu\|\Lambda^\beta\bar{u}^\sigma\|^2 + \kappa\|\hat{u}^{\sigma_{m-1}}\cdot\bar{u}^\sigma\|^2 - \zeta\|\bar{u}^\sigma\|^2 = 0. \tag{3.8}$$

Due to $\nu > 0,\ \kappa > 0$, we can deduce from (3.7) and (3.8) that

$$Re(\nabla_t^\alpha u^{n+\sigma}, \bar{u}^{n+\sigma}) \leq \zeta\cdot\|\bar{u}^{n+\sigma}\|^2, \quad 0 \leq n \leq N-1. \tag{3.9}$$

If $\zeta \leq 0$, by using Lemma 3.2, we can further deduce from (3.9) that

$$\sum_{s=1}^{n+1} c_{n+1-s}^{(n,\sigma)}(\|u^s\|^2 - \|u^{s-1}\|^2) \leq 0. \tag{3.10}$$

Here we use mathematical induction to prove the conclusion $\|u^n\| \leq \|u^0\|$. For $n = 0$, it obvious follows from (3.10) that $\|u^1\| \leq \|u^0\|$. Assuming that

$$\|u^s\| \leq \|u^0\| \text{ for } s \leq n, \tag{3.11}$$

then we need to prove that $\|u^{n+1}\| \leq \|u^0\|$. With the help of Lemma 3.1 and (3.11), we can derive from (3.10) that

$$\begin{aligned} c_0^{(n,\sigma)}\|u^{n+1}\|^2 &\leq \sum_{s=1}^{n}(c_{n-s}^{(n,\sigma)} - c_{n+1-s}^{(n,\sigma)})\|u^s\|^2 + c_n^{(n,\sigma)}\|u^0\|^2 \\ &\leq \Big(\sum_{s=1}^{n}(c_{n-s}^{(n,\sigma)} - c_{n+1-s}^{(n,\sigma)}) + c_n^{(n,\sigma)}\Big)\|u^0\|^2 \\ &\leq c_0^{(n,\sigma)}\|u^0\|^2, \end{aligned} \tag{3.12}$$

which further indicates that $\|u^{n+1}\| \leq \|u^0\|$. Therefore, we can derive that

$$\|u^n\| \leq \|u^0\|, \ \ 0 \leq n \leq N, \ \ \text{for } \ \zeta \leq 0. \tag{3.13}$$

On the other hand, if $\zeta > 0$, it holds from Lemma 3.2 and (3.9) that

$$\frac{1}{\tau^\alpha}\sum_{s=1}^{n+1} c_{n+1-s}^{(n,\sigma)}(\|u^s\|^2 - \|u^{s-1}\|^2) \le 2\zeta \cdot \|\bar{u}^{n+\sigma}\|^2 \le 2\zeta(\sigma\|u^{n+1}\| + (1-\sigma)\|u^n\|)^2. \tag{3.14}$$

By using Lemma 3.3, we can deduce from (3.14) that

$$\|u^{n+1}\| \le 2E_\alpha(11\zeta t_{n+1}^\alpha)\|u^0\|, \ \ 0 \le n \le N, \ \ \text{for} \ \ \zeta > 0. \tag{3.15}$$

In view of (3.13) and (3.15), we have completed the proof of this theorem.

3.3. Convergence analysis

Let $U_j^n = u(x_j, t_n)$ $(n = 1, 2, \cdots, N)$, $\hat{U}_j^m = u(x_j, t_1)$ and $\hat{U}_j^{\sigma_m} = \sigma\hat{U}_j^m + (1-\sigma)U_j^0$ for $m = 1, 2, \cdots, m_\alpha$. To discuss the local truncation error of the proposed method, we first deduce from (2.6) that

$$\frac{c_0^{(0,\sigma)}(\hat{U}_j^m - U_j^0)}{\tau^\alpha} + (\nu + i\xi)\Delta_h^\beta\hat{U}_j^{\sigma_m} + (\kappa + i\gamma)|\hat{U}_j^{\sigma_{m-1}}|^2\hat{U}_j^{\sigma_m} - \zeta\hat{U}_j^{\sigma_m} = \hat{F}_j^m, \quad m = 1, 2, \cdots, m_\alpha, \tag{3.16}$$

where $\hat{U}_j^{\sigma_0} = U_j^0$ and

$$\begin{aligned}\hat{F}_j^m =& \Big(\frac{c_0^{(0,\sigma)}(\hat{U}_j^m - U_j^0)}{\tau^\alpha} - {}_0D_t^\alpha u(x_j, t_\sigma)\Big) + (\nu + i\xi)\Big(\Delta_h^\beta\hat{U}_j^{\sigma_m} + \frac{\partial^\beta u(x_j, t_\sigma)}{\partial|x|^\beta}\Big)\\ &+ (\kappa + i\gamma)\Big(|\hat{U}_j^{\sigma_{m-1}}|^2\hat{U}_j^{\sigma_m} - |u(x_j, t_\sigma)|^2 u(x_j, t_\sigma)\Big) - \zeta\Big(\hat{U}_j^{\sigma_m} - u(x_j, t_\sigma)\Big).\end{aligned} \tag{3.17}$$

With the help of Taylor's formula, we obtain

$$\hat{U}_j^{\sigma_m} = u(x_j, t_\sigma) + O(\tau^2), \qquad \frac{\partial^\beta\hat{U}_j^{\sigma_m}}{\partial|x|^\beta} = \frac{\partial^\beta u(x_j, t_\sigma)}{\partial|x|^\beta} + O(\tau^2),$$
$$|\hat{U}_j^{\sigma_0}|^2\hat{U}_j^{\sigma_m} = |u(x_j, t_\sigma)|^2 u(x_j, t_\sigma) + O(\tau), \ \ |\hat{U}_j^{\sigma_{m-1}}|^2\hat{U}_j^{\sigma_m} = |u(x_j, t_\sigma)|^2 u(x_j, t_\sigma) + O(\tau^2) \text{ for } m > 1.$$

Combined with Lemmas 2.1 and 2.2, we can further deduce that

$$\tau\|\hat{F}^1\| + \|\hat{F}^m\| \le C_F(\tau^2 + h^4), \tag{3.18}$$

where C_F is a positive constant independent of τ and h.

Now we start to discuss the error estimate of the discrete scheme (2.5)–(2.7). First of all, we denote that

$$\hat{e}_j^m = \hat{U}_j^m - \hat{u}_j^m, \quad \hat{e}^m = (\hat{e}_1^m, \hat{e}_2^m, \cdots, \hat{e}_{M-1}^m), \ \ 1 \le m \le m_\alpha, \tag{3.19}$$
$$e_j^n = U_j^n - u_j^n, \quad e^n = (e_1^n, e_2^n, \cdots, e_{M-1}^n), \quad 1 \le n \le N. \tag{3.20}$$

Theorem 3.3. *Let $\{\hat{u}^s\}_{s=1}^{m_\alpha}$ be solutions of the discrete scheme* (2.6) *, then there is a positive τ^* such that when $\tau \le \tau^*$, we have the error estimates*

$$\|e^1\| \le C_F(\tau^2 + h^4), \quad \|e^1\|_\infty \le \tilde{C}_F(\tau^2 + h^4), \tag{3.21}$$

where C_F and $\tilde{C}_F$ are positive constants which are independent of τ and h.

Proof. Firstly, subtracting (2.6) from (3.16) for $m = 1$, then we have the error equation

$$\frac{c_0^{(0,\sigma)}\hat{e}_j^1}{\tau^\alpha} + (\nu + i\xi)\sigma\Delta_h^\beta\hat{e}_j^1 + (\kappa + i\gamma)\hat{G}_j^1 - \zeta\sigma\hat{e}_j^1 = \hat{F}_j^1, \quad 1 \le j \le M-1, \tag{3.22}$$

where

$$\hat{G}_j^1 = |U_j^0|^2\hat{U}_j^{\sigma_1} - |u_j^0|^2\hat{u}_j^{\sigma_1}. \tag{3.23}$$

Here mathematical induction is used to prove that

$$\|e^k\| \le C_F(\tau^{\frac{m\alpha}{2}} + h^4), \quad \|e^k\|_\infty \le \tilde{C}_F(\tau^{\frac{m\alpha}{2}} + h^4), \quad k = 1, 2, \cdots, m_\alpha. \tag{3.24}$$

To this end, computing the inner product of (3.22) with $\hat{e}^1$, and it follows that

$$\frac{c_0^{(0,\sigma)}\|\hat{e}^1\|^2}{\tau^\alpha} + (\nu + i\xi)\sigma\|\Lambda^\beta\hat{e}^1\|^2 + (\kappa + i\gamma)(\hat{G}^1, \hat{e}^1) - \zeta\sigma\|\hat{e}^1\|^2 = (\hat{F}^1, \hat{e}^1). \tag{3.25}$$

Considering the real part of (3.25), and combined with $\nu > 0$, we have

$$\frac{c_0^{(0,\sigma)}\|\hat{e}^1\|^2}{\tau^\alpha} \le -Re\{(\kappa + i\gamma)(\hat{G}^1, \hat{e}^1)\} + \zeta\sigma\|\hat{e}^1\|^2 + (\hat{F}^1, \hat{e}^1), \tag{3.26}$$

this together with the Cauchy-Schwarz inequality further implies that

$$c_0^{(0,\sigma)}\|\hat{e}^1\| \le \tau^\alpha(\sqrt{\kappa^2 + \gamma^2}\|\hat{G}^1\| + |\zeta|\sigma\|\hat{e}^1\| + \|\hat{F}^1\|). \tag{3.27}$$

Set $C_u = \max_{0\le t\le T}\|u(\cdot, t)\|_\infty$, $C_G = 2(C_u + 1)^2$. From the definition of $\|\hat{G}^1\|$, we obtain

$$\|\hat{G}^1\| = \Big\||U^0|^2\hat{U}^1 - |u^0|^2\hat{u}^1\Big\| \le \sigma\|U^0\|_\infty^2\|\hat{e}^1\| \le \sigma C_G\|\hat{e}^1\|, \tag{3.28}$$

Combining (3.27) and (3.28), we have

$$\Big(c_0^{(0,\sigma)} - \sigma(C_G\sqrt{\kappa^2 + \gamma^2} + |\zeta|)\tau^\alpha\Big)\|\hat{e}^1\| \le \tau^\alpha\|\hat{F}^1\|. \tag{3.29}$$

When $\tau \le \tau_1 = \min\Big\{\sqrt[\alpha]{\frac{c_0^{(0,\sigma)}}{2\sigma(C_G\sqrt{\kappa^2+\gamma^2}+|\zeta|)}},\ \sqrt[\alpha/2]{\frac{c_0^{(0,\sigma)}}{2}},\ 1\Big\}$, it follows from (3.18) and (3.29) that

$$\|\hat{e}^1\| \le \frac{2\tau^{\alpha/2}}{c_0^{(0,\sigma)}}\tau^{\alpha/2}\|\hat{F}^1\| \le C_F(\tau^{1+\frac{\alpha}{2}} + h^4). \tag{3.30}$$

Taking the discrete inner product of (3.22) with $\frac{1}{\nu - i\xi}\hat{e}^1$, and considering the real part of resulting equation yields

$$\frac{\nu c_0^{(0,\sigma)}\|\hat{e}^1\|^2}{(\nu^2 + \xi^2)\tau^\alpha} + \sigma\|\Lambda^\beta\hat{e}^1\|^2 = -Re\Big\{\frac{\kappa + i\gamma}{\nu + i\xi}(\hat{G}^1, \hat{e}^1)\Big\} + \frac{\nu\zeta\sigma}{\nu^2 + \xi^2}\|\hat{e}^1\|^2 + Re\Big\{\frac{1}{\nu + i\xi}(\hat{F}^1, \hat{e}^1)\Big\}. \tag{3.31}$$

Furthermore, by using the Cauchy-Schwarz inequality and the Young inequality, we have

$$-Re\Big\{\frac{\kappa + i\gamma}{\nu + i\xi}(\hat{G}^1, \hat{e}^1)\Big\} \le \frac{\sqrt{\kappa^2 + \gamma^2}}{\sqrt{\nu^2 + \xi^2}}\|\hat{G}^1\| \cdot \|\hat{e}^1\| \le \frac{(\kappa^2 + \gamma^2)\tau^\alpha}{2\nu c_0^{(0,\sigma)}}\|\hat{G}^1\|^2 + \frac{\nu c_0^{(0,\sigma)}}{2(\nu^2 + \xi^2)\tau^\alpha}\|\hat{e}^1\|^2, \tag{3.32}$$

$$Re\Big\{\frac{1}{\nu+i\xi}(\hat{F}^1,\hat{e}^1)\Big\}\leq\frac{1}{\sqrt{\nu^2+\xi^2}}\|\hat{F}^1\|\cdot\|\hat{e}^1\|\leq\frac{\tau^\alpha}{2\nu c_0^{(0,\sigma)}}\|\hat{F}^1\|^2+\frac{\nu c_0^{(0,\sigma)}}{2(\nu^2+\xi^2)\tau^\alpha}\|\hat{e}^1\|^2. \tag{3.33}$$

It holds from (3.18), (3.28) and (3.30)–(3.33) that

$$\|\Lambda^\beta\hat{e}^1\|^2\leq\frac{(\kappa^2+\gamma^2)\tau^\alpha}{2\sigma\nu c_0^{(0,\sigma)}}\|\hat{G}^1\|^2+\frac{\nu|\zeta|}{\nu^2+\xi^2}\|\hat{e}^1\|^2+\frac{\tau^\alpha}{2\sigma\nu c_0^{(0,\sigma)}}\|\hat{F}^1\|^2\leq C_1C_F^2(\tau^{1+\frac{\alpha}{2}}+h^4)^2, \tag{3.34}$$

where $C_1=\frac{\sqrt{(\kappa^2+\gamma^2)}C_G}{\nu}+\frac{\nu|\zeta|}{\nu^2+\xi^2}+\frac{1}{2\sigma\nu c_0^{(0,\sigma)}}$. By using Lemmas 2.3 and 2.4, it follows from (3.30) and (3.34) that

$$\|\hat{e}^1\|_\infty\leq C_{\beta/2}\|\hat{e}^1\|_{H^{\beta/2}}\leq\tilde{C}_F(\tau^{1+\frac{\alpha}{2}}+h^4), \tag{3.35}$$

where $\tilde{C}_F=C_{\beta/2}C_F\sqrt{1+(\frac{\pi}{2})^2C_1}$.

Therefore, it follows from (3.30) and (3.35) that the conclusion (3.24) is true for $m=1$. Assume that the conclusion (3.24) is valid for $1\leq k\leq m-1$, that is

$$\|\hat{e}^k\|\leq C_F(\tau^{1+\frac{k\alpha}{2}}+h^4),\ \ \|\hat{e}^k\|_\infty\leq\tilde{C}_F(\tau^{1+\frac{k\alpha}{2}}+h^4),\ \ k=1,2,\cdots,m-1. \tag{3.36}$$

We start to prove that the conclusion (3.24) is still valid for $k=m$. Subtracting (2.6) from (3.16) yields the error equation

$$\frac{c_0^{(0,\sigma)}\hat{e}_j^m}{\tau^\alpha}+(\nu+i\xi)\sigma\Delta_h^\beta\hat{e}_j^m+(\kappa+i\gamma)\hat{G}_j^m-\zeta\sigma\hat{e}_j^m=\hat{F}_j^m, \tag{3.37}$$

where

$$\hat{G}_j^m=|\hat{U}_j^{\sigma_{m-1}}|^2\hat{U}_j^{\sigma_m}-|\hat{u}_j^{\sigma_{m-1}}|^2\hat{u}_j^{\sigma_m}=|\hat{U}_j^{\sigma_{m-1}}|^2\hat{e}_j^{\sigma_m}+(U_j^{\sigma_{m-1}}\overline{\hat{e}^{\sigma_{m-1}}}+\hat{e}_j^{\sigma_{m-1}}\overline{u_j^{\sigma_{m-1}}})\hat{U}_j^{\sigma_m}. \tag{3.38}$$

It follows from the induction assumption (3.36) that when τ and h are taken to be sufficiently small such that $\tau^{1+\frac{k\alpha}{2}}+h^4\leq1/\tilde{C}_F$, we have

$$\|\hat{u}^k\|_\infty\leq\|U^1\|_\infty+\|\hat{e}^k\|_\infty\leq\max_{0\leq t\leq T}\|u(\cdot,t)\|_\infty+1=C_u+1,\ \ 1\leq k\leq m-1. \tag{3.39}$$

Noticing the definition of $\hat{G}_j^m$, we have $\|\hat{G}^m\|\leq\sigma C_G(\|\hat{e}^m\|+\|\hat{e}^{m-1}\|)$. Taking the inner product of (3.37) with $\hat{e}^m$ gives

$$\frac{c_0^{(0,\sigma)}\|\hat{e}^m\|^2}{\tau^\alpha}+(\nu+i\xi)\sigma\|\Lambda^\beta\hat{e}^m\|^2+(\kappa+i\gamma)(\hat{G}^m,\hat{e}^m)-\zeta\sigma\|\hat{e}^m\|^2=(\hat{F}^m,\hat{e}^m). \tag{3.40}$$

Similarly, considering the real part of (3.40) as well as the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}c_0^{(0,\sigma)}\|\hat{e}^m\|&\leq\tau^\alpha(\sqrt{\kappa^2+\gamma^2}\|\hat{G}^m\|+|\zeta|\sigma\cdot\|\hat{e}^m\|+\|\hat{F}^m\|)\\&\leq\tau^\alpha(\sigma C_G\sqrt{\kappa^2+\gamma^2}+|\zeta|\sigma)\|\hat{e}^m\|+\tau^\alpha\sigma C_G\sqrt{\kappa^2+\gamma^2}\|\hat{e}^{m-1}\|+\tau^\alpha\|\hat{F}^m\|.\end{aligned} \tag{3.41}$$

Therefore when $\tau\leq\tau_2=\Big\{\sqrt[\alpha]{\frac{c_0^{(0,\sigma)}}{2\sigma(C_G\sqrt{\kappa^2+\gamma^2}+|\zeta|)}},\ \sqrt[\alpha/2]{\frac{c_0^{(0,\sigma)}}{2(\sigma C_G\sqrt{\kappa^2+\gamma^2}+1)}}\Big\}$, we have

$$\|\hat{e}^m\|\leq\frac{2\tau^{\alpha/2}(\sigma C_G\sqrt{\kappa^2+\gamma^2}\tau^{\alpha/2}\|\hat{e}^{m-1}\|+\tau^{\alpha/2}\|\hat{F}^m\|)}{c_0^{(0,\sigma)}}$$

$$\leq \frac{2\tau^{\alpha/2}(\sigma C_G\sqrt{\kappa^2+\gamma^2}+1)C_F(\tau^{1+\frac{m\alpha}{2}}+h^4)}{c_0^{(0,\sigma)}}$$
$$\leq C_F(\tau^{1+\frac{m\alpha}{2}}+h^4). \tag{3.42}$$

Taking the discrete inner product of (3.37) with $\frac{1}{\nu-i\xi}\hat{e}^m$, then considering the real part of the resulting equation gives

$$\frac{\nu c_0^{(0,\sigma)}\|\hat{e}^m\|^2}{(\nu^2+\xi^2)\tau^\alpha}+\sigma\|\Lambda^\beta\hat{e}^m\|^2 \leq -Re\Big\{\frac{\kappa+i\gamma}{\nu+i\xi}(\hat{G}^m,\hat{e}^m)\Big\}+\frac{\nu|\zeta|\sigma}{\nu^2+\xi^2}\|\hat{e}^m\|^2+Re\Big\{\frac{1}{\nu+i\xi}(\hat{F}^m,\hat{e}^m)\Big\}. \tag{3.43}$$

For the first and the third items in the right hand of (3.43), the following estimates hold

$$-Re\Big\{\frac{\kappa+i\gamma}{\nu+i\xi}(\hat{G}^m,\hat{e}^m)\Big\}\leq\frac{\sqrt{\kappa^2+\gamma^2}}{\sqrt{\nu^2+\xi^2}}\|\hat{G}^m\|\cdot\|\hat{e}^m\|\leq\frac{(\kappa^2+\gamma^2)\tau^\alpha}{2\nu c_0^{(0,\sigma)}}\|\hat{G}^m\|^2+\frac{\nu c_0^{(0,\sigma)}}{2(\nu^2+\xi^2)\tau^\alpha}\|\hat{e}^m\|^2, \tag{3.44}$$

$$Re\Big\{\frac{1}{\nu+i\xi}(\hat{F}^m,\hat{e}^m)\Big\}\leq\frac{1}{\sqrt{\nu^2+\xi^2}}\|\hat{F}^m\|\cdot\|\hat{e}^m\|\leq\frac{\tau^\alpha}{2\nu c_0^{(0,\sigma)}}\|\hat{F}^m\|^2+\frac{\nu c_0^{(0,\sigma)}}{2(\nu^2+\xi^2)\tau^\alpha}\|\hat{e}^m\|^2. \tag{3.45}$$

Since $\|\hat{G}^m\|^2\leq 2\sigma^2C_G^2(\|\hat{e}^m\|^2+\|\hat{e}^{m-1}\|^2)$, we can deduce from (3.42)–(3.45) that

$$\|\Lambda^\beta\hat{e}^m\|^2\leq\frac{(\kappa^2+\gamma^2)\tau^\alpha}{2\sigma\nu c_0^{(0,\sigma)}}\|\hat{G}^m\|^2+\frac{\nu|\zeta|}{\nu^2+\xi^2}\|\hat{e}^m\|^2+\frac{\tau^\alpha}{2\sigma\nu c_0^{(0,\sigma)}}\|\hat{F}^m\|^2\leq C_1C_F^2(\tau^{1+\frac{m\alpha}{2}}+h^4)^2. \tag{3.46}$$

In view of (3.42) and (3.46) as well as Lemmas 2.3 and 2.4, we can also arrive at

$$\|\hat{e}^m\|_\infty\leq C_{\beta/2}\|\hat{e}^m\|_{H^{\beta/2}}\leq C_{\beta/2}\sqrt{\|\hat{e}^m\|^2+(\frac{\pi}{2})^\beta\|\Lambda^\beta\hat{e}^m\|^2}\leq\tilde{C}_F(\tau^{1+\frac{m\alpha}{2}}+h^4). \tag{3.47}$$

From (3.42) and (3.47), we can conclude that (3.24) still holds for $k=m$. Therefore, the induction is closed. Noticing that $u_j^1=\hat{u}_j^{m_\alpha}$ ($m_\alpha=\lceil\frac{2}{\alpha}\rceil$), then we further derive from (3.24) that when $\tau\leq\min\{\tau_1,\ \tau_2\}$, it holds

$$\|e^1\|=\|\hat{e}^{m_\alpha}\|\leq C_F(\tau^2+h^4),\quad \|e^1\|_\infty=\|\hat{e}^{m_\alpha}\|_\infty\leq\tilde{C}_F(\tau^2+h^4). \tag{3.48}$$

Therefore, we have completed the proof of this theorem.

Lemma 3.4. *([49]) (Discrete fractional Grönwall inequality) For any sequence of complex functions $\{\omega^n\}_{n=0}^N$, if $\|\omega^0\|\leq C\vartheta$, $\|\omega^1\|\leq C\vartheta$ and*

$$Re\Big(\nabla_t^\alpha\omega^{n+\sigma},\ \bar{\omega}^{n+\sigma}\Big)\leq\lambda_1\|\bar{\omega}^{n+\sigma}\|^2+\lambda_2\|\tilde{\omega}^{n+\sigma}\|^2+\vartheta^2,$$

where $\lambda_1,\ \lambda_2,\ \vartheta$ are nonnegative constants, then there is a positive constant τ^* such that when $\tau\leq\tau^*$, it holds

$$\|\omega^n\|\leq C^*\vartheta,\quad 1\leq n\leq N,$$

where C^* is a positive constant which is independent of time step τ.

To discuss the local truncation error of proposed method (2.5)–(2.7), we first deduce from (2.5) that

$$\nabla_t^\alpha U_j^{n+\sigma} + (\nu + i\xi)\Delta_h^\beta \bar{U}_j^{n+\sigma} + (\kappa + i\gamma)|\tilde{U}_j^{n+\sigma}|^2\bar{U}_j^{n+\sigma} - \zeta\bar{U}_j^{n+\sigma} = F_j^n, \ \ n = 1, \cdots, N-1, \tag{3.49}$$

where

$$\begin{aligned} F_j^n =& \Big(\nabla_t^\alpha U_j^{n+\sigma} - {}_0D_t^\alpha u(x_j, t_{n+\sigma})\Big) + (\nu + i\xi)\Big(\Delta_h^\beta \bar{U}_j^{n+\sigma} + \frac{\partial^\beta u(x_j, t_{n+\sigma})}{\partial |x|^\beta}\Big) \\ &+ (\kappa + i\gamma)\Big(|\tilde{U}_j^{n+\sigma}|^2\bar{U}_j^{n+\sigma} - |u(x_j, t_{n+\sigma})|^2 u(x_j, t_{n+\sigma})\Big) - \zeta\Big(\bar{U}_j^{n+\sigma} - u(x_j, t_{n+\sigma})\Big). \end{aligned} \tag{3.50}$$

Using Taylor's expansion, Lemmas 2.1 and 2.2, it obvious from (3.50) that

$$\|F^n\| \le C_F(\tau^2 + h^4), \quad 1 \le n \le N-1. \tag{3.51}$$

Theorem 3.4. *Suppose that the solution of TSFGLE* (1.1)–(1.3) *is sufficiently smooth, and let* $\{u^n\}_{n=1}^N$ *be the solutions of the numerical scheme* (2.5)–(2.6)*, then there is a positive* τ^* *such that when* $\tau \le \tau^*$*, we have error estimates*

$$\|e^n\| \le \hat{C}(\tau^2 + h^4), \quad \|e^n\|_\infty \le \tilde{C}(\tau^2 + h^4), \ \ 1 \le n \le N, \tag{3.52}$$

where $\hat{C}$ *and* $\tilde{C}$ *are positive constants independent of* τ *and h.*

Proof. Subtracting (2.5) from (3.49) yields the error equation

$$\nabla_t^\alpha e_j^{n+\sigma} + (\nu + i\xi)\Delta_h^\beta \bar{e}_j^{n+\sigma} + (\kappa + i\gamma)G_j^n - \zeta\bar{e}_j^{n+\sigma} = F_j^n, \tag{3.53}$$

where

$$G_j^n = |\tilde{U}_j^{n+\sigma}|^2\bar{U}_j^{n+\sigma} - |\tilde{u}_j^{n+\sigma}|^2\bar{u}_j^{n+\sigma}. \tag{3.54}$$

Now we use mathematical induction to prove this theorem. We can deduce from Theorem 3.3 that the convergence result (3.52) is valid for $n = 1$. Assuming that the conclusion (3.52) holds for $n \le k$, then when $\tau^2 + h^4 \le 1/\tilde{C}$, we have

$$\|u^n\|_\infty \le \|U^n\|_\infty + \|e^n\|_\infty \le C_u + \tilde{C}(\tau^2 + h^4) \le C_u + 1, \ \ n \le k. \tag{3.55}$$

Now we turn to prove that the estimate (3.52) is still valid for $n = k + 1$. Taking inner product of (3.53) for $n = k$ with $\bar{e}^{k+\sigma}$ gives

$$(\nabla_t^\alpha e^{k+\sigma}, \bar{e}^{k+\sigma}) + (\nu + i\xi)\|\Lambda^\beta \bar{e}^{k+\sigma}\|^2 + (\kappa + i\gamma)(G^k, \bar{e}^{k+\sigma}) - \zeta\|\bar{e}^{k+\sigma}\|^2 = (F^k, \bar{e}^{k+\sigma}). \tag{3.56}$$

Notice that

$$\begin{aligned} G_j^k = |\tilde{U}_j^{k+\sigma}|^2\bar{U}_j^{k+\sigma} - |\tilde{u}_j^{k+\sigma}|^2\bar{u}_j^{k+\sigma} &= \bar{u}_j^{k+\sigma}(|\tilde{U}_j^{k+\sigma}|^2 - |\tilde{u}_j^{k+\sigma}|^2) + |\tilde{U}_j^{k+\sigma}|^2\bar{e}_j^{k+\sigma} \\ &= |\tilde{U}_j^{k+\sigma}|^2\bar{e}_j^{k+\sigma} + \bar{u}_j^{k+\sigma}(\tilde{e}_j^{k+\sigma}\overline{\tilde{U}_j^{k+\sigma}} + \tilde{u}_j^{k+\sigma}\overline{\tilde{e}_j^{k+\sigma}}). \end{aligned} \tag{3.57}$$

Thus we can get

$$\|G^k\| \le C_G(\|\bar{e}^{k+\sigma}\| + \|\tilde{e}^{k+\sigma}\|). \tag{3.58}$$

Since $\nu > 0$, considering the real part of (3.56) and Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} Re(\nabla_t^\alpha e^{k+\sigma}, \bar{e}^{k+\sigma}) \le & \sqrt{\kappa^2+\gamma^2}\|G^k\|\cdot\|\bar{e}^{k+\sigma}\| + |\zeta|\cdot\|\bar{e}^{k+\sigma}\|^2 + \|F^k\|\cdot\|\bar{e}^{k+\sigma}\| \\ \le & \frac{\sqrt{\kappa^2+\gamma^2}}{2}(\|G^k\|^2 + \|\bar{e}^{k+\sigma}\|^2) + |\zeta|\cdot\|\bar{e}^{k+\sigma}\|^2 + \frac{1}{2}(\|F^k\|^2 + \|\bar{e}^{k+\sigma}\|^2) \\ \le & \Big(\sqrt{\kappa^2+\gamma^2}(C_G^2+\frac{1}{2}) + |\zeta| + \frac{1}{2}\Big)\|\bar{e}^{k+\sigma}\|^2 + \sqrt{\kappa^2+\gamma^2}C_G^2\|\tilde{e}^{k+\sigma}\|^2 + \frac{1}{2}\|F^k\|^2. \end{aligned} \tag{3.59}$$

Let $\lambda_1 = \sqrt{\kappa^2+\gamma^2}(C_G^2+\frac{1}{2})+|\zeta|+\frac{1}{2},\ \lambda_2 = \sqrt{\kappa^2+\gamma^2}C_G^2$. According to Lemma 3.4 , we can deduce from (3.51) and (3.59) that there is a constant τ_3 such that when $\tau \le \tau_3$, it follows that

$$\|e^{k+1}\| \le \hat{C}(\tau^2 + h^4). \tag{3.60}$$

To further establish the estimate of e^{k+1} in pointwise sense, we take the inner product of (3.53) for $n = k$ with $\frac{1}{(\nu-i\xi)}\nabla_t^\alpha e^{k+\sigma}$, then taking the real part yields

$$\begin{aligned} & \frac{\nu}{\nu^2+\xi^2}\|\nabla_t^\alpha e^{k+\sigma}\|^2 + Re(\Lambda^\beta \bar{e}^{k+\sigma}, \nabla_t^\alpha \Lambda^\beta e^{k+\sigma}) \\ = & -Re\Big\{\frac{\kappa+i\gamma}{\nu+i\xi}(G^k, \nabla_t^\alpha e^{k+\sigma})\Big\} + Re\Big\{\frac{\zeta}{\nu+i\xi}(\bar{e}^{k+\sigma}, \nabla_t^\alpha e^{k+\sigma})\Big\} + Re\Big\{\frac{1}{\nu+i\xi}(F^k, \nabla_t^\alpha e^{k+\sigma})\Big\}. \end{aligned} \tag{3.61}$$

In view of the Cauchy-Schwarz inequality as well as $Re(u, v) = Re(v, u)$, we can further deduce from (3.61) that

$$\begin{aligned} & \frac{\nu}{\nu^2+\xi^2}\|\nabla_t^\alpha e^{k+\sigma}\|^2 + Re(\nabla_t^\alpha \Lambda^\beta e^{k+\sigma}, \Lambda^\beta \bar{e}^{k+\sigma}) \\ \le & \frac{\sqrt{\kappa^2+\gamma^2}}{\sqrt{\nu^2+\xi^2}}\|G^k\|\cdot\|\nabla_t^\alpha e^{k+\sigma}\| + \frac{|\zeta|}{\sqrt{\nu^2+\xi^2}}\|\bar{e}^{k+\sigma}\|\cdot\|\nabla_t^\alpha e^{k+\sigma}\| + \frac{1}{\sqrt{\nu^2+\xi^2}}\|F^k\|\cdot\|\nabla_t^\alpha e^{k+\sigma}\|. \end{aligned} \tag{3.62}$$

For three items in the right hand of the inequality (3.62), we have

$$\frac{\sqrt{\kappa^2+\gamma^2}}{\sqrt{\nu^2+\xi^2}}\|G^k\|\cdot\|\nabla_t^\alpha e^{k+\sigma}\| \le \frac{3(\kappa^2+\gamma^2)}{4\nu}\|G^k\|^2 + \frac{\nu}{3(\nu^2+\xi^2)}\|\nabla_t^\alpha e^{k+\sigma}\|^2, \tag{3.63}$$

$$\frac{|\zeta|}{\sqrt{\nu^2+\xi^2}}\|\bar{e}^{k+\sigma}\|\cdot\|\nabla_t^\alpha e^{k+\sigma}\| \le \frac{3\zeta^2}{4\nu}\|\bar{e}^{k+\sigma}\|^2 + \frac{\nu}{3(\nu^2+\xi^2)}\|\nabla_t^\alpha e^{k+\sigma}\|^2, \tag{3.64}$$

$$\frac{1}{\sqrt{\nu^2+\xi^2}}\|F^k\|\cdot\|\nabla_t^\alpha e^{k+\sigma}\| \le \frac{3}{4\nu}\|F^k\|^2 + \frac{\nu}{3(\nu^2+\xi^2)}\|\nabla_t^\alpha e^{k+\sigma}\|^2. \tag{3.65}$$

From (3.58) and Lemma 2.5, we get

$$\|G^k\|^2 \le 2C_G^2(\|\bar{u}^{k+\sigma}\|^2 + \|\tilde{e}^{k+\sigma}\|^2) \le 2C_G^2\hat{C}_\beta^2(\|\Lambda^\beta \bar{u}^{k+\sigma}\|^2 + \|\Lambda^\beta \tilde{e}^{k+\sigma}\|^2). \tag{3.66}$$

It is obvious from (3.62)–(3.66) that

$$Re(\nabla_t^\alpha \Lambda^\beta e^{k+\sigma}, \Lambda^\beta \bar{e}^{k+\sigma}) \le \frac{3(\kappa^2+\gamma^2)}{4\nu}\|G^k\|^2 + \frac{3\zeta^2}{4\nu}\|\bar{e}^{k+\sigma}\|^2 + \frac{3}{4\nu}\|F^k\|^2$$

$$\leq \frac{6(\kappa^2+\gamma^2)C_G^2\hat{C}_\beta^2+3\zeta^2\hat{C}_\beta^2}{4\nu}\|\Lambda^\beta \bar{e}^{k+\sigma}\|^2+\frac{3(\kappa^2+\gamma^2)C_G^2\hat{C}_\beta^2}{2\nu}\|\Lambda^\beta \tilde{e}^{k+\sigma}\|^2+\frac{3}{4\nu}\|F^k\|^2. \tag{3.67}$$

By using Lemma 3.4, it follows from (3.51) and (3.67) that there is a constant τ_4 such that when $\tau \leq \tau_4$, we obatin

$$\|\Lambda^\beta e^{k+1}\|^2 \leq C^*(\tau^2+h^4)^2. \tag{3.68}$$

In view of (3.60), (3.68) and Lemma 2.3, we can further deduce that

$$\|e^{k+1}\|_\infty \leq \tilde{C}(\tau^2+h^4), \tag{3.69}$$

where $\tilde{C} = C_{\beta/2}\sqrt{\hat{C}+(\frac{\pi}{2})^2C^*}$ is a positive constant independent of τ and h.

Combining (3.60) and (3.69), it indicates that the stated result (3.52) still holds for $n = k+1$. Therefore, the induction is closed, and we have completed the proof of this theorem.

3.4. The stability of the linearized difference method

Denote $\{v_j^n \,|1 \leq j \leq M-1,\ 0 \leq n \leq N\}$ as the solution of the following difference scheme for TSFGLE with another initial value function, i.e.,

$$\nabla_t^\alpha v_j^{n+\sigma} + (\nu+i\xi)\Delta_h^\beta \bar{v}_j^{n+\sigma} + (\kappa+i\gamma)|\tilde{v}_j^{n+\sigma}|^2\bar{v}_j^{n+\sigma} - \zeta\bar{v}_j^{n+\sigma} = 0,\ 1 \leq n \leq N-1, \tag{3.70}$$

$$\frac{c_0^{(0,\sigma)}(\hat{v}_j^m - v_j^0)}{\tau^\alpha} + (\nu+i\xi)\Delta_h^\beta \hat{v}_j^{\sigma_m} + (\kappa+i\gamma)|\hat{v}_j^{\sigma_{m-1}}|^2\hat{v}_j^{\sigma_m} - \zeta\hat{v}_j^{\sigma_m} = 0, \quad m = 1,2,\cdots,m_\alpha, \tag{3.71}$$

with initial value $v_j^0 = \varphi(x_j)$, $1 \leq j \leq M-1$, then the stability theorem of the discrete scheme (2.5)–(2.7) can be established by a similar analysis as in Theorem 3.4.

Theorem 3.5. *Let u_j^n and v_j^n as the difference solutions of TSFGLE corresponding to initial value functions $\psi(x)$ and $\varphi(x)$ respectively. When h and τ are taken sufficiently small, it holds that*

$$\|u^n - v^n\| \leq \bar{C}\|\psi-\varphi\|, \quad 0 \leq n \leq N, \tag{3.72}$$

where $\bar{C}$ is a positive constant independent of τ and h.

Proof. Since the proof process of this theorem is similar to Theorem 3.4, we omit it here.

4. Numerical simulation for the two-dimensional problem

Similar to the literature [51], the proposed difference method can be extended to solve multi-dimensional TSFGLE. This section is devoted to constructing a difference scheme for the following two-dimensional problem

$$\frac{\partial^\alpha u}{\partial t^\alpha} - (\nu+i\xi)\Big(\frac{\partial^{\beta_1}u}{\partial|x|^{\beta_1}} + \frac{\partial^{\beta_2}u}{\partial|y|^{\beta_2}}\Big) + (\kappa+i\gamma)|u|^2u - \zeta u = 0, \quad t \in (0,T], \tag{4.1}$$

$$u(x,y,0) = \psi(x,y), \quad (x,y) \in \Omega = (a,b)\times(c,d), \tag{4.2}$$

$$u(x,y,t) = 0, \quad (x,y) \in \mathbb{R}^2 \setminus \Omega,\ t \in [0,T], \tag{4.3}$$

with parameters $0 < \alpha \leq 1$, $1 < \beta_1,\ \beta_2 \leq 2$, $\nu > 0$, $\kappa > 0$.

4.1. Linearized difference method for 2D TSFGLE

For given integers M_1 and M_2, we define $x_j = a + jh_x$ $(0 \le j \le M_1)$, $y_k = c + kh_y$ $(0 \le k \le M_2)$, with $h_x = \frac{b-a}{M_1}$ and $h_y = \frac{d-c}{M_2}$. Denote u_{jk}^n as the numerical approximation of $u(x_j, y_k, t_n)$. In view of Lemmas 2.1 and 2.2, a three-level linearized difference method is derived for numerically solving the TSFGLE (4.1)–(4.3)

$$\nabla_t^{\alpha} u_{jk}^{n+\sigma} + (\nu + i\xi)(\Delta_{h_x}^{\beta_1} + \Delta_{h_y}^{\beta_2})\bar{u}_{jk}^{n+\sigma} + (\kappa + i\gamma)|\tilde{u}_{jk}^{n+\sigma}|^2\tilde{u}_{jk}^{n+\sigma} - \zeta\bar{u}_{jk}^{n+\sigma} = 0, \ 1 \le n \le N-1. \tag{4.4}$$

The numerical solution $u_{jk}^1 = \hat{u}_{jk}^{m_\alpha}$ $(m_\alpha = \lceil \frac{2}{\alpha} \rceil)$ can be obtained by the iterative processes

$$\frac{c_0^{(0,\sigma)}(\hat{u}_{jk}^m - u_{jk}^0)}{\tau^\alpha} + (\nu + i\xi)(\Delta_{h_x}^{\beta_1} + \Delta_{h_y}^{\beta_2})\hat{u}_{jk}^{\sigma_m} + (\kappa + i\gamma)|\hat{u}_{jk}^{\sigma_{m-1}}|^2\hat{u}_{jk}^{\sigma_{m-1}} - \zeta\hat{u}_{jk}^{\sigma_{m-1}} = 0, \quad m = 1, 2, \cdots, m_\alpha, \tag{4.5}$$

with initial value

$$u_{jk}^0 = \psi(x_j, y_k). \tag{4.6}$$

4.2. Convergence analysis

Let $\mathbb{V}_h = \Big\{v \mid v = \{v_{jk}\},\ 1 \le j \le M_1 - 1, 1 \le k \le M_2 - 1\Big\}$. For any $v, w \in \mathbb{V}_h$, the discrete inner product and l^2-norm can defined similarly as in the one-dimensional case (2.8). From Ref. [39] and Lemma 2.4, it is easy to show that there exists a symmetric positive operator $\Lambda^{\frac{1}{2}}$ such that

$$\Big((\Delta_{h_x}^{\beta_1} + \Delta_{h_y}^{\beta_2})v, v\Big) = (\Lambda^{\frac{1}{2}}v, \Lambda^{\frac{1}{2}}v). \tag{4.7}$$

Now we study the local truncation error of difference method (4.4)–(4.6), and it holds from (4.4) that

$$\nabla_t^{\alpha} U_{jk}^{n+\sigma} + (\nu + i\xi)(\Delta_{h_x}^{\beta_1} + \Delta_{h_y}^{\beta_2})\bar{U}_{jk}^{n+\sigma} + (\kappa + i\gamma)|\tilde{U}_{jk}^{n+\sigma}|^2\tilde{U}_{jk}^{n+\sigma} - \zeta\bar{U}_{jk}^{n+\sigma} = F_{jk}^n, \quad n = 1, \cdots, N-1, \tag{4.8}$$

where

$$\begin{aligned} F_{jk}^n =& \Big(\nabla_t^{\alpha} U_{jk}^{n+\sigma} - {}_0D_t^{\alpha} u(x_j, y_k, t_{n+\sigma})\Big) + (\nu + i\xi)\Big((\Delta_{h_x}^{\beta_1} + \Delta_{h_y}^{\beta_2})\bar{U}_{jk}^{n+\sigma} + (\frac{\partial^{\beta_1}}{\partial|x|^{\beta_1}} + \frac{\partial^{\beta_2}}{\partial|y|^{\beta_2}})u(x_j, y_k, t_{n+\sigma})\Big) \\ &+ (\kappa + i\gamma)\Big(|\tilde{U}_{jk}^{n+\sigma}|^2\tilde{U}_{jk}^{n+\sigma} - |u(x_j, y_k, t_{n+\sigma})|^2u(x_j, y_k, t_{n+\sigma})\Big) - \zeta\Big(\bar{U}_{jk}^{n+\sigma} - u(x_j, y_k, t_{n+\sigma})\Big). \end{aligned} \tag{4.9}$$

Similarly, we can deduce from (4.5) that

$$\frac{c_0^{(0,\sigma)}(\hat{U}_{jk}^m - U_{jk}^0)}{\tau^\alpha} + (\nu + i\xi)(\Delta_{h_x}^{\beta_1} + \Delta_{h_y}^{\beta_2})\hat{U}_{jk}^{\sigma_m} + (\kappa + i\gamma)|\hat{U}_{jk}^{\sigma_{m-1}}|^2\hat{U}_{jk}^{\sigma_{m-1}} - \zeta\hat{U}_{jk}^{\sigma_{m-1}} = \hat{F}_{jk}^m, \ m = 1, \cdots, m_\alpha, \tag{4.10}$$

where $\hat{U}_{jk}^{\sigma_0} = U_{jk}^0$ and

$$\begin{aligned} \hat{F}_{jk}^m =& \Big(\frac{c_0^{(0,\sigma)}(\hat{U}_{jk}^m - U_{jk}^0)}{\tau^\alpha} - {}_0D_t^{\alpha} u(x_j, y_k, t_\sigma)\Big) + (\nu + i\xi)\Big((\Delta_{h_x}^{\beta_1} + \Delta_{h_y}^{\beta_2})\hat{U}_{jk}^{\sigma_m} + (\frac{\partial^{\beta_1}}{\partial|x|^{\beta_1}} + \frac{\partial^{\beta_2}}{\partial|y|^{\beta_2}})u(x_j, y_k, t_\sigma)\Big) \\ &+ (\kappa + i\gamma)\Big(|\hat{U}_{jk}^{\sigma_{m-1}}|^2\hat{U}_{jk}^{\sigma_{m-1}} - |u(x_j, y_k, t_\sigma)|^2u(x_j, y_k, t_\sigma)\Big) - \zeta\Big(\hat{U}_{jk}^{\sigma_{m-1}} - u(x_j, y_k, t_\sigma)\Big). \end{aligned} \tag{4.11}$$

By using Lemmas 2.1, 2.2 and Taylor's formula, we have

$$\tau\|\hat{F}^1\| + \|\hat{F}^m\| \le \breve{C}_F(\tau^2 + h_x^4 + h_y^4), \quad \|F^n\| \le \breve{C}_F(\tau^2 + h_x^4 + h_y^4). \tag{4.12}$$

Denote $e_{jk}^n = U_{jk}^n - u_{jk}^n$, then we have the error estimate for discrete scheme (4.4)–(4.6).

Theorem 4.1. *Let U^n_{jk} and u^n_{jk} be solutions of model* (4.1)–(4.3) *and discrete scheme* (4.4)–(4.6) *respectively. Suppose $\tau \le \breve{C}_0 h^{\frac{1}{2}+\varepsilon}$, where ε is a positive constant and $h = \max\{h_x, h_y\}$. When h and τ are taken sufficiently small, we have the error estimate*

$$\|e^n\| \le \breve{C}(\tau^2 + h_x^4 + h_y^4), \quad 1 \le n \le N, \tag{4.13}$$

where $\breve{C}$ is a positive constant which is independent of τ, h_x and h_y.

Proof. Subtracting (4.5) from (4.10) yields the error equation

$$\frac{c_0^{(0,\sigma)}\hat{e}^m_{jk}}{\tau^\alpha} + (\nu + i\xi)\sigma(\Delta^{\beta_1}_{h_x} + \Delta^{\beta_2}_{h_y})\hat{e}^m_{jk} + (\kappa + i\gamma)\hat{G}^m_{jk} - \zeta\sigma\hat{e}^{\sigma_{m-1}}_{jk} = \hat{F}^m_{jk}, \tag{4.14}$$

where

$$\hat{G}^m_{jk} = |\hat{U}^{\sigma_{m-1}}_{jk}|^2\hat{U}^{\sigma_{m-1}}_{jk} - |\hat{u}^{\sigma_{m-1}}_{jk}|^2\hat{u}^{\sigma_{m-1}}_{jk}. \tag{4.15}$$

By similar analysis as Theorem 3.3, we can conclude that

$$\|\hat{e}^k\| \le \breve{C}_1(\tau^{\frac{k\alpha}{2}} + h_x^4 + h_y^4), \quad k = 1, 2, \cdots, m_\alpha. \tag{4.16}$$

Therefore, we have $\|e^1\| = \|\hat{e}^{m_\alpha}\| \le \breve{C}(\tau^2 + h_x^4 + h_y^4)$, and it implies that the conclusion is valid for $n = 1$. Assuming that conclusion (4.13) is true for $0 \le n \le l$, then we need to prove the estimate (4.13) still holds for $n = l + 1$. To this end, subtracting (4.4) from (4.8) gives the error equation

$$\nabla^\alpha_t e^{n+\sigma}_{jk} + (\nu + i\xi)(\Delta^{\beta_1}_{h_x} + \Delta^{\beta_2}_{h_y})\bar{e}^{n+\sigma}_{jk} + (\kappa + i\gamma)G^n_{jk} - \zeta\bar{e}^{n+\sigma}_{jk} = F^n_{jk}, \tag{4.17}$$

where

$$G^n_{jk} = |\tilde{U}^{n+\sigma}_{jk}|^2\tilde{U}^{n+\sigma}_{jk} - |\tilde{u}^{n+\sigma}_{jk}|^2\tilde{u}^{n+\sigma}_{jk}. \tag{4.18}$$

Taking inner product of (4.17) for $n = l$ with $\bar{e}^{l+\sigma}$ gives

$$(\nabla^\alpha_t e^{l+\sigma}, \bar{e}^{l+\sigma}) + (\nu + i\xi)\|\Lambda^{\frac{1}{2}}\bar{e}^{l+\sigma}\|^2 + (\kappa + i\gamma)(G^l, \bar{e}^{l+\sigma}) - \zeta\|\bar{e}^{l+\sigma}\|^2 = (F^l, \bar{e}^{l+\sigma}). \tag{4.19}$$

Under the inductive hypothesis $\tau \le \breve{C}_0 h^{\frac{1}{2}+\varepsilon}$ and applying the inverse inequality, it holds when h is taken sufficiently small that

$$\|e^n\|_\infty \le h^{-1}\|e^n\| \le \breve{C}_2(h^{2\varepsilon} + h^3) \le 1, \ \ 0 \le n \le l, \tag{4.20}$$

which further indicates that

$$\|u^n\|_\infty \le \|U^n\|_\infty + \|e^n\|_\infty \le C_u + \breve{C}_2(h^{2\varepsilon} + h^3) \le C_u + 1 = \breve{C}_u, \ \ 0 \le n \le l. \tag{4.21}$$

Noticing the definition of G^l_{jk}, we have

$$\begin{aligned} \left|G^l_{jk}\right| = \left||\tilde{U}^{l+\sigma}_{jk}|^2\tilde{U}^{l+\sigma}_{jk} - |\tilde{u}^{l+\sigma}_{jk}|^2\tilde{u}^{l+\sigma}_{jk}\right| &= \tilde{u}^{l+\sigma}_{jk}(|\tilde{U}^{l+\sigma}_{jk}|^2 - |\tilde{u}^{l+\sigma}_{jk}|^2) + |\tilde{U}^{l+\sigma}_{jk}|^2\tilde{e}^{l+\sigma}_{jk} \\ &= |\tilde{U}^{l+\sigma}_{jk}|^2\tilde{e}^{l+\sigma}_{jk} + \tilde{u}^{l+\sigma}_{jk}(\tilde{e}^{l+\sigma}_{jk}\overline{\tilde{U}^{l+\sigma}_{jk}} + \tilde{u}^{l+\sigma}_{jk}\overline{\tilde{e}^{l+\sigma}_{jk}}) \\ &\le 3\breve{C}_u^2|\tilde{e}^{l+\sigma}_{jk}|, \end{aligned} \tag{4.22}$$

Notice that

$$Re(\kappa + i\gamma)(G^l, \bar{e}^{l+\sigma}) \le \sqrt{\kappa^2+\gamma^2}\|G^l\| \cdot \|\bar{e}^{l+\sigma}\| \le \frac{3\check{C}_u^2\sqrt{\kappa^2+\gamma^2}}{2}(\|\tilde{e}^{l+\sigma}\|^2 + \|\bar{e}^{l+\sigma}\|^2). \tag{4.23}$$

Since $\nu > 0$, considering the real part of (4.19) and Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} Re(\nabla_t^{\alpha} e^{l+\sigma}, \bar{e}^{l+\sigma}) &\le \frac{3\check{C}_u^2\sqrt{\kappa^2+\gamma^2}}{2}(\|\tilde{e}^{l+\sigma}\|^2 + \|\bar{e}^{l+\sigma}\|^2) + |\zeta| \cdot \|\bar{e}^{l+\sigma}\|^2 + \frac{1}{2}(\|F^l\|^2 + \|\bar{e}^{l+\sigma}\|^2) \\ &\le \Big(\frac{3\check{C}_u^2\sqrt{\kappa^2+\gamma^2}}{2} + |\zeta| + \frac{1}{2}\Big)\|\bar{e}^{l+\sigma}\|^2 + \frac{3\check{C}_u^2\sqrt{\kappa^2+\gamma^2}}{2}\|\tilde{e}^{l+\sigma}\|^2 + \frac{1}{2}\|F^l\|^2. \end{aligned} \tag{4.24}$$

According to Lemma 3.4 , we can drive from (4.12) and (4.24) that when τ is taken sufficiently small , it follows that

$$\|e^{l+1}\| \le \hat{C}(\tau^2 + h_x^4 + h_y^4). \tag{4.25}$$

Therefore, we have completed the proof of this theorem.

We mention that the well-posedness, boundness property and stability theorem of the difference scheme (4.4)–(4.6) can be also established by similar analysis as in the one-dimensional case.

5. Numerical experiment

Some numerical examples will be performed to show the effectiveness of the proposed scheme and the correctness of theoretical analysis. Denote U^N and u^N as the exact solution and numerical approximation at $t = T$ respectively. Define the error function $e(\tau, h) := U^N - u^N$, then corresponding convergence rates in time and space are respectively obtained as follows

$$\text{Rate}(\tau) = \log_2\Big(\frac{\|e(2\tau, h)\|_\infty}{\|e(\tau, h)\|_\infty}\Big), \quad \text{Rate}(h) = \log_2\Big(\frac{\|e(\tau, 2h,)\|_\infty}{\|e(\tau, h)\|_\infty}\Big). \tag{5.1}$$

The convergence orders in l^2-norm can be computed similarly.

Example 1. Set $\nu = 0.3,\ \xi = 0.5,\ \kappa = -\frac{\nu(3\sqrt{1+4\nu^2}-1)}{4+18\nu^2},\ \gamma = -1,\ \zeta = 0$. For $\alpha = 1$, $\beta = 2$, the analytical solution of the model (1.1) is given explicitly as

$$u(x,t) = \phi(x)\exp(iR\ln(\phi(x)) - i\rho t), \tag{5.2}$$

where

$$R = \frac{\sqrt{1+4\nu^2}-1}{2\nu},\ \ \rho = \sqrt{\frac{R\sqrt{1+4\nu^2}}{-2\kappa}},\ \ \phi(x) = R\text{sech}(x),\ \ \rho = -\frac{R(1+4\nu^2)}{2\nu}.$$

Since the exact solution (5.2) decays exponentially as $|x| \to +\infty$, we take a sufficiently large computational domain $\Omega = [-20, 20]$ such that the truncation error can be ignored. The main task is to check the convergence rates of the derived method. To this end, we first fix $h = 0.01$, and plot the errors at $t = 1$ in the maximum norm with different time steps in the left side of Figure 1. It follows that the difference method (2.5)–(2.7) has second-order accuracy in time. Similarly, we take $\tau = 0.0002$, and plot errors in the maximum norm with different space steps in the right side of Figure 1. We can observe that the proposed method is fourth-order convergent in space.

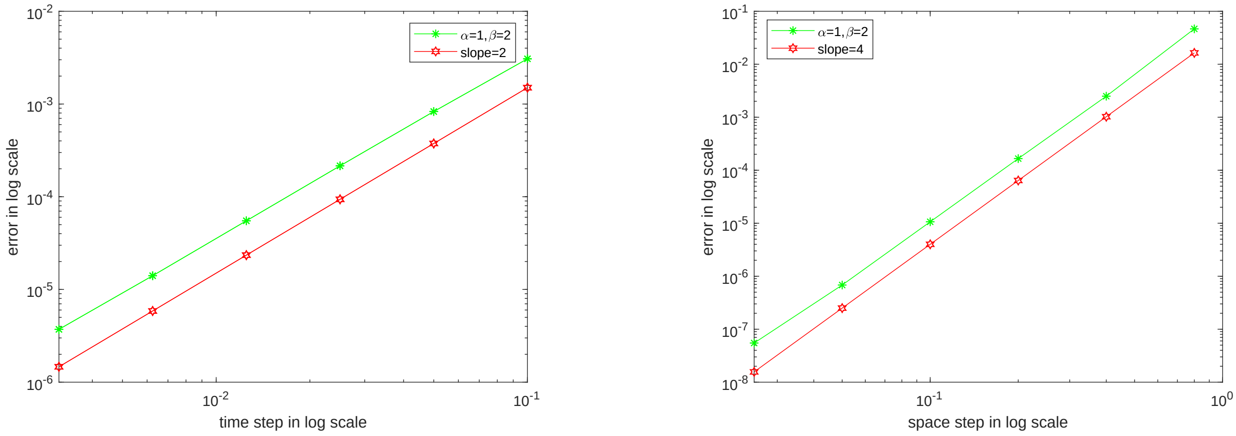

Figure 1. Errors versus τ with fixed $h = 0.01$ (left) and errors versus h with fixed $\tau = 0.0002$ (right).

Example 2. Consider the nonlinear time-space fractional Ginzburg-Landau equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} - (\nu + i\xi)\frac{\partial^\beta u}{\partial |x|^\beta} + (\kappa + i\gamma)|u|^2 u - \zeta u = f(x,t), \quad -1 \le x \le 1, \ 0 < t \le 1, \tag{5.3}$$

The parameters are taken as $\nu = \xi = \kappa = \gamma = 1$, $\zeta = -1$. The initial value $\psi(x)$, boundary condition and source term $f(x,t)$ are chosen such that $u(x,t) = t^3(1-x^2)^{4+\frac{\beta}{2}}$.

Now we continue to validate the convergence rates of proposed method (2.5)–(2.7). First, by fixing $h = 1/400$, numerical errors at $t = 1$ and convergence rates with various time step for different α and β are depicted in Table 1. Besides, we fix $\tau = 1/5000$, and present errors in maximum norm as well as convergence rates of the proposed method for various space steps in Table 2. These results show that numerical solutions are convergent with second-order accuracy in time and fourth-order accuracy in space, and also validate the correctness of theoretical analysis in Theorem 3.4. Besides, the graph of the numerical solution is displayed in Figure 2, and we can find that numerical solution approximates the analytical solution very well.

Table 1. Errors and corresponding convergence rates in time of the proposed method with $h = 1/400$.

	$\alpha = 0.2,\ \beta = 1.3$		$\alpha = 0.4,\ \beta = 1.6$		$\alpha = 0.6,\ \beta = 1.7$		$\alpha = 0.9,\ \beta = 1.9$	
τ	$\|e(\tau,h)\|_\infty$	Rate	$\|e(\tau,h)\|_\infty$	Rate	$\|e(\tau,h)\|_\infty$	Rate	$\|e(\tau,h)\|_\infty$	Rate
1/12	7.2327e-3	–	3.6810e-3	–	1.6020e-3	–	2.1935e-3	–
1/24	1.8672e-3	2.0554	9.6408e-4	1.9329	4.2751e-4	1.9057	5.2772e-4	2.0554
1/48	4.7599e-4	2.0273	2.4714e-4	1.9638	1.1032e-4	1.9543	1.2945e-4	2.0274
1/96	1.2024e-4	2.0167	6.2601e-5	1.9810	2.8020e-5	1.9772	3.1989e-5	2.0167
1/192	3.0227e-5	2.0125	1.5759e-5	1.9899	7.0620e-6	1.9883	7.9281e-6	2.0125

Example 3. Consider the following nonlinear time-space fractional Ginzburg-Landau equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} - (\nu + i\xi)\frac{\partial^\beta u}{\partial |x|^\beta} + (\kappa + i\gamma)|u|^2 u - \zeta u = 0, \quad -20 \le x \le 20. \tag{5.4}$$

Table 2. Errors and corresponding convergence rates in space of the proposed method with $\tau = 1/5000$.

	$\alpha = 0.2,\ \beta = 1.3$		$\alpha = 0.4,\ \beta = 1.6$		$\alpha = 0.6,\ \beta = 1.7$		$\alpha = 0.9,\ \beta = 1.9$	
h	$\|\|e(\tau,h)\|\|_\infty$	Rate	$\|\|e(\tau,h)\|\|_\infty$	Rate	$\|\|e(\tau,h)\|\|_\infty$	Rate	$\|\|e(\tau,h)\|\|_\infty$	Rate
1/8	4.4206e-3	–	5.6949e-3	–	7.0771e-3	–	8.3894e-3	–
1/16	2.9248e-4	3.9178	3.6459e-4	3.9653	4.5374e-4	3.9632	5.4028e-4	3.9749
1/32	1.8928e-5	3.9497	2.3919e-5	3.9300	2.9243e-5	3.9556	3.4361e-5	4.0132
1/64	1.1465e-6	4.0451	1.4558e-6	4.0383	1.7865e-6	4.0328	2.1279e-6	4.0405
1/128	8.0713e-8	3.8283	8.9513e-8	4.0236	1.0299e-7	4.1165	1.2931e-7	4.0595

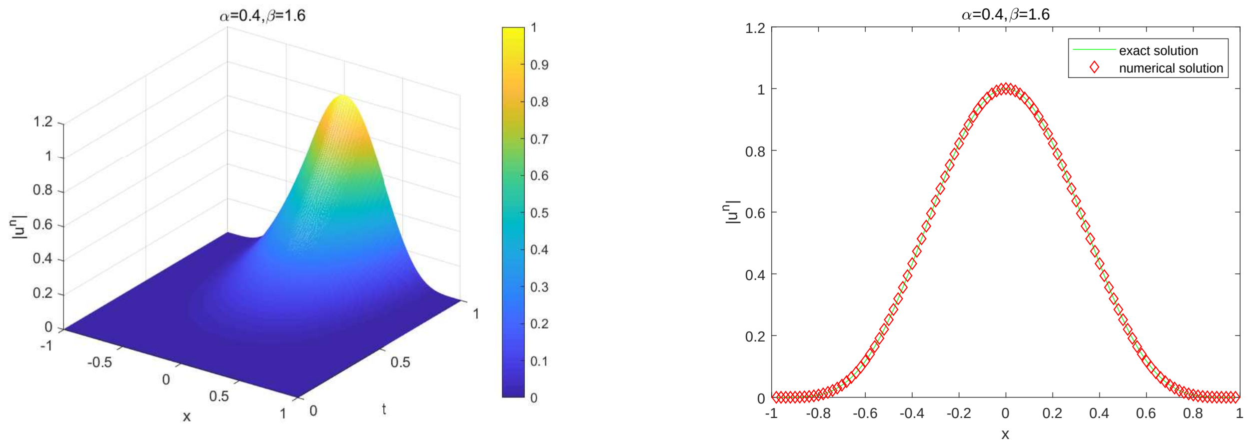

Figure 2. Graphs of numerical solution (left) and the numerical solution as well as the exact solution at $t = 1$ (right).

Here we take parameters $\nu = 1,\ \xi = -1,\ \kappa = 2,\ \gamma = \frac{1}{2}$, and the initial value is taken as

$$\psi(x) = \text{sech}(x)\exp(-ix).$$

We take $\tau = 0.004,\ h = 0.05$, and present the evolution of $||u^n||$ with time t for different ζ in Figure 3. It shows that $||u^n||$ is bounded in finite time, and indicates that the stated conclusion in Theorem 3.2 is valid. In particular, we find that when $\zeta < 0$, the numerical approximation solution in l^2-norm decays quickly to zero as time t increases, and the decay trend is more rapidly with ζ going smaller. Besides, the graphs of the numerical solutions for $\zeta = 0.4$ is presented in Figure 4, and it indicates that the values of α and β have significant influence on the evolution of numerical solutions.

Example 4. Consider two-dimensional time-space fractional Ginzburg-Landau equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} - (\nu + i\xi)\left(\frac{\partial^{\beta_1} u}{\partial |x|^{\beta_1}} + \frac{\partial^{\beta_2} u}{\partial |y|^{\beta_2}}\right) + (\kappa + i\gamma)|u|^2 u - \zeta u = f(x,y,t),\ -1 \le x \le 1,\ -1 \le y \le 1, \quad (5.5)$$

The parameters are taken as $\nu = \xi = \frac{1}{2},\ \kappa = \gamma = 1.5,\ \zeta = -1$. The initial value $\psi(x,y)$, boundary condition and source term $f(x,y,t)$ are chosen such that $u(x,y,t) = t^3(1-x^2)^{4+\frac{\beta_1}{2}}(1-y^2)^{4+\frac{\beta_2}{2}}$.

The l^2- errors and corresponding convergence rates of scheme (4.4)–(4.6) with the relationship of τ and $h = h_x = h_y$ are displayed in Tables 3 and 4. It can be observed again that the proposed method

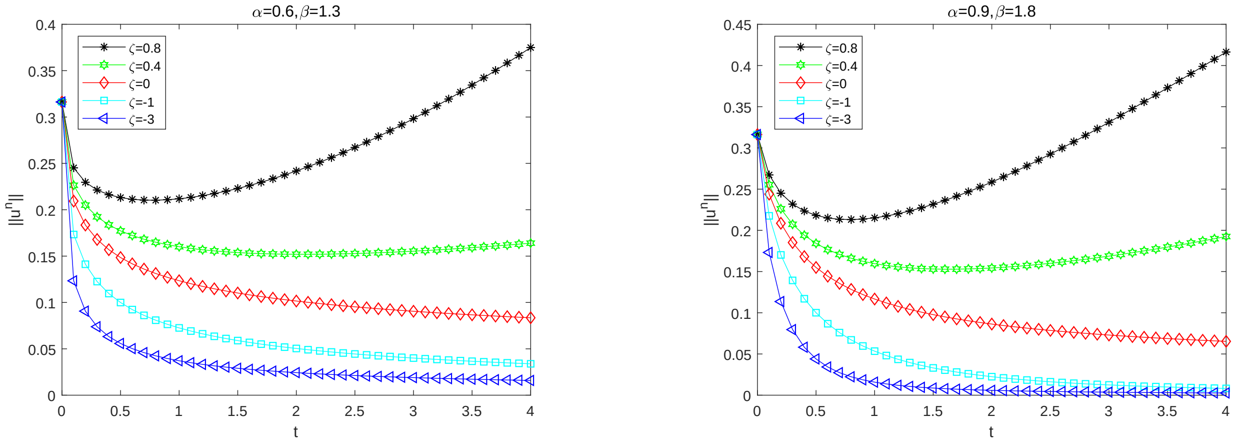

Figure 3. The evolution of $\|u^n\|$ for different ζ with $\tau = 0.04$ and $h = 0.01$.

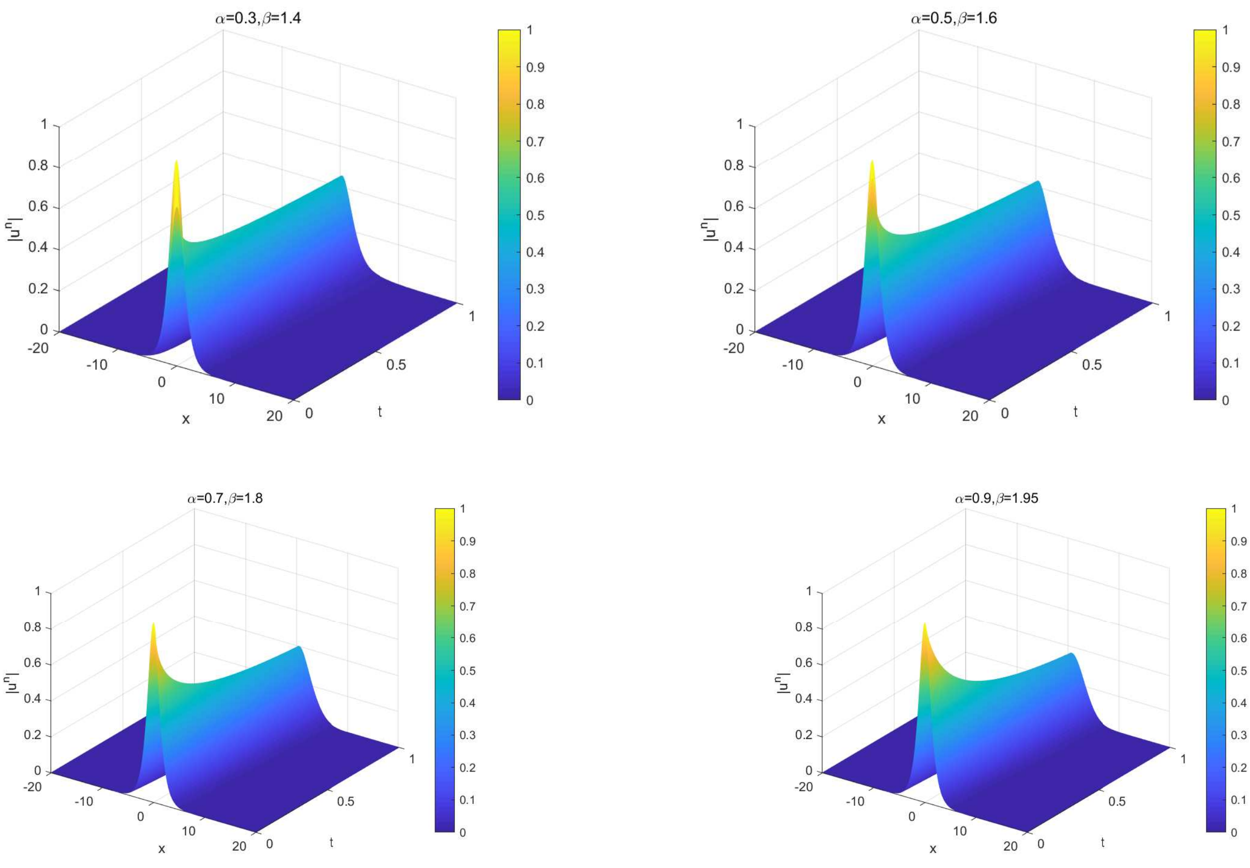

Figure 4. Graphs of numerical solution with $\tau = 0.01$ and $\tau = 0.01$ for $\zeta = 0.4$.

has second-order accuracy in time direction and fourth-order accuracy in space direction, which is accordance with the result as stated in Theorem 4.1.

Table 3. Errors and corresponding convergence rates in time of the difference method with $h = 1/120$.

	$\alpha = 0.2, \beta_1 = 1.3, \beta_2 = 1.5$		$\alpha = 0.5, \beta_1 = 1.7, \beta_2 = 1.4$		$\alpha = 0.8, \beta_1 = 1.9, \beta_2 = 1.6$	
τ	$\|\|e(\tau, h)\|\|$	Rate	$\|\|e(\tau, h)\|\|$	Rate	$\|\|e(\tau, h)\|\|$	Rate
1/12	4.1844e-3	–	1.8265e-3	–	1.4807e-3	–
1/24	1.1008e-3	1.9265	4.7859e-4	1.9322	3.6566e-4	2.0177
1/48	2.8168e-4	1.9664	1.2367e-4	1.9523	9.0843e-5	2.0090
1/96	7.1212e-5	1.9839	3.1366e-5	1.9792	2.2562e-5	2.0094

Table 4. Errors and corresponding convergence rates in space of the difference method with $\tau = 1/2000$.

	$\alpha = 0.3, \beta_1 = 1.2, \beta_2 = 1.4$		$\alpha = 0.6, \beta_1 = 1.5, \beta_2 = 1.7$		$\alpha = 0.9, \beta_1 = 1.8, \beta_2 = 1.3$	
h	$\|\|e(\tau, h)\|\|$	Rate	$\|\|e(\tau, h)\|\|$	Rate	$\|\|e(\tau, h)\|\|$	Rate
1/8	5.1382e-3	–	7.1841e-3	–	7.1450e-3	–
1/16	3.5184e-4	3.8683	4.9838e-4	3.8495	4.7853e-4	3.9003
1/32	2.3325e-5	3.9150	3.3049e-5	3.9145	3.1699e-5	3.9160
1/64	1.4998e-6	3.9591	2.1038e-6	3.9735	2.0270e-6	3.9669

6. Conclusions

In this article, based on $L2$-1_σ formula together with a second-order extrapolation technique for time discretization and a fourth-order difference method for space discretization, a high-order linearized discrete method is proposed for solving the time-space fractional Ginzburg-Landau equation. We show that the discrete scheme is uniquely solvable, and obtain a priori bound of numerical solution in the discrete L^2-norm. The difference scheme is rigorously proved to be convergent in the pointwise sense with the accuracy of $O(\tau^2 + h^2)$. Besides, the difference method for the two-dimensional problem is constructed, and the convergence analysis is also analyzed. Numerical results are also presented to confirm our theoretical results.

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Conflict of interest

The authors declare there is no conflicts of interest.

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