Electronic Research Archive

ERA, 30 (10): 3572–3585.
DOI: 10.3934/era.2022182
Received: 16 April 2022
Revised: 15 June 2022
Accepted: 10 July 2022
Published: 28 July 2022

http://www.aimspress.com/journal/ERA

Technical note

A guide for using integration by parts: *Pet-LoPo-InPo*

William Guo*

School of Engineering and Technology, Central Queensland University, Bruce Highway, North Rockhampton, QLD 4702, Australia

* **Correspondence:** Email: w.guo@cqu.edu.au; Tel: +61-7-49309687.

Abstract: Based on the tutorial cases accumulated in the past several years, by reclassifying the arithmetic functions (A) in LIATE into the polynomial function (*P*), the standard power function (*Po*), and the integer power function (*nPo*), a new guide, comprising three sub-guides, *Pet*, *LoPo* and *InPo*, or *Pet-LoPo-InPo*, is summarized in this note to guide practicing integration by parts. This new guide removes many incompatible combinations included in LIATE, rationalizes the relationship between the exponential and trigonometric functions in LIATE, and expands the coverage of the *P*-functions beyond the traditional definitions. Hence, the new guide can reduce potential confusions that students may experience in using LIATE for their practices of integration by parts. The advantages of this new guide are demonstrated by many worked examples in this note.

Keywords: integration by parts; combinative integrands; LIATE; LIPTE; *Pet-LoPo-InPo*

1. Introduction

Integration by parts is a powerful technique for obtaining solutions for many types of integration commonly involved in differential equations [1–5], integral transforms and other advanced mathematics [6–8], hence it is a necessary part included in all calculus textbooks for most undergraduate STEM programs in the world [9–13]. Integration by parts is originated from the inverse operation of the product rule of differentiation expressed as follows

$$\int udv = uv - \int vdu , \tag{1}$$

where both u and v can be different types of functions. In practice, once u and dv on the left side of formula (1) are identified, v and du can be obtained through

$$\int u dv = uv - \int v du \longleftarrow v = \int dv, \quad du = u' dx . \tag{2}$$

This formula seems expressed simply but choosing the order for u and v to make the integral $\int v du$ simpler than $\int u dv$ depends more on individual's experience.

In teaching applied calculus courses to undergraduate engineering and education students for more than a decade at a regional university in Australia, through answering numerous questions from different cohorts of students on how to better decide the order of u and v in using integration by parts, the author also observed the practices of students who adopted different approaches to solve the same questions at different times. For example, several years ago a few students cited the 'LIATE rule' for choosing the order of u and v in using integration by parts to solve the following indefinite integral

$$\int (9x^2 \ln x - x \cos x) dx,$$

as shown in Figure 1. Their attempts were successful in finding the integral for the question, which led the author to find Kasube's classic note on LIATE published in 1983 [14] where L refers to the logarithmic functions; I refers to the inverse trigonometric functions; A refers to the algebraic functions; T refers to the trigonometric functions, and E refers to the exponential functions.

Find $\int (9x^2 \ln x - x \cos x) dx$

$= \int 9x^2 \ln x \, dx - \int x \cos x \, dx$	Split terms using integration as a linear operator
$\int 9x^2 \ln x \, dx = \int u dv = uv - \int v du$	Integrate first term by parts
$= \ln x \times 3x^3 - \int 3x^3 \frac{dx}{x}$ $= \ln x \times 3x^3 - 3\int x^2 dx$ $= \ln x \times 3x^3 - x^3 + c$ $= x^3(3 \ln x - 1) + c$	Choosing term u by order of L.I.A.T.E Let $u = \ln x$, $\frac{du}{dx} = \frac{1}{x}$, $\frac{dx}{x} = du$ Let $dv = 9x^2, v = 3x^3$

Figure 1. Part of a student's work citing the LIATE rule to conduct integration by parts.

Further investigations revealed that LIATE is more a rough '*rule of thumb*' to guide choosing the order of u and v in integration by parts for some types of combined integrands, far from a universal rule considered by many students as their default guide in using integration by parts. Such misunderstanding caused more confusions for the students who tried to apply LIATE to some seemingly simple integrals but actually without exact solutions. For example, at the time when students were learning how to use numeric methods to obtain approximate results for $\int_0^1 \frac{\sin x}{x} dx$, some students tried to use LIATE to obtain an exact solution for the integral. Even after they were told this seemingly simple integral has no exact solution, they believed that the integrand as a combination of A and T was perfect for applying LIATE.

Nevertheless, LIATE seems a useful guide for students to decide the order in integration by parts for many common combinations of functions given the fact no other similar rules are available. Hence, from that time onwards, LIATE was introduced to the students as a rough rule to guide their practices of using integration by parts whereas the early draft of this technical note was also prepared to balance the potential over-reliance of LIATE to which students may tend. With the accumulation of more incompatible cases with LIATE in teaching integrations to undergraduate students in the recent years, this technical note has been gradually enriched to the current status ready to be shared with the mathematics teaching and learning community in the world.

In Section 2, the LIATE rule is reviewed using examples to not only highlight its strengths in guiding the use of integration by parts but also reveal its weaknesses or limits. Section 3 presents the *Pet-LoPo-InPo* guide for using integration by parts with examples. Concluding remarks are summarized in Section 4. It should be noted that the new guide reported in this note only intends to provide students with a practical guideline that can reduce potential confusions caused by the incompatibility of some types of combinative integrands indicated in LIATE. It is not a universal rule to cover all possible integrable integrands through integration by parts.

2. The LIATE rule and weaknesses

2.1. The LIATE rule

In 1983, Kasube [14] proposed LIATE as a *rule of thumb* for choosing the u function that comes first in $\int u dv = uv - \int v du$. LIATE stands for

- L – logarithmic functions, for example, $\ln x$, $\log_b x$
- I – inverse trigonometric functions, such as $\arcsin x$, $\arctan x$
- A – algebraic functions, for example, $2x^3$, $3x$, -5
- T – trigonometric functions, for example, $\sin x$, $\cos 2x$, $4\tan 3x$
- E – exponential functions, for example, e^x, e^{-3x}, 5^{2x}.

LIATE indicates the order of selecting the first function (first part) u from the combined integrand by integration by parts. Once u is selected, the remainder of the combined term become the differential dv.

For example, in $\int xe^{-x}dx$, the combined integrand is xe^{-x}. By LIATE, x is classified as A and should be chosen as u ahead of e^{-x} (E), or $u = x$ and $dv = e^{-x}dx$.

Example 1: Evaluate $\int xe^{-x}dx$.

In the integrand xe^{-x}, x is an algebraic function (A) and e^{-x} is an exponential function (E). By formula (2) and LIATE, let $u = x$ and $dv = e^{-x}dx$.

$$\int \underbrace{x}_{u}\underbrace{e^{-x}dx}_{dv} = (\underbrace{x}_{u})(\underbrace{-e^{-x}}_{v}) - \int (\underbrace{-e^{-x}}_{v})(\underbrace{dx}_{du}) \longleftarrow v = \int dv = \int e^{-x}dx = -e^{-x},\quad du = u'dx = (x)'dx = dx$$
$$= -xe^{-x} + \int e^{-x}dx = -xe^{-x} + (-e^{-x}) + c = -xe^{-x} - e^{-x} + c = -e^{-x}(x+1) + c.$$

Example 2: Evaluate $\int x^3 \ln x dx$.

In the integrand $x^3 e^{-x}$, x^3 is an algebraic function (A) and ln$x$ is a logarithmic function (L). By formula (2) and LIATE, let $u = \ln x$ and $dv = x^3 dx$.

$$\int \underbrace{\ln x}_{u} \underbrace{x^3 dx}_{dv} = \underbrace{(\ln x)}_{u} \left(\underbrace{\frac{1}{4} x^4}_{v} \right) - \int \left(\underbrace{\frac{1}{4} x^4}_{v} \right) \left(\underbrace{\frac{dx}{x}}_{du} \right) \longleftarrow v = \int x^3 dx = \frac{1}{4} x^4, \quad du = (\ln x)' dx = \frac{dx}{x}$$

$$= \frac{1}{4} x^4 \ln x - \frac{1}{4} \int x^3 dx = \frac{1}{4} x^4 \ln x - \frac{1}{16} x^4 + c = \frac{1}{16} x^4 (4 \ln x - 1) + c.$$

Example 3: Evaluate $\int 2x \cos x dx$.

In the integrand $2x \cos x$, $2x$ is an algebraic function (A) and cos$x$ is a trigonometric function (T). By formula (2) and LIATE, let $u = 2x$ and $dv = \cos x dx$.

$$\int \underbrace{2x}_{u} \underbrace{\cos x dx}_{dv} = (\underbrace{2x}_{u})(\underbrace{\sin x}_{v}) - \int (\underbrace{\sin x}_{v})(\underbrace{2dx}_{du}) \longleftarrow v = \int \cos x dx = \sin x, \quad du = (2x)' dx = 2dx$$

$$= 2x \sin x - \int 2 \sin x dx = 2x \sin x - 2(-\cos x) + c = 2x \sin x + 2 \cos x + c.$$

Example 4: Evaluate $\int 2 \arcsin x dx$.

In the integrand $2 \arcsin x$, 2 is a constant (A) and arcsin$x$ is an inverse trigonometric function (I). By formula (2) and LIATE, let $u = \arcsin x$ and $dv = 2dx$.

$$\int \underbrace{\arcsin x}_{u} \underbrace{(2dx)}_{dv} = (\underbrace{\arcsin x}_{u})(\underbrace{2x}_{v}) - \int (\underbrace{2x}_{v}) \left(\underbrace{\frac{dx}{\sqrt{1-x^2}}}_{du} \right) \longleftarrow v = 2x, \quad du = (\arcsin x)' dx = \frac{dx}{\sqrt{1-x^2}}$$

$$= 2x \arcsin x - \int \frac{2x dx}{\sqrt{1-x^2}} \longleftarrow t = 1 - x^2, dt = -2x dx \leftrightarrow 2x dx = -dt$$

$$= 2x \arcsin x - \int \frac{-dt}{\sqrt{t}} = 2x \arcsin x + 2\sqrt{t} + c = 2x \arcsin x + 2\sqrt{1-x^2} + c.$$

Example 5: Evaluate $\int x^2 \arctan x dx$.

In the integrand $x^2 \arctan x$, x^2 is an algebraic function (A) and arctan$x$ is an inverse trigonometric function (I). By formula (2) and LIATE, let $u = \arctan x$ and $dv = x^2 dx$.

$$\int \underbrace{\arctan x}_{u} \underbrace{(x^2 dx)}_{dv} = (\underbrace{\arctan x}_{u})(\underbrace{\frac{1}{3} x^3}_{v}) - \int \left(\underbrace{\frac{1}{3} x^3}_{v} \right) \left(\underbrace{\frac{dx}{1+x^2}}_{du} \right) \longleftarrow v = \frac{1}{3} x^3, \quad du = (\arctan x)' dx = \frac{dx}{1+x^2}$$

$$= \frac{1}{3}x^3 \arctan x - \frac{1}{3}\int \frac{x^3}{1+x^2}dx \longleftarrow \frac{x^3}{1+x^2} = x - \frac{x}{1+x^2} \quad \textit{by long division}$$
$$= \frac{1}{3}x^3 \arctan x - \frac{1}{3}\int \left(x - \frac{x}{1+x^2} \right) dx = \frac{1}{3}x^3 \arctan x - \frac{1}{3}\left(\frac{1}{2}x^2 - \int \frac{x}{1+x^2}dx \right)$$
$$= \frac{1}{3}x^3 \arctan x - \frac{1}{6}x^2 + \frac{1}{3}\int \frac{x}{1+x^2}dx \longleftarrow u = 1+x^2, du = 2xdx \leftrightarrow xdx = \frac{1}{2}du$$
$$= \frac{1}{3}x^3 \arctan x - \frac{1}{6}x^2 + \frac{1}{3}\int \frac{1}{2}\frac{du}{u} = \frac{1}{3}x^3 \arctan x - \frac{1}{6}x^2 + \frac{1}{6}\ln u + c$$
$$= \frac{1}{6}[2x^3 \arctan x - x^2 + \ln(1+x^2)] + c.$$

2.2. Weaknesses of LIATE

Although LIATE works well in finding out the integrals for some types of combinative integrands, more types of combinative integrands that are within the span of LIATE cannot be integrated using integration by parts under the guidance of LIATE. For example, the combinations of LI (e.g., $\ln x \cdot \arctan x$), LT (e.g., $\ln x \cdot \cos x$), LE (e.g., $\ln x \cdot e^x$), IT (e.g., $\arcsin x \cdot \cos x$), IE (e.g., $\arctan x \cdot e^x$), and some AT (e.g., $\frac{\sin x}{x}$) are indicated by LIATE, but none of these is integrable by integration by parts

Example 6: Evaluate $\int \ln x \cos x dx$.

In the integrand $\ln x \cos x$, ln$x$ is a logarithmic function (L) and cos$x$ is a trigonometric function (T). By formula (2) and LIATE, let $u = \ln x$ and $dv = \cos x dx$.

$$\int \underbrace{\ln x}_{u}\underbrace{\cos x dx}_{dv} = (\underbrace{\ln x}_{u})(\underbrace{\sin x}_{v}) - \int (\underbrace{\sin x}_{v})\left(\underbrace{\frac{1}{x}dx}_{du} \right) = \ln x \sin x - \int \frac{\sin x}{x}dx.$$

Here $\int \frac{\sin x}{x}dx$ is not integrable by analytical methods.

Example 7: Evaluate $\int \arctan x \cdot e^x dx$.

In the integrand arctan$x$·e^x, $\arctan x$ is an inverse trigonometric function (I) and e^x is an exponential function (E). By formula (2) and LIATE, let u = arctan$x$ and $dv = e^x dx$.

$$\int \underbrace{\arctan x}_{u}\underbrace{e^x dx}_{dv} = (\underbrace{\arctan x}_{u})(\underbrace{e^x}_{v}) - \int (\underbrace{e^x}_{v})\left(\underbrace{\frac{dx}{1+x^2}}_{du} \right) = e^x \arctan x - \int \frac{e^x}{1+x^2}dx.$$

Here $\int \frac{e^x}{1+x^2}dx$ is not integrable by analytical methods.

Hence, without an explicit explanation on the limit or applicability of LIATE, students might be misled to some extent.

Another ambiguous issue with LIATE is that its term for algebraic functions (A) in LIATE is too loose compared with other functions that are specified for particular operations. For example, logarithmic functions are defined for any positive number to any positive base except 1. In mathematics, algebraic functions cover all common functions, including power functions, polynomials, and fractions. In most cases, A in LIATE only represents polynomial functions where the exponent only takes zero or positive integers. This issue has been noticed by some educators and one suggestion was to replace A in LIATE by P for polynomial functions, which makes LIATE to LIPTE [15]. By using P for A, however, LIPTE would exclude some combinative integrands involving power functions with negative or fractional values for the exponent, which is demonstrated by the following examples.

Example 8: Evaluate $\int \frac{1}{\sqrt{x}} \ln x dx$.

In the integrand $\frac{1}{\sqrt{x}} \ln x$, ln$x$ is a logarithmic function (L) and $\frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$ is an algebraic function (A). By formula (2) and LIATE, let $u = \ln x$ and $dv = x^{-\frac{1}{2}} dx$.

$$\int \underbrace{\ln x}_{u} \underbrace{\frac{1}{\sqrt{x}} dx}_{dv} = (\underbrace{\ln x}_{u})(\underbrace{2\sqrt{x}}_{v}) - \int (\underbrace{2\sqrt{x}}_{v}) \left(\underbrace{\frac{1}{x} dx}_{du} \right) = 2\sqrt{x} \ln x - \int \frac{2}{\sqrt{x}} dx$$
$$= 2\sqrt{x} \ln x - 4\sqrt{x} + c = 2\sqrt{x} (\ln x - 2) + c.$$

However, by LIPTE, $\frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$ is not a polynomial function; hence LIPTE is not applicable to it.

Example 9: Evaluate $\int \frac{1}{x^3} \ln x dx$.

In the integrand $\frac{1}{x^3} \ln x$, ln$x$ is a logarithmic function (L) and $\frac{1}{x^3} = x^{-3}$ is an algebraic function (A). By formula (2) and LIATE, let $u = \ln x$ and $dv = \frac{1}{x^3} dx$.

$$\int \underbrace{\ln x}_{u} \underbrace{\frac{1}{x^3} dx}_{dv} = (\underbrace{\ln x}_{u}) \left(\underbrace{-\frac{1}{2x^2}}_{v} \right) - \int \left(\underbrace{-\frac{1}{2x^2}}_{v} \right) \left(\underbrace{\frac{1}{x} dx}_{du} \right) = -\frac{1}{2x^2} \ln x + \int \frac{1}{2x^3} dx = -\frac{1}{2x^2} \ln x - \frac{1}{4x^2} + c$$
$$= -\frac{1}{4x^2} (2 \ln x + 1) + c.$$

However, by LIPTE, $\frac{1}{x^3} = x^{-3}$ is not a polynomial function; hence LIPTE is not applicable to this case.

Another minor issue with LIATE is that the order for a combinative integrand with a trigonometric function (T) and an exponential function (E) is interchangeable in most cases if it is integrable. Hence, TE can be as ET too, which is demonstrated by the following example.

Example 10: Evaluate $\int e^x \cos x dx$.

In the integrand $e^x\cos x$, $\cos x$ is a trigonometric function (T) and e^x is an exponential function (E). By formula (2) and LIATE, let $u = \cos x$ and $dv = e^x dx$.

$$I = \int \underbrace{\cos x}_{u} \underbrace{e^x dx}_{dv} = (\underbrace{\cos x}_{u})(\underbrace{e^x}_{v}) - \int (\underbrace{e^x}_{v})(\underbrace{-\sin x dx}_{du}) = e^x \cos x + \int \sin x e^x dx$$
$$= e^x \cos x + \left(\sin x e^x - \int \cos x e^x dx\right) = e^x \cos x + e^x \sin x - \underbrace{\int \cos x e^x dx}_{I}$$
$$I = e^x \sin x + e^x \cos x - I \longrightarrow 2I = e^x \sin x + e^x \cos x$$
$$I = \int \cos x e^x dx = \frac{1}{2} e^x (\sin x + \cos x) + c.$$

Alternatively, the same result can be found by exchanging the order of the exponential function e^x (E) with the trigonometric function $\cos x$ (T) as follows.

$$I = \int \underbrace{e^x}_{u} \underbrace{\cos x dx}_{dv} = (\underbrace{e^x}_{u})(\underbrace{\sin x}_{v}) - \int (\underbrace{\sin x}_{v})(\underbrace{e^x dx}_{du}) = e^x \sin x - \int \sin x e^x dx$$
$$= e^x \sin x - \left(e^x(-\cos x) + \int \cos x e^x dx\right) = e^x \sin x + e^x \cos x - \underbrace{\int e^x \cos x dx}_{I}$$
$$I = e^x \sin x + e^x \cos x - I \longrightarrow 2I = e^x (\sin x + \cos x)$$
$$I = \int \cos x e^x dx = \frac{1}{2} e^x (\sin x + \cos x) + c.$$

This kind of technique is sometimes referred to as the embedded integration by parts [13,16]. Even though, in most cases, only sine or cosine functions combined with either exponential functions or polynomial functions are integrable by the embedded integration by parts, rather than all trigonometric functions, which is demonstrated by the following example.

Example 11: Evaluate $\int x^2 \tan x dx$.

In the integrand $x^2\tan x$, $\tan x$ is a trigonometric function (T) and x^2 is a polynomial function (P). By formula (2) and LIPTE, let $u = x^2$ and $dv = \tan x dx$.

$$\int \underbrace{x^2}_{u} \underbrace{\tan x dx}_{dv} = (\underbrace{x^2}_{u})(\underbrace{-\ln \cos x}_{v}) - \int (\underbrace{-\ln \cos x}_{v})(\underbrace{2x dx}_{du}) = -x^2 \ln \cos x + \int 2x \ln \cos x dx.$$

The new integration has an integrand $2x \ln \cos x$, a combination of a logarithmic function $\ln\cos x$ and a polynomial function $2x$, which by LIPTE should go with $u = \ln \cos x$ and $dv = 2x dx$. This will result in

$$\int \underbrace{x^2}_{u} \underbrace{\tan x dx}_{dv} = -x^2 \ln \cos x + \int 2x \ln \cos x dx = -x^2 \ln \cos x + (\ln \cos x)(x^2) - \int x^2 (-\tan x dx)$$

or $$\int \underbrace{x^2}_{u} \underbrace{\tan x dx}_{dv} = -x^2 \ln \cos x + x^2 \ln \cos x + \int x^2 \tan x dx = \int x^2 \tan x dx.$$

This means such integration is not resolvable by integration by parts, despite indicated in LIPTE.

3. The *Pet-LoPo-InPo* guide

By examining LIATE and LIPTE, the algebraic function (A) or the polynomial function (P) plays a pivotal role in both of them. As aforementioned, either A or P is not properly defined in LIATE or LIPTE and simply using A or P would not be a proper reflection of the integrability of the combined integrands and the function types of A or P. Hence, a new guide based on the properties of the P-functions, *Pet-LoPo-InPo*, is summarized in this note for conducting integration by parts. The *Pet-LoPo-InPo* guide is based on the following classified definitions.

- ***P***: Polynomial functions in the form of ax^n where n can take zero and positive integers, for example, 5, $2x^2$, $-4x^5$, $(3x^2 - 4x^5 + 1)$.
- ***Po***: Power functions in the form of ax^n where n can take any real number (zero, positive and negative), for example, 3, $2x^3$, x^{-2}, $\sqrt{x}$, $\frac{2}{\sqrt[3]{x}}$. In most cases, n cannot take -1.
- ***nPo***: Integer power functions in the form of ax^n where n can only take integers, for example, 3, $2x^3$, x^{-2}. In most cases, n cannot take -1.
- ***Lo***: Logarithmic functions, for example, $\ln x$, $\log_b x$.
- ***I***: Inverse trigonometric functions, such as $\arcsin x$, $\arctan x$.
- ***e***: Exponential functions, for example, e^x, e^{-3x}, 5^{2x}.
- ***t***: The primary trigonometric (or sinusoidal) functions, for example, $\sin x$, $\cos 2x$, $4\sin 3x$.

The three sub-guides, *Pet*, *LoPo* and *InPo*, are independent from each other, i.e., each sub-guide only works within its group. *Pet* means that combinations of any two functions from polynomial functions, exponential functions, and primary trigonometric functions are possibly integrable by integration by parts. *LoPo* indicates that combinations of logarithmic functions only with power functions are possibly integrable by integration by parts. Similarly, *InPo* implies that combinations of inverse trigonometric functions only with integer power functions are possibly integrable by integration by parts. This is a stark contrast to LIATE that does not exclude those incompatible combinations for using integration by parts.

Within a sub-guide, the process works the same as using LIATE, i.e., the order of selecting the first function (first part) u from the combined integrand for integration by parts is from the left letter to the right letter. For example, in the integrand $3x^2\cos x$, $3x^2$ is a polynomial function (P) and $\cos x$ is a primary trigonometric function (t). By the *Pet* guide, P ($3x^2$) is on the left of t ($\cos x$), hence, $u = 3x^2$.

In *Pet*, capital P means that any polynomial function should come first before either the exponential or primary trigonometric functions in integration by parts. The use of small letters e and t implies that the order of these two functions is interchangeable in integration by parts. This is to relax the order indicated in LIATE. The use of *Pet-LoPo-InPo* is demonstrated by the following examples.

3.1. Pet

The *Pet* guide is largely similar to ATE in LIATE or PTE in LIPTE demonstrated in Examples 1, 3 and 10, except the interchangeability of the order between the exponential and primary trigonometric functions in their combinations.

Example 12: Evaluate $\int (2x^2-4x+3)\sin 2x dx$.

In $(2x^2-4x+3)\sin 2x$, $(2x^2-4x+3)$ is a second-order polynomial function (*P*) and sin2*x* is a primary trigonometric function (*t*). By *Pet,* let $u=2x^2-4x+3$ and $dv=\sin 2x dx$.

$$\int \underbrace{(2x^2-4x+3)}_{u}\underbrace{\sin 2x dx}_{dv} = \underbrace{(2x^2-4x+3)}_{u}\left(\underbrace{-\frac{1}{2}\cos 2x}_{v}\right) - \int\left(\underbrace{-\frac{1}{2}\cos 2x}_{v}\right)\underbrace{(4x-4)dx}_{du}$$
$$= -\left(x^2-2x+\frac{3}{2}\right)\cos 2x + \int 2(x-1)\cos 2x dx$$
$$= -\left(x^2-2x+\frac{3}{2}\right)\cos 2x + 2(x-1)\left(\frac{1}{2}\sin 2x\right) - \int \sin 2x dx$$
$$= \left(-x^2+2x-\frac{3}{2}\right)\cos 2x + (x-1)\sin 2x + \frac{1}{2}\cos 2x + c$$
$$= (-x^2+2x-1)\cos 2x + (x-1)\sin 2x + c = (x-1)\sin 2x - (x-1)^2\cos 2x + c.$$

Example 13: Evaluate $\int (3x^2-4x)(e^x-e^{-x})dx$.

In $(3x^2-4x)(e^x-e^{-x})$, $(3x^2-4x)$ is a second-order polynomial function (*P*) and (e^x-e^{-x}) is a combined exponential function (*e*). By *Pet,* let $u=3x^2-4x$ and $dv=(e^x-e^{-x})dx$.

$$\int \underbrace{(3x^2-4x)}_{u}\underbrace{(e^x-e^{-x})dx}_{dv} = \underbrace{(3x^2-4x)}_{u}\underbrace{(e^x+e^{-x})}_{v} - \int \underbrace{(e^x+e^{-x})}_{v}\underbrace{(6x-4)dx}_{du}$$
$$= (3x^2-4x)(e^x+e^{-x}) - \int \underbrace{(6x-4)}_{u}\underbrace{(e^x+e^{-x})dx}_{dv}$$
$$= (3x^2-4x)(e^x+e^{-x}) - (6x-4)(e^x-e^{-x}) + \int 6(e^x-e^{-x})dx$$
$$= (3x^2-4x)(e^x+e^{-x}) - (6x-4)(e^x-e^{-x}) + 6(e^x+e^{-x}) + c$$
$$= (3x^2-4x+6)(e^x+e^{-x}) - (6x-4)(e^x-e^{-x}) + c$$
$$= (3x^2-10x+10)e^x + (3x^2+2x+2)e^{-x} + c$$

The interchangeability of the order between the exponential and primary trigonometric functions in their combinations was already demonstrated in Example 10.

3.2. LoPo

The *LoPo* guide is for integrands only involving both logarithmic functions (*Lo*) and power functions (*Po*) because logarithmic functions are not compatible with any other functions except power functions for using integration by parts. In such a situation, the logarithmic function should always come first before the power function. The *LoPo* guide not only excludes the unnecessary relationships between logarithmic functions and other functions in LIATE but also broadens the coverage of the *P*-functions in LIPTE. Examples 8 and 9 solved previously already demonstrated the power of *LoPo* for such cases. The following three examples demonstrate both the special and general cases of using *LoPo* for integration by parts, respectively.

Example 14: Evaluate $I = \int \frac{\ln x}{x} dx$.

As the integrand involves ln*x*, by *LoPo*, let $u = \ln x$ and $dv = \frac{1}{x} dx$.

$$I = \int \underbrace{\ln x}_{u} \underbrace{\frac{1}{x} dx}_{dv} = \underbrace{\ln x}_{u} \underbrace{\ln x}_{v} - \int \underbrace{\ln x}_{v} \underbrace{\left(\frac{dx}{x}\right)}_{du} = (\ln x)^2 - \int \frac{\ln x}{x} dx = (\ln x)^2 - I$$

$$2I = (\ln x)^2 \longrightarrow I = \frac{1}{2} (\ln x)^2 + c.$$

Alternatively, this integral can be resolved by substitution but it is not the focus of this note.

Example 15: Evaluate $I(n) = \int x^n \ln x dx$ where *n* is any real number except –1.

In $x^n \ln x$, x^n is a power function (*Po*). By *LoPo*, let $u = \ln x$ and $dv = x^n dx$.

$$I(n) = \int \underbrace{\ln x}_{u} \underbrace{x^n dx}_{dv} = \underbrace{\ln x}_{u} \underbrace{\left(\frac{1}{n+1} x^{n+1}\right)}_{v} - \int \underbrace{\left(\frac{1}{n+1} x^{n+1}\right)}_{v} \underbrace{\left(\frac{dx}{x}\right)}_{du} = \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{n+1} \int x^n dx$$
$$= \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{(n+1)^2} x^{n+1} + c = \frac{1}{n+1} x^{n+1} \left(\ln x - \frac{1}{n+1}\right) + c.$$

This is a general solution to all integrands in the form of $x^n \ln x$ where *n* can take any real number, except –1. For example, taking *n* = 3, the solution is the same as that in Example 2; taking *n* = –3, the solution is the same as that in Example 9; taking *n* = –1/2, the solution is the same as that in Example 8. By taking *n* = 1/3, the solution becomes

$$I(\frac{1}{3}) = \int x^{\frac{1}{3}} \ln x dx = \frac{1}{\frac{1}{3}+1} x^{\frac{1}{3}+1} \left(\ln x - \frac{1}{\frac{1}{3}+1} \right) + c = \frac{3}{4} x^{\frac{4}{3}} \left(\ln x - \frac{3}{4} \right) + c.$$

This formula is not applicable for *n* = –1, which is the special case solved in Example 14.

The following example is to derive a reduction formula using integration by parts.

Example 16: Evaluate $I(m,n) = \int (\ln x)^m x^n dx$ where *m* is any positive integer and *n* is any real number except –1.

The integrand is a combination of the logarithmic function (*Lo*) and the power function (*Po*). By *LoPo*, let $u = (\ln x)^m$ and $dv = x^n dx$.

$$I(m,n) = \int \underbrace{(\ln x)^m}_{u} \underbrace{x^n dx}_{dv} = \underbrace{(\ln x)^m}_{u} \underbrace{\left(\frac{1}{n+1} x^{n+1}\right)}_{v} - \int \underbrace{\left(\frac{1}{n+1} x^{n+1}\right)}_{v} \underbrace{\left(m(\ln x)^{m-1} \frac{dx}{x}\right)}_{du}$$
$$= \frac{1}{n+1}(\ln x)^m x^{n+1} - \frac{m}{n+1}\int (\ln x)^{m-1} x^n dx = \frac{1}{n+1}(\ln x)^m x^{n+1} - \frac{m}{n+1} I(m-1,n).$$

This is actually the general reduction formula to all integrands in the form of $(\ln x)^m x^n$ where m is any positive integer and n can take any real number except –1. This means that Examples 2, 8, 9 and 15 are special cases of $I(1, 3)$, $I(1, -½)$, $I(1, -3)$, and $I(1, n)$, respectively. To obtain the integral for $I(2, 3)$, we can use the above reduction formula as follows.

$$I(2,3) = \int (\ln x)^2 x^3 dx = \frac{1}{3+1}(\ln x)^2 x^{3+1} - \frac{2}{3+1} I(2-1,3) = \frac{1}{4}(\ln x)^2 x^4 - \frac{1}{2} I(1,3).$$

$I(1, 3)$ was obtained in Example 2, hence,

$$I(2,3) = \int (\ln x)^2 x^3 dx = \frac{1}{4}(\ln x)^2 x^4 - \frac{1}{2} I(1,3) = \frac{1}{4}(\ln x)^2 x^4 - \frac{1}{2}\left[\frac{1}{16} x^4 (4\ln x - 1)\right] + c$$
$$= \frac{1}{4}(\ln x)^2 x^4 - \frac{1}{32} x^4 (4\ln x - 1) + c = \frac{1}{32} x^4 \left[8(\ln x)^2 - 4\ln x + 1\right] + c.$$

Similarly, the integral for $(\ln x)^3$ can be found using this reduction formula as flows.

$$I(0,0) = \int (\ln x)^0 x^0 dx = \int dx = x$$
$$I(1,0) = \int (\ln x) dx = \frac{1}{0+1}(\ln x) x^{0+1} - \frac{1}{0+1} I(1-1,0) = (\ln x)x - I(0,0) = (\ln x)x - x$$
$$I(2,0) = \int (\ln x)^2 dx = \frac{1}{0+1}(\ln x)^2 x^{0+1} - \frac{2}{0+1} I(2-1,0) = (\ln x)^2 x - 2I(1,0)$$
$$= (\ln x)^2 x - 2\left[(\ln x)x - x\right] = x\left[(\ln x)^2 - 2(\ln x) + 2\right]$$
$$I(3,0) = \int (\ln x)^3 dx = \frac{1}{0+1}(\ln x)^3 x^{0+1} - \frac{3}{0+1} I(3-1,0) = (\ln x)^3 x - 3I(2,0)$$
$$= (\ln x)^3 x - 3x\left[(\ln x)^2 - 2(\ln x) + 2\right] + c = x\left[(\ln x)^3 - 3(\ln x)^2 + 6(\ln x) - 6\right] + c.$$

3.3. InPo

InPo is for integrands involving some combinations of the inverse trigonometric functions (*I*) and the integer power functions (*nPo*) where the exponent can only take negative or positive integers, except –1. In such a situation, the inverse trigonometric function should always come first before the power function. The *InPo* guide not only excludes the unnecessary relationships between the inverse trigonometric functions and other functions in LIATE but also expands the coverage of the polynomial functions in LIPTE to the negative half. Examples 4 and 5 solved previously can also be solved exactly the same way using *InPo*. The following two examples demonstrate solving integrals involving power functions with negative integer exponents by *InPo* through integration by parts.

Example 17: Evaluate $\int \frac{\arcsin x}{x^3}dx = \int x^{-3}\arcsin x dx$.

The integrand is a combination of an inverse trigonometric function and a power function with a negative index. By *InPo*, let $u = \arcsin x$ and $dv = x^{-3}dx$.

$$\int \underbrace{\arcsin x}_{u}\,\underbrace{x^{-3}dx}_{dv} = \underbrace{(\arcsin x)}_{u}\underbrace{\left(\frac{1}{-2}x^{-2}\right)}_{v} - \int \underbrace{\left(\frac{1}{-2}x^{-2}\right)}_{v}\underbrace{\left(\frac{dx}{\sqrt{1-x^2}}\right)}_{du} = -\frac{1}{2x^2}\arcsin x + \frac{1}{2}\int \frac{dx}{x^2\sqrt{1-x^2}}.$$

Since

$$\int \frac{dx}{x^2\sqrt{1-x^2}} = \int \frac{\cos t dt}{\sin^2 t\sqrt{1-\sin^2 t}} = \int \frac{\cos t dt}{\sin^2 t \cos t} \longleftarrow x = \sin t, dx = \cos t dt$$

$$= \int \frac{dt}{\sin^2 t} = -\cot t = -\frac{\cos t}{\sin t} = -\frac{\sqrt{1-x^2}}{x},$$

$$\int \frac{\arcsin x}{x^3}dx = -\frac{1}{2x^2}\arcsin x - \frac{1}{2}\frac{\sqrt{1-x^2}}{x} + c = -\frac{1}{2x^2}(\arcsin x + x\sqrt{1-x^2}) + c.$$

Example 18: Evaluate $\int \frac{\arctan x}{x^4}dx = \int x^{-4}\arctan x dx$.

The integrand is a combination of an inverse trigonometric function and a power function with a negative index. By *InPo*, let $u = \arctan x$ and $dv = x^{-4}dx$.

$$\int \underbrace{\arctan x}_{u}\,\underbrace{x^{-4}dx}_{dv} = \underbrace{(\arctan x)}_{u}\underbrace{\left(\frac{1}{-3}x^{-3}\right)}_{v} - \int \underbrace{\left(\frac{1}{-3}x^{-3}\right)}_{v}\underbrace{\left(\frac{dx}{1+x^2}\right)}_{du} = -\frac{1}{3x^3}\arctan x + \frac{1}{3}\int \frac{dx}{x^3(1+x^2)}.$$

By partial fractions

$$\frac{1}{x^3(1+x^2)} = \frac{ax^2+bx+c}{x^3} + \frac{dx+e}{1+x^2} = \frac{(ax^2+bx+c)(1+x^2)+x^3(dx+e)}{x^3(1+x^2)}$$

$$\frac{1}{x^3(1+x^2)} = \frac{ax^2+bx+c+ax^4+bx^3+cx^2+dx^4+ex^3}{x^3(1+x^2)} = \frac{(a+d)x^4+(b+e)x^3+(a+c)x^2+bx+c}{x^3(1+x^2)}$$

Hence, by comparing the coefficients on both sides,

$$\begin{cases} a+d=0 \\ b+e=0 \\ a+c=0 \\ b=0 \\ c=1 \end{cases} \longrightarrow \begin{cases} d=-a=1 \\ e=-b=0 \\ a=-c=-1 \\ b=0 \\ c=1 \end{cases} \longrightarrow \frac{1}{x^3(1+x^2)}=\frac{1-x^2}{x^3}+\frac{x}{1+x^2}=\frac{1}{x^3}-\frac{1}{x}+\frac{x}{1+x^2}.$$

Therefore,

$$\int \frac{\arctan x}{x^4}dx=-\frac{1}{3x^3}\arctan x+\frac{1}{3}\int \frac{dx}{x^3(1+x^2)}=-\frac{1}{3x^3}\arctan x+\frac{1}{3}\int\left(\frac{1}{x^3}-\frac{1}{x}+\frac{x}{1+x^2}\right)dx$$

$$=-\frac{1}{3x^3}\arctan x+\frac{1}{3}\left(-\frac{1}{2x^2}-\ln x+\frac{1}{2}\ln(1+x^2)\right)+c$$

$$=-\frac{1}{3x^3}\arctan x-\frac{1}{6x^2}-\frac{1}{3}\ln x+\frac{1}{6}\ln(1+x^2)+c=\frac{1}{6}\left[\ln\frac{1+x^2}{x^2}-\frac{1}{x^3}(2\arctan x+x)\right]+c.$$

4. Conclusions

This note is based on the author’s tutorial notes on integration by parts for the undergraduate students accumulated over the past several years at a regional university in Australia. It was triggered by the citation of the LIATE rule by a few students in their assignments prior to the preparation of this note on integration by parts. Although further investigations found some problems with LIATE for conducting integration by parts, LIATE remains a useful rule of thumb for undergraduate students in their attempts to solve integrals through integration by parts. Nevertheless, it is desirable to amend those problems with LIATE so as to reduce the possible causes that may confuse or mislead students in utilizing integration by parts. It would be even better to expand the span of LIATE to cover more types of integrands solvable through integration by parts or by mixed approaches.

By reclassifying the algebraic functions (A) in LIATE into the polynomial functions (*P*), the standard power functions (*Po*), and the integer power functions (*nPo*), this note has largely achieved the goals by modifying LIATE or LIPTE to *Pet-LoPo-InPo* to guide practices of using integration by parts. The *Pet* guide applies to integrands involving any combination of two functions from polynomial, exponential, and primary trigonometric functions, indicating that the polynomial function should come first before either the exponential or the primary trigonometric function whereas the order of the exponential and primary trigonometric functions are interchangeable. The *LoPo* guide applies to the integrands only involving logarithmic and power functions where the logarithmic function is always the first choice. The *InPo* guide applies to the integrands involving only the inverse trigonometric and the integer power functions where the inverse trigonometric function is always the first choice.

This new guide removes many incompatible combinative integrands included in LIATE, rationalizes the relationship between the exponential and trigonometric functions in LIATE, and expands the coverage of the *P*-functions beyond the traditional definition in LIPET. Hence, the new guide can reduce confusions experienced by students in using LIATE for their practices of integration by parts. The advantages of this new guide have been demonstrated by many worked examples in this note.

However, this new guide is not as concise as the LIATE rule. Furthermore, although this new guide has expanded the span of LIATE for using integration by parts, it is still unsure how much more the *Pet-LoPo-InPo* guide could reach out to other integrands by using integration by parts. As demonstrated in the worked examples, particularly for *InPo* where the inverse trigonometric functions are involved, integration by parts alone would not be able to find the final result. Mixed approaches, demonstrated in Examples 4, 5, 17 and 18, would be required to obtain the integrals for sophisticated integrands.

Conflict of interest

The author declares there is no conflict of interest.

References

1. W. Guo, W. Li, C. C. Tisdell, Effective pedagogy of guiding undergraduate engineering students solving first-order ordinary differential equations, *Mathematics*, **9** (2021), 1623. https://doi.org/10.3390/math9141623
2. R. K. Nagle, E. B. Saff, *Fundamentals of Differential Equations*, 3rd ed. 1993, USA: Addison-Wesley.
3. D. G. Zill, *A First Course in Differential Equations with Modeling Applications*, 10th ed. 2013, Boston, USA: Cengage Learning.
4. W. Guo, Unification of the common methods for solving the first-order linear ordinary differential equations, *STEM Educ.*, **1** (2021), 127–140. https://doi.org/10.3934/steme.2021010
5. P. Revathy, R. Prabakaran, S. Muthukumar, Contemporary issues in teaching and learning techniques of differential equations: A review among engineering students, *Int. J. Adv. Sci. Technol.*, **29** (2020), 1313–1330.
6. A. S. Firdous, Z. L. Waseem, S. N. Kottakkaran, S. K. Amany, Analytical solutions of generalized differential equations using quadratic-phase Fourier transform, *AIMS Math.*, **7** (2022), 1925–1940. https://doi.org/10.3934/math.2022111
7. W. Guo, The Laplace transform as an alternative general method for solving linear ordinary differential equations, *STEM Educ.*, **1** (2021), 309–329. https://doi.org/10.3934/steme.2021020
8. E. Kreyszig, *Advanced Engineering Mathematics*, 10th ed. 2011, USA: Wiley.
9. J. Stewart, *Calculus: Concepts and Contexts*, 4th ed. 2019. USA: Cengage.
10. D. Trim, *Calculus for Engineers*. 4th ed. 2008, Toronto, Canada: Pearson.
11. A. Croft, R. Davison, M. Hargreaves, J. Flint, *Engineering Mathematics*, 5th ed. 2017, Harlow, UK: Pearson.
12. J. Bird, *Higher Engineering Mathematics*. 7th ed. 2014, UK: Routledge.
13. W. Guo, *Essentials and Examples of Applied Mathematics*, 2nd ed. 2021, Melbourne, Australia: Pearson.
14. H. E. Kasube, A technique for integration by parts, *Am. Math. Mon.*, **90** (1983), 210–211. https://doi.org/10.1080/00029890.1983.11971195
15. C. Taylor, *The LIPET Strategy for Integration by Parts*. 2019. Retrieved on the 16th of July 2022 from https://www.thoughtco.com/liPetstrategy-for-integration-by-parts-3126211.
16. W. Guo, Streamlining applications of integration by parts in teaching applied calculus, *STEM Educ.*, **2** (2022), 73–83. https://doi.org/10.3934/steme.2022005

AIMS Press

©2022 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)