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Research article

Stability of mild solutions of the fractional nonlinear abstract Cauchy problem

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Abstract: Since the first work on Ulam-Hyers stabilities of differential equation solutions to date, many important and relevant papers have been published, both in the sense of integer order and fractional order differential equations. However, when we enter the field of fractional calculus, in particular, involving fractional differential equations, the path that is still long to be traveled, although there is a range of published works. In this sense, in this paper, we investigate the Ulam-Hyers and Ulam-Hyers-Rassias stabilities of mild solutions for fractional nonlinear abstract Cauchy problem in the intervals $[0, T]$ and $[0, \infty)$ using Banach fixed point theorem.

Keywords: fractional nonlinear abstract Cauchy; Ulam-Hyers stabilities; semigroup theory; mild solution; Banach fixed point theorem

1. Introduction

One of the most active themes of differential equations has been the Ulam-Hyers stability. The theme came in 1940 by Ulam in a lecture on unresolved issues at the University of Wisconsin [1, 2]. The issue raised by Ulam was partially answered the following year by Hyers in the case of the Banach spaces. Thus, on the theory of stabilities, came to be called Ulam-Hyers. However, in 1978 [3], Rassias presented a generalization of the version presented by Hyers. In this sense, due to this breakthrough and novelty in mathematical analysis, many researchers have investigated the stability of solutions of functional differential equations. The idea of Ulam-Hyers stability for functional equations, is the substitution of the functional equation for a given inequality that acts as a perturbation of the equation. We suggest some monographs and papers that allow a more thorough search of the subjects [4–7].

With the beginning of the fractional calculus and over the years his theory being well consolidated and grounded, many researchers began to look in a different way for the area, especially researchers working with differential equations [8–14]. In this sense, today it is more than proven that investigating and analyzing certain physical problems, through fractional derivatives, ensures more accurate and consistent results with reality. On the other hand, moving to a more theoretical side, investigating the existence, uniqueness and Ulam-Hyers stability of solution for fractional differential equations has gained increasing prominence in the scientific community, although there are a range of works, the theory is still being built with good results [15–18].

In 2020, Inc et al. [19] investigated the solution of the fractional Burger-Huxley equation in the Caputo fractional derivative sense. In this sense, some examples were presented in order to elucidate the investigated results. On the other hand, we can also highlight the interesting work carried out by Ahmad et al. [20], on the new analyzing technique for nonlinear time-fractional Cauchy reaction-diffusion model equations. The present work aims to investigate numerical solutions of equations of the nonlinear fractional order Cauchy reaction-diffusion model in time. Interestingly, the approach discussed in this work can be used without the use of any transformation, Adomian polynomials, small perturbations, discretization or linearization. In this sense, solutions numerically are compared with exact solutions. In 2021, Chu et al. [21], evaluated a fractional Cauchy diffusion-reaction equation via the homotopy perturbation transform and iterative transform method. In this sense, the authors concluded that by the current investigated technique, they indicate that the approach is easy to implement and accurate. Other interesting works, in particular, with a numerical approach see [22–26] and the references therein.

In 2012, Wang and Zhou [15], investigated several kind of stabilities of mild solution for fractional evolution equation in Banach space, namely: Mittag-Leffler-Ulam stability, Mittag-Leffler-Ulam-Leffler-Ulam-Hyers stability, Mittag-Leffler-Ulam-Hyers-Rassias stability and generalized Mittag-Leffler-Ulam-Hyers-Rassias stability. In 2014, Abbas [27] investigated the existence, uniqueness and stability of mild solution for integrodifferential equation with nonlocal conditions through Holder inequality, Schauder fixed point theorem, and Gronwall inequality in Banach space. On the other hand, Zhou and Jiao [11], using fractional operators and some fixed point theorems, investigated the existence and uniqueness of mild solutions for fractional neutral evolution equations and made some applications in order to elucidate the obtained results. In this sense, Saadati et al. [28], presented results on the existence of mild solutions for fractional abstract equations with non-instantaneous impulses. In order to obtain such results, the authors used non-compactness measure and the Darbo-Sadovskii and Tychonoff fixed point theorems. See too [12, 29–35] and the references therein.

In 2017, Zhou et al. [36] investigated sufficient conditions for the existence of non-oscillatory solutions for a neutral functional differential equation. We can also highlight the important work addressed by Wang et al. [37], on the existence and Ulam stability of the new class for fractional order differential switched systems with coupled nonlocal initial and impulsive conditions in $\mathbb{R}^n$.

Although we are faced with a significant amount of work dealing with solution properties of fractional differential equations, there is still much work to be done. In order to propose new results and provide new materials on Ulam-Hyers stability in order to contribute positively to the area, the present paper has as main purpose to investigate some Ulam-Hyers stabilities in the intervals $[0, T]$ and $[0, \infty)$.

So, we consider the fractional nonlinear abstract Cauchy problem given by

$$\left\{ \begin{array}{rcll} {}^{H}\mathbb{D}_{0^+}^{\alpha,\beta}\xi(t) & = & \mathcal{A}\xi(t) + u(t)\mathcal{H}(t,\xi(t)), & t \in I \\ I_{0^+}^{1-\gamma}\xi(0) & = & \xi_0 \end{array} \right. \tag{1.1}$$

where ${}^{H}\mathbb{D}_{0^+}^{\alpha,\beta}(\cdot)$ is the Hilfer fractional derivative of order $0 < \alpha \leq 1$ and type $0 \leq \beta \leq 1$, $I = [0,T]$ or $[0,\infty)$, $\xi \in \Omega$, Ω Banach space, $t \in I$, $: \Omega \to \Omega$ is the infinitesimal generator of a C_0-semigroup $(\mathbb{S}(t))_{t\geq 0}$ and $\mathcal{H} : I \times \Omega \to \Omega$ is a given continuous function.

We highlight below the main points that motivated us to investigate the mild solution stability for the fractional abstract Cauchy problem:

1) A new class of Ulam-Hyers type stabilities for fractional abstract Cauchy problem;
2) At the limit of $\beta \to 1$ in the mild solution for abstract Cauchy problem with $0 < \alpha < 1$, we have a sub-class of Ulam-Hyers stabilities for the Riemann-Liouville fractional derivative;
3) At the limit of $\beta \to 0$ in the mild solution for abstract Cauchy problem with $0 < \alpha < 1$, we have a sub-class of Ulam-Hyers stabilities for the Caputo fractional derivative;
4) When $\alpha = 1$, we have as particular case, the integer version;
5) An important consequence of the obtained results are the possible future applications through the Ulam-Hyers stabilities in engineering, biology and especially in mathematics;

The paper is organized as follows. In Section 2 we introduce the ξ-Riemann-Liouville fractional integral, the ξ-Hilfer fractional derivative and fundamental concept of the operator (α,β)-resolvent. In this sense, it is presented the mild solution for fractional Cauchy problem as well as the Ulam-Hyers stability. In Section 3, it is directed to the first result of this paper, that is, we investigate the Ulam-Hyers and Ulam-Hyers-Rassias stabilities in the interval $[0,T]$ and discuss some particular cases. In Section 4, we discuss the Ulam-Hyers and Ulam-Hyers-Rassias stabilities in the interval $[0,\infty)$. Concluding remarks close the paper.

2. Preliminaries

In this Section, we introduce some important definitions and results in order to assist in the development of this paper.

Let $T > 0$ be a given positive real number. The weighted space of continuous functions $\xi \in I_1 = (0,T]$ and $I = [0,T]$ is given by reference [38]

$$C_{1-\gamma}(I_1,\Omega) = \left\{\xi \in C(I_1,\Omega),\ t^{1-\gamma}\xi(t) \in C(I_1,\Omega)\right\}$$

where $0 \leq \gamma \leq 1$, with norm

$$\|\xi\|_{C_{1-\gamma}} = \sup_{t\in I} \|\xi(t)\|_{C_{1-\gamma}}$$

and

$$\|\xi - \phi\|_{C_{1-\gamma}} = \mathrm{d}_{1-\gamma}(\xi,\phi) := \sup_{t\in I} \|\xi(t) - \phi(t)\|_{C_{1-\gamma}}.$$

Let $(\Omega, \|\cdot\|_{C_{1-\gamma}})$ be a given Banach space and $I = [0,+\infty)$ or $I = [0,T]$ where T and $\mathscr{L}(\Omega)$ the set of bounded linear maps from Ω to Ω.

Let (a,b) $(-\infty \leq a < b \leq \infty)$ be a finite interval (or infinite) of the real line $\mathbb{R}$ and let $\alpha > 0$. Also let $\psi(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(x)$ (we denote first derivative as $\dfrac{d}{dx}\psi(x) = \psi'(x)$) on (a,b). The left-sided fractional integral of a function f with respect to a function ψ on $[a,b]$ is defined by references [8, 38]

$$\mathcal{I}_{a+}^{\alpha;\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(s)\,(\psi(x) - \psi(s))^{\alpha-1} f(s)\,ds. \tag{2.1}$$

On the other hand, let $n-1 < \alpha < n$ with $n \in \mathbb{N}$, let $J = [a,b]$ be an interval such that $-\infty \leq a < b \leq \infty$ and let $f, \psi \in C^n[a,b]$ be two functions such that ψ is increasing and $\psi'(x) \neq 0$, for all $x \in J$. The left-sided $\psi-$Hilfer fractional derivative ${}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi}(\cdot)$ of a function f of order α and type $0 \leq \beta \leq 1$, is defined by references [8, 9]

$${}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(x) = \mathcal{I}_{a+}^{\beta(n-\alpha);\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathcal{I}_{a+}^{(1-\beta)(n-\alpha);\psi} f(x). \tag{2.2}$$

Next, we present the definition of the operator (α,β)-resolvent, fundamental in the construction of mild solution for fractional abstract Cauchy problem Eq (1.1).

Definition 2.1. Let $\alpha > 0$ and $\beta \geq 0$ [34]. A function $\mathbb{S}_{\alpha,\beta} : \mathbb{R}_+ \to \mathscr{L}(\Omega)$ is called a β-times integrated α-resolvent operator function of an (α,β)-resolvent operator function (ROF) if the following conditions are satisfied:

1) $\mathbb{S}_{\alpha,\beta}(\cdot)$ is strongly continuous on $\mathbb{R}_+$ and $\mathbb{S}_{\alpha,\beta}(0) = g_{\beta+1}(0)I$;
2) $\mathbb{S}_{\alpha,\beta}(s)\mathbb{S}_{\alpha,\beta}(t) = \mathbb{S}_{\alpha,\beta}(t)\mathbb{S}_{\alpha,\beta}(s)$ for all $t, s \geq 0$;
3) The function equation $\mathbb{S}_{\alpha,\beta}(s)I_t^{\alpha}\mathbb{S}_{\alpha,\beta}(t) - I_s^{\alpha}\mathbb{S}_{\alpha,\beta}(s)\mathbb{S}_{\alpha,\beta}(t) = g_{\beta+1}(s)I_t^{\alpha}\mathbb{S}_{\alpha,\beta}(t) - g_{\beta+1}(t)I_s^{\alpha}\mathbb{S}_{\alpha,\beta}(s)$ for all $t, s \geq 0$.

The generator $\mathcal{A}$ of $\mathbb{S}_{\alpha,\beta}$ is defined by

$$D(\mathcal{A}) := \left\{ x \in \Omega : \lim_{t \to 0^+} \frac{\mathbb{S}_{\alpha,\beta}(t)\,x - g_{\beta+1}(t)\,x}{g_{\alpha+\beta+1}(t)} \text{ exists} \right\} \tag{2.3}$$

and

$$\mathcal{A}\,x := \lim_{t \to 0^+} \frac{\mathbb{S}_{\alpha,\beta}(t)\,x - g_{\beta+1}(t)\,x}{g_{\alpha+\beta+1}(t)}, \quad x \in D(\mathcal{A}), \tag{2.4}$$

where $g_{\alpha+\beta+1}(t) := \dfrac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta)}$ $(\alpha + \beta > 0)$.

An (α,β)-ROF $\mathbb{S}_{\alpha,\beta}$ is said to be exponentially bounded if there exist constants $\delta \geq 1$, $w \geq 0$ such that $\|\mathbb{T}_{\alpha}(t)\|_{C_{1-\gamma}} \leq \delta\, e^{wt}$ and $\|\mathbb{S}_{\alpha,\beta}(t)\|_{C_{1-\gamma}} \leq \delta\, e^{wt}$, $\quad t \geq 0\cdot$.

Now, we consider the continuous function given $\mathcal{H} : I \times \Omega \to \Omega$ such that, for almost all $t \in I$, yields

$$\|\mathcal{H}(t,x) - \mathcal{H}(t,y)\|_{C_{1-\gamma}} \leq \ell(t)\|x - y\|_{C_{1-\gamma}}, \quad x, y \in \Omega \tag{2.5}$$

where $\ell : [0,T] \to \mathbb{R}^+$ and $u : [0,T] \to \mathbb{R}$ are two given measurable functions such that ℓ, u and ℓu are locally integrable on I.

The following is the definition of the Mainardi function, fundamental in mild solution of Eq (1.1). Then, the Wright function, denoted by $M_\alpha(Q)$, is defined by references [39, 40]

$$M_\alpha(Q) = \sum_{n=1}^{\infty} \frac{(-Q)^{n-1}}{(n-1)!\Gamma(1-\alpha n)}, \quad 0 < \alpha < 1, \quad Q \in \mathbb{C}$$

satisfying the relation

$$\int_0^\infty \theta^{\bar{\delta}} M_\alpha(\theta)\,\mathrm{d}\theta = \frac{\Gamma(1+\bar{\delta})}{\Gamma(1+\alpha\bar{\delta})}, \quad \text{for } \theta, \bar{\delta} \geq 0\cdot$$

Lemma 2.2. *The fractional nonlinear differential equation* [39, 40], Eq (1.1)*, is equivalent to the integral equation*

$$\xi(t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)}\xi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[\mathcal{A}\xi(s) + u(s)\mathcal{H}(s,\xi(s))\right]\mathrm{d}s, \ t \in [0,T]\cdot \tag{2.6}$$

A function $\xi \in C_{1-\gamma}(I,\Omega)$ is called a mild solution of Eq (1.1)*, if the integral equation,* Eq (2.6) *holds, yields*

$$\xi(t) = \mathbb{S}_{\alpha,\beta}(t)\xi(0) + \int_0^t \mathbb{T}_\alpha(t-s)u(s)\mathcal{H}(s,\xi(s))\,\mathrm{d}s, \quad t \in I \tag{2.7}$$

where $\mathbb{T}_\alpha(t) = t^{\alpha-1}G_\alpha(t)$, $G_\alpha(t) = \displaystyle\int_0^\infty \alpha\theta M_\alpha(\theta)\mathbb{S}(t^{\alpha\theta})\,\mathrm{d}\theta$ and $\mathbb{S}_{\alpha,\beta}(t) = \mathcal{I}_0^{\beta(1-\alpha)}\mathbb{T}_\beta(t)$.

For a given $\xi_0 \in \Omega$ and any $\xi \in C_{1-\gamma}(I,\Omega)$, we set

$$\Lambda(\xi)(t) := \mathbb{S}_{\alpha,\beta}(t)\xi_0 + \int_0^t \mathbb{T}_\alpha(t-s)u(s)\mathcal{H}(s,\xi(s))\,\mathrm{d}s \tag{2.8}$$

for all $t \in I$.

For the procedure in this paper, ℓ, u are measurable functions such that ℓ, u and the product ℓu are locally integrable. Moreover, it is easy to see that the application $\xi \to \Lambda(\xi)$ is a self-mapping of the space $C_{1-\gamma}(I,\Omega)$.

On the other hand, for $\xi_0 \in \Omega$ and ε, we consider

$$\xi(t) = \Lambda(\xi(t)), \quad t \in I \tag{2.9}$$

and the following inequalities

$$\|\xi(t) - \Lambda(\xi(t))\|_{C_{1-\gamma}} \leq \varepsilon, \quad t \in I \tag{2.10}$$

and

$$\|\xi(t) - \Lambda(\xi(t))\|_{C_{1-\gamma}} \leq G(t), \quad t \in I \tag{2.11}$$

where $\xi \in C_{1-\gamma}(I,\Omega)$ and $G \in C_{1-\gamma}(I,(0,+\infty))$.

The following are the definitions of the main results to be investigated in this paper. The definitions were adapted to the problem version of fractional differential equations.

Definition 2.3. The Eq (2.9) is Ulam-Hyers stable if there exists a real number $c > 0$ such that for each $\varepsilon > 0$ and for each solution $\xi \in C_{1-\gamma}(I,\Omega)$ of Eq (2.9) such that [15, 41]

$$\|\xi(t) - v(t)\|_{C_{1-\gamma}} \leq \varepsilon, \quad t \in I\cdot \tag{2.12}$$

Definition 2.4. The Eq (2.9) is generalized Ulam-Hyers stable if there exists $\theta \in C_{1-\gamma}([0,+\infty),[0,+\infty))$, $\theta(0)=0$, such that for each $\varepsilon > 0$ and for each solution $\xi \in C_{1-\gamma}(I,\Omega)$ of Eq (2.10) there exists a solutions $v \in C_{1-\gamma}(I,\Omega)$ of Eq (2.9) such that [15, 41]

$$||\xi(t) - v(t)||_{C_{1-\gamma}} \leq \theta(\varepsilon)\,, \quad t \in I\cdot \tag{2.13}$$

Definition 2.5. The Eq (2.9) is generalized Ulam-Hyers-Rassias stable with respect to $G \in C_{1-\gamma}([0,+\infty),[0,+\infty))$, if there exists $c_G > 0$ such that for each solution $\xi \in C_{1-\gamma}(I,\Omega)$ of Eq (2.11) there exists a solution $v \in C_{1-\gamma}(I,\Omega)$ of Eq (2.9) such that [15, 41]

$$||\xi(t) - v(t)||_{C_{1-\gamma}} \leq c_G G(t)\,, \quad t \in I\cdot \tag{2.14}$$

3. Ulam-Hyers and Ulam-Hyers-Rassias stabilities of mild on $[0,T]$.

In this Section, we investigate the first of the main results of this paper, i.e., the Ulam-Hyers and Ulam-Hyers-Rassias stabilities of Eq (2.9) in the interval $[0,T]$ using the Banach fixed point theorem.

Let $\left(\mathbb{S}_{\alpha,\beta}(t)\right)_{t\geq 0}$ the (α,β)-resolvent operator function on a Banach space $(\Omega, ||\cdot||_{C_{1-\gamma}})$ and the continuous function $\xi : [0,T] \to \Omega$, given by

$$\Lambda(\xi)(t) := \mathbb{S}_{\alpha,\beta}(t)\xi_0 + \int_0^t \mathbb{T}_\alpha(t-s)u(s)\mathcal{H}(s,\xi(s))\,\mathrm{d}s\,, \quad t \in [0,T) \tag{3.1}$$

for $\xi_0 \in \Omega$ fixed.

Then, we have the theorem that gives certain conditions, guarantees the Ulam-Hyers stability to Eq (2.9) on the finite interval $[0,T]$.

Theorem 3.1. *Let* $\left(\mathbb{S}_{\alpha,\beta}(t)\right)_{t\geq 0}$ *the* (α,β)*-resolvent operator function on a Banach space* $(\Omega, ||\cdot||_{C_{1-\gamma}})$, *with* $0 \leq \gamma \leq 1$ *and let* $T > 0$ *be a positive real number. We set*

$$\widetilde{\lambda} := \delta \int_0^T e^{w(T-s)}|u(s)|\ell(s)\,\mathrm{d}s\cdot \tag{3.2}$$

If $\widetilde{\lambda} < 1$, *then the* Eq (2.9) *is stable in the Ulam-Hyers sense.*

Proof. Admit that $\widetilde{\lambda} < 1$ and let $\varepsilon > 0$ be given. For $\phi, \xi \in C_{1-\gamma}(I, \Omega)$, yields

$$\begin{aligned}
\|(\Lambda\phi)(t) - (\Lambda\xi)(t)\|_{C_{1-\gamma}} &= \left\| \mathbb{S}_{\alpha,\beta}(t)\xi_0 + \int_0^t \mathbb{T}_\alpha(t-s)u(s)\mathcal{H}(s,\phi(s))\,\mathrm{d}s \right. \\
&- \left. \mathbb{S}_{\alpha,\beta}(t)\xi_0 - \int_0^t \mathbb{T}_\alpha(t-s)u(s)\mathcal{H}(s,\xi(s))\,\mathrm{d}s \right\|_{C_{1-\gamma}} \\
&= \left\| \int_0^t \mathbb{T}_\alpha(t-s)u(s)\left(\mathcal{H}(s,\phi(s)) - \mathcal{H}(s,\xi(s))\right)\,\mathrm{d}s \right\|_{C_{1-\gamma}} \\
&\leq \int_0^t \|\mathbb{T}_\alpha(t-s)\|_{C_{1-\gamma}} |u(s)| \|\mathcal{H}(s,\phi(s)) - \mathcal{H}(s,\xi(s))\|_{C_{1-\gamma}}\,\mathrm{d}s \\
&\leq \int_0^t \|\mathbb{T}_\alpha(t-s)\|_{C_{1-\gamma}} |u(s)|\,\ell(s)\,\|\phi(s) - \xi(s)\|_{C_{1-\gamma}}\,\mathrm{d}s \\
&\leq \int_0^T \delta\, e^{w(T-s)} |u(s)|\,\ell(s)\,\|\phi(s) - \xi(s)\|_{C_{1-\gamma}}\,\|\phi(s) - \xi(s)\|_{C_{1-\gamma}}\,\mathrm{d}s \\
&= \delta \int_0^T e^{w(T-s)} |u(s)|\,\ell(s)\,\mathrm{d}s\,\|\phi(s) - \xi(s)\|_{C_{1-\gamma}} \\
&= \widetilde{\lambda}\|\phi - \xi\|_{C_{1-\gamma}}, \quad t \in [0,T]\cdot
\end{aligned}$$

So, one has

$$\mathrm{d}_{1-\gamma}(\Lambda\phi, \Lambda\xi) \leq \widetilde{\lambda}\,\mathrm{d}_{1-\gamma}(\xi, \xi).$$

What implies, that Λ is a contradiction. On the other hand, consider $\theta, \phi \in C_{1-\gamma}(I, \Omega)$, such that

$$\mathrm{d}_{1-\gamma}(\Lambda\theta, \theta) \leq \varepsilon\cdot$$

and

$$\mathrm{d}_{1-\gamma}(\phi, \xi) \leq \frac{\varepsilon}{1-\widetilde{\lambda}}.$$

Thus, we obtain

$$\begin{aligned}
\mathrm{d}_{1-\gamma}(\Lambda\phi, \theta) &\leq \mathrm{d}_{1-\gamma}(\theta, \Lambda\theta) + \mathrm{d}_{1-\gamma}(\Lambda\theta, \Lambda\phi) \\
&\leq \varepsilon + \frac{\varepsilon}{1-\widetilde{\lambda}} \leq \frac{\varepsilon}{1-\widetilde{\lambda}}\cdot
\end{aligned}$$

In this sense, we have that the ball closed $\overline{B}_{C_{1-\gamma}}\left(\theta, \dfrac{\varepsilon}{1-\widetilde{\lambda}}\right)$ of the Banach space $C_{1-\gamma}(I, \Omega)$ is invariant by the map Λ, that is,

$$\Lambda\left(\overline{B}_{C_{1-\gamma}}\left(\theta, \frac{\varepsilon}{1-\widetilde{\lambda}}\right)\right) \subset \overline{B}_{C_{1-\gamma}}\left(\theta, \frac{\varepsilon}{1-\widetilde{\lambda}}\right)\cdot$$

Then, applying the Banach fixed point theorem in the Λ acting $\overline{B}_{C_{1-\gamma}}\left(\theta, \dfrac{\varepsilon}{1-\widetilde{\lambda}}\right)$, we have that there is only one element such that $\xi = \Lambda(\xi)$. So we have to ξ is a solution of Eq (2.9), which satisfies

$$d_{1-\gamma} = \|\theta(t) - \xi(t)\|_{C_{1-\gamma}} \leq c\,\varepsilon, \quad t \in [0,T]$$

where $c := 1/(1-\widetilde{\lambda})$. Thus, we conclude that the integral equation (2.9) is stable in the Ulam-Hyers sense . □

Thus, we conclude the first part of the result. Next, we will investigate the Ulam-Hyers-Rassias stability by completing the first purpose of this paper.

Theorem 3.2. *Let* $(\Omega, \|\cdot\|_{C_{1-\gamma}})$ *be a Banach space and let* $(\mathbb{S}_{\alpha,\beta}(t))_{t\geq 0}$ *be a* (α,β)*-resolvent operator function on* Ω*. Let* $\delta \geq 1$*,* $w \geq 0$ *be constants such that*

$$\|\mathbb{S}_{\alpha,\beta}(t)\|_{C_{1-\gamma}} \leq \delta\, e^{wt} \text{ and } \|\mathbb{T}_{\alpha}(t)\|_{C_{1-\gamma}} \leq \delta\, e^{wt} \tag{3.3}$$

for all $t \geq 0$*. Let* $\xi_0 \in \Omega$*,* $T > 0$ *and* $G : [0,T] \to (0,\infty)$ *be a continuous function.*

Suppose that a continuous function $f : [0,T] \to \Omega$ *satisfies*

$$\left\| f(t) - \mathbb{S}_{\alpha,\beta}(t)\xi_0 - \int_0^t \mathbb{T}_{\alpha}(t-s)u(s)\mathcal{H}(s, f(s))\,\mathrm{d}s \right\|_{C_{1-\gamma}} \leq G(t) \tag{3.4}$$

for all $t \in [0,T]$*.*

Suppose that there exists a positive constant ρ *such that*

$$\ell(s)|u(s)|e^{w(T-s)} \leq \rho \tag{3.5}$$

for almost all $s \in [0,T]$*. Then,* $\exists C_G > 0$ *(constant) and a unique continuous function* $v : [0,T] \to\to \Omega$ *such that*

$$v(t) = \mathbb{S}_{\alpha,\beta}(t)\xi_0 + \int_0^t \mathbb{T}_{\alpha}(t-s)u(s)\mathcal{H}(s, v(s))\,\mathrm{d}s\,, \quad t \in [0,T] \tag{3.6}$$

and

$$\|f(t) - v(t)\|_{C_{1-\gamma}} \leq C_G G(t)\,, \quad t \in [0,T]\,\cdot \tag{3.7}$$

Proof. Consider $K > 0$ be such that $K\delta\rho < 1$ and continuous function $\phi : [0,T] \to (0,\infty)$ as follows,

$$\int_0^t \phi(s)\,\mathrm{d}s \leq K\,\phi(t)\,, \quad t \in [0,T]. \tag{3.8}$$

Now let, f, G satisfy the inequality (3.4) and $\widetilde{\alpha}_G, \widetilde{\beta}_G > 0$ such that

$$\widetilde{\alpha}_G\phi(t) \leq G(t) \leq \widetilde{\beta}_G\phi(t)\,, \quad t \in [0,T]\,\cdot \tag{3.9}$$

On the other hand, for all $h, g \in C_{1-\gamma}(I,\Omega)$, consider the following set

$$\mathrm{d}_{\phi,1-\gamma}(h,g) := \inf\left\{C \in [0,\infty) : \|h(t) - g(t)\|_{C_{1-\gamma}} \leq C\phi(t)\,, \quad t \in [0,T]\right\}\,\cdot$$

It is easy to see that $(C_{1-\gamma}(I,\Omega), \mathrm{d}_{\phi,1-\gamma})$ is a metric and that $(C_{1-\gamma}(I,\Omega), \mathrm{d}_{\phi,1-\gamma})$ is a complete metric space.

Now, consider the operator $\Lambda : C_{1-\gamma}(I,\Omega) \to C_{1-\gamma}(I,\Omega)$ defined by

$$(\Lambda h)(t) := \mathbb{S}_{\alpha,\beta}(t)\xi_0 + \int_0^t \mathbb{T}_{\alpha}(t-s)u(s)\mathcal{H}(s, h(s))\,\mathrm{d}s\,, \quad t \in [0,T]\cdot$$

The next step is to show that Λ is a contraction in the metric space $C_{1-\gamma}(I,\Omega)$ induced by metric $\mathrm{d}_{\phi,1-\gamma}$. Then, let $h, g \in C_{1-\gamma}(I,\Omega)$ and $C(h,g) \in [0,\infty)$ a constant such that

$$\|h(t) - g(t)\|_{C_{1-\gamma}} \leq C(h,g)\phi(t)\,, \quad t \in [0,T]\,\cdot$$

Then, using Eqs (3.3), (3.5) and (3.8), we obtain

$$\begin{aligned}
\|(\Lambda h)(t)-(\Lambda g)(t)\|_{C_{1-\gamma}} &= \left\|\int_0^t \mathbb{T}_\alpha(t-s)u(s)\left(\mathcal{H}(s,h(s))-\mathcal{H}(s,g(s))\right)\mathrm{d}s\right\|_{C_{1-\gamma}}\\
&\leq \int_0^t \|\mathbb{T}_\alpha(t-s)\|_{C_{1-\gamma}}|u(s)|\|\mathcal{H}(s,h(s))-\mathcal{H}(s,g(s))\|_{C_{1-\gamma}}\,\mathrm{d}s\\
&\leq \int_0^t \delta\, e^{w(t-s)}|u(s)|\ell(s)\|h(s))-g(s\|_{C_{1-\gamma}}\,\mathrm{d}s\\
&\leq \delta C(h,g)\int_0^t e^{w(t-s)}\phi(s)|u(s)|\ell(s)\,\mathrm{d}s\\
&\leq \delta C(h,g)\rho\int_0^t \phi(s)\,\mathrm{d}s \leq \delta C(h,g)\rho K\phi(t)\\
&= C(h,g)\delta\rho K\phi(t)\,,\quad t\in[0,T]\cdot
\end{aligned}$$

Therefore, we have $d_{\phi,1-\gamma}(\Lambda(h),\Lambda(g)) \leq \delta\rho K C(h,g)$ from which we deduce that

$$d_{\phi,1-\gamma}(\Lambda(h),\Lambda(g)) \leq \delta\rho K d_{\phi,1-\gamma}(h,g)\cdot$$

Using the fact that $\delta\rho K < 1$, we have that Λ is a contraction in $(C_{1-\gamma}(I,\Omega), d_{\phi,1-\gamma})$. In this sense, through Banach fixed point theorem, we have that there is a unique function $v \in C_{1-\gamma}(I,\Omega)$ such that $v = \Lambda(v)$. Now, using By the triangle inequality, yields

$$\begin{aligned}
d_{\phi,1-\gamma}(f,v) &\leq d_{\phi,1-\gamma}(f,\Lambda(f)) + d_{\phi,1-\gamma}(\Lambda(f),\Lambda(v))\\
&\leq \beta_G + \delta\rho K d_{\phi,1-\gamma}(f,v)
\end{aligned}$$

which implies that

$$d_{\phi,1-\gamma}(f,v) \leq \frac{\beta_G}{1-\delta\rho K}. \tag{3.10}$$

Which in turn, one has

$$\begin{aligned}
\|f(t)-v(t)\|_{C_{1-\gamma}} &\leq \frac{\beta_G}{1-\delta\rho K}\phi(t)\\
&\leq \frac{\beta_G}{1-\delta\rho K}\frac{G(t)}{\alpha_G} = C_G G(t)\,,\quad t\in[0,T]
\end{aligned} \tag{3.11}$$

where $C_G := \dfrac{\beta_G}{(1-\delta\rho K)\alpha_G}$, which is the desired inequality (3.7). □

Remark 3.3. *From* Theorem 3.1 *and* Theorem 3.2*, we have some particular cases, that is, by taking the boundaries with $\beta \to 1$ and $\beta \to 0$. Also, we have the whole case when $\alpha = 1$. So we have the following versions:*

(1) *For $\beta \to 0$ in Theorem 3.1, we have the particular case in the sense of the Riemann-Liouville fractional derivative, given by:*

Theorem 3.4. *Let $\left(\mathbb{S}_{\alpha,0}(t)\right)_{t\geq 0}$ the $(\alpha,0)-$ resolvent operator function on Banach space $\left(\Omega, \|\cdot\|_{C_\alpha}\right)$ with and let $T > 0$ be a positive real number. We set*

$$\widetilde{\lambda} := \delta\int_0^T e^{\omega(T-s)}\,|u(s)|\,\ell(s)\,ds.$$

If $\widetilde{\lambda} < 1$ them Eq (2.9) *is stable in the Ulam-Hyers sense.*

(2) *Taking limit* $\beta \to 1$ *in* Eq (2.9), *we have the version of Theorem 3.1 for the Caputo fractional derivative, ensuring that* Eq (2.9) *is Ulam-Hyers stable.*

(3) *Taking limit* $\beta \to 1$ *in* Eq (2.9), *we have as particular case the version of* Theorem 3.2 *for the Caputo fractional derivative given by the following theorem (Ulam-Hyers-Rassias):*

Theorem 3.5. *Let* $(\Omega, \|\cdot\|)$ *be a Banach space and* $(\mathbb{S}_{\alpha,1}(t))_{t\geq 0}$ *be* $(\alpha, 1)-$ *resolvent operator function on* Ω. *Let* $\delta \geq 1$, $\omega \geq 0$ *be constants such that* $\left\|\mathbb{S}_{\alpha,1}(t)\right\| \leq \delta e^{\omega t}$ *and* $\|\mathbb{T}_{\alpha}(t)\| \leq \delta e^{\omega t}$ *for all* $t \geq 0$. *Let* $\xi_0 \in \Omega$ *be fixed,* $T > 0$ *and* $G : [0,T] \to (0,\infty)$ *be a continuous function. Suppose that a continuous function* $f : [0,T] \to \Omega$ *satisfies*

$$\left\|f(t) - \mathbb{S}_{\alpha,1}(t)\xi_0 - \int_0^t \mathbb{T}_{\alpha}(t-s)\,u(s)\,H(s, f(s))\,ds\right\| \leq G(t)$$

for all $t \in [0,T]$.

Suppose that exists a positive constant ρ *such that*

$$\ell(s)\,|u(s)|\,e^{\omega(T-s)} \leq \rho$$

for almost all $s \in [0,T]$. *Then, exist the constant* $C_G > 0$ *and a unique continuous functions* $v : [0,T] \to \Omega$ *such that*

$$v(t) = \mathbb{S}_{\alpha,1}(t)\xi_0 + \int_0^t \mathbb{T}_{\alpha}(t-s)\,u(s)\,H(s, f(s))\,ds,\ t \in [0,T]$$

and

$$\|f(t) - v(t)\| \leq C_G G(t),\ \forall t \in [0,T].$$

(4) *Taking limit* $\beta \to 1$ *or* $\beta \to 0$ *and choosing* $\alpha = 1$, *we have the version of the* Theorem 3.1 *and* Theorem 3.2, *for integer case.*

4. Ulam-Hyers and Ulam-Hyers-Rassias stabilities of mild solution on $[0, +\infty)$

As in Section 3, we investigate the Ulam-Hyers and Ulam-Hyers-Rassias stabilities in the interval $[0, +\infty]$, in fact completing the main results investigated in this paper. So we start with the following theorem:

Theorem 4.1. *Let* $\xi_0 \in \Omega$ *be fixed and let* $\varepsilon > 0$ *be a given positive number. Suppose that a continuous function* $f : [0, +\infty) \to \Omega$ *satisfies*

$$\left\|f(t) - \mathbb{S}_{\alpha,\beta}(t)\xi_0 - \int_0^t \mathbb{T}_{\alpha}(t-s)u(s)\mathcal{H}(s, f(s))\,\mathrm{d}s\right\|_{C_{1-\gamma}} \leq \varepsilon \tag{4.1}$$

for all $t \in [0, +\infty)$.

Suppose that

$$\widetilde{\lambda}_{\alpha,1-\gamma} = \sup_{t\geq 0} \int_0^t \ell(s)|u(s)|\|\mathbb{T}_{\alpha}(t-s)\|_{C_{1-\gamma}}\,\mathrm{d}s < 1 \tag{4.2}$$

with $0 < \alpha \leq 1$ *and* $0 \leq \gamma \leq 1$.

Then, there exists a unique continuous function $v : [0,+\infty) \to \Omega$ *such that*

$$v(t) = \mathbb{S}_{\alpha,\beta}(t)\xi_0 + \int_0^t u(s)\mathbb{T}_\alpha(t-s)\mathcal{H}(s, v(s))\,\mathrm{d}s\,, \quad t \in [0,+\infty) \tag{4.3}$$

and

$$\|f(t) - v(t)\|_{C_{1-\gamma}} \leq \frac{\varepsilon}{1-\widetilde{\lambda}_{\alpha,1-\gamma}}\,, \quad t \in [0,+\infty)\cdot \tag{4.4}$$

Proof. Consider that $\widetilde{\lambda}_{\alpha,1-\gamma} < 1$, $\varepsilon > 0$ be given and $f \in C_{1-\gamma}([0,+\infty),\Omega)$ satisfy the inequality (4.1). On the other hand, we consider the set $\widetilde{\mathcal{E}}_{f,1-\gamma}$, given by

$$\widetilde{\mathcal{E}}_{f,1-\gamma} := \left\{ g \in C_{1-\gamma}([0,+\infty),\Omega);\ \sup_{t\geq 0} \|g(t) - f(t)\|_{C_{1-\gamma}} < +\infty \right\}. \tag{4.5}$$

The set $\widetilde{\mathcal{E}}_{f,1-\gamma}$ is no-empty, because it contains f and $\Lambda(f)$. Now, consider the functions $h, g \in \widetilde{\mathcal{E}}_{f,1-\gamma}$, such that

$$d_{1-\gamma}(h,g) := \sup_{t\geq 0} \|h(t) - g(t)\|_{C_{1-\gamma}}.$$

Then, $d_{1-\gamma}$ is a distance and the metric space $(\widetilde{\mathcal{E}}_{f,1-\gamma}, d_{1-\gamma})$ is complete.

For any functions $h, g \in \widetilde{\mathcal{E}}_{f,1-\gamma}$, yields

$$\begin{aligned}
\|(\Lambda h)(t) - (\Lambda g)(t)\|_{C_{1-\gamma}} &= \left\| \int_0^t u(s)\mathbb{T}_\alpha(t-s)\left[\mathcal{H}(s,h(s)) - \mathcal{H}(s,g(s))\right]\,\mathrm{d}s \right\|_{C_{1-\gamma}} \\
&\leq \int_0^t \|\mathbb{T}_\alpha(t-s)\|_{C_{1-\gamma}} |u(s)| \|\mathcal{H}(s,h(s)) - \mathcal{H}(s,g(s))\|_{C_{1-\gamma}}\,\mathrm{d}s \\
&\leq \int_0^t \|\mathbb{T}_\alpha(t-s)\|_{C_{1-\gamma}} |u(s)| \ell(s) \|h(s) - g(s)\|_{C_{1-\gamma}}\,\mathrm{d}s \\
&\leq \int_0^t \|\mathbb{T}_\alpha(t-s)\|_{C_{1-\gamma}} |u(s)| \ell(s)\,\mathrm{d}s\, d_{1-\gamma}(h,g) \\
&\leq \widetilde{\lambda}_{\alpha,1-\gamma} d_{1-\gamma}(h,g)\,, \quad t \in [0,+\infty).
\end{aligned}$$

Therefore, follows that

$$d_{1-\gamma}(\Lambda h, \Lambda g) \leq \widetilde{\lambda}_{\alpha,1-\gamma} d_{1-\gamma}(h,g).$$

Moreover, it is easy to show that $\Lambda(h) \in \widetilde{\mathcal{E}}_{f,1-\gamma}$ for any function $h \in \widetilde{\mathcal{E}}_{f,1-\gamma}$. Thus, we have Λ is a contraction in $(\widetilde{\mathcal{E}}_{f,1-\gamma}, d_{1-\gamma})$. In this sense, by Banach fixed point theorem, we have that there is only one element $v \in \widetilde{\mathcal{E}}_{f,1-\gamma}$ such that $v = \Lambda(v)$. By the triangle inequality, yields

$$\begin{aligned}
d_{1-\gamma}(f,v) &\leq \leq d_{1-\gamma}(f,\Lambda(f)) + d_{1-\gamma}(\Lambda(f),\Lambda(v)) \\
&\leq \varepsilon + \widetilde{\lambda}_{\alpha,1-\gamma} d_{1-\gamma}(f,v)
\end{aligned}$$

that implies

$$d_{1-\gamma}(f,v) \leq \frac{\varepsilon}{1-\widetilde{\lambda}_{\alpha,1-\gamma}},$$

this is,

$$\|f(t) - v(t)\|_{C_{1-\gamma}} \leq c\,\varepsilon\,, \quad t \in [0,+\infty) \tag{4.6}$$

where $c := \dfrac{1}{1-\widetilde{\lambda}_{\alpha,1-\gamma}}$.

The inequality (4.6) shows that the Eq (2.9) is Ulam-Hyers stable. □

With the following result aimed at investigating the Ulam-Hyers-Rassias stability, complete the second main result of this paper.

Theorem 4.2. *Let Ω be a Banach space, $(\mathbb{S}_{\alpha,\beta}(t))_{t\geq 0}$ be a (α,β)-resolvent operator function on Ω and $\phi_0 \in \Omega$ be fixed. Let $K > 0$ be given and $\phi : [0,+\infty) \to (0,+\infty)$ be a continuous function such that*

$$\int_0^t \phi(s)\,\mathrm{d}s \leq K\,\phi(t)\,, \quad t \in [0,+\infty)\cdot \tag{4.7}$$

Suppose that a continuous function $f : [0,+\infty) \to \Omega$ satisfies

$$\left\| f(t) - \mathbb{S}_{\alpha,\beta}(t)\xi_0 - \int_0^t u(s)\mathbb{T}_\alpha(t-s)\mathcal{H}(s, f(s))\,\mathrm{d}s \right\|_{C_{1-\gamma}} \leq \phi(t) \tag{4.8}$$

for all $t \in [0,+\infty)$.

Suppose that there exists a positive constant $\rho > 0$ such that

$$\ell(s)|u(s)|\|\mathbb{T}_\alpha(t-s)\|_{C_{1-\gamma}} \leq \rho \tag{4.9}$$

for almost all $(s,t) \in [0,+\infty)$ with $0 \leq s \leq t$ and suppose that

$$K\rho < 1\cdot \tag{4.10}$$

Then, there exists a unique continuous function $v : [0,+\infty) \to \omega$ such that

$$v(t) = \mathbb{S}_{\alpha,\beta}(t)\xi_0 + \int_0^t u(s)\mathbb{T}_\alpha(t-s)\mathcal{H}(s, v(s))\,\mathrm{d}s\,, \quad t \in [0,+\infty) \tag{4.11}$$

and

$$\|f(t) - v(t)\|_{C_{1-\gamma}} \leq \frac{1}{1-K\rho}\phi(t)\,, \quad t \in [0,+\infty)\cdot \tag{4.12}$$

Proof. Let $f \in C_{1-\gamma}([0,+\infty))$ satisfy the inequality (4.8) and the following set, defined by

$$\widetilde{\mathcal{E}}_{f,1-\gamma} := \left\{ g \in C_{1-\gamma}([0,+\infty),\Omega) : \exists C \geq 0 : \|g(t) - f(t)\|_{C_{1-\gamma}} \leq C\phi(t)\,, \quad t \in [0,+\infty) \right\}.$$

The set $\widetilde{\mathcal{E}}_{f,1-\gamma}$ is not empty, because it contains f and $\Lambda(f)$.

Now, for $h, g \in \widetilde{\mathcal{E}}_{f,1-\gamma}$, we define the following set

$$d_{\phi,1-\gamma}(h,g) := \inf\left\{ C \in [0,+\infty) : \|h(t) - g(t)\|_{C_{1-\gamma}} \leq C\phi(t)\,, \quad t \in [0,+\infty) \right\}.$$

Note that it is easy to see that $(\widetilde{\mathcal{E}}_{f,1-\gamma}, d_{\phi,1-\gamma})$ is a complete metric space satisfying $\Lambda(\widetilde{\mathcal{E}}_{f,1-\gamma}) \subset \widetilde{\mathcal{E}}_{f,1-\gamma}$, where $\Lambda : \widetilde{\mathcal{E}}_{f,1-\gamma} \to \widetilde{\mathcal{E}}_{f,1-\gamma}$ is defined by

$$(\Lambda h)(t) := \mathbb{S}_{\alpha,\beta}(t)\xi_0 + \int_0^t u(s)\mathbb{T}_\alpha(t-s)\mathcal{H}(s, h(s))\,\mathrm{d}s\,, \quad t \in [0,+\infty).$$

The idea is to prove that in fact the Λ application is a contraction on the metric space $(\widetilde{\mathcal{E}}_{f,1-\gamma}, \mathrm{d}_{\phi,1-\gamma})$. Then, let $h, g \in \widetilde{\mathcal{E}}_{f,1-\gamma}$ and $C(h,g) \in [0,+\infty)$ be an arbitrary constant such that

$$\|h(t) - g(t)\|_{C_{1-\gamma}} \leq C(h,g)\phi(t)\,, \quad t \in [0,+\infty).$$

In this sense, we have the following inequality

$$\begin{aligned}
\|(\Lambda h)(t) - (\Lambda g)(t)\|_{C_{1-\gamma}} &= \left\| \int_0^t u(s)\mathbb{T}_\alpha(t-s)\left(\mathcal{H}(s,h(s)) - \mathcal{H}(s,g(s))\right) \mathrm{d}s \right\|_{C_{1-\gamma}} \\
&\leq \int_0^t |u(s)| \|\mathbb{T}_\alpha(t-s)\|_{C_{1-\gamma}} \|\mathcal{H}(s,h(s)) - \mathcal{H}(s,g(s))\|_{C_{1-\gamma}} \,\mathrm{d}s \\
&\leq \int_0^t |u(s)| \|\mathbb{T}_\alpha(t-s)\|_{C_{1-\gamma}} \ell(s) \|h(s)) - g(s\|_{C_{1-\gamma}} \,\mathrm{d}s \\
&\leq C(h,g) \int_0^t |u(s)| \ell(s) \|\mathbb{T}_\alpha(t-s)\|_{C_{1-\gamma}} \phi(s) \,\mathrm{d}s \\
&\leq C(h,g) d \int_0^t \phi(s) \,\mathrm{d}s \\
&\leq C(h,g) dK\phi(t)\,, \quad t \in [0,T]\cdot
\end{aligned}$$

Therefore, yields $\mathrm{d}_{\phi,1-\gamma}(\Lambda(h), \Lambda(g)) \leq C(h,g)\rho K$, that implies in

$$\mathrm{d}_{\phi,1-\gamma}(\Lambda(h), \Lambda(g)) \leq \rho K \mathrm{d}_{\phi,1-\gamma}(h,g).$$

Using the fact that $\rho K < 1$, we get Λ is strictly contractive on the $(\widetilde{\mathcal{E}}_{f,1-\gamma}, \mathrm{d}_{\phi,1-\gamma})$. Thus, through the Banach fixed point theorem, there is a unique function $v \in \widetilde{\mathcal{E}}_{f,1-\gamma}$ such that $v = \Lambda(v)$. Using the triangle inequality, yields

$$\begin{aligned}
d_{\phi,1-\gamma}(f,v) &\leq d_{\phi,1-\gamma}(f,\Lambda(f)) + d_{\phi,1-\gamma}(\Lambda(f),\Lambda(v)) \\
&\leq 1 + \rho K d_{\phi,1-\gamma}(f,v)
\end{aligned}$$

which implies that

$$d_{\phi,1-\gamma} \leq \frac{1}{1-\rho K}.$$

Therefore, we conclude that

$$\|f(t) - v(t)\|_{C_{1-\gamma}} \leq C_\phi \phi(t)\,, \quad t \in [0,+\infty)$$

where $C_\phi := \dfrac{1}{1-\rho K}$. □

Remark 4.3. *In the same way that we highlight the particular cases for Theorem 3.1 and Theorem 3.2, here also the remark made according to Remark 3.3 is valid.*

5. Conclusions

We conclude this paper with the objectives achieved, i.e., we investigate the Ulam-Hyers and Ulam-Hyers-Rassias stabilities of mild solution for fractional nonlinear abstract nonlinear Cauchy problem:

the first part was destined to the bounded interval $[0, T]$ and the second part to the interval $[0, \infty)$. It is important to emphasize the fundamental role of the Banach fixed point theorem in the results obtained.

Although, the results presented here, contribute to the growth of the theory; some questions still need to be answered. The first question is about the possibility of investigating the existence and uniqueness of mild solutions for fractional differential equations formulated via ψ-Hilfer fractional derivative. Consequently, the second allows us to question the Ulam-Hyers stabilities.

As highlighted in the introduction, the Ulam-Hyers stabilities theory is indeed interesting and of paramount importance for the theory of fractional differential equations, and that there are still open problems, in particular as highlighted above. Certainly, the results investigated here contribute to the area, since the limited number of works involving more general fractional derivatives are still little addressed. We are working on other work on the existence and controllability of mild solutions to fractional differential equations, in particular involving the ψ-Hilfer fractional derivative.

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Conflict of interest

The authors declare there is no conflicts of interest.

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