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Research article

Sign-changing solutions for Schrödinger system with critical growth

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Abstract: We consider the following Schrödinger system

$$\begin{cases} -\Delta u_j = \sum_{i=1}^{k} \beta_{ij}|u_i|^3|u_j|u_j + \lambda_j|u_j|^{q-2}u_j, & \text{in } \Omega, \\ u_j = 0 \quad \text{on } \partial\Omega, \ \ j = 1, \cdots, k \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary. Assume $5 < q < 6$, $\lambda_j > 0$, $\beta_{jj} > 0$, $j = 1, \cdots, k$, $\beta_{ij} = \beta_{ji}$, $i \neq j, i, j = 1, \cdots, k$. Note that the nonlinear coupling terms are of critical Sobolev growth in dimension 3. We prove that under an additional condition on the coupling matrix the problem has infinitely many sign-changing solutions. The result is obtained by combining the method of invariant sets of descending flow with the approach of using approximation of systems of subcritical growth.

Keywords: Schrödinger system; critical growth; sign-changing solutions

1. Introduction

In this paper, we consider the Schrödinger system with critical growth

$$\begin{cases} -\Delta u_j = \sum_{i=1}^{k} \beta_{ij}|u_i|^3|u_j|u_j + \lambda_j|u_j|^{q-2}u_j, & \text{in } \Omega, \\ u_j = 0 \quad \text{on } \partial\Omega, \ \ j = 1, \cdots, k \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary. This type of coupled systems have applications in some physical problem. In physics literatures the signs of the coupling constants β_{ij}, $i \neq j$

being positive or negative determine the nature of the system being attractive or repulsive. We refer to [1–7] for further references on subcritical and critical problems therein. Our main result is the following.

Theorem 1.1. *Assume* $5 < q < 6$, $\lambda_j > 0$, $j = 1, \cdots, k$, $\beta_{jj} > 0$, $j = 1, \cdots, k$, $\beta_{ij} = \beta_{ji}$, $i, j = 1, \cdots, k$, *and there exists* $\beta_0 > 0$ *such that*

$$\sum_{i,j=1}^{k} \beta_{ij}\xi_i\xi_j \geq \beta_0|\xi|^2 \quad \text{for } \xi = (\xi_1, \cdots, \xi_k) \in \mathbb{R}_+^k.$$

Then the problem (P) has infinitely many solutions with all components being sign-changing.

It is worth mentioning that our Theorem 1.1 allows some pairs of components to be attractive, and others repulsive.

Example 1.1. *(attractive)* $\beta_{ij} \geq 0$, $i \neq j$, $i, j = 1, \cdots, k$.

Example 1.2. *(repulsive)* $\beta_{ij} \leq 0$, $i \neq j$, $i, j = 1, \cdots, k$ *and the matrix* $B = (\beta_{ij})$ *is positive definite.*

The problem (P) has a variational structure given by the functional

$$I(U) = \frac{1}{2}\int_\Omega \sum_{j=1}^{k} |\nabla u_j|^2 dx - \frac{1}{6}\int_\Omega \sum_{i,j=1}^{k} \beta_{ij}|u_i|^3|u_j|^3\, dx - \frac{1}{q}\int_\Omega \sum_{j=1}^{k} \lambda_j |u_j|^q\, dx$$

for $U = (u_1, \cdots, u_k) \in X = H_0^1(\Omega) \times \cdots \times H_0^1(\Omega)$, the k-fold product of $H_0^1(\Omega)$. In order to prove Theorem 1.1, we shall use a subcritical approximation scheme together with the method of invariant sets of descending flow, in particular the abstract theorem from [8,9]. We briefly outline our approaches here.

When we try to apply the abstract theorem to the functional I, we are faced with some difficulties. Firstly being in dimension 3 the problem (P) is of critical Sobolev growth, and the functional I fails to satisfy the Palais-Smale condition. From the classical work of [10] (see also for p-Laplacian in [11] and for coupled systems in [9, 12]), using a subcritical approximation is an effective method to overcome this difficulty. Since we also need to study the nodal property, we shall use an alternative scheme of subcritical approximation as done in [13]. Our approach avoids passing to the limit of the subcritical problems and is easier to deal with nodal property of the solutions. Secondly in order to deal with nodal property of the solutions we employ the method of invariant sets of descending flow which has become a well developed method for constructing multiple nodal solutions, we refer [8, 9, 14–20] for further references therein. In particular we rely on the recent developments in [8,9]. To use the method of invariant set of descending flow one needs to construct certain invariant sets. This usually requires some additional property of the gradient flow (or a pseudo gradient flow). Our approximation approach also accommodates this issue well. More precisely, we consider the perturbed functionals

$$I_p^{(\varepsilon)}(U) = \frac{1}{2}\int_\Omega \sum_{j=1}^{k} |\nabla u_j|^2\, dx - \frac{1}{2p} g_\varepsilon\Big(\int_\Omega \sum_{i,j=1}^{k} \beta_{ij}|u_i|^p|u_j|^p\, dx\Big) - \frac{1}{q}\int_\Omega \sum_{j=1}^{k} \lambda_j|u_j|^q\, dx$$

for $U = (u_1, \cdots, u_k) \in X$, where $2 < p < 3$, $0 < \varepsilon \leq 1$ and g_ε is a smooth function satisfying

$$g_\varepsilon(t) = t \text{ for } 0 \leq t \leq \frac{1}{\varepsilon}, \quad g_\varepsilon(t) = c_\varepsilon t^{\frac{1}{2}} \text{ for } t \geq \frac{2}{\varepsilon}, \tag{1.1}$$

here c_ε is a constant depending on ε. The critical points of $I_p^{(\varepsilon)}$ will be used as approximate solutions of the problem (P) and it turns out that the approximate solutions converge to the solutions of the problem (P) as the parameters $\varepsilon \downarrow 0$, $p \uparrow 3$. Technically, we will first construct sign-changing critical points U of $I_p^{(\varepsilon)}$, then send ε to zero while holding p fixed to get sign-changing critical points U with $\|U\|_\infty < \frac{1}{\varepsilon}$ for the functionals I_p defined by

$$I_p(U) = \frac{1}{2}\int_\Omega \sum_{j=1}^k |\nabla u_j|^2\,dx - \frac{1}{2p}\int_\Omega \sum_{i,j=1}^k \beta_{ij}|u_i|^p|u_j|^p\,dx - \frac{1}{q}\int_\Omega \sum_{j=1}^k \lambda_j|u_j|^q\,dx.$$

Then using the profile decomposition of approximation solutions we obtain solutions of (P) by passing limit of $p \to 3$.

The paper is organized as follows. In Section 2 we will work on the perturbed functionals and construct multiple nodal solutions as approximating solutions to the original problem. Section 3 is devoted to the convergence analysis of the approximating solutions, then the proof of our main result follows.

2. Critical points of the perturbed functionals

First we give the exact definition of the function g_ε. Let φ be a smooth function such that $\varphi(t) = 1$ for $0 \le t \le 1$, $\varphi(t) = \frac{1}{2}$ for $t \ge 2$ and $\varphi(t)$ is decreasing in t. Define $g(t) = \exp\left\{\int_1^t \frac{\varphi(\tau)}{\tau}\,d\tau\right\}$ and, for $\varepsilon > 0$, $g_\varepsilon(t) = \frac{1}{\varepsilon}g(\varepsilon t)$. Then g_ε satisfies (1.1) and

$$\frac{1}{2}g_\varepsilon(t) \le g_\varepsilon'(t)t \le g_\varepsilon(t), \quad g_\varepsilon'(t)t^{\frac{1}{2}} \le c_\varepsilon \text{ for } t \ge 0. \tag{2.1}$$

To construct multiple nodal solutions for the subcritical approximating problems we will make use of the following abstract theorem from [8,9]. In the following, $\gamma(A)$ denotes the genus of a symmetric and closed subset A (see [21] for properties of the genus theory).

Theorem 2.1. *Let X be a Banach space, f be an even C^1-functional on X, A be an odd mapping from X to X, and P_j, Q_j, $j = 1,\cdots,k$ be open convex subsets of X with $Q_j = -P_j$. Denote $W = \bigcup_{j=1}^k (P_j \cup Q_j)$, $\Sigma = \bigcap_{j=1}^k (\partial P_j \cap \partial Q_j)$. Assume*

(I_1) f is an even C^1-functional on X, and satisfies the Palais-Smale condition.

(I_2) there exists $c^* > 0$ such that

$$f(x) \ge c^*, \quad \textit{for } x \in \Sigma.$$

(A_1) given $b_0 > 0$, $c_0 > 0$, there exists $b = b(b_0, c_0)$ such that if $|f(x)| \le c_0$, $\|Df(x)\| \ge b_0$, then

$$\langle Df(x),\ x - Ax\rangle \ge b\|x - Ax\| > 0.$$

(A_2) $A(\partial P_j) \subset P_j$, $A(\partial Q_j) \subset Q_j$, $j = 1,\cdots,k$.

Introduce a sequence of families of subsets of X.

$$\Gamma_l = \{B|\, B \subset X,\ B\,compact,\ -B = B\,and\,\gamma(B \cap \sigma^{-1}(\Sigma)) \geq l\,for\,\sigma \in \Lambda\}$$

$$\Lambda = \{\sigma|\, \sigma \in C(X, X),\ \sigma\,odd,\ \sigma(P_j) \subset P_j,\ \sigma(Q_j) \subset Q_j,\ j = 1, \cdots, k\,and\ \sigma(x) = x\ if\,f(x) \leq 0\}.$$

Define

$$c_l = \inf_{B\in\Gamma_l} \sup_{x\in B\setminus W} f(x).$$

Assume

(Γ) Γ_l, $l = 1, 2, \cdots$ *are nonempty.*

Then

(1) $c_l \geq c^*$, $l = 1, 2, \cdots$ *are critical values of* f.

(2) $c_1 \leq c_2 \leq \cdots$ *and* $c_l \to +\infty$ *as* $l \to \infty$.

(3) *The critical set* $K^*_{c_l}$ *is nonempty, where*

$$K^*_c = \{x|\, x \in X \setminus W,\ f(x) = c,\ Df(x) = 0\}.$$

(4) *If* $c_l = c_{l+1} = \cdots = c_{l+k-1}$ *for some integer k, then* $\gamma(K^*_{c_l}) \geq k$.

(5) *If X is a Hilbert space, f is a* C^2*-functional and for every critical point x of f,* $D^2 f(x)$ *is a Fredholm operator, then there exists* $x \in K^*_{c_l}$ *with* $m^*(x, f) \geq l$*, where* m^* *is the augmented Morse index of x.*

Lemma 2.1. *The functional* $I_p^{(\varepsilon)}$ *is of* C^2*-class, satisfies the Palais-Smale condition and* $D^2 I_p^{(\varepsilon)}(U)$ *is a Fredholm operator for every critical point U of the functional* $I_p^{(\varepsilon)}$.

Proof. Note that $2 < p < q < 6$. Thus it is easy to verify that $I_p^{(\varepsilon)}$ is a C^2-functional. Moreover

$$\begin{aligned}\langle DI_p^{(\varepsilon)}(U), \phi\rangle = &\int_\Omega \sum_{j=1}^k \nabla u_j \nabla \varphi_j\, dx - g'_\varepsilon\Big(\int_\Omega \sum_{i,j=1}^k \beta_{ij}|u_i|^p|u_j|^p\, dx\Big)\int_\Omega \sum_{i,j=1}^k \beta_{ij}|u_i|^p|u_j|^{p-2}u_j\varphi_j\, dx\\ &- \int_\Omega \sum_{j=1}^k \lambda_j|u_j|^{q-2}u_j\varphi_j\, dx \quad \text{for } \phi = (\varphi_1, \cdots, \varphi_k) \in X.\end{aligned} \tag{2.2}$$

We have

$$\begin{aligned}&I_p^{(\varepsilon)}(U) - \frac{1}{p}\langle DI_p^{(\varepsilon)}(U), U\rangle\\ =&\Big(\frac{1}{2} - \frac{1}{p}\Big)\int_\Omega \sum_{i,j=1}^k |\nabla u_j|^2\, dx + \frac{1}{p}g'_\varepsilon\Big(\int_\Omega \sum_{i,j=1}^k \beta_{ij}|u_i|^p|u_j|^p\, dx\Big)\int_\Omega \sum_{i,j=1}^k \beta_{ij}|u_i|^p|u_j|^p\, dx\\ &- \frac{1}{2p}g_\varepsilon\Big(\int_\Omega \sum_{i,j=1}^k \beta_{ij}|u_i|^p|u_j|^p\, dx\Big) + \Big(\frac{1}{p} - \frac{1}{q}\Big)\int_\Omega \sum_{j=1}^k \lambda_j|u_j|^q\, dx\\ \geq&\Big(\frac{1}{2} - \frac{1}{p}\Big)\int_\Omega \sum_{j=1}^k |\nabla u_j|^2\, dx = \Big(\frac{1}{2} - \frac{1}{p}\Big)\|U\|^2.\end{aligned} \tag{2.3}$$

In the above we have used the fact that $g'_\varepsilon(t)t \geq \frac{1}{2}g_\varepsilon(t)$. By (2.3), any Palais-Smale sequence of $I_p^{(\varepsilon)}$ is bounded in X. Since the functional $I_p^{(\varepsilon)}$ is of subcritical growth, by a standard argument $I_p^{(\varepsilon)}$ satisfies the Palais-Smale condition.

Let U be a critical point of $I_p^{(\varepsilon)}$. By the regularity theory, U is continuous on $\bar{\Omega}$ and therefore is uniformly bounded. Then the operator $D^2 I_p^{(\varepsilon)}(U)$ is a compact perturbation of the Laplacian operator, hence a Fredholm operator. □

Lemma 2.2. *Denote* $\beta_{ij}^{\pm} = \max(\pm\beta_{ij},\ 0)$. *Define a compact and odd operator A:* $U = (u_1, \cdots, u_k) \in X \mapsto V = (v_1, \cdots, v_k) = AU \in X$ *by the equations*

$$\begin{aligned}&\int_\Omega \nabla v_j \nabla\varphi_j\,dx + g'_\varepsilon\Big(\int_\Omega \sum_{i,j=1}^k \beta_{ij}|u_i|^p|u_j|^p\,dx\Big)\int_\Omega \sum_{i=1}^k \beta_{ij}^-|u_i|^p|u_j|^{p-2}v_j\varphi_j\,dx\\ =&g'_\varepsilon\Big(\int_\Omega \sum_{i,j=1}^k \beta_{ij}|u_i|^p|u_j|^p\,dx\Big)\int_\Omega \sum_{i=1}^k \beta_{ij}^+|u_i|^p|u_j|^{p-2}u_j\varphi_j\,dx + \lambda_j\int_\Omega |u_j|^{q-2}u_j\varphi_j\,dx\end{aligned} \tag{2.4}$$

for $\phi = (\varphi_1, \cdots, \varphi_k) \in X$. *Then the following property holds: if* $\|I_p^{(\varepsilon)}(U)\| \leq c_0$ *and* $\|DI_p^{(\varepsilon)}(U)\| \geq b_0 > 0$, *there exists* $b = b(b_0, c_0)$ *such that*

$$\langle DI_p^{(\varepsilon)}(U), U - AU\rangle \geq b\|U - AU\| > 0.$$

Proof. By (2.2) and (2.4), we have

$$\begin{aligned}\langle DI_p^{(\varepsilon)}(U), \phi\rangle =& \int_\Omega \sum_{j=1}^k \nabla(u_j - v_j)\nabla\varphi_j\,dx\\ &+ g'_\varepsilon\Big(\int_\Omega \sum_{i,j=1}^k \beta_{ij}|u_i|^p|u_j|^p\,dx\Big)\int_\Omega \sum_{i=1}^k \beta_{ij}^-|u_i|^p|u_j|^{p-2}(u_j - v_j)\varphi_j\,dx\end{aligned} \tag{2.5}$$

for $\phi = (\varphi_1, \cdots, \varphi_k) \in X$. Choose $\phi = U - V$, we obtain

$$\begin{aligned}\langle DI_p^{(\varepsilon)}(U), U - V\rangle =&\|U - V\|^2\\ &+ g'_\varepsilon\Big(\int_\Omega \sum_{i,j=1}^k \beta_{ij}|u_i|^p|u_j|^p\,dx\Big)\int_\Omega \sum_{i=1}^k \beta_{ij}^-|u_i|^p|u_j|^{p-2}(u_j - v_j)^2\,dx.\end{aligned}$$

Hence,

$$\langle DI_p^{(\varepsilon)}(U), U - V\rangle \geq \|U - V\|^2 \tag{2.6}$$

and

$$\langle DI_p^{(\varepsilon)}(U), U - V\rangle \geq g'_\varepsilon\Big(\int_\Omega \sum_{i,j=1}^k \beta_{ij}|u_i|^p|u_j|^p\,dx\Big)\int_\Omega \sum_{i=1}^k \beta_{ij}^-|u_i|^p|u_j|^{p-2}(u_j - v_j)^2\,dx. \tag{2.7}$$

It follows from (2.1), (2.5) and (2.7) that

$$
\begin{aligned}
&|\langle DI_p^{(\varepsilon)}(U),\phi\rangle|\\
\le&\|U-V\|\|\phi\|+g_\varepsilon'\Big(\int_\Omega\sum_{i,j=1}^k\beta_{ij}|u_i|^p|u_j|^p\,dx\Big)\left(\int_\Omega\sum_{i=1}^k\beta_{ij}^-|u_i|^p|u_j|^{p-2}\varphi_j^2\,dx\right)^{\frac12}\\
&\cdot\left(\int_\Omega\sum_{i=1}^k\beta_{ij}^-|u_i|^p|u_j|^{p-2}(u_j-v_j)^2\,dx\right)^{\frac12}\\
\le&\|U-V\|\|\phi\|+cg_\varepsilon'\Big(\int_\Omega\sum_{i,j=1}^k\beta_{ij}|u_i|^p|u_j|^p\,dx\Big)\|U\|_{L^{2p}(\Omega)}^{p-1}\|\phi\|_{L^{2p}(\Omega)}\\
&\cdot\left(\int_\Omega\sum_{i=1}^k\beta_{ij}^-|u_i|^p|u_j|^{p-2}(u_j-v_j)^2\,dx\right)^{\frac12}\\
\le&\|U-V\|\|\phi\|+c\left(g_\varepsilon'\Big(\int_\Omega\sum_{i,j=1}^k\beta_{ij}|u_i|^p|u_j|^p\,dx\Big)\|U\|_{L^{2p}(\Omega)}^p\right)^{\frac12}\|U\|_{L^{2p}(\Omega)}^{\frac{p-2}{2}}\|\phi\|_{L^{2p}(\Omega)}\\
&\cdot\left(g_\varepsilon'\Big(\int_\Omega\sum_{i,j=1}^k\beta_{ij}|u_i|^p|u_j|^p\,dx\Big)\int_\Omega\sum_{i=1}^k\beta_{ij}^-|u_i|^p|u_j|^{p-2}(u_j-v_j)^2\,dx\right)^{\frac12}\\
\le&\|U-V\|\|\phi\|+c\beta_0^{-\frac14}\sqrt{c_\varepsilon}\sqrt{\langle DI_p^{(\varepsilon)}(U),U-V\rangle}\|U\|_{L^{2p}(\Omega)}^{\frac{p-2}{2}}\|\phi\|_{L^{2p}(\Omega)},
\end{aligned}
\tag{2.8}
$$

which implies that

$$
\|\langle DI_p^{(\varepsilon)}(U)\|\le\|U-V\|+C_\varepsilon\|U\|_{L^{2p}(\Omega)}^{\frac{p-2}{2}}(\langle DI_p^{(\varepsilon)}(U),U-V\rangle)^{\frac12}.
\tag{2.9}
$$

Choose s such that $2<s<p<q$. Then we deduce from (2.1) and (2.5) that

$$
\begin{aligned}
&I_p^{(\varepsilon)}(U)-\frac1s\langle U-V,U\rangle\\
=&I_p^{(\varepsilon)}(U)-\frac1s\langle DI_p^{(\varepsilon)}(U),U\rangle\\
&+\frac1s g_\varepsilon'\Big(\int_\Omega\sum_{i,j=1}^k\beta_{ij}|u_i|^p|u_j|^p\,dx\Big)\int_\Omega\sum_{i=1}^k\beta_{ij}^-|u_i|^p|u_j|^{p-2}(u_j-v_j)u_j\,dx\\
=&\left(\frac12-\frac1s\right)\|U\|^2+\frac1s g_\varepsilon'\Big(\int_\Omega\sum_{i,j=1}^k\beta_{ij}|u_i|^p|u_j|^p\,dx\Big)\int_\Omega\sum_{i=1}^k\beta_{ij}|u_i|^p|u_j|^p\,dx\\
&+\frac1s g_\varepsilon'\Big(\int_\Omega\sum_{i,j=1}^k\beta_{ij}|u_i|^p|u_j|^p\,dx\Big)\int_\Omega\sum_{i=1}^k\beta_{ij}^-|u_i|^p|u_j|^{p-2}(u_j-v_j)u_j\,dx\\
&-\frac1{2p}g_\varepsilon\Big(\int_\Omega\sum_{i,j=1}^k\beta_{ij}|u_i|^p|u_j|^p\,dx\Big)+\left(\frac1s-\frac1q\right)\int_\Omega\sum_{j=1}^k\lambda_j|u_j|^q\,dx\\
\ge&\left(\frac12-\frac1s\right)\|U\|^2+\left(\frac1{2s}-\frac1{2p}\right)g_\varepsilon\Big(\int_\Omega\sum_{i,j=1}^k\beta_{ij}|u_i|^p|u_j|^p\,dx\Big)\\
&+\frac1s g_\varepsilon'\Big(\int_\Omega\sum_{i,j=1}^k\beta_{ij}|u_i|^p|u_j|^p\,dx\Big)\int_\Omega\sum_{i=1}^k\beta_{ij}^-|u_i|^p|u_j|^{p-2}(u_j-v_j)u_j\,dx.
\end{aligned}
\tag{2.10}
$$

Notice that the matrix $B = (\beta_{ij})$ is positively definite, we have

$$\sum_{i,j=1}^{k} \beta_{ij}^{-}|u_i|^p|u_j|^p \le C|U|^{2p} \le C\sum_{i,j=1}^{k} \beta_{ij}|u_i|^p|u_j|^p.$$

It implies from (2.1) that

$$g_\varepsilon'\Big(\int_\Omega \sum_{i,j=1}^{k} \beta_{ij}|u_i|^p|u_j|^p\,dx\Big)\int_\Omega \sum_{i=1}^{k} \beta_{ij}^{-}|u_i|^p|u_j|^p\,dx \le Cg_\varepsilon\Big(\int_\Omega \sum_{i,j=1}^{k} \beta_{ij}|u_i|^p|u_j|^p\,dx\Big). \tag{2.11}$$

From (2.7), (2.10) and (2.11), for sufficiently small $\sigma > 0$, we obtain

$$\begin{aligned}
&\left(\frac{1}{2}-\frac{1}{s}\right)\|U\|^2 + \left(\frac{1}{2s}-\frac{1}{2p}\right)g_\varepsilon\Big(\int_\Omega \sum_{i,j=1}^{k} \beta_{ij}|u_i|^p|u_j|^p\,dx\Big)\\
\le & I_p^{(\varepsilon)}(U) - \frac{1}{s}\langle U-V, U\rangle - \frac{1}{s}g_\varepsilon'\Big(\int_\Omega \sum_{i,j=1}^{k} \beta_{ij}|u_i|^p|u_j|^p\,dx\Big)\int_\Omega \sum_{i=1}^{k} \beta_{ij}^{-}|u_i|^p|u_j|^{p-2}(u_j-v_j)u_j\,dx\\
\le & I_p^{(\varepsilon)}(U) + \frac{1}{s}|\langle U-V, U\rangle| + C\left(g_\varepsilon'\Big(\int_\Omega \sum_{i,j=1}^{k} \beta_{ij}|u_i|^p|u_j|^p\,dx\Big)\int_\Omega \sum_{i=1}^{k} \beta_{ij}^{-}|u_i|^p|u_j|^p\,dx\right)^{\frac{1}{2}}\\
&\cdot\left(g_\varepsilon'\Big(\int_\Omega \sum_{i,j=1}^{k} \beta_{ij}|u_i|^p|u_j|^p\,dx\Big)\int_\Omega \sum_{i=1}^{k} \beta_{ij}^{-}|u_i|^p|u_j|^{p-2}(u_j-v_j)^2\,dx\right)^{\frac{1}{2}}\\
\le & |I_p^{(\varepsilon)}(U)| + \frac{1}{s}|\langle U-V, U\rangle| + C\left(g_\varepsilon\Big(\int_\Omega \sum_{i,j=1}^{k} \beta_{ij}|u_i|^p|u_j|^p\,dx\Big)\right)^{\frac{1}{2}}\sqrt{\langle DI_p^{(\varepsilon)}(U), U-V\rangle}\\
\le & |I_p^{(\varepsilon)}(U)| + C\|U-V\|^2 + C\langle DI_p^{(\varepsilon)}(U), U-V\rangle + \sigma\|U\|^2 + \sigma g_\varepsilon\Big(\int_\Omega \sum_{i,j=1}^{k} \beta_{ij}|u_i|^p|u_j|^p\,dx\Big)
\end{aligned}$$

Therefore,

$$g_\varepsilon\Big(\int_\Omega \sum_{i,j=1}^{k} \beta_{ij}|u_i|^p|u_j|^p\,dx\Big) \le C(|I_p^{(\varepsilon)}(U)| + \|U-V\|^2 + \langle DI_p^{(\varepsilon)}(U), U-V\rangle). \tag{2.12}$$

According to (1.1), there exists $C_\varepsilon > 0$ such that $t^{\frac{1}{2}} < C_\varepsilon(1 + g_\varepsilon(t))$. Hence

$$\begin{aligned}
\|U\|_{L^{2p}(\Omega)}^p \le & C\left(\int_\Omega \sum_{i,j=1}^{k} \beta_{ij}|u_i|^p|u_j|^p\,dx\right)^{\frac{1}{2}}\\
\le & C_\varepsilon\left(1 + g_\varepsilon\Big(\int_\Omega \sum_{i,j=1}^{k} \beta_{ij}|u_i|^p|u_j|^p\,dx\Big)\right)\\
\le & C_\varepsilon\left(1 + |I_p^{(\varepsilon)}(U)| + \|U-V\|^2 + \langle DI_p^{(\varepsilon)}(U), U-V\rangle\right).
\end{aligned} \tag{2.13}$$

Combining (2.9) with (2.13), we obtain

$$\begin{aligned}
&\|\langle DI_p^{(\varepsilon)}(U)\| \le \|U-V\|\\
&\qquad + C_\varepsilon\left(1 + |I_p^{(\varepsilon)}(U)| + \|U-V\|^2 + \langle DI_p^{(\varepsilon)}(U), U-V\rangle\right)^{\frac{p-2}{2p}}(\langle DI_p^{(\varepsilon)}(U), U-V\rangle)^{\frac{1}{2}}.
\end{aligned}$$

For $\sigma > 0$ small enough, we have

$$\begin{aligned}&\left(1+|I_p^{(\varepsilon)}(U)|+\|U-V\|^2\right)^{\frac{p-2}{2p}}(\langle DI_p^{(\varepsilon)}(U),U-V\rangle)^{\frac{1}{2}}\\ \le& C\left(1+|I_p^{(\varepsilon)}(U)|+\|U-V\|^2\right)^{\frac{p-2}{p}}\|U-V\|+\sigma\|DI_p^{(\varepsilon)}(U)\|\\ \le& C\left(1+|I_p^{(\varepsilon)}(U)|^{\frac{p-2}{p}}+\|U-V\|^{\frac{2(p-2)}{p}}\right)\|U-V\|+\sigma\|DI_p^{(\varepsilon)}(U)\|\end{aligned}$$

and

$$\begin{aligned}\left(\langle DI_p^{(\varepsilon)}(U),U-V\rangle\right)^{\frac{p-2}{2p}}(\langle DI_p^{(\varepsilon)}(U),U-V\rangle)^{\frac{1}{2}}\le&\left(\|DI_p^{(\varepsilon)}(U)\|\|U-V\|\right)^{1-\frac{1}{p}}\\ \le& C\|U-V\|^{p-1}+\sigma\|DI_p^{(\varepsilon)}(U)\|.\end{aligned}$$

By the above inequalities, we have

$$\|\langle DI_p^{(\varepsilon)}(U)\|\le C_\varepsilon\left(1+|I_p^{(\varepsilon)}(U)|^{\frac{p-2}{p}}+\|U-V\|^{p-2}\right)\|U-V\|. \tag{2.14}$$

If $\|I_p^{(\varepsilon)}(U)\|\le c_0$ and $\|DI_p^{(\varepsilon)}(U)\|\ge b_0>0$, we deduce from (2.13) that there exists $b=b(b_0,c_0)$ such that $\|U-V\|>b$. It follows from (2.6) that

$$\langle DI_p^{(\varepsilon)}(U),U-AU\rangle\ge b\|U-AU\|>0.$$

That A is odd is obvious. The compactness of A follows the regularity theory and the subcritical growth. □

Lemma 2.3. *Let $P_j, Q_j,\ j=1,\cdots,k$ be open convex subsets of X, defined by*

$$P_j=P_j(\delta)=\{U|\,U=(u_1,\cdots,u_k)\in X,\ \|u_j^-\|_{L^6(\Omega)}<\delta\}$$

$$Q_j=Q_j(\delta)=\{U|\,U=(u_1,\cdots,u_k)\in X,\ \|u_j^+\|_{L^6(\Omega)}<\delta\}.$$

Then there exists a constant $\delta_1>0$ such that for $0<\delta<\delta_1$ it holds that

$$A(\partial P_j)\subset P_j,\quad A(\partial Q_j)\subset Q_j,\ \ j=1,\cdots,k.$$

Proof. Choose $\phi=V^+=(v_1^+,\cdots,v_k^+)$ as test function in (2.4). Then we have

$$\begin{aligned}&c\|v_j^+\|^2_{L^6(\Omega)}\\ \le&\int_\Omega\nabla v_j\nabla v_j^+\,dx\\ \le& g_\varepsilon'\Big(\int_\Omega\sum_{i,j=1}^k\beta_{ij}|u_i|^p|u_j|^p\,dx\Big)\int_\Omega\sum_{i=1}^k\beta_{ij}^+|u_i|^p(u_j^+)^{p-1}v_j^+\,dx+\lambda_j\int_\Omega(u_j^+)^{q-1}v_j^+\,dx\\ \le& g_\varepsilon'\Big(\int_\Omega\sum_{i,j=1}^k\beta_{ij}|u_i|^p|u_j|^p\,dx\Big)\cdot c\|U\|^p_{L^{2p}(\Omega)}\|u_j^+\|^{p-1}_{L^{2p}(\Omega)}\|v_j^+\|_{L^{2p}(\Omega)}+c\|u_j^+\|^{q-1}_{L^q(\Omega)}\|v_j^+\|_{L^q(\Omega)}\\ \le& c_\varepsilon'(\|u_j^+\|^{p-2}_{L^6(\Omega)}+\|u_j^+\|^{q-2}_{L^6(\Omega)})\|u_j^+\|_{L^6(\Omega)}\cdot\|v_j^+\|_{L^6(\Omega)},\end{aligned} \tag{2.15}$$

where c'_ε is a constant depending on ε but independent of $p \in [\frac{5}{2}, 3]$. By (2.15) we have

$$\|v_j^+\|_{L^6(\Omega)} \leq c'_\varepsilon(\|u_j^+\|^{p-2}_{L^6(\Omega)} + \|u_j^+\|^{q-2}_{L^6(\Omega)})\|u_j^+\|_{L^6(\Omega)}. \tag{2.16}$$

Choose δ_1 such that $c'_\varepsilon(\delta_1^{p-2} + \delta_1^{q-2}) \leq \frac{1}{2}$. Then for $0 < \delta < \delta_1$ and $U \in \partial Q_j$ we have $\|u_j^+\|_{L^6(\Omega)} = \delta$ and

$$\|v_j^+\|_{L^6(\Omega)} \leq c'_\varepsilon(\delta^{p-2} + \delta^{q-2})\|u_j^+\|_{L^6(\Omega)} \leq \frac{1}{2}\|u_j^+\|_{L^6(\Omega)} = \frac{1}{2}\delta.$$

That is for $U \in \partial Q_j$, we have $V = AU \in Q_j$ and $A(\partial Q_j) \subset Q_j$. Similarly $A(\partial P_j) \subset P_j$, $j = 1, \cdots, k$. □

Lemma 2.4. *There exists $\delta_2 > 0$ such that for $0 < \delta < \delta_2$ there exists $c^* > 0$ independent of $\varepsilon \in (0, 1]$, $p \in [2, 3]$ such that*

$$I_p^{(\varepsilon)}(U) \geq c^* > 0 \quad \text{for } U \in \Sigma.$$

Proof.

$$\begin{aligned} I_p^{(\varepsilon)} =& \frac{1}{2}\int_\Omega \sum_{j=1}^k |\nabla u_j|^2\,dx - \frac{1}{2p}g_\varepsilon\Big(\int_\Omega \sum_{i,j=1}^k \beta_{ij}|u_i|^p|u_j|^p\,dx\Big) - \frac{1}{q}\int_\Omega \sum_{j=1}^k \lambda_j|u_j|^q\,dx \\ \geq& \frac{1}{2}\|U\|^2 - c(\|U\|^{2p}_{L^{2p}(\Omega)} + \|U\|^q_{L^q(\Omega)}) \\ \geq& c_1\|U\|^2_{L^6(\Omega)} - c_2(\|U\|^{2p}_{L^6(\Omega)} + \|U\|^q_{L^6(\Omega)}). \end{aligned}$$

For $U \in \Sigma = \bigcap_{j=1}^k (\partial P_j \cap \partial Q_j)$,

$$\|U\|^6_{L^6(\Omega)} = \int_\Omega \sum_{j=1}^k ((u_j^+)^6 + (u_j^-)^6)dx = 2k\delta^6.$$

Hence

$$\begin{aligned} I_p^{(\varepsilon)}(U) &\geq c_1\|U\|^2_{L^6(\Omega)} - c_2(\|U\|^{2p}_{L^6(\Omega)} + \|U\|^q_{L^6(\Omega)}) \\ &\geq \frac{1}{2}c_1\|U\|^2_{L^6(\Omega)} = \frac{1}{2}c_1(\sqrt[6]{2k}\delta)^2 := c^* > 0 \end{aligned}$$

provided $c_2((\sqrt[6]{2k}\delta_2)^{2p-2} + (\sqrt[6]{2k}\delta_2)^{q-2}) \leq \frac{1}{2}c_1$ and $0 < \delta < \delta_2$. □

Now we define a sequence of critical values of the perturbed functional $I_p^{(\varepsilon)}$

$$c_l(\varepsilon, p) = \inf_{B\in\Gamma_l} \sup_{U\in B\setminus W} I_p^{(\varepsilon)}(U),\ l = 1, 2, \cdots$$

where $W = \bigcup_{j=1}^k (P_j \cup Q_j)$ and for $l = 1, 2, \cdots$

$$\Gamma_l = \{B | B \subset X,\ B \text{ compact}, -B = B,\ \gamma(B \cap \sigma^{-1}(\Sigma)) \geq l \text{ for } \sigma \in \Lambda\},$$

$$\begin{aligned} \Lambda = \{\sigma | \sigma \in C(X, X),\ \sigma \text{ odd } \sigma(P_j) \subset P_j,\ \sigma(Q_j) \subset Q_j,\ j = 1, \cdots, k \\ \text{and } \sigma(U) = U \text{ if } I_p^{(\varepsilon)}(U) \leq 0\}. \end{aligned}$$

Proposition 2.1. *$c_l(\varepsilon, p)$, $l = 1, 2, \cdots$ are critical values of the functional $I_p^{(\varepsilon)}$. There exists $U_l(\varepsilon, p) \in X$ such that $I_p^{(\varepsilon)}(U_l(\varepsilon, p)) = c_l(\varepsilon, p)$, $DI_p^{(\varepsilon)}(U_l(\varepsilon, p)) = 0$, $U_l(\varepsilon, p)$ is sign-changing and the augmented Morse index $m^*(U_l(\varepsilon, p), I_p^{(\varepsilon)}) \geq l$. Moreover there exists a constant L_l, independent of ε, p, such that*

$$I_p^{(\varepsilon)}(U_l(\varepsilon, p)) = c_l(\varepsilon, p) \leq L_p \quad \varepsilon \in (0, 1],\ p \in (2, 3).$$

Proof. We apply Theorem 2.1 to our functional $I_p^{(\varepsilon)}$. We have verified the conditions (I_1)(Lemma 2.1), (I_2)(Lemma 2.4), (A_1)(Lemma 2.2) and (A_2)(Lemma 2.3). We need only to verify the condition Γ. Denote $n = l + k$. Choose nk functions $v_i \in C_0^\infty(\Omega)$, $i = 1, \cdots, nk$ with disjoint supports. Denote

$$F_l = \Big\{U|\, U = \Big(\sum_{i=1}^{n} t_i v_i,\ \sum_{i=n+1}^{2n} t_i v_i, \cdots, \sum_{i=n(k-1)+1}^{nk} t_i v_i\Big) \in X, \\ t = (t_1, t_2, \cdots, t_{nk}) \in \mathbb{R}^{nk},\ |t| \leq R\Big\}.$$

By Lemma 4.2 in [9] for R sufficiently large $F_l \in \Gamma_l$, Γ_l is nonempty. Now we have

$$c_l(\varepsilon, p) \leq \sup_{U \in F_l} I_p^{(\varepsilon)}(U) \leq \sup_{U \in E_l} J(U) := L_l,$$

where

$$J(U) = \frac{1}{2}\int_\Omega \sum_{j=1}^{k} |\nabla u_j|^2\, dx - \frac{1}{q}\int_\Omega \sum_{j=1}^{k} \lambda_j |u_j|^q\, dx,\ \ U = (u_1, \cdots, u_k) \in X,$$

$$E_l = \Big\{U|\, U = (\sum_{i=1}^{n} t_i v_i,\ \sum_{i=n+1}^{2n} t_i v_i, \cdots, \sum_{i=n(k-1)+1}^{nk} t_i v_i) \in X \\ t = (t_1, t_2, \cdots, t_{nk}) \in \mathbb{R}^{nk}\Big\}.$$

□

3. Convergence of the approximate solutions

As we have mentioned that the critical points of the perturbed functional $I_p^{(\varepsilon)}$ will be used as approximate solutions of the original problem (P). Now we prove that these approximate solutions converge to solutions of the original problem. More precisely, we show for any given integer k, we can find $\epsilon > 0$ small so that the functional $I_p^{(\varepsilon)}$ has k nodal critical points whose L^∞ norm all less than $\frac{1}{\varepsilon}$ (therefore they are critical points of the functional I_p). Then we send p to 3 to get solutions of the original problem.

Lemma 3.1. *Assume that $U \in X$ satisfies $I_p^{(\varepsilon)}(U) \leq L$, $DI_p^{(\varepsilon)}(U) = 0$, where L is independent of ε, p. Then there exists a constant $\bar{\varepsilon} = \bar{\varepsilon}(L)$ such that if $0 < \varepsilon < \bar{\varepsilon}$, $I_p(U) = I_p^{(\varepsilon)}(U)$, $DI_p(U) = DI_p^{(\varepsilon)}(U) = 0$, U is a critical point of I_p.*

Proof. By (2.3), we have

$$L \geq I_p^{(\varepsilon)}(U) = I_p^{(\varepsilon)}(U) - \frac{1}{p}\langle DI_p^{(\varepsilon)}(U),\ U\rangle \geq (\frac{1}{2} - \frac{1}{p})\|U\|^2.$$

There exists a constant M, independent of $\varepsilon,\ p$, such that

$$\|U\|^2 \le M, \quad \int_\Omega \sum_{i,j=1}^k \beta_{ij}|u_i|^p|u_j|^p\,dx \le M.$$

Choose $\bar\varepsilon = \frac{1}{2M}$. Then for $0 < \varepsilon < \bar\varepsilon$,

$$g_\varepsilon\Big(\int_\Omega \sum_{i,j=1}^k \beta_{ij}|u_i|^p|u_j|^p\,dx\Big) = \int_\Omega \sum_{i,j=1}^k \beta_{ij}|u_i|^p|u_j|^p\,dx,\ \ g'_\varepsilon\Big(\int_\Omega \sum_{i,j=1}^k \beta_{ij}|u_i|^p|u_j|^p\,dx\Big) = 1.$$

Hence for $0 < \varepsilon < \bar\varepsilon$, $I_p(U) = I_p^{(\varepsilon)}(U)$, $DI_p(U) = DI_p^{(\varepsilon)}(U) = 0$. □

Lemma 3.2. *Assume $p_n \in (2,3]$, $p_n \to 3$, $U_n \in X$, $n = 1,2,\cdots$ such that $I_{p_n}(U_n) \le L, DI_p(U_n) = 0$, where L is independent of p, and U_n is sign-changing. Then U_n is bounded in X. Assume $U_n \rightharpoonup U$ in X. Then $U_n \to U$ in X, $I(U) \le L$, $DI(U) = 0$ and U is sign-changing.*

Proof. Again we have

$$\begin{aligned} L \ge I_{p_n}(U_n) &= I_{p_n}(U_n) - \frac{1}{p_n}\langle DI_{p_n}(U_n),\ U_n\rangle \\ &\ge (\frac{1}{2} - \frac{1}{2p_n})\|U_n\|^2 \ge \frac{1}{4}\|U_n\|^2. \end{aligned}$$

So U_n is bounded in X. Assume $U_n \rightharpoonup U$ in X. We have the following profile decomposition [22]

$$U_n = U + \sum_{k\in\Lambda} \sigma_{n,k}^{\frac12} V_k(\sigma_{n,k}(\cdot - x_{n,k})) + R_n \tag{3.1}$$

where $V_k \in \mathcal{D} = \mathcal{D}(\mathbb{R}^3)$, $R_n \in \mathcal{D}$, $x_{n,k} \in \overline{\Omega}$, $\sigma_{n,k} \to +\infty$, $R_n \to 0$ in $L^6(\Omega)$ as $n \to \infty$.

Assume $\sigma_n = \sigma_{n,1} = \min\{\sigma_{n,k}| k \in \Lambda\}$, $x_n = x_{n,1}$. The following claim can be proved as in [9–11].

Claim. There exist positive constant c, $\overline{c}$ such that

$$|U_n(x)| \le c \text{ for } x \in \mathcal{A}_n, \quad \int_{\mathcal{A}_n} |\nabla U_n|^2\,dx \le c\sigma_n^{-\frac12} \tag{3.2}$$

where $\mathcal{A}_n$ is called a safe region and defined by

$$\mathcal{A}_n = \{x|\, x \in \mathbb{R}^3,\ (\bar c + 2)\sigma_n^{-\frac12} < |x - x_n| < (\bar c + 3)\sigma_n^{-\frac12}\}.$$

Let $U_n = (u_1, \cdots, u_k) \in X$ be a critical point of I_{p_n}. The following local Pohožaev identity holds (e.g., [9]):

$$\begin{aligned} &(\frac{3}{2p_n} - \frac12)\int_{D_n} \sum_{j=1}^k |Du_j|^2\,dx + (\frac{3}{q} - \frac{3}{2p_n})\int_{D_n} \lambda_j|u_j|^p\,dx \\ =&\frac12 \int_{D_n} \sum_{i,j=1}^k |\nabla u_j|^2(x - x^*, \nabla\eta)dx - \int_{D_n} \sum_{j=1}^k (\nabla u_j,\ \nabla\eta)dx - \int_{D_n} \sum_{j=1}^k (\nabla u_j,\ x - x^*)(\nabla u_j, \nabla\eta)dx \\ &- \frac{3}{2p_n}\int_{D_n} \sum_{j=1}^k (\nabla u_j, \nabla\eta)u_j\,dx - \int_{D_n} (\frac1q \sum_{j=1}^k \lambda_j|u_j|^q + \frac{1}{2p_n}\sum_{i,j=1}^k \beta_{ij}|u_i|^{p_n}|u_j|^{p_n})(x - x^*, \nabla\eta)dx \\ &+ \frac12 \int_{\partial_e D_n} \sum_{j=1}^k |\nabla u_j|^2(x - x^*, n)\eta\,d\sigma \end{aligned} \tag{3.3}$$

where $D_n = B_{(\bar{c}+3)\sigma_n^{-\frac{1}{2}}}(x_n)$, $\partial_e D_n = \partial D_n \cap \partial\Omega$, n is the outward normal to $\partial\Omega$, $x^* \in \mathbb{R}^N$, $\eta \in C_0^\infty(\mathbb{R}^3)$ such that $\eta(x) = 1$ for $|x - x^*| \le (\bar{c}+2)\sigma_n^{-\frac{1}{2}}$, $\eta(x) = 0$ for $|x - x^*| \ge (\bar{c}+3)\sigma_n^{-\frac{1}{2}}$ and $|\nabla\eta| \le 2\sigma_n^{\frac{1}{2}}$.

Choose x^* such that $|x^* - x| \le (\bar{c}+8)\sigma_n^{-\frac{1}{2}}$ and $(x - x^*, n) \le 0$ for all $x \in \partial_e D_n$. If $\partial_e D_n = \emptyset$ we simply choose $x^* = x_n$. With this choice of x^* and the fact $2p_n \le 6$ we have

$$
\begin{aligned}
&(\frac{N}{q} - \frac{N}{2p_n}) \int_{D_n} \sum_{j=1}^k \lambda_j |u_j|^q \eta\, dx \\
\le& \frac{1}{2} \int_{D_n} \sum_{j=1}^k |\nabla u_j|^2 (x - x^*, \nabla\eta) dx - \int_{D_n} \sum_{j=1}^k (\nabla u_j, x - x^*)(\nabla u_j, \nabla\eta) dx \\
&- \frac{N}{2p_n} \int_{D_n} \sum_{j=1}^k (\nabla u_j, \nabla\eta) u_j\, dx - \int_{D_n} (\frac{1}{q} \sum_{j=1}^k \lambda_j |u_j|^q + \frac{1}{2p_n} \sum_{i,j=1}^k \beta_{ij} |u_i|^{p_n} |u_j|^{p_n})(x - x^*, \nabla\eta) dx.
\end{aligned}
\tag{3.4}
$$

The integrals of the right hand side of (3.4) are taken over the domain $\mathcal{A}_n$ due to the fact that $\nabla\eta = 0$ outside $\mathcal{A}_n$. Hence by the claim (3.2), we have

$$\text{RHS of (3.4)} \le c\sigma_n^{-\frac{1}{2}}.$$

For the left hand side of (3.4), we have, keeping the profile decomposition (3.2) in mind

$$
\begin{aligned}
\text{LHS of (3.4)} =& \Big(\frac{N}{q} - \frac{N}{2p_n}\Big) \int_{D_n} \sum_{j=1}^k \lambda_j |u_j|^q \eta\, dx \ge c \int_{D_n} |U_n|^q\, dx \\
\ge& c\sigma_n^{\frac{q}{2}-3} \int_{|y|\le L} \left|\sigma_n^{-\frac{1}{2}} U_n(\sigma_n^{-1} y + x_n)\right|^q dx \ge c\sigma_n^{\frac{q}{2}-3}.
\end{aligned}
$$

In the above we assume $\sigma_n^{-\frac{1}{2}} U_n(\sigma_n^{-1} \cdot + x_n) \rightharpoonup V$ in $\mathcal{D}$ and choose $L > 0$ such that $\int_{|y|\le L} |V|^q\, dx > 0$. Because $q > 5$ we arrive at a contradiction for n large in

$$\sigma_n^{\frac{q}{2}-3} \le c\sigma_n^{-\frac{1}{2}}.$$

Hence the index set Λ in the profile decomposition (3.1) is empty and $U_n \to U$ in $L^6(\Omega)$, which implies that $U_n \to U$ in X due to the fact $DI_{p_n}(U_n) = 0$. Therefore $I(U) = \lim_{n\to\infty} I_{p_n}(U_n) \le L$ and $DI(U) = \lim_{n\to\infty} DI_{p_n}(U_n) = 0$.

Finally we prove that U is sign-changing. Denote $U_n = (u_{1n}, \cdots, u_{kn})$, $U = (u_1, \cdots, u_k)$. We have

$$\int_\Omega \nabla u_{jn} \nabla\varphi_j\, dx = \int_\Omega \sum_{i=1}^k \beta_{ij} |u_{in}|^{p_n} |u_{jn}|^{p_n - 2} u_{jn} \varphi_j\, dx + \lambda_j \int_\Omega |u_{jn}|^{q-2} u_{jn} \varphi_j\, dx, \ \ \varphi_j \in H_0^1(\Omega).$$

Choosing $\varphi_j = u_{jn}^+$, we have

$$
\begin{aligned}
c\|u_{jn}^+\|_{L^6(\Omega)}^2 \le& \int_\Omega \nabla u_{jn} \nabla u_{jn}^+ \, dx \\
=& \int_\Omega \sum_{i=1}^k \beta_{ij} |u_{in}|^{p_n} |u_{jn}|^{p_n-2} u_{jn} u_{jn}^+ \, dx + \lambda_j \int_\Omega |u_{jn}|^{q-2} u_{jn} u_{jn}^+ \, dx \\
\le& \int_\Omega \sum_{i=1}^k \beta_{ij} |u_{in}|^{p_n} (u_{jn}^+)^{p_n} dx + \lambda_j \int_\Omega (u_{jn}^+) q \, dx \\
\le& \|U_n\|_{L^{2p_n}(\Omega)}^{p_n} \|u_{jn}^+\|_{L^{2p_n}(\Omega)}^{p_n} + \lambda_j \|u_{jn}^+\|_{L^q(\Omega)}^q \\
\le& (\|U_n\|^{2p_n - \frac{5}{2}} + \|U_n\|^{q-\frac{5}{2}}) \|u_{jn}^+\|_{L^6(\Omega)}^{\frac{5}{2}} \\
\le& c(L) \|u_{jn}^+\|_{L^6(\Omega)}^{\frac{5}{2}},
\end{aligned}
$$

in which we used that $\|U_n\|^2$ is bounded by $4L$. Hence there exists $\delta > 0$ such that $\|u_{jn}^+\|_{L^6(\Omega)} \ge \delta$ and $\|u_j\|_{L^6(\Omega)} = \lim_{n\to\infty} \|u_{jn}^+\|_{L^6(\Omega)} \ge \delta > 0$. Similarly $\|u_j^-\|_{L^6(\Omega)} \ge \delta > 0$, $j = 1, \cdots, k$ and we have $U = (u_1, \cdots, u_k)$ is sign-changing. □

Proof of Theorem 1.1. First, obviously functions in $X\backslash W$ are sign-changing. Given an integer l, by Proposition 2.1 the functional $I_p^{(\varepsilon)}$, $0 < \varepsilon \le 1$, $2 < p < 3$ has a sign-changing critical point $U_l(\varepsilon, p)$ with the augmented Morse index $m^*(U_l(\varepsilon, l),\ I_p^{(\varepsilon)}) \ge l$. Moreover, there exists a constant L_l, independent of ε, p, such that $I_p^{(\varepsilon)}(U_l(\varepsilon, p)) \le L_l$.

By Lemma 3.1 there exists $\bar\varepsilon = \bar\varepsilon(L_l)$, independent of p, such that $0 < \varepsilon < \bar\varepsilon$, $I_p(U_l(\varepsilon, p)) = I_p^{(\varepsilon)}(U_l(\varepsilon, p)) \le L_l$, $DI_p(U_l(\varepsilon, p)) = DI_p^{(\varepsilon)}(U_l(\varepsilon, p)) = 0$. Denote $U_l(p) = U_l(\varepsilon, p)$. Then $U_l(p)$ is a sign-changing critical point of the functional I_p with the augmented Morse index $m^*(U_l(p),\ I_p) \ge l$. Moreover $I_p(U_l(p)) \le L_p$ for $2 < p < 3$.

Choose $p_n \in (2, 3)$, $p_n \to 3$. By Lemma 3.2, $U_l(p_n)$ is bounded in X. Assume $U_l(p_n) \rightharpoonup U_l$ in X. Then $U_l(p_n) \to U_l$ in X, $I(U_l) \le L_l$, $DI(U_l) = 0$, U_l is sign-changing, and the augmented Morse index

$$m^*(U_l, I) \ge \overline{\lim_{n\to\infty}}\, m^*(U_l(p_n),\ I_{p_n}) \ge l.$$

U_l is a sign-changing critical point of the functional I with $m^*(U_l,\ I) \ge l$. Since the integer l is arbitrary, I has infinitely many sign-changing critical points, that is the problem (P) has infinitely many sign-changing solutions. Finally we prove $I(U_l) \to +\infty$ as $l \to \infty$. Otherwise $I(U_l) \le L$, $DI(U_l) = 0$. By Lemma 3.2 U_l is bounded in X. Assume $U_{l_n} \rightharpoonup U$ in X as $l_n \to \infty$. Then by Lemma 3.2, $U_{l_n} \to U$ in X, $DI(U) = 0$. Therefore

$$+\infty > m^*(U, I) \ge \overline{\lim_{n\to\infty}}\, m^*(U_{l_n}, I) \ge \lim_{n\to\infty} l_n = +\infty,$$

we arrive at a contradiction. □

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Conflict of interest

The authors declare there is no conflict of interest.

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