Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 30(1): 221–241.
DOI: 10.3934/era.2022012
Received: 23 May 2021
Revised: 05 September 2021
Accepted: 29 October 2021
Published: 24 December 2021

Research article

On the construction of $\mathbb{Z}_2^n$−grassmannians as homogeneous $\mathbb{Z}_2^n$−spaces

Mohammad Mohammadi*and Saad Varsaie

Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), No. 444, Prof. Yousef Sobouti Blvd. P. O. Box 45195-1159 Zanjan, Postal Code 45137-66731, Iran

* **Correspondence:** Email: moh.mohamady@iasbs.ac.ir.

Abstract: In this paper, we construct the $\mathbb{Z}_2^n$−grassmannians by gluing of the $\mathbb{Z}_2^n$−domains and give an explicit description of the action of the $\mathbb{Z}_2^n$−Lie group $GL(\overrightarrow{\mathbf{m}})$ on the $\mathbb{Z}_2^n$−grassmannian $G_{\overrightarrow{\mathbf{k}}}(\overrightarrow{\mathbf{m}})$ in the functor of points language. In particular, we give a concrete proof of the transitively of this action, and the gluing of the local charts of the $\mathbb{Z}_2^n$−grassmannian.

Keywords: $\mathbb{Z}_2^n$−Lie group; Homogeneous superspace; $\mathbb{Z}_2^n$−manifold; $\mathbb{Z}_2^n$− grassmannian

1. Introduction

There is growing interest in studying generalized supergeometry, that is, geometry of graded manifolds where the grading group is not $\mathbb{Z}_2$, but $\mathbb{Z}_2^n = \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$. The foundational aspects of the theory of $\mathbb{Z}_2^n$−manifolds were recently studied in [1–4]. This generalization is used in physics, see [5]. Also in Mathematics, there exist many examples of $\mathbb{Z}_2^n$−graded $\mathbb{Z}_2^n$−commutative algebras: quaternions and Clifford algebras, the algebra of Deligne differential superforms, etc. Moreover, there exist interesting examples of $\mathbb{Z}_2^n$−manifolds. In this paper, we study the $\mathbb{Z}_2^n$−grassmannians as $\mathbb{Z}_2^n$−manifolds and their constructions.

In the context of manifolds, homogeneous superspaces have been defined and investigated extensively using the functor of points approach in [6–8]. In this paper, we show that $\mathbb{Z}_2^n$−grassmannians $G_{\overrightarrow{\mathbf{k}}}(\overrightarrow{\mathbf{m}})$ are homogeneous, c.f. section 3. To this end, we show that the $\mathbb{Z}_2^n$−Lie group $GL(\overrightarrow{\mathbf{m}})$, c.f. section 2, acts transitively on $\mathbb{Z}_2^n$−grassmannian $G_{\overrightarrow{\mathbf{k}}}(\overrightarrow{\mathbf{m}})$, c.f. section 3.

In the first section, we recall briefly all necessary basic concepts such as $\mathbb{Z}_2^n$− grading spaces, $\mathbb{Z}_2^n$−manifolds, $\mathbb{Z}_2^n$−Lie groups and an action of a $\mathbb{Z}_2^n$−Lie group on a $\mathbb{Z}_2^n$−manifold. We use these concepts in the case of $\mathbb{Z}_2$−geometry in [6] and [8].

In section 2, we study the $\mathbb{Z}_2^n$−grassmannians extensively. The supergrassmannians are introduced by Manin in [9], but here by developing an efficient formalism, we fill in the details of the proof of this statement.

In section 3, by a functor of points approach, an action of the $\mathbb{Z}_2^n$−Lie group $GL(\overrightarrow{\mathbf{m}})$ on the $\mathbb{Z}_2^n$−grassmannian $G_{\overrightarrow{\mathbf{k}}}(\overrightarrow{\mathbf{m}})$ is defined by gluing local actions. Finally it is shown that this action is transitive.

2. Preliminaries

Let $\mathbb{Z}_2^n = \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ be the n−fold Cartesian product of $\mathbb{Z}_2$. From now on, we set $\mathbf{q} := 2^n - 1$ and by $\overrightarrow{\mathbf{k}}$, we mean $(k_0, k_1, \ldots, k_{\mathbf{q}})$ such that $k_i \in \mathbb{N}$. Consider the bi-additive map

$$\begin{aligned} \langle .\,, . \rangle &: \mathbb{Z}_2^n \times \mathbb{Z}_2^n \to \mathbb{Z}_2 \\ \langle a, b \rangle &= \sum_{i=1}^{n} a_i b_i (mod 2). \end{aligned} \tag{2.1}$$

The even subgroup $(\mathbb{Z}_2^n)_0$ consists of elements $\gamma \in \mathbb{Z}_2^n$ such that $\langle \gamma, \gamma \rangle = 0$, and the set $(\mathbb{Z}_2^n)_1$ consists of odd elements $\gamma \in \mathbb{Z}_2^n$ such that $\langle \gamma, \gamma \rangle = 1$.

One can fix an ordering on $\mathbb{Z}_2^n$; based on this ordering, each even element is smaller than each odd element. Given two even (odd) elements $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$, the first one is smaller than the second one for the lexicographical order, if $a_i < b_i$, for the first i where a_i and b_i differ. For example, the lexicographical ordering on $\mathbb{Z}_2^3$ is

$$(0,0,0) < (0,1,1) < (1,0,1) < (1,1,0) < (0,0,1) < (0,1,0) < (1,0,0) < (1,1,1).$$

Obviously, $\mathbb{Z}_2^n$ with lexicographical ordering is totally ordered set. Thus it may be diagrammed as an ascending chain as follows

$$\gamma_0 < \gamma_1 < \ldots < \gamma_{\mathbf{q}}.$$

In the supergeometry, the sign rules between generators of the algebra are completely determined by their parity. One can define a grading by (2.1) such that $\epsilon(a, b) = (-1)^{\langle a,b \rangle}$ will be a sign rule what will lead to $\mathbb{Z}_2^n$−geometry. Also, it has been shown that any other sign rule for finite number of coordinates is obtained from the above sign rule for sufficiently big n. See [1] for more details.

2.1. $\mathbb{Z}_2^n$−geometry

The $\mathbb{Z}_2^n$−graded objects like $\mathbb{Z}_2^n$−algebras, $\mathbb{Z}_2^n$−ringed spaces, $\mathbb{Z}_2^n$−domains and $\mathbb{Z}_2^n$−manifolds have been studied in [1, 3, 4]. In the following, we recall the necessary definitions from these references.

By definition, a $\mathbb{Z}_2^n$−*vector space* is a direct sum $V = \bigoplus_{\gamma \in \mathbb{Z}_2^n} V_\gamma$ of vector spaces V_γ over a field $\mathbb{K}$ (with characteristic 0). For each $\gamma \in \mathbb{Z}_2^n$, the elements of V_γ is called homogeneous with degree γ. If $x \in V_\gamma$ be a homogeneous element of V, then the degree of x is represented by $\tilde{x} = \gamma$.

A $\mathbb{Z}_2^n$−*ring* $\mathcal{R} = \bigoplus_{\gamma \in \mathbb{Z}_2^n} \mathcal{R}_\gamma$ is a ring such that its multiplication satisfies

$$\mathcal{R}_{\gamma_1} \mathcal{R}_{\gamma_2} \subset \mathcal{R}_{\gamma_1 + \gamma_2}.$$

A $\mathbb{Z}_2^n$−ring $\mathcal{R}$ is called $\mathbb{Z}_2^n$−commutative, if for any homogeneous elements $a, b \in \mathcal{R}$

$$a.b = (-1)^{\langle \tilde{a}, \tilde{b} \rangle} b.a.$$

For any $\mathbb{Z}_2^n-$algebra $\mathcal{R}$, let J be an ideal of $\mathcal{R}$ generated by all homogeneous elements of R having nonzero degree. If $f : R \to S$ is a morphism of $\mathbb{Z}_2^n-$algebras, then $f(J_R) \subseteq J_S$. Let M be an $\mathcal{R}$-module. The collection of sets $\{x + J^k M\}_{k=0}^{\infty}$ can be considered as a basis for a topology on M. This topology is called $J-$adic topology. Note that x runs over all elements of M. The $J-$adic topology plays a fundamental role in $\mathbb{Z}_2^n-$geometry.

A $\mathbb{Z}_2^n-$ring $\mathcal{R}$ with respect to $J-$adic topology is Hausdorff complete if the natural ring morphism $\mathcal{R} \to \varprojlim_{k\in\mathbb{N}} \mathcal{R}/J^k$ is an isomorphism.

Example 2.1. Let R be a ring and $\xi_1, \ldots, \xi_{\mathbf{q}}$ be indeterminates with degree $\gamma_1, \ldots, \gamma_{\mathbf{q}} \in \mathbb{Z}_2^n$ respectively such that

$$\xi_i \xi_j = (-1)^{\langle \gamma_i, \gamma_j \rangle} \xi_j \xi_i.$$

Then $R[[\xi_1, \ldots, \xi_{\mathbf{q}}]]$ is the $\mathbb{Z}_2^n-$commutative associative unital $R-$algebra of formal power series in the ξ_a with coefficients in R. If J be an ideal generated by all formal power series $\sum_{\vec{\mathbf{k}}} a_{\vec{\mathbf{k}}} \xi_1^{k_0} \ldots \xi_q^{k_q}$ whose first term $a_{\vec{\mathbf{k}}}$ is equal to zero, then One can see $R[[\xi_1, \ldots, \xi_{\mathbf{q}}]]$ is $J-$adically Hausdorff complete.

By a $\mathbb{Z}_2^n$*−ringed space*, we mean a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of $\mathbb{Z}_2^n-$commutative $\mathbb{Z}_2^n-$graded rings on X. A morphism between two $\mathbb{Z}_2^n-$ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a pair $\psi := (\overline{\psi}, \psi^*)$ such that $\overline{\psi} : X \to Y$ is a continuous map and $\psi^* : \mathcal{O}_Y \to \overline{\psi}_* \mathcal{O}_X$ is a homomorphism of weight zero between the sheaves of $\mathbb{Z}_2^n-$commutative $\mathbb{Z}_2^n-$graded rings. Let $\mathbf{q} = 2^n - 1$, the $\mathbb{Z}_2^n$-ringed space

$$\mathbb{R}^{\vec{\mathbf{m}}} := \left(\mathbb{R}^{m_0}, C^{\infty}_{\mathbb{R}^{m_0}}(-)[[\xi_1^1, \ldots, \xi_1^{m_1}, \xi_2^1, \ldots, \xi_2^{m_2}, \ldots, \xi_3^1, \ldots, \xi_{\mathbf{q}}^{m_{\mathbf{q}}}]]\right),$$

is called $\mathbb{Z}_2^n$*-domain* such that $C^{\infty}_{\mathbb{R}^{m_0}}$ is the sheaf of smooth functions on $\mathbb{R}^{m_0}$. By evaluation of $f = \sum f_I \xi^I$ at $x \in U$, denoted by $ev_x(f)$, we mean $f_{\emptyset}(x)$. Also for each open $U \subset \mathbb{R}^{m_0}$,

$$\mathcal{O}_{\mathbb{R}^{\vec{\mathbf{m}}}}(U) := C^{\infty}_{\mathbb{R}^{m_0}}(U)[[\xi_1^1, \ldots, \xi_1^{m_1}, \xi_2^1, \ldots, \xi_2^{m_2}, \ldots, \xi_3^1, \ldots, \xi_{\mathbf{q}}^{m_{\mathbf{q}}}]],$$

is the $\mathbb{Z}_2^n-$commutative associative unital $\mathbb{Z}_2^n-$algebra of formal power series in formal variables ξ_i^j's of degrees γ_i which commuting as follows:

$$\xi_i^j \xi_k^l = (-1)^{\langle \gamma_i, \gamma_k \rangle} \xi_k^l \xi_i^j.$$

Let $\mathcal{J}(U)$ be the ideal generated by all homogeneous formal power series of nonzero degree. Then it is easily seen that $\mathcal{O}_{\mathbb{R}^{\vec{\mathbf{m}}}}(U)$ is Hausdorff complete with respect to $\mathcal{J}(U)$-adic topology. Equivalently there is a canonical ring isomorphism between $\mathcal{O}(U)$ and $\varprojlim_k \mathcal{O}(U)/\mathcal{J}^k(U)$. Let $V \subset U$. Then the following diagram is commutative

$$\begin{array}{ccc}
\mathcal{O}(U) & \longrightarrow & \varprojlim_k \mathcal{O}(U)/\mathcal{J}^k(U) \\
{\scriptstyle r_{UV}}\downarrow & & \downarrow{\scriptstyle R_{UV}} \\
\mathcal{O}(V) & \longrightarrow & \varprojlim_k \mathcal{O}(V)/\mathcal{J}^k(V)
\end{array}$$

where $r_{UV} : \mathcal{O}(U) \to \mathcal{O}(V)$ is the restriction map and R_{UV} is induced morphism. This shows that the notion of adic topology may be extended for sheaf $\mathcal{O}$ such that each stalk $\mathcal{O}_p$, $p \in \mathbb{R}^{m_0}$, is Hausdorff complete with respect to $\mathcal{J}_p-$adic topology.

A $\mathbb{Z}_2^n$-*manifold* of dimension $\vec{\mathbf{m}}$ is a $\mathbb{Z}_2^n$-ringed space $(\overline{M}, \mathcal{O}_M)$ that is locally isomorphic to $\mathbb{R}^{\vec{\mathbf{m}}}$. In addition $\overline{M}$ is a second countable and Hausdorff topological space. A morphism between two $\mathbb{Z}_2^n$-manifolds $M = (\overline{M}, \mathcal{O}_M)$ and $N = (\overline{N}, \mathcal{O}_N)$ is a local morphism between two local $\mathbb{Z}_2^n$-ringed spaces.

Let $(M, \mathcal{O}_M)$ be a $\mathbb{Z}_2^n$−manifold, and let $\mathcal{J}$ be the ideal sheaf $\mathcal{J}$, defined by

$$\mathcal{J}(U) = \Big\langle f \in \mathcal{O}_M(U) \quad | \quad \deg f \neq 0 \Big\rangle.$$

The structure sheaf $\mathcal{O}_M$ is $\mathcal{J}$−adically Hausdorff complete as a sheaf of $\mathbb{Z}_2^n$− commutative $\mathbb{Z}_2^n$−rings. See Proposition 6.9 in [3] for more details.

Analogous with supergeometry, one can obtain a $\mathbb{Z}_2^n$−manifold by gluing $\mathbb{Z}_2^n$− domains. We will use this method to construct the $\mathbb{Z}_2^n$−grassmannian as a $\mathbb{Z}_2^n$− manifold in section 3.

In order to introduce the concept of Jacobian, we need a few definitions and a proposition from [10].

Definition 2.2. Let M be a $\mathbb{Z}_2^n$-manifold and $m \in \overline{M}$. The *tangent space of M at m*, denoted by $T_m M$, is the real $\mathbb{Z}_2^n$-vector space of $\mathbb{R}$−derivations $\mathcal{O}_m \to \mathbb{R}$.

Definition 2.3. Let $\psi : M \to N$ be a morphism of $\mathbb{Z}_2^n$-manifolds. Assume $\overline{\psi}(m) = n \in \overline{N}$ for a point $m \in \overline{M}$. The *tangent map of ψ at m* is a morphism of $\mathbb{Z}_2^n$-vector spaces denoted by $(d\psi)_m : T_m M \to T_n N$ and defined by

$$(d\psi)_m(v)\Big([f]_n\Big) = v\Big(\psi_m^*([f]_n)\Big), \qquad \forall v \in T_m M, \quad \forall [f]_n \in \mathcal{O}_{N,n}.$$

Let $\psi : M \to N$ and $\varphi : N \to Q$ be two morphisms of $\mathbb{Z}_2^n$-manifolds, one can use the above definition to show that for any point $m \in \overline{M}$,

$$d(\varphi \circ \psi)_m = d\varphi_{\overline{\psi}(m)} \circ d\psi_m. \tag{2.2}$$

Here we state the chain rule for $\mathbb{Z}_2^n$-manifolds and use it to relate the tangent map to the $\mathbb{Z}_2^n$-graded Jacobian matrix. See Proposition 2.10 in [10].

Proposition 2.4. *Let $U^{\vec{p}}$, $U^{\vec{r}}$ be $\mathbb{Z}_2^n$-domains, with coordinates u^α, v^β respectively. Let $\psi : U^{\vec{p}} \to U^{\vec{r}}$ be a morphism of $\mathbb{Z}_2^n$-manifolds. Then,*

$$\frac{\partial \psi^*(f)}{\partial u^\alpha} = \sum_\beta \frac{\partial \psi^*(v^\beta)}{\partial u^\alpha} \psi^* \left(\frac{\partial f}{\partial v^\beta} \right), \qquad \forall f \in \mathcal{O}(U^{\vec{r}}).$$

Let us study the matrix representation of the equation (2.2). Assume $\psi = \Big(v^\gamma(u)\Big)$ and $\varphi = \Big(w^\alpha(v)\Big)$ are local representations of $\mathbb{Z}_2^n$-morphisms $\psi : M \to N$ and $\varphi : N \to S$ around m and $\overline{\psi}(m)$ respectively, one has

$$\begin{aligned}
\partial_{u^\beta} w^\alpha &= \sum_\gamma \partial_{u^\beta} v^\gamma \partial_{v^\gamma} w^\alpha \\
&= \sum_\gamma (-1)^{\langle \deg(u^\beta) + \deg(v^\gamma), \deg(v^\gamma) + \deg(w^\alpha) \rangle} \partial_{v^\gamma} w^\alpha \partial_{u^\beta} v^\gamma.
\end{aligned}$$

Definition 2.5. Let ψ be a $\mathbb{Z}_2^n$-morphism between $\mathbb{Z}_2^n$-domains U and V. If $u \mapsto \Big(v^i(u)\Big)$ is a representation of ψ, then the *Jacobian of ψ* is a $\mathbb{Z}_2^n$-matrix of degree zero as follow

$$Jac\psi = \Big((-1)^{\langle \deg(v^i) + \deg(u^j), \deg(u^j) \rangle} \partial_{u^j} v^i \Big)_{ij}.$$

At the end of this section, it is worth mentioning that some of the required concepts of category theory are given in the Appendix. At there, among other things, we talked about Yoneda lemma (Lemma A.1). According to this lemma two objects $X, Y \in Obj(C)$ are isomorphic if and only if the functor of points associated to them are isomorphic. Indeed, the Yoneda embedding is an equivalence between C and a subcategory of representable functors in $[C, \mathbf{SET}]$ since not all functors are representable. See (Appendix A.1) for more details.

2.2. $\mathbb{Z}_2^n$−Lie groups

Let **ZSM** be the category of $\mathbb{Z}_2^n$−manifolds. This is a category whose objects are $\mathbb{Z}_2^n$−manifolds whose morphisms are morphisms between two $\mathbb{Z}_2^n$−manifolds. **ZSM** is a locally small category and has finite product property, see [11] for more details. In addition it has a terminal object $\mathbb{R}^{\vec{\mathbf{0}}}$, that is the constant sheaf $\mathbb{R}$ on a singleton $\{0\}$.

Let $M = (\overline{M}, \mathcal{O}_M)$ be a $\mathbb{Z}_2^n$−manifold and $p \in \overline{M}$. There is a map $j_p = (\bar{j}_p, {j_p}^*)$ where:

$$\bar{j}_p : \{0\} \to \overline{M} \qquad , \qquad \begin{array}{l} {j_p}^* : \mathcal{O}_M \to \mathbb{R} \\ g \mapsto \tilde{g}(p) =: ev_p(g). \end{array}$$

So, for each $\mathbb{Z}_2^n$−manifold T, one can define the morphism

$$\hat{p}_{_T} : T \to \mathbb{R}^{\vec{\mathbf{0}}} \xrightarrow{j_p} M, \tag{2.3}$$

as a composition of j_p and the unique morphism $T \to \mathbb{R}^{\vec{\mathbf{0}}}$.

By $\mathbb{Z}_2^n$−Lie group, we mean a group-object in the category **ZSM**. This group is a $\mathbb{Z}_2^n$−graded group. Graded Lie groups are extensively studied in [12]. The category **ZSM** has group-object because of existence categorical products and terminal object. Also, one can show that any $\mathbb{Z}_2^n$−Lie group G induced a group structure over its T-points for any arbitrary $\mathbb{Z}_2^n$−manifold T. This means that the functor $T \to G(T)$ takes values in category of groups. Moreover, for any other $\mathbb{Z}_2^n$−manifold S and morphism $T \to S$, the corresponding map $G(S) \to G(T)$ is a homomorphism of groups. See Appendix for more details. As another form, one can also define a $\mathbb{Z}_2^n$−Lie group as a representable functor $T \to G(T)$ from category **ZSM** to category of groups.

Example 2.6. Consider the $\mathbb{Z}_2^n$−domain $\mathbb{R}^{\vec{\mathbf{m}}}$ and an arbitrary $\mathbb{Z}_2^n$−manifold T. Let $f_i^j \in \mathcal{O}(T)_{\gamma_i}$, $0 \leq i \leq \mathbf{q}$, $1 \leq j \leq m_j$, be γ_i−degree elements. By Theorem 6.8 in [1] (Fundamental theorem of $\mathbb{Z}_2^n$-morphisms), One may define a unique morphism $\psi : T \to \mathbb{R}^{\vec{\mathbf{m}}}$, by setting $\xi_i^j \mapsto f_i^j$ where (ξ_i^j) is a global coordinates system on $\mathbb{R}^{\vec{\mathbf{m}}}$. Thus ψ may be represented by (f_i^j).

One can see a degree zero square matrix with entries in the standard block format

$$\left[\begin{array}{c|c|c} B_{00} & \dots & B_{0\mathbf{q}} \\ \hline \dots & & \dots \\ \hline B_{\mathbf{q}0} & \dots & B_{\mathbf{qq}} \end{array}\right].$$

is invertible if and only if B_{ii} is an invertible matrix for all $0 \leq i \leq \mathbf{q}$, see Proposition 1.5 and Proposition 6.1 in [10].

Let V be a finite dimensional $\mathbb{Z}_2^n$-vector space of dimension $\overrightarrow{\mathbf{m}} = m_0|m_1|\ldots|m_{\mathbf{q}}$. One can define the $\mathbb{Z}_2^n$−Lie group $GL(V)$ which is denoted by $GL(\overrightarrow{\mathbf{m}})$ if $V = \mathbb{R}^{\overrightarrow{\mathbf{m}}}$. For more details, see (Appendix A.2). It can be shown that T−points of $GL(\overrightarrow{\mathbf{m}})$ are the $\overrightarrow{\mathbf{m}} \times \overrightarrow{\mathbf{m}}$ invertible $\mathbb{Z}_2^n$−matrices of weight zero where the elements of the $m_k \times m_u$ block B_{ku} have degree $\gamma_k + \gamma_u$ and the multiplication is the matrix product.

Let $x \in \overline{G}$, one can define the left and right translation by x as

$$r_x := \mu \circ (1_G \times \hat{x}_G) \circ \Delta_G, \tag{2.4}$$

$$l_x := \mu \circ (\hat{x}_G \times 1_G) \circ \Delta_G, \tag{2.5}$$

where Δ_G is the diagonal map on G and $\hat{x}_G$ is as above. One can show that pullbacks of above morphisms are as following

$$r_x^* := (1_{O(G)} \otimes ev_x) \circ \mu^*, \tag{2.6}$$

$$l_x^* := (ev_x \otimes 1_{O(G)}) \circ \mu^*. \tag{2.7}$$

One may also use the language of functor of points to describe two morphisms (2.4) and (2.5).

One can see the definition of vector fields on a graded Lie group G in [12]. According to the standard superspace, we have the following definition:

Definition 2.7. Let G be a $\mathbb{Z}_2^n$−Lie group. A Vector field X on G is called *right invariant vector field*, if we have

$$(X \otimes 1) \circ \mu^* = \mu^* \circ X.$$

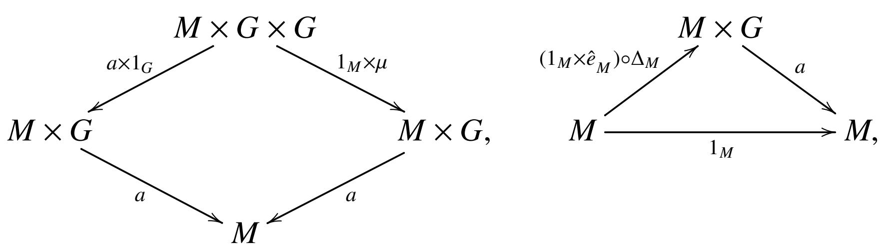
Similarly, one can use $(1 \otimes X) \circ \mu^* = \mu^* \circ X$, for a left invariant vector field X.

The bracket of two right invariant vector fields is right invariant. So we have

Definition 2.8. Let G be a $\mathbb{Z}_2^n$− Lie group. The set of all right invariant vector fields is denoted by $\mathfrak{g}$ and is called the *$\mathbb{Z}_2^n$−Lie algebra associated with the $\mathbb{Z}_2^n$−Lie group G*, and we write $\mathfrak{g} = Lie(G)$

Similar to standard supergeometry, one can show that $\mathfrak{g} = Lie(G)$ is a finite dimensional $\mathbb{Z}_2^n$−vector space canonically identified with the tangent space at the identity of the $\mathbb{Z}_2^n$−Lie group G.

Definition 2.9. Let M be a $\mathbb{Z}_2^n$−manifold and let G be a $\mathbb{Z}_2^n$−Lie group with μ, i and e as its multiplication, inverse and unit morphisms respectively. A morphism $a : M \times G \to M$ is called a (right) *action of G on M*, if the following diagrams commute



where $\hat{e}_M, \Delta_M$ are as above. In this case, we say G acts from right on M. One can define left action analogously.

According to the above diagrams, one has:

$$a \circ (1_M \times \mu) = a \circ (a \times 1_G), \tag{2.8}$$

$$a \circ (1_M \times \hat{e}_M) \circ \Delta_M = 1_M. \tag{2.9}$$

By Yoneda lemma (Lemma A.1), one may consider, equivalently, the action of G as a natural transformation:

$$a(.) : M(.) \times G(.) \to M(.).$$

As in [13], for each $p \in M$ and $g \in G$, one can define the morphisms a_p and a^g and use the functor point language to show that the maps a_p and a^g satisfy the relations

$$a^g \circ a^{g^{-1}} = id_M \qquad \forall g \in \overline{G} \tag{2.10}$$

$$a^g \circ a_p = a_p \circ r_g \qquad \forall g \in \overline{G}, p \in \overline{M}. \tag{2.11}$$

where r_g is the right translation on Lie $\mathbb{Z}_2^n$-group G.

For proof of the following proposition, see (Appendix A.3).

Proposition 2.10. *Let $a : M \times G \to M$ be an action of a Lie $\mathbb{Z}_2^n$-group G on a $\mathbb{Z}_2^n$-manifold M. Then*

(1) a^g is a $\mathbb{Z}_2^n$-diffeomorphism for all $g \in \overline{G}$.
(2) a_p has constant rank for all $p \in \overline{M}$.

Before next definition, we recall that a morphism between $\mathbb{Z}_2^n$−manifolds, say $\psi : M \to N$ is a submersion at $x \in \overline{M}$, if $(d\psi)_x$ is surjective and ψ is called submersion, if it is surjective at each point. (For more details, refer to [10]). Also ψ is a *surjective submersion*, if in addition ψ_0 is surjective.

Definition 2.11. Let G acts on M with action $a : M \times G \to M$. The action a is called *transitive*, if there exist $p \in \overline{M}$ such that a_p is a surjective submersion.

It is shown that, if a_p is a submersion for one $p \in \overline{M}$, then it is a submersion for all point in $\overline{M}$. Also one can show that a is transitive if and only if $\overline{a}$ is a transitive action in classical geometry and $(da_p)_e$ is a surjective because of Proposition 2.10. The following proposition will be required in the last section.

Proposition 2.12. *Let $a : M \times G \to M$ be an action of a Lie $\mathbb{Z}_2^n$-group G on a $\mathbb{Z}_2^n$-manifold M. Let $p \in \overline{M}$ and* $\dim G = \overrightarrow{\boldsymbol{r}} = (r_0, r_1, \dots, r_q)$ *and* $\overrightarrow{\boldsymbol{r}}' := (0, r_1, \dots, r_q)$. *If the map*

$$(a_p)_{\mathbb{R}^{\vec{\mathbf{r}}'}} : G(\mathbb{R}^{\overrightarrow{\mathbf{r}}'}) \to M(\mathbb{R}^{\overrightarrow{\mathbf{r}}'})$$

is surjective, then a is a transitive action.

Proof. According to above argument, it is enough to show that $\overline{a}$ is a transitive action in classical geometry and $(da_p)_e$ is surjective.

Let $(a_p)_{\mathbb{R}^{\vec{\mathbf{r}}'}}$ be surjective. Looking at the reduced part of each morphism in

$$(a_p)_{\mathbb{R}^{\vec{\mathbf{r}}'}}\Big(G(\mathbb{R}^{\overrightarrow{\mathbf{r}}'})\Big) = M(\mathbb{R}^{\overrightarrow{\mathbf{r}}'}),$$

we have that

$$(a_p)_{\mathbb{R}^{\vec{\mathbf{0}}}} = \overline{a_p} : \overline{G} \to \overline{M} \tag{2.12}$$

is surjective. So $\overline{a}$ is a classical transitive action. Let now $\{t, \xi_1, \ldots, \xi_{\mathbf{q}}\}$ be coordinates in a neighbourhood U of $m \in \overline{M}$. Consider the element $\Phi \in M(\mathbb{R}^{\vec{\mathbf{r}}'})$ defined by

$$\begin{aligned}\Phi^* : \mathcal{O}(U) \to \mathcal{O}(\mathbb{R}^{\vec{\mathbf{r}}'}) &= S^+(\eta_1, \ldots, \eta_{\mathbf{q}}) \\ t^i &\mapsto \overline{t^i}(m) \\ \xi^j_s &\mapsto \eta^j_s \qquad \forall j, \quad \forall 1 \le s \le \mathbf{q}.\end{aligned}$$

By surjectivity of $(a_p)_{\mathbb{R}^{\vec{\mathbf{r}}'}}$, there exists $\psi \in G(\mathbb{R}^{\vec{\mathbf{r}}'})$ such that $(a_p)_{\mathbb{R}^{\vec{\mathbf{r}}'}}(\psi) = \Phi$ and we have

$$\begin{aligned}\psi^* \circ a^*_p(t^i) &= \overline{t^i}(m) \\ \psi^* \circ a^*_p(\xi^j_s) &= \eta^j_s, \qquad \forall 1 \le j \le \mathbf{q}.\end{aligned}$$

This implies that $(T_m M)_{\gamma \neq 0}$ is in the image of $(da_p)_{\overline{\psi}}$. Since, by our previous considerations, $\overline{a_p}$ is a submersion, $(T_m M)_0$ is in the image $\overline{a_p}$. Hence, due to Proposition 8.1.5 we are done. □

In the following, after introducing the concept of stabilizer, we state some related results without any proofs, since the proofs are the same as the proofs of the similar results in supergeometry with appropriate modifications without any extra difficulties regarding $\mathbb{Z}_2^n$−geometry. For more details see [6] and [8].

Definition 2.13. Let G be a $\mathbb{Z}_2^n$−Lie group and let a be an action of G on $\mathbb{Z}_2^n$−manifold M. By *stabilizer* of $p \in \overline{M}$, we mean a $\mathbb{Z}_2^n$−manifold G_p equalizing the diagram

$$G \underset{\hat{p}_G}{\overset{a_p}{\rightrightarrows}} M.$$

Proposition 2.14. *Let $a : M \times G \to M$ be an action, then*

1. *The following diagram admits an equalizer G_p*

$$G \underset{\hat{p}_G}{\overset{a_p}{\rightrightarrows}} M.$$

2. *G_p is a $\mathbb{Z}_2^n$−sub Lie group of G.*
3. *The functor $T \to \big(G(T)\big)_{\hat{p}_T}$ is represented by G_p, where $\big(G(T)\big)_{\hat{p}_T}$ is the stabilizer in $\hat{p}_T$ of the action of $G(T)$ on $M(T)$.*

Proposition 2.15. *Suppose G acts transitively on M. There exists a G-equivariant isomorphism*

$$\frac{G}{G_p} \xrightarrow{\cong} M.$$

3. $\mathbb{Z}_2^n$-grassmannian

Supergrassmannians $G_{k|l}(m|n)$ are introduced and studied by Manin in [9] and [14]. Also the authors have studied them in more details in [13] and [15]. In this section, we introduce the $\mathbb{Z}_2^n$-grassmannian which is denoted by $G_{\overrightarrow{\mathbf{k}}}(\overrightarrow{\mathbf{m}})$ shortly, or $G_{k_0|k_1|\ldots|k_{\mathbf{q}}}(m_0|m_1|\ldots|m_{\mathbf{q}})$. For convenience from now, we set

$$\begin{aligned}\beta_0 &:= \textstyle\sum_{\gamma_i+\gamma_j=\gamma_0} k_i(m_j-k_j),\\ \beta_1 &:= \textstyle\sum_{\gamma_i+\gamma_j=\gamma_1} k_i(m_j-k_j),\\ &\ldots\\ \beta_{\mathbf{q}} &:= \textstyle\sum_{\gamma_i+\gamma_j=\gamma_{\mathbf{q}}} k_i(m_j-k_j),\end{aligned}$$

and also decompose any $\mathbb{Z}_2^n$−matrix into $2^n\times 2^n$ blocks

$$\left[\begin{array}{c|c|c|c} B_{00} & B_{01} & \ldots & B_{0\mathbf{q}}\\ \hline \ldots & & & \ldots\\ \hline B_{\mathbf{q}0} & B_{\mathbf{q}1} & \ldots & B_{\mathbf{qq}}\end{array}\right].$$

such that the elements of block B_{ku} have degree $\gamma_k+\gamma_u$. By a $\mathbb{Z}_2^n$-grassmannian, $G_{\overrightarrow{\mathbf{k}}}(\overrightarrow{\mathbf{m}})$, we mean a $\mathbb{Z}_2^n$-manifold which is constructed by gluing the following $\mathbb{Z}_2^n$- domains

$$\mathbb{R}^{\overrightarrow{\beta}} = \left(\mathbb{R}^{\beta_0},\ C^\infty_{\mathbb{R}^{\beta_0}}(-)[[\xi_1^1,\ldots,\xi_1^{\beta_1},\xi_2^1,\ldots,\xi_2^{\beta_2},\ldots,\xi_{\mathbf{q}}^1,\ldots,\xi_{\mathbf{q}}^{\beta_{\mathbf{q}}}]]\right)$$

For $i=0,1,\ldots,\mathbf{q}$, let $I_i\subset\{1,\ldots,m_i\}$ be a sorted subset in ascending order with k_i elements. The elements of I_i are called γ_i-degree indices. The multi-index $\overrightarrow{\mathbf{I}}=(I_1,...,I_{\mathbf{q}})$ is called $\overrightarrow{\mathbf{k}}$-index. Set $\mathcal{U}_{\overrightarrow{\mathbf{I}}} := (\overline{U}_{\overrightarrow{\mathbf{I}}}, O_{\overrightarrow{\mathbf{I}}})$, where

$$\overline{U}_{\overrightarrow{\mathbf{I}}} = \mathbb{R}^{\beta_0} \quad , \quad O_{\overrightarrow{\mathbf{I}}} = C^\infty_{\mathbb{R}^{\beta_0}}(-)[[\xi_1^1,\ldots,\xi_1^{\beta_1},\xi_2^1,\ldots,\xi_2^{\beta_2},\ldots,\xi_{\mathbf{q}}^1,\ldots,\xi_{\mathbf{q}}^{\beta_{\mathbf{q}}}]].$$

Let each $\mathbb{Z}_2^n$-domain $\mathcal{U}_{\overrightarrow{\mathbf{I}}}$ be labeled by a $\mathbb{Z}_2^n$−matrix $\overrightarrow{\mathbf{k}}\times\overrightarrow{\mathbf{m}}$ of weight zero, say $A_{\overrightarrow{\mathbf{I}}}$, with $2^n\times 2^n$ blocks B_{ij} each of which is a $k_i\times m_j$ matrix. In addition, except for columns with indices in $I_0\cup I_1\cup\ldots\cup I_{\mathbf{q}}$, which together form a $\mathbb{Z}_2^n$−submatrix denoted by $M_{\overrightarrow{\mathbf{I}}}A_{\overrightarrow{\mathbf{I}}}$, the matrix is filled from up to down and left to right by $x_a^{\overrightarrow{\mathbf{I}}},\xi_b^{\overrightarrow{\mathbf{I}}}$, the free generators of $O_{\overrightarrow{\mathbf{I}}}(\mathbb{R}^{\beta_0})$ each of them sits in a block with same degree. This process impose an ordering on the set of generators. In addition $M_{\overrightarrow{\mathbf{I}}}A_{\overrightarrow{\mathbf{I}}}$ is supposed to be the identity matrix.

For example, consider $G_{1|2|1|1}2|2|2|2$. Then let $I_0=\{1\}, I_1=\{1,2\}, I_2=\{1\}, I_3=\{2\}$, so $\overrightarrow{\mathbf{I}}$ is a $1|2|1|1$-index. In this case the set of generators of $O_{\overrightarrow{\mathbf{I}}}(\mathbb{R}^{\beta_0})$ is

$$\{x^1,x^2,x^3,\xi_1^1,\xi_1^2,\xi_1^3,\xi_1^4,\xi_2^1,\xi_2^2,\xi_2^3,\xi_2^4,\xi_3^1,\xi_3^2,\xi_3^3,\xi_3^4\},$$

and $A_{\overrightarrow{\mathbf{I}}}$ is:

$$\left[\begin{array}{cc|cc|cc|cc} 1 & x^1 & 0 & 0 & 0 & \xi_2^2 & \xi_3^4 & 0\\ \hline 0 & \xi_1^1 & 1 & 0 & 0 & \xi_3^2 & \xi_2^3 & 0\\ 0 & \xi_1^2 & 0 & 1 & 0 & \xi_3^3 & \xi_2^4 & 0\\ \hline 0 & \xi_2^1 & 0 & 0 & 1 & x_2 & \xi_1^4 & 0\\ \hline 0 & \xi_3^1 & 0 & 0 & 0 & \xi_1^3 & x^3 & 1\end{array}\right].$$

Note that, in this example,

$$\{x^1,\xi_1^1,\xi_1^2,\xi_2^1,\xi_3^1,\xi_2^2,\xi_3^2,\xi_3^3,x_2,\xi_1^3,\xi_3^4,\xi_2^3,\xi_2^4,\xi_1^4,x^3\} \tag{3.1}$$

is corresponding total ordered set of generators.

By $\widetilde{U}_{\overrightarrow{\mathbf{I}},\overrightarrow{\mathbf{J}}}$, we mean the set of all points of $\overline{U}_{\overrightarrow{\mathbf{I}}}$, on which $M_{\overrightarrow{\mathbf{J}}}A_{\overrightarrow{\mathbf{I}}}$ is invertible. Obviously $\widetilde{U}_{\overrightarrow{\mathbf{I}},\overrightarrow{\mathbf{J}}}$ is an open set. The transition map between the two $\mathbb{Z}_2^n$-domains $U_{\overrightarrow{\mathbf{I}}}$ and $U_{\overrightarrow{\mathbf{J}}}$ is denoted by

$$g_{\overrightarrow{\mathbf{I}},\overrightarrow{\mathbf{J}}} : \left(\widetilde{U}_{\overrightarrow{\mathbf{J}},\overrightarrow{\mathbf{I}}}, \mathcal{O}_{\overrightarrow{\mathbf{J}}}|_{\widetilde{U}_{\overrightarrow{\mathbf{J}},\overrightarrow{\mathbf{I}}}}\right) \longrightarrow \left(\widetilde{U}_{\overrightarrow{\mathbf{I}},\overrightarrow{\mathbf{J}}}, \mathcal{O}_{\overrightarrow{\mathbf{I}}}|_{\widetilde{U}_{\overrightarrow{\mathbf{I}},\overrightarrow{\mathbf{J}}}}\right).$$

Note that $g_{\overrightarrow{\mathbf{I}},\overrightarrow{\mathbf{J}}} = (\overline{g}_{\overrightarrow{\mathbf{I}},\overrightarrow{\mathbf{J}}}, g^*_{\overrightarrow{\mathbf{I}},\overrightarrow{\mathbf{J}}})$, where $g^*_{\overrightarrow{\mathbf{I}},\overrightarrow{\mathbf{J}}}$ is an isomorphism between sheaves determined by defining on each entry of $D_{\overrightarrow{\mathbf{I}}}(A_{\overrightarrow{\mathbf{I}}})$ as a rational expression which appears as the corresponding entry provided by the pasting equation

$$D_{\overrightarrow{\mathbf{I}}}\Big((M_{\overrightarrow{\mathbf{I}}}A_{\overrightarrow{\mathbf{J}}})^{-1}A_{\overrightarrow{\mathbf{J}}}\Big) = D_{\overrightarrow{\mathbf{I}}}(A_{\overrightarrow{\mathbf{I}}}), \tag{3.2}$$

where $D_{\overrightarrow{\mathbf{I}}}(A_{\overrightarrow{\mathbf{I}}})$ is a matrix which is remained after omitting $M_{\overrightarrow{\mathbf{I}}}A_{\overrightarrow{\mathbf{I}}}$. Clearly, the left hand side of (3.2) is defined whenever $M_{\overrightarrow{\mathbf{I}}}A_{\overrightarrow{\mathbf{J}}}$ is invertible. The morphism $g^*_{\overrightarrow{\mathbf{I}},\overrightarrow{\mathbf{J}}}$ induces the continuous map $\overline{g}_{\overrightarrow{\mathbf{I}},\overrightarrow{\mathbf{J}}}$ (in the case $n = 1$, see [16], lemma 3.1).

For example in $G_{1|2|1|1}(2|2|2|2)$ suppose

$$I_0 = \{1\}, I_1 = \{1,2\}, I_2 = \{1\}, I_3 = \{2\},$$
$$J_0 = \{2\}, J_1 = \{1,2\}, J_2 = \{2\}, J_3 = \{1\},$$

so $\overrightarrow{\mathbf{I}}, \overrightarrow{\mathbf{J}}$ are 1|2|1|1-indices. We have:

$$A_{\overrightarrow{\mathbf{I}}} = \left[\begin{array}{cc|cc|cc|cc} 1 & x^1 & 0 & 0 & 0 & \xi_2^2 & \xi_3^4 & 0 \\ \hline 0 & \xi_1^1 & 1 & 0 & 0 & \xi_3^2 & \xi_2^3 & 0 \\ 0 & \xi_1^2 & 0 & 1 & 0 & \xi_3^3 & \xi_2^4 & 0 \\ \hline 0 & \xi_2^1 & 0 & 0 & 1 & x^2 & \xi_1^4 & 0 \\ \hline 0 & \xi_3^1 & 0 & 0 & 0 & \xi_1^3 & x^3 & 1 \end{array}\right], A_{\overrightarrow{\mathbf{J}}} = \left[\begin{array}{cc|cc|cc|cc} x^1 & 1 & 0 & 0 & \xi_2^2 & 0 & 0 & \xi_3^4 \\ \hline \xi_1^1 & 0 & 1 & 0 & \xi_3^2 & 0 & 0 & \xi_2^3 \\ \xi_1^2 & 0 & 0 & 1 & \xi_3^3 & 0 & 0 & \xi_2^4 \\ \hline \xi_2^1 & 0 & 0 & 0 & x^2 & 1 & 0 & \xi_1^4 \\ \hline \xi_3^1 & 0 & 0 & 0 & \xi_1^3 & 0 & 1 & x^3 \end{array}\right]$$

$$M_{\overrightarrow{\mathbf{J}}}A_{\overrightarrow{\mathbf{I}}} = \left[\begin{array}{c|cc|c|c} x^1 & 0 & 0 & \xi_2^2 & \xi_3^4 \\ \hline \xi_1^1 & 1 & 0 & \xi_3^2 & \xi_2^3 \\ \xi_1^2 & 0 & 1 & \xi_3^3 & \xi_2^4 \\ \hline \xi_2^1 & 0 & 0 & x^2 & \xi_1^4 \\ \hline \xi_3^1 & 0 & 0 & \xi_1^3 & x^3 \end{array}\right].$$

The maps $g_{\overrightarrow{\mathbf{I}},\overrightarrow{\mathbf{J}}}$ are gluing morphisms. In fact, a straightforward computation shows the following proposition holds.

Proposition 3.1. *Let* $g_{\overrightarrow{I},\overrightarrow{J}} = (\overline{g}_{\overrightarrow{I},\overrightarrow{J}}, g^*_{\overrightarrow{I},\overrightarrow{J}})$ *be as above, then*

1. $g^*_{\overrightarrow{I},\overrightarrow{I}} = id$.
2. $g^*_{\overrightarrow{J},\overrightarrow{I}} \circ g^*_{\overrightarrow{I},\overrightarrow{J}} = id$.
3. $g^*_{\overrightarrow{S},\overrightarrow{I}} \circ g^*_{\overrightarrow{J},\overrightarrow{S}} \circ g^*_{\overrightarrow{I},\overrightarrow{J}} = id$.

Proof. For first equality, note that the map $g^*_{\overrightarrow{\mathbf{I}},\overrightarrow{\mathbf{I}}}$ is obtained from the following equality:

$$D_{\overrightarrow{\mathbf{I}}}\Big((M_{\overrightarrow{\mathbf{I}}}A_{\overrightarrow{\mathbf{I}}})^{-1}A_{\overrightarrow{\mathbf{I}}}\Big) = D_{\overrightarrow{\mathbf{I}}}A_{\overrightarrow{\mathbf{I}}},$$

where the matrix $M_{\overrightarrow{\mathbf{I}}}A_{\overrightarrow{\mathbf{I}}}$ is identity. So $g^*_{\overrightarrow{\mathbf{I}},\overrightarrow{\mathbf{I}}}$ is defined by the following equality:

$$D_{\overrightarrow{\mathbf{I}}}A_{\overrightarrow{\mathbf{I}}} = D_{\overrightarrow{\mathbf{I}}}A_{\overrightarrow{\mathbf{I}}}.$$

This shows the first equality. For second equality, let $\overrightarrow{\mathbf{J}}$ be an another $\overrightarrow{\mathbf{k}}$-index, so $g^*_{\overrightarrow{\mathbf{J}},\overrightarrow{\mathbf{I}}}$ is obtained by the following equality:

$$D_{\overrightarrow{\mathbf{J}}}\Big((M_{\overrightarrow{\mathbf{J}}}A_{\overrightarrow{\mathbf{I}}})^{-1}A_{\overrightarrow{\mathbf{I}}}\Big) = D_{\overrightarrow{\mathbf{J}}}A_{\overrightarrow{\mathbf{J}}}.$$

One may see that $g^*_{\overrightarrow{\mathbf{J}},\overrightarrow{\mathbf{I}}} \circ g^*_{\overrightarrow{\mathbf{I}},\overrightarrow{\mathbf{J}}}$ is obtained by following equality:

$$D_{\overrightarrow{\mathbf{I}}}\bigg(\Big(M_{\overrightarrow{\mathbf{I}}}\big((M_{\overrightarrow{\mathbf{J}}}A_{\overrightarrow{\mathbf{I}}})^{-1}A_{\overrightarrow{\mathbf{I}}}\big)\Big)^{-1}(M_{\overrightarrow{\mathbf{J}}}A_{\overrightarrow{\mathbf{I}}})^{-1}A_{\overrightarrow{\mathbf{I}}}\bigg) = D_{\overrightarrow{\mathbf{I}}}A_{\overrightarrow{\mathbf{I}}}.$$

For left side, we have

$$\begin{aligned}
&= D_{\overrightarrow{\mathbf{I}}}\bigg(\Big((M_{\overrightarrow{\mathbf{J}}}A_{\overrightarrow{\mathbf{I}}})^{-1}M_{\overrightarrow{\mathbf{I}}}A_{\overrightarrow{\mathbf{I}}}\Big)^{-1}(M_{\overrightarrow{\mathbf{J}}}A_{\overrightarrow{\mathbf{I}}})^{-1}A_{\overrightarrow{\mathbf{I}}}\bigg)\\
&= D_{\overrightarrow{\mathbf{I}}}\bigg(\Big((M_{\overrightarrow{\mathbf{J}}}A_{\overrightarrow{\mathbf{I}}})^{-1}\Big)^{-1}(M_{\overrightarrow{\mathbf{J}}}A_{\overrightarrow{\mathbf{I}}})^{-1}A_{\overrightarrow{\mathbf{I}}}\bigg)\\
&= D_{\overrightarrow{\mathbf{I}}}\Big((M_{\overrightarrow{\mathbf{J}}}A_{\overrightarrow{\mathbf{I}}})(M_{\overrightarrow{\mathbf{J}}}A_{\overrightarrow{\mathbf{I}}})^{-1}A_{\overrightarrow{\mathbf{I}}}\Big) = D_{\overrightarrow{\mathbf{I}}}(A_{\overrightarrow{\mathbf{I}}}).
\end{aligned}$$

Accordingly the map $g^*_{\overrightarrow{\mathbf{J}},\overrightarrow{\mathbf{I}}} \circ g^*_{\overrightarrow{\mathbf{I}},\overrightarrow{\mathbf{J}}}$ is obtained by $D_{\overrightarrow{\mathbf{I}}}A_{\overrightarrow{\mathbf{I}}} = D_{\overrightarrow{\mathbf{I}}}A_{\overrightarrow{\mathbf{I}}}$ and it shows that this map is identity. For third equality, it is sufficient to show that the map $g^*_{\overrightarrow{\mathbf{S}},\overrightarrow{\mathbf{I}}} \circ g^*_{\overrightarrow{\mathbf{J}},\overrightarrow{\mathbf{S}}} \circ g^*_{\overrightarrow{\mathbf{I}},\overrightarrow{\mathbf{J}}}$ is obtained from

$$D_{\overrightarrow{\mathbf{I}}}A_{\overrightarrow{\mathbf{I}}} = D_{\overrightarrow{\mathbf{I}}}A_{\overrightarrow{\mathbf{I}}}.$$

This case obtains from case 2 analogously. □

So the sheaves $(\overline{U}_{\overrightarrow{\mathbf{I}}}, \mathcal{O}_{\overrightarrow{\mathbf{I}}})$ may be glued through the $g_{\overrightarrow{\mathbf{I}},\overrightarrow{\mathbf{J}}}$ to construct the $\mathbb{Z}_2^n$−grassmannian $G_{\overrightarrow{\mathbf{k}}}(\overrightarrow{\mathbf{m}})$. Indeed, according to Lemma 3.1 in [10], the conditions of the above proposition are necessary and sufficient for gluing.

4. $\mathbb{Z}_2^n$−grassmannian as homogeneous $\mathbb{Z}_2^n$−space

In this section, we study construction of the quotients of $\mathbb{Z}_2^n$−Lie groups. Although this study is parallel its analogue in supergeometry, see [18] for $\mathbb{Z}_2$ graded case, but generalizing related results in $\mathbb{Z}_2^n$ setting necessitates working with $\mathcal{J}$−adic topology.

Let $G = (\overline{G}, \mathcal{O}_G)$ be a $\mathbb{Z}_2^n$−Lie group and $H = (\overline{H}, \mathcal{O}_H)$ be a closed $\mathbb{Z}_2^n$−Lie supgroup of G. One can define a $\mathbb{Z}_2^n$−manifold structure on the topological space $\overline{X} = \overline{G}/\overline{H}$ as follows:
Let $\mathfrak{g} = Lie(G)$ and $\mathfrak{h} = Lie(H)$ be the $\mathbb{Z}_2^n$−Lie algebras corresponding with G and H. For each $Z \in \mathfrak{g}$, let D_Z be the left invariant vector field on G associated with Z. For $\mathfrak{h}$, a $\mathbb{Z}_2^n$−subalgebra of $\mathfrak{g}$, set:

$$\forall U \subset \overline{G} \qquad \mathcal{O}_{\mathfrak{h}}(U) := \{f \in \mathcal{O}_G(U) \quad | \quad D_Z f = 0 \quad on\ U, \quad \forall Z \in \mathfrak{h}\}.$$

On the other hand, for any open subset $U \subset \overline{G}$ set:

$$\mathcal{O}_{inv}(U) := \{f \in \mathcal{O}_G(U) \quad | \quad \forall x_0 \in \overline{H},\ r^*_{x_0} f = f\},$$

where r_{x_0} is the right translation by x_0 in (2.5). If $\overline{H}$ is connected, then $O_{inv}(U) = O_{\mathfrak{h}}(U)$. Let $\overline{\pi} : \overline{G} \to \overline{X}$ be the natural projection. For each open subset $\overline{W} \subset \overline{X} = \overline{G}/\overline{H}$, the structure sheaf O_X is defined as following

$$O_X(\overline{W}) := O_{inv}(W) \cap O_{\mathfrak{h}}(W),$$

where $W = \overline{\pi}^{-1}(\overline{W})$. One can show that O_X is a sheaf on $\overline{X}$ and the ringed space $X = (\overline{X}, O_X)$ is a $\mathbb{Z}_2^n-$domain locally. Indeed, according to the definition of distribution in [19], we have a distribution spanned by the vector fields in $\mathfrak{h}$ which is denoted by $\mathfrak{D}_{\mathfrak{h}}$. This distribution is involutive, because $\mathfrak{h}$ is a $\mathbb{Z}_2^n-$Lie algebra. By using the local Frobenius theorem, $\mathfrak{D}_{\mathfrak{h}}$ is integrable. For more details about Frobenius theorem in the $\mathbb{Z}_2^n-$graded category, see [19]. So there is an open neighborhood U of identity element 1 and coordinates $(x, \xi_1, \ldots, \xi_{\mathbf{q}})$ such that $\mathfrak{D}_{\mathfrak{h}}$ is spanned on U by

$$\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{r_0}}, \frac{\partial}{\partial \xi_1^1}, \ldots, \frac{\partial}{\partial \xi_1^{r_1}}, \ldots, \frac{\partial}{\partial \xi_{\mathbf{q}}^1} \ldots, \frac{\partial}{\partial \xi_{\mathbf{q}}^{r_{\mathbf{q}}}}.$$

Let $\mathfrak{D}'_{\mathfrak{h}}$ be spanned by

$$\frac{\partial}{\partial x^{r_0+1}}, \ldots, \frac{\partial}{\partial x^{m_0}}, \frac{\partial}{\partial \xi_1^{r_1+1}}, \ldots, \frac{\partial}{\partial \xi_1^{m_1}}, \ldots, \frac{\partial}{\partial \xi_{\mathbf{q}}^{r_{\mathbf{q}}+1}} \ldots, \frac{\partial}{\partial \xi_{\mathbf{q}}^{m_{\mathbf{q}}}}$$

on U. One can use the local Frobenius theorem again, so $\mathfrak{D}'_{\mathfrak{h}}$ is involutive distribution. One can show that $\mathbb{Z}_2^n-$manifold X is obtained by gluing the integral $\mathbb{Z}_2^n-$manifolds of the local distributions $\mathfrak{D}'_{\mathfrak{h}}$. So X is a $\mathbb{Z}_2^n-$manifold and is called homogeneous $\mathbb{Z}_2^n-$space. The details of proof are similar to standard supergeometry in [6].

In this section, we want to show that the $\mathbb{Z}_2^n-$grassmannian $G_{\vec{\mathbf{k}}}(\vec{\mathbf{m}})$ is a homogeneous $\mathbb{Z}_2^n-$space. According to the section 1, it is enough to find a $\mathbb{Z}_2^n-$Lie group which acts on $G_{\vec{\mathbf{k}}}(\vec{\mathbf{m}})$ transitively. For this, we need the following remark and the next lemma

Remark 4.1. Let $\mathcal{X}$ be an element of $U_{\vec{\mathbf{I}}}(T)$ where $\vec{\mathbf{I}}$ is an arbitrary index. One can correspond to $\mathcal{X}$ a $\mathbb{Z}_2^n-$matrix $\vec{\mathbf{k}} \times \vec{\mathbf{m}}$ called $[\mathcal{X}]_{\vec{\mathbf{I}}}$ as follows: Except for columns with indices in $I_0 \cup I_1 \cup \ldots \cup I_{\mathbf{q}}$, the blocks are filled from up to down and left to right by f_i, g_j's where

$$f_i := \mathcal{X}(x_i), \quad g_j := \mathcal{X}(\xi_j),$$

according to the ordering (3.1), where $(x_i; \xi_j)$ is the global coordinates of the $\mathbb{Z}_2^n-$domain $U_{\vec{\mathbf{I}}}$. The columns with indices in $I_0 \cup I_1 \cup \ldots \cup I_{\mathbf{q}}$ form an identity matrix.

Lemma 4.2. *Let $\psi : T \to \mathbb{R}^{\vec{r}}$ be a T-point of $\mathbb{R}^{\vec{r}}$ and (z_{tu}) be a global coordinates of $\mathbb{R}^{\vec{r}}$ with ordering as the one introduced in (*3.1*). If $B = (\psi^*(z_{tu}))$ is the $\mathbb{Z}_2^n-$matrix corresponding to ψ, then the $\mathbb{Z}_2^n-$matrix corresponding to $(g_{\vec{I},\vec{J}})_T(\psi)$ is as follows:*

$$D_{\vec{I}}\Big((M_{\vec{I}}[B]_{\vec{J}})^{-1}[B]_{\vec{J}}\Big),$$

where $[B]_{\vec{J}}$ is introduced in Remark 4.1.

Proof. Note that $g^*_{\vec{\mathbf{I}},\vec{\mathbf{J}}}$ may be represented by a $\mathbb{Z}_2^n$−matrix as follows:

$$D_{\vec{\mathbf{I}}}\Big((M_{\vec{\mathbf{I}}}A_{\vec{\mathbf{J}}})^{-1}A_{\vec{\mathbf{J}}}\Big),$$

where $A_{\vec{\mathbf{J}}}$ is the label of $U_{\vec{\mathbf{J}}}$. Let $M_{\vec{\mathbf{I}}}A_{\vec{\mathbf{J}}} = (m_{tu})$ and $(M_{\vec{\mathbf{I}}}A_{\vec{\mathbf{J}}})^{-1} = (m^{tu})$. If $z = (z_{ij})$ be a coordinates system on $U_{\vec{\mathbf{I}}}$, then one has

$$g^*_{\vec{\mathbf{I}},\vec{\mathbf{J}}}(z_{tu}) = \sum m^{tk}(z).z_{ku}.$$

Then

$$\psi^* \circ g^*_{\vec{\mathbf{I}},\vec{\mathbf{J}}}(z_{tu}) = \psi^*(\sum m^{tk}(z).z_{ku}) = \sum m^{tk}(\psi^*(z)).\psi^*(z_{ku}).$$

For second equality one may note that ψ^* is a homomorphism of $\mathbb{Z}_2^n$−algebras and $m^{tk}(z)$ is a rational function of z. Obviously, the last expression is the (t,u)-entry of the matrix $D_{\vec{\mathbf{I}}}\Big((M_{\vec{\mathbf{I}}}[B]_{\vec{\mathbf{J}}})^{-1}[B]_{\vec{\mathbf{J}}}\Big)$. This completes the proof. □

Theorem 4.3. *The $\mathbb{Z}_2^n$−Lie group $GL(\vec{\mathbf{m}})$ acts on $\mathbb{Z}_2^n$−grassmannian $G_{\vec{k}}(\vec{\mathbf{m}})$.*

Proof. First, we have to define a morphism $a : G_{\vec{\mathbf{k}}}(\vec{\mathbf{m}})\times GL(\vec{\mathbf{m}}) \to G_{\vec{\mathbf{k}}}(\vec{\mathbf{m}})$. For this, by Yoneda lemma, it is sufficient to define a_T:

$$a_T : G_{\vec{\mathbf{k}}}(\vec{\mathbf{m}})(T) \times GL(\vec{\mathbf{m}})(T) \to G_{\vec{\mathbf{k}}}(\vec{\mathbf{m}})(T).$$

for each $\mathbb{Z}_2^n$−manifold T or equivalently define

$$(a_T)^{\mathcal{P}} : G_{\vec{\mathbf{k}}}(\vec{\mathbf{m}})(T) \to G_{\vec{\mathbf{k}}}(\vec{\mathbf{m}})(T).$$

where $\mathcal{P}$ is a fixed arbitrary element in $GL(\vec{\mathbf{m}})(T)$. For brevity, we denote $(a_T)^{\mathcal{P}}$ by $\mathbf{A}$. One may consider $GL(\vec{\mathbf{m}})(T)$, as the set of $\vec{\mathbf{m}} \times \vec{\mathbf{m}}$ invertible $\mathbb{Z}_2^n$−matrices with entries in $\mathcal{O}(T)$, but there is not such a description for $G_{\vec{\mathbf{k}}}(\vec{\mathbf{m}})(T)$, because it is not a $\mathbb{Z}_2^n$−domain. We know each $\mathbb{Z}_2^n$−grassmanian is constructed by gluing $\mathbb{Z}_2^n$−domains (c.f. section 2), so one may define the actions of $GL(\vec{\mathbf{m}})$ on $\mathbb{Z}_2^n$−domains $(\overline{U}_{\vec{\mathbf{I}}}, \mathcal{O}_{\vec{\mathbf{I}}})$ and then shows that these actions glued to construct a_T.

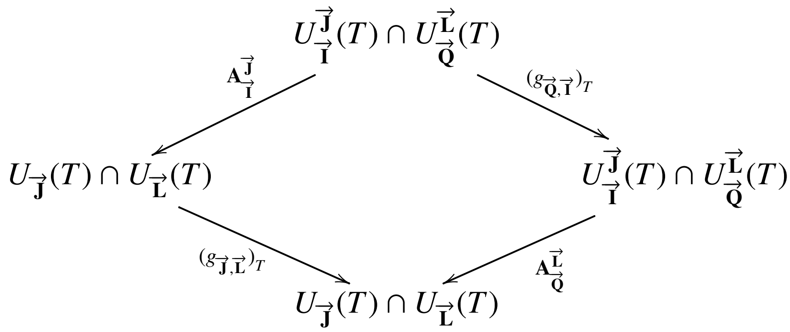
For defining $\mathbf{A}$, it is needed to refine the covering $\{U_{\vec{\mathbf{I}}}(T)\}_{\vec{\mathbf{I}}}$. Set

$$U^{\vec{\mathbf{J}}}_{\vec{\mathbf{I}}}(T) := \Big\{\psi \in U_{\vec{\mathbf{I}}}(T) \quad | \quad M_{\vec{\mathbf{J}}}\Big([\psi]_{\vec{\mathbf{I}}}[\mathcal{P}]\Big) \qquad \text{is invertible}\Big\},$$

where $[\mathcal{P}]$ is the matrix form of the fixed arbitrary element $\mathcal{P}$ in $GL(\vec{\mathbf{m}})(T)$, see [8] and [21]. One can show that $\{U^{\vec{\mathbf{J}}}_{\vec{\mathbf{I}}}(T)\}_{\vec{\mathbf{I}},\vec{\mathbf{J}}}$ is a covering for $G_{\vec{\mathbf{k}}}(\vec{\mathbf{m}})(T)$ and $\mathbf{A}\Big(U^{\vec{\mathbf{J}}}_{\vec{\mathbf{I}}}(T)\Big) \subseteq U_{\vec{\mathbf{J}}}(T)$. Now consider all maps

$$\begin{aligned}\mathbf{A}^{\vec{\mathbf{J}}}_{\vec{\mathbf{I}}} : U^{\vec{\mathbf{J}}}_{\vec{\mathbf{I}}}(T) &\to U_{\vec{\mathbf{J}}}(T)\\ \psi &\mapsto D_{\vec{\mathbf{J}}}\Big(\Big(M_{\vec{\mathbf{J}}}([\psi]_{\vec{\mathbf{I}}}[\mathcal{P}])\Big)^{-1}[\psi]_{\vec{\mathbf{I}}}[\mathcal{P}]\Big)\end{aligned}$$

where, $[\psi]_{\vec{\mathbf{I}}}$ is as above. We have to show that these maps may be glued to construct a global map on $G_{\vec{\mathbf{k}}}(\vec{\mathbf{m}})(T)$. For this, it is sufficient to show that the following diagram commutes:



where $(g_{\vec{\mathbf{I}},\vec{\mathbf{J}}})_T$ is the induced map from $g_{\vec{\mathbf{I}},\vec{\mathbf{J}}}$ on T-points. The following proposition is used to show commutativity of the above diagram. □

Proposition 4.4. *The last diagram commutes.*

Proof. We have to show that

$$(g_{\vec{\mathbf{L}},\vec{\mathbf{J}}})_T \circ \mathbf{A}^{\vec{\mathbf{J}}}_{\vec{\mathbf{I}}} = \mathbf{A}^{\vec{\mathbf{L}}}_{\vec{\mathbf{Q}}} \circ (g_{\vec{\mathbf{Q}},\vec{\mathbf{I}}})_T. \tag{4.1}$$

for arbitrary $\vec{\mathbf{k}}$-indices $\vec{\mathbf{I}}, \vec{\mathbf{J}}, \vec{\mathbf{Q}}, \vec{\mathbf{L}}$. Let $\psi \in U^{\vec{\mathbf{J}}}_{\vec{\mathbf{I}}}(T) \cap U^{\vec{\mathbf{L}}}_{\vec{\mathbf{Q}}}(T)$ be an arbitrary element. One has $\psi \in U^{\vec{\mathbf{J}}}_{\vec{\mathbf{I}}}(T)$, so

$$D_{\vec{\mathbf{J}}}\Big(\big(M_{\vec{\mathbf{J}}}([\psi]_{\vec{\mathbf{I}}}[\mathcal{P}])\big)^{-1}[\psi]_{\vec{\mathbf{I}}}[\mathcal{P}]\Big) \in U_{\vec{\mathbf{J}}}(T),$$

$$(g_{\vec{\mathbf{L}},\vec{\mathbf{J}}})_T\bigg(D_{\vec{\mathbf{J}}}\Big((M_{\vec{\mathbf{J}}}([\psi]_{\vec{\mathbf{I}}}[\mathcal{P}]))^{-1}[\psi]_{\vec{\mathbf{I}}}[\mathcal{P}]\Big)\bigg) \in U_{\vec{\mathbf{L}}}(T).$$

From left side of (4.1), we have:

$$\begin{aligned}
&(g_{\vec{\mathbf{L}},\vec{\mathbf{J}}})_T \circ \mathbf{A}^{\vec{\mathbf{J}}}_{\vec{\mathbf{I}}}(\psi)\\
&= (g_{\vec{\mathbf{L}},\vec{\mathbf{J}}})_T\bigg(D_{\vec{\mathbf{J}}}\Big((M_{\vec{\mathbf{J}}}([\psi]_{\vec{\mathbf{I}}}[\mathcal{P}]))^{-1}[\psi]_{\vec{\mathbf{I}}}[\mathcal{P}]\Big)\bigg)\\
&= D_{\vec{\mathbf{L}}}\bigg(\Big(M_{\vec{\mathbf{L}}}\big((M_{\vec{\mathbf{J}}}([\psi]_{\vec{\mathbf{I}}}[\mathcal{P}]))^{-1}[\psi]_{\vec{\mathbf{I}}}\mathcal{P}\big)\Big)^{-1}(M_{\vec{\mathbf{J}}}([\psi]_{\vec{\mathbf{I}}}\mathcal{P}))^{-1}[\psi]_{\vec{\mathbf{I}}}[\mathcal{P}]\bigg)\\
&= D_{\vec{\mathbf{L}}}\bigg(\Big((M_{\vec{\mathbf{J}}}([\psi]_{\vec{\mathbf{I}}}[\mathcal{P}]))^{-1}(M_{\vec{\mathbf{L}}}([\psi]_{\vec{\mathbf{I}}}[\mathcal{P}]))\Big)^{-1}(M_{\vec{\mathbf{J}}}([\psi]_{\vec{\mathbf{I}}}[\mathcal{P}]))^{-1}[\psi]_{\vec{\mathbf{I}}}[\mathcal{P}]\bigg)\\
&= D_{\vec{\mathbf{L}}}\bigg((M_{\vec{\mathbf{L}}}([\psi]_{\vec{\mathbf{I}}}[\mathcal{P}]))^{-1}M_{\vec{\mathbf{J}}}([\psi]_{\vec{\mathbf{I}}}[\mathcal{P}])(M_{\vec{\mathbf{J}}}([\psi]_{\vec{\mathbf{I}}}[\mathcal{P}]))^{-1}[\psi]_{\vec{\mathbf{I}}}[\mathcal{P}]\bigg)\\
&= D_{\vec{\mathbf{L}}}\bigg(\Big(M_{\vec{\mathbf{L}}}([\psi]_{\vec{\mathbf{I}}}\mathcal{P})\Big)^{-1}[\psi]_{\vec{\mathbf{I}}}\mathcal{P}\bigg).
\end{aligned}$$

For right side of equation (4.1), we have

$$\mathbf{A}^{\vec{\mathbf{L}}}_{\vec{\mathbf{Q}}} \circ (g_{\vec{\mathbf{Q}},\vec{\mathbf{I}}})_T(\psi)$$

$$= \mathbf{A}^{\vec{\mathbf{L}}}_{\vec{\mathbf{Q}}}\Big(D_{\vec{\mathbf{Q}}}\Big((M_{\vec{\mathbf{Q}}}[\psi]_{\vec{\mathbf{I}}})^{-1}[\psi]_{\vec{\mathbf{I}}}\Big)\Big)$$

$$= D_{\vec{\mathbf{L}}}\Big(\Big[M_{\vec{\mathbf{L}}}\Big((M_{\vec{\mathbf{Q}}}[\psi]_{\vec{\mathbf{I}}})^{-1}[\psi]_{\vec{\mathbf{I}}}[\mathcal{P}]\Big)\Big]^{-1}(M_{\vec{\mathbf{Q}}}[\psi]_{\vec{\mathbf{I}}})^{-1}[\psi]_{\vec{\mathbf{I}}}[\mathcal{P}]\Big)$$

$$= D_{\vec{\mathbf{L}}}\Big(\Big[(M_{\vec{\mathbf{Q}}}[\psi]_{\vec{\mathbf{I}}})^{-1}M_{\vec{\mathbf{L}}}([\psi]_{\vec{\mathbf{I}}}[\mathcal{P}])\Big]^{-1}(M_{\vec{\mathbf{Q}}}[\psi]_{\vec{\mathbf{I}}})^{-1}[\psi]_{\vec{\mathbf{I}}}[\mathcal{P}]\Big)$$

$$= D_{\vec{\mathbf{L}}}\Big(\Big(M_{\vec{\mathbf{L}}}([\psi]_{\vec{\mathbf{I}}}[\mathcal{P}])\Big)^{-1}(M_{\vec{\mathbf{Q}}}[\psi]_{\vec{\mathbf{I}}})(M_{\vec{\mathbf{Q}}}[\psi]_{\vec{\mathbf{I}}})^{-1}[\psi]_{\vec{\mathbf{I}}}[\mathcal{P}]\Big)$$

$$= D_{\vec{\mathbf{L}}}\Big(\Big(M_{\vec{\mathbf{L}}}([\psi]_{\vec{\mathbf{I}}}\mathcal{P})\Big)^{-1}[\psi]_{\vec{\mathbf{I}}}\mathcal{P}\Big).$$

This shows that the above diagram commutes. □

Therefore $GL(\vec{\mathbf{m}})$ acts on $G_{\vec{\mathbf{k}}}(\vec{\mathbf{m}})$ with action a. Now it is needed to show that this action is transitive.

Theorem 4.5. *$GL(\vec{m})$ acts on $G_{\vec{k}}(\vec{m})$ transitively.*

Proof. By proposition 2.12, it is sufficient to show that the map

$$(a_p)_{\mathbb{R}^{\vec{\mathbf{r}}'}} : GL(\vec{\mathbf{m}})(\mathbb{R}^{\vec{\mathbf{r}}'}) \to G_{\vec{\mathbf{k}}}(\vec{\mathbf{m}})(\mathbb{R}^{\vec{\mathbf{r}}'}),$$

is surjective, where $\vec{\mathbf{r}} = (r_0, r_1, \ldots, r_{\mathbf{q}})$ is dimension of $GL(\vec{\mathbf{m}})$ and one can consider $\vec{\mathbf{r}}' = (0, r_1, \ldots, r_{\mathbf{q}})$. Let

$$p = (p_0, p_1, \ldots, p_{\mathbf{q}}) \in \overline{U_{\vec{\mathbf{I}}}} \subset G_{k_0}(m_0) \times G_{k_1}(m_1) \times \ldots \times G_{k_{\mathbf{q}}(m_{\mathbf{q}})}$$

be an element and $\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_{\mathbf{q}}$ be the matrices corresponding to $p_0, p_1, \ldots, p_{\mathbf{q}}$ respectively as subspaces. As an element of $G_{\vec{\mathbf{k}}}(\vec{\mathbf{m}})(T)$, one may represent $\hat{p}_T$, as follows

$$\hat{p}_T = \left[\begin{array}{c|c|c|c} \overline{p}_0 & 0 & \ldots & 0 \\ \hline 0 & \overline{p}_1 & \ldots & 0 \\ \hline 0 & 0 & \ldots & 0 \\ \hline 0 & 0 & \ldots & \overline{p}_{\mathbf{q}} \end{array}\right]$$

where T is an arbitrary $\mathbb{Z}_2^n$−manifold. For surjectivity, let

$$W = \left[\begin{array}{c|c|c|c} W_{00} & W_{01} & \ldots & W_{0\mathbf{q}} \\ \hline W_{10} & W_{11} & \ldots & W_{1\mathbf{q}} \\ \hline \ldots & \ldots & \ldots & \ldots \\ \hline W_{\mathbf{q}0} & W_{\mathbf{q}1} & \ldots & W_{\mathbf{q}\mathbf{q}} \end{array}\right] \in U_{\vec{\mathbf{J}}}(\mathbb{R}^{\vec{\mathbf{r}}'}),$$

be an arbitrary element. We have to show that there exists an element $V \in GL(\vec{\mathbf{m}})(\mathbb{R}^{\vec{\mathbf{r}}'})$ such that $\hat{p}_T V = W$. Since the Lie group $GL(m_i)$ acts on manifold $G_{k_i}(m_i)$ transitively, then there exists an invertible matrix $H_{ii} \in GL(m_i)$ such that $\bar{p}_i H_{ii} = W_{ii}$. In addition, the equations $\bar{p}_i Z = W_{ij}$ have solutions since $rank(\bar{p}_i) = k_i$. Let H_{ij} be solutions of these equations respectively. Clearly, One can see

$$V = \left[\begin{array}{c|c|c|c} H_{00} & H_{01} & \dots & H_{0\mathbf{q}} \\ \hline H_{10} & H_{11} & \dots & H_{1\mathbf{q}} \\ \hline \dots & \dots & \dots & \dots \\ \hline H_{\mathbf{q}0} & H_{\mathbf{q}1} & \dots & H_{\mathbf{q}\mathbf{q}} \end{array}\right]_{\overrightarrow{\mathbf{m}}\times\overrightarrow{\mathbf{m}}}$$

satisfy in the equation $\hat{p}_{_T} V = W$. So $(a_p)_{\mathbb{R}^{\overrightarrow{\mathbf{r}}'}}$ is surjective. By Proposition 2.12, $GL(\overrightarrow{\mathbf{m}})$ acts on $G_{\overrightarrow{\mathbf{k}}}(\overrightarrow{\mathbf{m}})$ transitively. □

Thus according to Proposition 2.15, $G_{\overrightarrow{\mathbf{k}}}(\overrightarrow{\mathbf{m}})$ is a homogeneous $\mathbb{Z}_2^n$−space.

A. Basic Concepts

A.1. Category theory

By a locally small category, we mean a category such that the collection of all morphisms between any two of its objects is a set. Let X, Y are objects in a category and $\alpha, \beta : X \to Y$ are morphisms between these objects. An universal pair (E, ϵ) is called *equalizer* if the following diagram commutes:

$$E \xrightarrow{\epsilon} X \underset{\beta}{\overset{\alpha}{\rightrightarrows}} Y,$$

i.e., $\alpha \circ \epsilon = \beta \circ \epsilon$ and also for each object T and any morphism $\tau : T \to X$ which satisfy $\alpha \circ \tau = \beta \circ \tau$, there exists unique morphism $\sigma : T \to E$ such that $\epsilon \circ \sigma = \tau$. If equalizer existed then it is unique up to isomorphism. For example, in the category of sets, which is denoted by **SET**, the equalizer of two morphisms $\alpha, \beta : X \to Y$ is the set $E = \{x \in X | \alpha(x) = \beta(x)\}$ together with the inclusion map $\epsilon : E \hookrightarrow X$.

Let C be a locally small category, and X be an object in C. By T-points of X, we mean $X(T) := Hom_C(T, X)$ for any $T \in Obj(C)$. The functor of points of X is a functor which is denoted by $X(.)$ and is defined as follows:

$$\begin{aligned} X(.) : C &\to \mathbf{SET} \\ S &\mapsto X(S), \end{aligned}$$

$$\begin{aligned} X(.) : Hom_C(S, T) &\to Hom_{\mathbf{SET}}(X(T), X(S)) \\ \varphi &\mapsto X(\varphi), \end{aligned}$$

where $X(\varphi) : f \mapsto f \circ \varphi$. A functor $F : C \to \mathbf{SET}$ is called representable if there exists an object X in C such that F and $X(.)$ are isomorphic. Then one may say that F is represented by X. The category of functors from C to **SET** is denoted by $[C, \mathbf{SET}]$. It is shown that the category of all representable functors from C to **SET** is a subcategory of $[C, \mathbf{SET}]$.

Corresponding to each morphism $\psi : X \to Y$, there exists a natural transformation $\psi(.)$ from $X(.)$ to $Y(.)$. This transformation corresponds the mapping $\psi(T) : X(T) \to Y(T)$ with $\xi \mapsto \psi \circ \xi$ for each

$T \in Obj(C)$. Now set:

$$\mathcal{Y} : C \to [C, \mathbf{SET}]$$
$$X \mapsto X(.)$$
$$\psi \mapsto \psi(.).$$

Obviously, $\mathcal{Y}$ is a covariant functor and it is called ***Yoneda embedding***. The following lemma, is brought from [8] or the section $III.2$ in [20].

Lemma A.1. *The Yoneda embedding is full and faithful functor, i.e. the map*

$$Hom_C(X, Y) \longrightarrow Hom_{[C,\boldsymbol{SET}]}(X(.), Y(.)),$$

is a bijection for each $X, Y \in Obj(C)$.

Thus according to this lemma, $X, Y \in Obj(C)$ are isomorphic if and only if their functor of points are isomorphic. The Yoneda embedding is an equivalence between C and a subcategory of representable functors in $[C, \mathbf{SET}]$ since not all functors are representable.

A.2. The $\mathbb{Z}_2^n$−Lie group $GL(V)$

Let V be a finite dimensional $\mathbb{Z}_2^n$-vector space of dimension $\vec{\mathbf{m}} = m_0|m_1|\ldots|m_{\mathbf{q}}$ and let

$$\{R_1, \ldots, R_{m_0}, R_{m_0+1}, \ldots, R_{m_0+m_1}, \ldots, R_{m_0+\ldots+m_{\mathbf{q}}}\}$$

be a basis of V for which the elements $R_{m_0+\ldots+m_{i-1}+k}$, $1 \le k \le m_i$, are of weight γ_i for $0 \le i \le \mathbf{q}$.

Consider the functor F from the category **ZSM** to **GRP** the category of groups which maps each $\mathbb{Z}_2^n$−manifold T to $Aut_{\mathcal{O}(T)}(\mathcal{O}(T) \otimes V)$ the group of zero weight automorphisms of $\mathcal{O}(T) \otimes V$. Consider the $\mathbb{Z}_2^n$−manifold $\mathbf{End}(V) = \Big(\prod_i End(V_i), \mathcal{A}\Big)$ where $\mathcal{A}$ is the following sheaf

$$C^\infty_{\mathbb{R}^{m_0^2+\ldots+m_{\mathbf{q}}^2}}[[\xi_1^1, \ldots, \xi_1^{t_1}, \xi_2^1, \ldots, \xi_2^{t_2}, \ldots, \xi_3^1, \ldots, \xi_{\mathbf{q}}^{t_{\mathbf{q}}}]]. \tag{A.1}$$

where $t_k = \sum_{\gamma_i+\gamma_j=\gamma_k} m_i m_j$. Let F_{ij} be a linear transformation on V defined by $R_k \mapsto \delta_{ik} R_j$, then $\{F_{ij}\}$ is a basis for $\mathbf{End}(V)$. If $\{f_{ij}\}$ is the corresponding dual basis, then it may be considered as a global coordinates on $\mathbf{End}(V)$. Let X be the open submanifold of $\mathbf{End}(V)$ corresponding to the open set:

$$\overline{X} = \prod_i GL(V_i) \subset \prod_i End(V_i).$$

Thus, we have

$$X = \Big(\prod_i GL(V_i), \mathcal{A}|_{\prod_i GL(V_i)}\Big).$$

It can be shown that the functor F may be represented by X. For this, one may show that $Hom(T, X) \cong Aut_{\mathcal{O}(T)}(\mathcal{O}(T) \otimes V)$. To this end, first, one can notice Theorem 9 in [17] to see

$$Hom(T, X) = Hom\Big(\mathcal{A}(X), \mathcal{O}(T)\Big).$$

It is known that each $\psi \in Hom\Big(\mathcal{A}(X), \mathcal{O}(T)\Big)$ may be uniquely determined by $\{g_{ij}\}$ where $g_{ij} = \psi(f_{ij})$, see Theorem 6.8 in [1]. Now set $\Psi(R_j) := \Sigma g_{ij} R_i$. One may consider Ψ as an element of $Aut_{\mathcal{O}(T)}(\mathcal{O}(T) \otimes$

V). Obviously $\psi \mapsto \Psi$ is a bijection from $Hom(T, X)$ to $Aut_{O(T)}(O(T) \otimes V)$. Thus the $\mathbb{Z}_2^n$−manifold X is a $\mathbb{Z}_2^n$−Lie group and denoted it by $GL(V)$ or $GL(\overrightarrow{\mathbf{m}})$ if $V = \mathbb{R}^{\overrightarrow{\mathbf{m}}}$. Therefore T- points of $GL(\overrightarrow{\mathbf{m}})$ are the $\overrightarrow{\mathbf{m}} \times \overrightarrow{\mathbf{m}}$ invertible $\mathbb{Z}_2^n$−matrices of weight zero

$$\left[\begin{array}{c|c|c|c} B_{00} & B_{01} & \dots & B_{0\mathbf{q}} \\ \hline \dots & & & \dots \\ \hline B_{\mathbf{q}0} & B_{\mathbf{q}1} & \dots & B_{\mathbf{q}\mathbf{q}} \end{array}\right].$$

where the elements of the $m_k \times m_u$ block B_{ku} have degree $\gamma_k + \gamma_u$ and the multiplication is the matrix product.

A.3. Proof of (Proposition 2.10*)*

Proof. The proof of (1) is easy because of (2.10). For (2), let M be a $\mathbb{Z}_2^n$-manifold of dimension $\overrightarrow{\mathbf{m}}$ and G be a Lie $\mathbb{Z}_2^n$-group of dimension $\overrightarrow{\mathbf{k}}$. Let $\mathfrak{g}$ be the Lie $\mathbb{Z}_2^n$-algebra of G and let J_{a_p} be the Jacobian $\mathbb{Z}_2^n$-matrix of a_p in a neighborhood of a point $g \in \overline{G}$. Since

$$\overline{J_{a_p}}(g) = (da_p)_g = (da^g)_p \circ (da_p)_e \circ (dr_{g^{-1}})_g,$$

and a^g and $r_{g^{-1}}$ are diffeomorphisms, $\overline{J_{a_p}}(g)$ has rank equal to $\dim \mathfrak{g} - \dim \ker(da_p)_e$ for each $g \in \overline{G}$. Recall that if $X \in \mathfrak{g}$ we denote by $D_X^R := (X \otimes 1_{O_G(U)})\mu^*$ the right-invariant vector field associated with X. Using equation (2.8) we have, for each $X \in \ker(da_p)_e$,

$$D_X^R a_p^* = 0. \tag{A.2}$$

Because

$$D_X^R a_p^* = (X \otimes 1_{O_G(U)})\mu^*(ev_p \otimes 1_{O_G(U)})a^* = \Big((da_p)_e(X) \otimes 1_{O_G(U)}\Big)a^* = 0.$$

If $(\xi_0 = x, \xi_1, \dots, \xi_{\mathbf{q}})$ and $(\eta_0 = y, \eta_1, \dots, \eta_{\mathbf{q}})$ are coordinates in a neighborhood U of $e \in \overline{G}$, and in a neighborhood $V \supseteq \overline{a_p}(U)$ of $p \in \overline{M}$, respectively, then J_{a_p} may be represented as the following block matrix:

$$\begin{pmatrix} \frac{\partial a_p^*(\eta_0)}{\partial \xi_0} & \pm\frac{\partial a_p^*(\eta_0)}{\partial \xi_1} & \dots & \pm\frac{\partial a_p^*(\eta_0)}{\partial \xi_{\mathbf{q}}} \\ \pm\frac{\partial a_p^*(\eta_1)}{\partial \xi_0} & \pm\frac{\partial a_p^*(\eta_1)}{\partial \xi_1} & \dots & \pm\frac{\partial a_p^*(\eta_1)}{\partial \xi_{\mathbf{q}}} \\ \dots & \dots & \dots & \dots \\ \pm\frac{\partial a_p^*(\eta_{\mathbf{q}})}{\partial \xi_0} & \pm\frac{\partial a_p^*(\eta_{\mathbf{q}})}{\partial \xi_1} & \dots & \pm\frac{\partial a_p^*(\eta_{\mathbf{q}})}{\partial \xi_{\mathbf{q}}} \end{pmatrix} \in M_{\overrightarrow{\mathbf{m}},\overrightarrow{\mathbf{k}}}(O_G(U)), \tag{A.3}$$

where by $\dfrac{\partial a_p^*(\eta_t)}{\partial \xi_l}$, we mean a matrix as follows

$$\left(\frac{\partial a_p^*(\eta_t^i)}{\partial \xi_l^j}\right)_{1 \le i \le m_t, 1 \le j \le k_l},$$

and its sign in (A.3) is equal to

$$(-1)^{\langle \deg \eta_t + \deg \xi_l, \deg \xi_l \rangle}.$$

Now, we find a matrix $A \in GL_{\overrightarrow{\mathbf{k}}}(\mathcal{O}_G(U))$ such that $J_{a_p}A$ has a certain set of columns equal to zero. We are going to use equation (A.2). Let

$$\dim \ker(da_p)_e = \overrightarrow{\mathbf{n}},$$

and $\{X_{u_0}\}, \ldots$ and $\{X_{u_\mathbf{q}}\}$ be the bases of $\mathfrak{g}_0, \ldots$ and $\mathfrak{g}_\mathbf{q}$ such that for all n_r if $u_r \leq n_r$ then $X_{u_r} \in \ker(da_p)_e$. Consider a block matrix as follows:

$$A = \begin{pmatrix} \alpha^0_{u_0,i_0} & \pm\alpha^0_{u_1,i_0} & \ldots & \pm\alpha^0_{u_\mathbf{q},i_0} \\ \pm\alpha^1_{u_0,i_1} & \pm\alpha^1_{u_1,i_1} & \ldots & \pm\alpha^1_{u_\mathbf{q},i_1} \\ \ldots & \ldots & \ldots & \ldots \\ \pm\alpha^\mathbf{q}_{u_0,i_\mathbf{q}} & \pm\alpha^\mathbf{q}_{u_1,i_\mathbf{q}} & \ldots & \pm\alpha^\mathbf{q}_{u_\mathbf{q},i_\mathbf{q}} \end{pmatrix} \tag{A.4}$$

where, for each u_j, $0 \leq j \leq \mathbf{q}$, $\alpha_{u_j.i_s}$'s are the coefficients appear in the following equality:

$$D^R_{X_{u_j}} = \sum_{1\leq i_0 \leq k_0} \alpha^0_{u_j,i_0} \frac{\partial}{\partial \xi_0^{i_0}} + \sum_{1\leq i_1 \leq k_1} \alpha^1_{u_j,i_1} \frac{\partial}{\partial \xi_1^{i_1}} + \ldots + \sum_{1\leq i_\mathbf{q} \leq k_\mathbf{q}} \alpha^\mathbf{q}_{u_j,i_\mathbf{q}} \frac{\partial}{\partial \xi_\mathbf{q}^{i_\mathbf{q}}},$$

where $\gamma_j + \gamma_s$ is the degree of $\alpha^s_{u_j,i_s}$ and the sign of (t,l)-th block is

$$(-1)^{\langle \deg \gamma_t + \deg \gamma_l, \deg \gamma_l \rangle}.$$

Since X_{u_i} and X_{u_j} are linearly independent at each point, the reduced matrix $\overline{A}$ is invertible and $A \in GL_{\overrightarrow{\mathbf{k}}}(\mathcal{O}_G(U))$. Now by equation (A.2), one may see that the matrix

$$J_{a_p}A = \begin{pmatrix} D^R_{X_{u_0}} a_p^*(\xi_0) & \pm D^R_{X_{u_1}} a_p^*(\xi_0) & \ldots & \pm D^R_{X_{u_\mathbf{q}}} a_p^*(\xi_0) \\ \pm D^R_{X_{u_0}} a_p^*(\xi_1) & \pm D^R_{X_{u_1}} a_p^*(\xi_1) & \ldots & \pm D^R_{X_{u_\mathbf{q}}} a_p^*(\xi_1) \\ \ldots & \ldots & \ldots & \ldots \\ \pm D^R_{X_{u_0}} a_p^*(\xi_\mathbf{q}) & \pm D^R_{X_{u_1}} a_p^*(\xi_\mathbf{q}) & \ldots & \pm D^R_{X_{u_\mathbf{q}}} a_p^*(\xi_\mathbf{q}) \end{pmatrix},$$

has $|\overrightarrow{\mathbf{n}}| = n_0 + \ldots + n_\mathbf{q}$ zero columns. Thus $\overline{J_{a_p}}$ has rank equal to $\overrightarrow{\mathbf{k}} - \overrightarrow{\mathbf{n}}$ with entries in $\mathcal{O}_G(U)$. In addition the first $|\overrightarrow{\mathbf{k}}| - |\overrightarrow{\mathbf{n}}|$ columns are non-zero. Thus one may describe J_{a_p} by $\begin{pmatrix} z & 0 \\ w & 0 \end{pmatrix}$. Then it is not restrictive to assume that z invertible. Thus one has

$$GJ = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$G = \begin{pmatrix} z^{-1} & 0 \\ -wz^{-1} & I \end{pmatrix}.$$

For more explanation, assume J_{a_p} is a matrix of the form as below

$$\begin{pmatrix} 0 & \alpha_{11} & 0 & \alpha_{12} & \dots & 0 & \alpha_{1\mathbf{q}} \\ 0 & \alpha_{21} & 0 & \alpha_{22} & \dots & 0 & \alpha_{2\mathbf{q}} \\ 0 & \dots & 0 & \dots & \dots & 0 & \dots \\ 0 & \alpha_{\mathbf{q}1} & 0 & \alpha_{\mathbf{q}2} & \dots & 0 & \alpha_{\mathbf{qq}} \\ 0 & \beta_{11} & 0 & \beta_{12} & \dots & 0 & \beta_{1\mathbf{q}} \\ 0 & \beta_{21} & 0 & \beta_{22} & \dots & 0 & \beta_{2\mathbf{q}} \\ 0 & \dots & 0 & \dots & \dots & 0 & \dots \\ 0 & \beta_{\mathbf{q}1} & 0 & \beta_{\mathbf{q}2} & \dots & 0 & \beta_{\mathbf{qq}} \end{pmatrix}.$$

Since $\overline{J_{a_p}}$ has rank $\overrightarrow{\mathbf{k}} - \overrightarrow{\mathbf{n}}$, we can suppose that α_{ii} and β_{jj} are invertible. By section 4 in [10], we can rearrange the matrix so that it takes the form $\begin{pmatrix} z & 0 \\ w & 0 \end{pmatrix}$, with z invertible and the matrix $G = \begin{pmatrix} z^{-1} & 0 \\ -wz^{-1} & I \end{pmatrix}$ such that $GJ = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$. We can then conclude that J_{a_p} has constant rank in U and, by translation, in all of $\overline{G}$. □

Conflict of interest

The author declares there is no conflict of interest.

References

1. T. Covolo, J. Grabowski, N. Poncin, $\mathbb{Z}_2^n$−supergeometry I: manifolds and morphisms. arXiv:1408.2755.
2. T. Covolo, J. Grabowski, N. Poncin, Splitting theorem for $\mathbb{Z}_2^n$−supermanifolds, *J. Geom. Phys.*, **110** (2016), 393–401. https://doi.org/10.1016/j.geomphys.2016.09.006
3. T. Covolo, J. Grabowski, N. Poncin, The category of $\mathbb{Z}_2^n$−supermanifolds, *J. Math. Phys.*, **57** (2016), 073503. https://doi.org/10.1063/1.4955416
4. T. Covolo, V. Ovsienko, N. Poncin, Higher trace and Berezinian of matrices over a Clifford algebra, *J. Geom. Phys.*, **62** (2012), 2294–2319. https://doi.org/10.1016/j.geomphys.2012.07.004
5. W. M. Yang, S. C. Jing, A New Kind of Graded Lie Algebra and Parastatistical Supersymmetry, *Science in China (Series A)*, **44** (2001),1167–1173. https://doi.org/10.1007/BF02877435
6. L. Balduzzi, C. Carmeli, G. Cassinelli, Super G-spaces, *Symmetry in mathematics and physics,AMS: Providence, RI, USA*,(2009), 159–176. https://doi.org/10.1090/conm/490/09594
7. L. Balduzzi, C. Carmeli, R. Fioresi, Quotients in supergeometry, *Symmetry in mathematics and physics,AMS: Providence, RI, USA*, (2009), 177–187. https://doi.org/10.1090/conm/490/09595
8. C. Carmeli, L. Caston, R. Fioresi, Mathematical Foundations of Supersymmetry, European Mathematical Society, 2011. https://doi.org/10.4171/097

9. Y. I. Manin, *Gauge Field Theory and Complex Geometry*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 289, Springer-Verlag, Berlin, 1988, Translated from the Russian by N. Koblitz and J. R. King.
10. T. Covolo, S. Kwok, N. Poncin, Differential Calculus on $\mathbb{Z}_2^n$supermanifolds. *arXiv preprint arXiv:1608.00949*, 2016.
11. A. J. Bruce, N. Poncin, Products in the category of $\mathbb{Z}_2^n$−manifolds, *J. Nonlinear Math. Phys.*, **26** (2019), 420–453. https://doi.org/10.1080/14029251.2019.1613051
12. B. Jubin, A. Kotov, N. Poncin, V. Salnikov, Differential graded Lie groups and their differential graded Lie algebras, *arXiv:1906.09630*.
13. F. Bahadorykhalily, M. Mohammadi, S. Varsaie, A class of homogeneous superspaces associated to odd involutions, *Period Math. Hung.*, **82** (2021), 153–172. https://doi.org/10.1007/s10998-020-00364-9
14. O. Sanchez-Valenzuela, Remarks on grassmannian supermanifolds, *Trans. Am. Math. Soc.*, **307** (1988), 597–614. https://doi.org/10.2307/2001190
15. M. Mohammadi, S. Varsaie, Supergrassmannians as Homogeneous Superspaces, *Asian-European J. Math.*, **14** (2021), 2150053. https://doi.org/10.1142/S1793557121500534
16. M. Roshandelbana, S. Varsaie, Analytic Approach To ν-Classes. arXiv:1801.06633
17. A. J. Bruce, N. Poncin, Functional analytic issues in $\mathbb{Z}_2^n$−Geometry, *Revista de la Unión Matemática Argentina*, **60** (2019), 611–636. https://doi.org/10.33044/revuma.v60n2a21
18. R. Fioresi, M. A. Lledo, V. S. Varadarajan, The Minkowski and conformal superspaces, *J. Math. Phys.*, **48** (2007), 113505. https://doi.org/10.1063/1.2799262
19. T. Covolo, S. Kwok, N. Poncin, The Frobenius Theorem for $\mathbb{Z}_2^n$supermanifolds, arXiv:1608.00961.
20. S. Maclane, Categories for the Working Mathematician, Springer, 1978. https://doi.org/10.1007/978-1-4757-4721-8
21. V. S. Varadarajan, *Supersymmetry for Mathematicians: An Introduction*, Courant Lecture Notes in Mathematics, vol. 11, New York University CourantInstitute of Mathematical Sciences, New York, 2004. https://doi.org/10.1090/cln/011