Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 30(1): 204–220.
DOI: 10.3934/era.2022011
Received: 08 August 2021
Revised: 17 November 2021
Accepted: 17 November 2021
Published: 24 December 2021

Research article

Smash product construction of modular lattice vertex algebras

Qiang Mu*

School of Mathematical Sciences, Harbin Normal University, Harbin, Heilongjiang 150025, China

* **Correspondence:** Email:qmu520@gmail.com.

Abstract: Motivated by a work of Li, we study nonlocal vertex algebras and their smash products over fields of positive characteristic. Through smash products, modular vertex algebras associated with positive definite even lattices are reconstructed. This gives a different construction of the modular vertex algebras obtained from integral forms introduced by Dong and Griess in lattice vertex operator algebras over a field of characteristic zero.

Keywords: vertex operator algebra; modular vertex algebra

1. Introduction

It is well known that there are three important classes of rational vertex operator algebras over the field of complex numbers; namely, affine vertex operator algebras, Virasoro vertex operator algebras and lattice vertex operator algebras (see [1]). It is natural to study the corresponding vertex algebra structures over fields of prime characteristic first.

There have already been some works on modular vertex algebras and their representations. For example, modular $A(V)$ theory and $A_n(V)$ theory were studied in [2, 3], modular Virasoro vertex operator algebras were studied in [4], and framed vertex operator algebras were studied in [5]. Modular vertex algebras obtained from integral forms in some vertex operator algebras over the field of complex numbers were used to study modular moonshine in [6–8].

Dong and Griess introduced an integral form of the vertex algebras associated with positive definite even lattices over a field of characteristic zero in [9], and the related modular vertex algebras were studied in [10].

In a series of papers (see [11–13]), Li studied nonlocal vertex algebras over a field of characteristic zero. In particular, in [13] Li introduced a smash product construction of nonlocal vertex algebras and used smash product to give a different construction of lattice vertex algebras and their modules (cf. [1, 14]). Motivated by [13], in this paper we study nonlocal vertex algebras and the smash product construction over fields of prime characteristic. As an application, modular vertex algebras associated with positive definite even lattices are reconstructed by using smash products. This gives another construction of the modular vertex algebras obtained from integral forms introduced by Dong and Griess of lattice vertex operator algebras over a field of characteristic zero.

This paper is organized as follows: In Section 2, we present some basic results on modular nonlocal vertex algebras and give the smash product construction of nonlocal vertex algebras. In Section 3, we use the smash product to construct the vertex algebras associated with positive definite even lattices.

2. Nonlocal vertex algebras and smash products

In this section, we first present some basic results on modular nonlocal vertex algebras. Then we give the smash product construction of modular nonlocal vertex algebras. The proofs for most of the results in this section are the same as those for characteristic zero (see [11–13]).

Let $\mathbb{F}$ be an algebraically closed field of an odd prime characteristic p, which is fixed throughout this paper. All vector spaces, including algebras, are considered to be over $\mathbb{F}$. We use $\mathbb{Z}$ for the integers, $\mathbb{Z}_+$ for the positive integers, and $\mathbb{N}$ for the nonnegative integers.

Note that for any $m \in \mathbb{Z},\ k \in \mathbb{N}$,

$$\binom{m}{k} = \frac{m(m-1)\cdots(m+1-k)}{k!} \in \mathbb{Z}.$$

Then we shall also view $\binom{m}{k}$ as an element of $\mathbb{F}$. Furthermore, for $m \in \mathbb{Z}$ we have

$$(x \pm z)^m = \sum_{k\in\mathbb{N}} \binom{m}{k} (\pm 1)^k x^{m-k} z^k \in \mathbb{F}[x, x^{-1}][[z]].$$

The following definition is the same as in characteristic zero (see [11–13]).

Definition 2.1. A *nonlocal vertex algebra* is a vector space V endowed with a distinguished vector $\mathbf{1}$, called the *vacuum vector*, and endowed with a linear map

$$\begin{aligned} Y(\cdot, x) : V &\to (\mathrm{End}\, V)[[x, x^{-1}]] \\ v &\mapsto Y(v, x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1} \end{aligned} \tag{2.1}$$

such that for $u, v \in V$,

$$u_n v = 0 \quad \text{for } n \text{ sufficiently large,} \tag{2.2}$$

$$Y(\mathbf{1}, x) = 1, \tag{2.3}$$

$$Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x\to 0} Y(v, x)\mathbf{1} = v \quad \text{for } v \in V \tag{2.4}$$

and for $u, v, w \in V$, there exists a nonnegative integer l such that

$$(x_0 + x_2)^l Y(u, x_0 + x_2) Y(v, x_2) w = (x_0 + x_2)^l Y(Y(u, x_0)v, x_2) w. \tag{2.5}$$

As in [15], define $\mathcal{B}$ to be the bialgebra with a basis $\{\mathcal{D}^{(r)} \mid r \in \mathbb{N}\}$, where

$$\mathcal{D}^{(m)} \cdot \mathcal{D}^{(n)} = \binom{m+n}{n} \mathcal{D}^{(m+n)}, \qquad \mathcal{D}^{(0)} = 1,$$
$$\Delta(\mathcal{D}^{(n)}) = \sum_{i=0}^{n} \mathcal{D}^{(n-i)} \otimes \mathcal{D}^{(i)}, \qquad \varepsilon(\mathcal{D}^{(n)}) = \delta_{n,0}$$

for $m, n \in \mathbb{N}$. Set

$$e^{x\mathcal{D}} = \sum_{n \in \mathbb{N}} x^n \mathcal{D}^{(n)} \in \mathcal{B}[[x]]. \tag{2.6}$$

The bialgebra structure of $\mathcal{B}$ can be described in terms of the generating functions as

$$e^{x\mathcal{D}} e^{z\mathcal{D}} = e^{(x+z)\mathcal{D}}, \qquad \Delta(e^{x\mathcal{D}}) = e^{x\mathcal{D}} \otimes e^{x\mathcal{D}}, \qquad \varepsilon(e^{x\mathcal{D}}) = 1, \tag{2.7}$$

in particular, we have $e^{x\mathcal{D}} e^{-x\mathcal{D}} = 1$.

Remark 2.2. Let U be any vector space. Then $U[[x, x^{-1}]]$ is a $\mathcal{B}$-module with $\mathcal{D}^{(n)}$ for $n \in \mathbb{N}$ acting as the *n-th Hasse differential operator* $\partial_x^{(n)}$ with respect to x, which is defined by

$$\partial_x^{(n)} x^m = \binom{m}{n} x^{m-n} \quad \text{for } m \in \mathbb{Z}. \tag{2.8}$$

Set $e^{z\partial_x} = \sum_{n \in \mathbb{N}} z^n \partial_x^{(n)}$. Then

$$f(x+z) = e^{z\partial_x} f(x) \quad \text{for } f(x) \in U[[x, x^{-1}]]. \tag{2.9}$$

As in the case of characteristic zero, we have (cf. [13]):

Lemma 2.3. *Let V be a nonlocal vertex algebra. Then V is naturally a $\mathcal{B}$-module with*

$$\mathcal{D}^{(n)} u = u_{-n-1} \mathbf{1} \tag{2.10}$$

for $u \in V$ and $n \in \mathbb{N}$. Furthermore,

$$e^{x_0\mathcal{D}} Y(u, x_2) e^{-x_0\mathcal{D}} = Y(e^{x_0\mathcal{D}} u, x_2) = Y(u, x_2 + x_0) = e^{x_0 \partial_{x_2}} Y(u, x_2). \tag{2.11}$$

Proof. Let $u \in V$. By (2.4) and (2.10), we have

$$e^{x\mathcal{D}} u = Y(u, x)\mathbf{1}, \tag{2.12}$$

and in particular, $\mathcal{D}^{(0)} = 1$ on V.

Let $l \in \mathbb{N}$ be such that

$$(x_0 + x_2)^l Y(Y(u, x_0)\mathbf{1}, x_2)\mathbf{1} = (x_0 + x_2)^l Y(u, x_0 + x_2) Y(\mathbf{1}, x_2)\mathbf{1}.$$

As $Y(\mathbf{1}, x) = 1$ and $Y(u, x)\mathbf{1}$ only involves nonnegative powers of x, we have

$$(x_0 + x_2)^l Y(Y(u, x_0)\mathbf{1}, x_2)\mathbf{1} = (x_0 + x_2)^l Y(u, x_0 + x_2)\mathbf{1}$$

$$= (x_0 + x_2)^l Y(u, x_2 + x_0)\mathbf{1}.$$

Multiplying both sides by $(x_2 + x_0)^{-l}$ we obtain

$$Y(Y(u, x_0)\mathbf{1}, x_2)\mathbf{1} = Y(u, x_2 + x_0)\mathbf{1}.$$

Using (2.12) and the equation above, we have

$$\begin{aligned} e^{x_2\mathcal{D}}(e^{x_0\mathcal{D}}u) &= Y(e^{x_0\mathcal{D}}u, x_2)\mathbf{1} = Y(Y(u, x_0)\mathbf{1}, x_2)\mathbf{1} \\ &= Y(u, x_2 + x_0)\mathbf{1} = e^{(x_2+x_0)\mathcal{D}}u. \end{aligned}$$

Therefore V is a $\mathcal{B}$-module.

Let $u, v \in V$. Then there is $l \in \mathbb{N}$ such that

$$\begin{aligned} (x_0 + x_2)^l Y(Y(u, x_0)\mathbf{1}, x_2)v &= (x_0 + x_2)^l Y(u, x_0 + x_2)Y(\mathbf{1}, x_2)v \\ &= (x_0 + x_2)^l Y(u, x_0 + x_2)v. \end{aligned}$$

We may assume that $x^l Y(u, x)v \in V[[x]]$ by replacing l with a bigger integer if necessary, so that

$$(x_0 + x_2)^l Y(u, x_0 + x_2)v = (x_0 + x_2)^l Y(u, x_2 + x_0)v.$$

Then

$$(x_2 + x_0)^l Y(Y(u, x_0)\mathbf{1}, x_2)v = (x_2 + x_0)^l Y(u, x_2 + x_0)v.$$

Multiplying both sides by $(x_2 + x_0)^{-l}$ we have

$$Y(e^{x_0\mathcal{D}}u, x_2)v = Y(Y(u, x_0)\mathbf{1}, x_2)v = Y(u, x_2 + x_0)v = e^{x_0\partial_{x_2}} Y(u, x_2)v.$$

Let $u, v \in V$ and let $l \in \mathbb{N}$ be such that

$$(x_0 + x_2)^l Y(Y(u, x_0)v, x_2)\mathbf{1} = (x_0 + x_2)^l Y(u, x_0 + x_2)Y(v, x_2)\mathbf{1}.$$

Since $Y(Y(u, x_0)v, x_2)\mathbf{1}$ involves only nonnegative powers of x_2, we can multiply both sides by $(x_0+x_2)^{-l}$ to get

$$Y(Y(u, x_0)v, x_2)\mathbf{1} = Y(u, x_0 + x_2)Y(v, x_2)\mathbf{1}.$$

Then

$$\begin{aligned} e^{x_2\mathcal{D}}Y(u, x_0)v &= Y(Y(u, x_0)v, x_2)\mathbf{1} = Y(u, x_0 + x_2)Y(v, x_2)\mathbf{1} \\ &= Y(u, x_0 + x_2)e^{x_2\mathcal{D}}v, \end{aligned}$$

that is,

$$e^{x_2\mathcal{D}}Y(u, x_0) = Y(u, x_0 + x_2)e^{x_2\mathcal{D}}$$

on V. Applying $e^{-x_2\mathcal{D}}$ from left, as $e^{x_2\mathcal{D}}e^{-x_2\mathcal{D}} = 1$, we have

$$e^{x_2\mathcal{D}}Y(u, x_0)e^{-x_2\mathcal{D}} = Y(u, x_0 + x_2).$$

Thus (2.11) holds. □

The following two results of [11] are valid here with the same proof.

Lemma 2.4. *For any subset S of a nonlocal vertex algebra V, the subalgebra $\langle S \rangle$ generated by S is linearly spanned by vectors*

$$v^{(1)}_{n_1} v^{(2)}_{n_2} \cdots v^{(r)}_{n_r} \mathbf{1}$$

for $r \in \mathbb{N}$, $v^{(i)} \in S$, $n_i \in \mathbb{Z}$.

Let V be a nonlocal vertex algebra. For $u, v \in V$, we say u, v are *mutually local* if there exists $k \in \mathbb{N}$ such that

$$(x_1 - x_2)^k Y(u, x_1)Y(v, x_2) = (x_1 - x_2)^k Y(v, x_2)Y(u, x_1). \tag{2.13}$$

A subset S of V is said to be *local* if every pair of elements of S are mutually local.

Lemma 2.5. *Let V be a nonlocal vertex algebra and let S be a local subset of V. Then the subalgebra $\langle S \rangle$ of V generated by S is a vertex algebra.*

Definition 2.6. Let V be a nonlocal vertex algebra. A *V-module* is a vector space W endowed with a linear map

$$\begin{aligned} Y_W(\cdot, x) : V &\to (\mathrm{End}\, W)[[x, x^{-1}]] \\ v &\mapsto Y_W(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \end{aligned} \tag{2.14}$$

such that for $v \in V$ and $w \in W$,

$$v_n w = 0 \quad \text{for } n \text{ sufficiently large}, \tag{2.15}$$

$$Y_W(\mathbf{1}, x) = 1, \tag{2.16}$$

and for $u, v \in V$ and $w \in W$, there exists a nonnegative integer l such that

$$(x_0 + x_2)^l Y_W(u, x_0 + x_2)Y_W(v, x_2)w = (x_0 + x_2)^l Y_W(Y(u, x_0)v, x_2)w. \tag{2.17}$$

A unital associative algebra A is called a *$\mathcal{B}$-module algebra* if A is a $\mathcal{B}$-module such that

$$h \cdot (ab) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b), \qquad h \cdot 1 = \varepsilon(h)1 \tag{2.18}$$

for $h \in \mathcal{B}$ and $a, b \in A$, where $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ is in the Sweedler notation.

Example 2.7. Let A be a $\mathcal{B}$-module algebra. Then A has a nonlocal vertex algebra structure with 1 as the vacuum vector and

$$Y(a, x)b = (e^{x\mathcal{D}} a)b \quad \text{for } a, b \in A.$$

Furthermore, on any module W for A as an associative algebra, there exists a module structure Y_W for A with $Y_W(a, x)w = (e^{x\mathcal{D}} a)w$ for $a \in A$, $w \in W$.

Remark 2.8. Let A and B be $\mathcal{B}$-module algebras. Then $A \otimes B$ is a $\mathcal{B}$-module algebra with $e^{x\mathcal{D}} = e^{x\mathcal{D}_A} \otimes e^{x\mathcal{D}_B}$.

Let V and U be nonlocal vertex algebras. A linear map f from V to U is a *homomorphism* of nonlocal vertex algebras if

$$\begin{aligned} f(\mathbf{1}) &= \mathbf{1}, \\ fY(u,x)v &= Y(f(u),x)f(v) \quad \text{for } u,v \in V. \end{aligned}$$

It is straightforward to show that $fe^{x\mathcal{D}_V} = e^{x\mathcal{D}_U}f$ if f is a homomorphism from V to U. A homomorphism of nonlocal vertex algebras from V to V is called an *endomorphism* of V.

Remark 2.9. Let A and B be $\mathcal{B}$-module algebras. Then a linear map f from A to B is a homomorphism of nonlocal vertex algebras from A to B if and only if f is a homomorphism of algebras and of $\mathcal{B}$-modules, that is, $fe^{x\mathcal{D}_A} = e^{x\mathcal{D}_B}f$.

As in the case of characteristic zero, we have:

Lemma 2.10. *Let V and U be nonlocal vertex algebras and let f be a linear map from V to U. If*

$$\begin{aligned} f(\mathbf{1}) &= \mathbf{1}, \\ fY(u,x)v &= Y(f(u),x)f(v) \quad \textit{for } u \in T, v \in V, \end{aligned}$$

where T is a generating subset of V, then f is a homomorphism of nonlocal vertex algebras.

Proof. Set

$$S = \{u \in V \mid f(Y(u,x)w) = Y(f(u),x)f(w) \text{ for } w \in V\}.$$

We must show that $V = S$. Since $T \subset S$ and T generates V, it suffices to show that S is a nonlocal vertex subalgebra of V. As $\mathbf{1} \in S$, it remains to show that for $u, v \in S$ and $n \in \mathbb{Z}$ we have $u_n v \in S$, that is,

$$f(Y(Y(u,x_0)v,x_2)w) = Y(f(Y(u,x_0)v),x_2)f(w) \tag{2.19}$$

for $w \in V$. Let $l \in \mathbb{N}$ be such that

$$\begin{aligned} (x_0+x_2)^l Y(u,x_0+x_2)Y(v,x_2)w &= (x_0+x_2)^l Y(Y(u,x_0)v,x_2)w, \\ (x_0+x_2)^l Y(f(u),x_0+x_2)Y(f(v),x_2)f(w) &= (x_0+x_2)^l Y(Y(f(u),x_0)f(v),x_2)f(w). \end{aligned}$$

Since $u, v \in S$, we have

$$\begin{aligned} (x_0+x_2)^l fY(Y(u,x_0)v,x_2)w &= (x_0+x_2)^l fY(u,x_0+x_2)Y(v,x_2)w \\ &= (x_0+x_2)^l Y(f(u),x_0+x_2)fY(v,x_2)w \\ &= (x_0+x_2)^l Y(f(u),x_0+x_2)Y(f(v)v,x_2)f(w) \\ &= (x_0+x_2)^l Y(Y(f(u),x_0)f(v),x_2)f(w) \\ &= (x_0+x_2)^l Y(f(Y(u,x_0)v),x_2)f(w). \end{aligned}$$

Multiplying both sides by $(x_2+x_0)^{-l}$, we obtain (2.19). □

Let V be a nonlocal vertex algebra and let A be a $\mathcal{B}$-module algebra. Following [13], we consider V as a subalgebra of $V \otimes A$ by the natural embedding, and consider $V \otimes A$ as an A-module with A acting on the second factor. Then

$$\mathrm{Hom}_{\mathbb{F}}(V, V \otimes A) = \mathrm{End}_A(V \otimes A)$$

as A-modules and as spaces. We also consider any linear map from V to $V \otimes A$ or from V to V as an A-linear endomorphism of $V \otimes A$. Then we consider the vertex operator map Y of V as an A-linear map from $V \otimes A$ to $(\mathrm{End}(V \otimes A))[[x, x^{-1}]]$, that is,

$$Y(v \otimes a, x) = Y(v, x) \otimes a \quad \text{for } v \in V,\ a \in A.$$

We denote by Y_{ten} the vertex operator map of $V \otimes A$.

Lemma 2.11. *Let V be a nonlocal vertex algebra, let A be a $\mathcal{B}$-module algebra, and let f be a linear map from V to $V \otimes A$. Then f is a homomorphism of nonlocal vertex algebras if and only if*

$$f(\mathbf{1}) = \mathbf{1},$$
$$fY(v, x) = Y((1 \otimes e^{x\mathcal{D}})f(v), x)f \quad \text{for } v \in V.$$

Proof. For any $u \in V$ and $a \in A$, we see that

$$Y_{\mathrm{ten}}(u \otimes a, x) = Y(u, x) \otimes Y(a, x) = Y(u, x) \otimes (e^{x\mathcal{D}}a) = Y((1 \otimes e^{x\mathcal{D}})(u \otimes a), x).$$

It follows that

$$Y_{\mathrm{ten}}(f(v), x) = Y((1 \otimes e^{x\mathcal{D}})f(v), x) \quad \text{for } v \in V.$$

Then for $v \in V$, we see $fY(v, x) = Y_{\mathrm{ten}}(f(v), x)f$ is equivalent to

$$fY(v, x) = Y((1 \otimes e^{x\mathcal{D}})f(v), x)f.$$

Thus our assertion follows. □

We denote by $\mathbb{F}((x))^-$ the $\mathcal{B}$-module algebra $\mathbb{F}((x))$ with $\mathcal{D}^{(n)}$ acting as $(-1)^n\partial_x^{(n)}$ for $n \in \mathbb{N}$. Now we consider the special case with $A = \mathbb{F}((x))^-$.

Definition 2.12. Let V be a nonlocal vertex algebra. We define $\mathrm{PEnd}(V)$ to be the subspace of $\mathrm{Hom}_{\mathbb{F}}(V, V \otimes \mathbb{F}((x))^-)$ consisting of the elements $f(x)$ such that

$$f(x)\mathbf{1} = \mathbf{1},$$
$$f(x_1)Y(u, x_2) = Y(f(x_1 - x_2)u, x_2)f(x_1) \quad \text{for } u \in V.$$

Lemma 2.13. *Let V be a nonlocal vertex algebra and let $f(x) \in \mathrm{Hom}_{\mathbb{F}}(V, V \otimes \mathbb{F}((x))^-)$. Then $f(x) \in \mathrm{PEnd}(V)$ if and only if $f(x)$ is a homomorphism of nonlocal vertex algebras from V to $V \otimes \mathbb{F}((x))^-$. Furthermore,*

$$e^{z\mathcal{D}_V} f(x)e^{-z\mathcal{D}_V} = f(x + z) \quad \text{for } f(x) \in \mathrm{PEnd}(V). \tag{2.20}$$

Proof. Since $e^{-x_2\partial_x}g(x) = g(x - x_2)$ for $g(x) \in \mathbb{F}((x))^-$, we have

$$f(x - x_2)u = (1 \otimes e^{-x_2\partial_x})f(x)u \quad \text{for } u \in V.$$

Thus the first assertion follows from Lemma 2.11.

For $f(x) \in \mathrm{PEnd}(V)$, since $f(x)$ is a homomorphism of nonlocal vertex algebras, we have

$$f(x)e^{-z\mathcal{D}_V} = (e^{-z\mathcal{D}_V} \otimes e^{z\partial_x})f(x) = e^{-z\mathcal{D}_V} f(x + z).$$

Then (2.20) holds. □

Let V be a nonlocal vertex algebra. We say a subset U of $\mathrm{Hom}(V, V \otimes \mathbb{F}((x))^{-})$ is Δ*-closed* if for every $a(x) \in U$, there exist elements $a_{(1i)}(x), a_{(2i)}(x) \in U$, $i = 1, 2, \ldots, r$, such that

$$a(x_1)Y(u, x_2) = \sum_{i=1}^{r} Y(a_{(1i)}(x_1 - x_2)u, x_2)a_{(2i)}(x_1) \quad \text{for } u \in V. \tag{2.21}$$

We denote by $B(V)$ the sum of all the Δ-closed subspaces U of $\mathrm{Hom}(V, V \otimes \mathbb{F}((x))^{-})$ such that

$$a(x)\mathbf{1} \in \mathbb{F}\mathbf{1} \quad \text{for } a(x) \in U.$$

Using the same arguments in [13, Proposition 3.4] we have:

Lemma 2.14. *Let V be a nonlocal vertex algebra. Then $B(V)$ is Δ-closed and $B(V)$ is a $\mathcal{B}$-module algebra with $\mathcal{D}^{(n)}$ acting as $\partial_x^{(n)}$ for $n \in \mathbb{N}$. Furthermore, V is a module for $B(V)$ as a nonlocal vertex algebra with $Y_V(a(x), x_0) = a(x_0)$ for $a(x) \in B(V)$.*

The following notion is the modular counterpart of the notion of differential bialgebra in [13].

Definition 2.15. A *$\mathcal{B}$-module bialgebra* is a bialgebra (B, Δ, ε) endowed with a $\mathcal{B}$-module structure such that $\varepsilon e^{x\mathcal{D}} = \varepsilon$ and $\Delta e^{x\mathcal{D}} = (e^{x\mathcal{D}} \otimes e^{x\mathcal{D}})\Delta$.

We shall need the following notion (see [13]).

Definition 2.16. A nonlocal vertex algebra V endowed with a coalgebra structure (V, Δ, ε) is called a *vertex bialgebra* if Δ and ε are homomorphisms of nonlocal vertex algebras.

Let (B, Δ, ε) be a $\mathcal{B}$-module bialgebra. Then B is a $\mathcal{B}$-module algebra, and we have a nonlocal vertex algebra B by Example 2.7. From definition we have $\Delta(1) = 1 \otimes 1$ and $\varepsilon(1) = 1$. Furthermore,

$$\begin{aligned} \varepsilon(Y(a, x)b) &= \varepsilon((e^{x\mathcal{D}}a)b) = \varepsilon(e^{x\mathcal{D}}a)\varepsilon(b) = \varepsilon(a)\varepsilon(b) = Y(\varepsilon(a), x)\varepsilon(b), \\ \Delta(Y(a, x)b) &= \Delta((e^{x\mathcal{D}}a)b) = \Delta(e^{x\mathcal{D}}a)\Delta(b) = ((e^{x\mathcal{D}} \otimes e^{x\mathcal{D}})\Delta(a))\Delta(b) \\ &= Y(\Delta(a), x)\Delta(b) \end{aligned}$$

for $a, b \in B$. Then Δ and ε are nonlocal vertex algebra homomorphisms. Thus B is a vertex bialgebra.

Definition 2.17. Let H be a vertex bialgebra. A *nonlocal vertex H-module-algebra* is a nonlocal vertex algebra V endowed with an H-module structure on V such that

$$\begin{aligned} Y(h, x)u &\in V \otimes \mathbb{F}((x)), \\ Y(h, x)\mathbf{1}_V &= \varepsilon(h)\mathbf{1}_V, \\ Y(h, x_1)Y(v, x_2)u &= \sum Y(Y(h_{(1)}, x_1 - x_2)v, x_2)Y(h_{(2)}, x_1)u \end{aligned}$$

for $h \in H$, $u, v \in V$.

The following results of [13] hold with the same arguments.

Lemma 2.18. *Let H be a vertex bialgebra, let T be a generating subset of H as a nonlocal vertex algebra, let V be a nonlocal vertex algebra, and let (V, Y_V^H) be an H-module. Suppose that*

$$Y_V^H(h, x) \in \mathrm{Hom}(V, V \otimes \mathbb{F}((x))^{-}),$$

$$Y_V^H(h,x)\mathbf{1} = \varepsilon(h)\mathbf{1},$$
$$Y_V^H(h,x_1)Y(v,x_2)u = \sum Y(Y_V^H(h_{(1)},x_1-x_2)v,x_2)Y_V^H(h_{(2)},x_1)u$$

for $h \in T$ and $u, v \in V$. Then V is a nonlocal vertex H-module-algebra.

Theorem 2.19. *Let H be a vertex bialgebra, and let V be a nonlocal vertex H-module-algebra. We define $V\sharp H = V \otimes H$ as a vector space and define*

$$Y^\sharp(u\otimes h,x)(v\otimes k) = \sum Y(u,x)Y(h_{(1)},x)v \otimes Y(h_{(2)},x)k.$$

for $u, v \in V$ and $h, k \in H$. Then $(V\sharp H, Y^\sharp)$ is a nonlocal vertex algebra. Furthermore,

$$Y^\sharp(h,x_1)Y^\sharp(v,x_2) = \sum Y^\sharp(Y(h_{(1)},x_1-x_2)v,x_2)Y^\sharp(h_{(2)},x_1)$$

for $h \in H$ and $v \in V$.

3. Lattice vertex algebras

In [9], Dong and Griess introduced an integral form of vertex operator algebras associated to even lattices over the complex field, from which one can define modular lattice vertex algebras. In this section, we construct the modular lattice vertex algebras through smash product.

Let L be a positive definite even lattice with a basis $\{\gamma_1,\dots,\gamma_d\}$ and let L° be the dual lattice of L. Let A_L denote the $d\times d$ matrix $(\langle\gamma_i,\gamma_j\rangle)_{1\le i,j\le d}$. Note that $\det(A_L)$ is independent of the choice of a basis for L. Let $\epsilon : L\times L \to \mathbb{F}^\times$ be a map such that

$$\epsilon(\alpha,0) = \epsilon(0,\alpha) = 1,$$
$$\epsilon(\alpha,\beta+\gamma)\epsilon(\beta,\gamma) = \epsilon(\alpha+\beta,\gamma)\epsilon(\alpha,\beta)$$

for $\alpha,\beta,\gamma \in L$. Denote by $\mathbb{F}_\epsilon[L]$ the ϵ-twisted group algebra of L with $\mathbb{F}$-basis $\{e_\alpha \mid \alpha \in L\}$ and multiplication

$$e_\alpha e_\beta = \epsilon(\alpha,\beta)e_{\alpha+\beta} \quad \text{for } \alpha,\beta \in L.$$

Next, recall from [9] the ring $M(1)_{\mathbb{Z}}$. Denote by $M(1)$ the polynomial algebra generated by $s_{\alpha,n}$ for $\alpha \in \{\gamma_1,\dots,\gamma_d\}$ and $n \in \mathbb{Z}_+$. Set $s_{\alpha,0} = 1$ for $\alpha \in \{\gamma_1,\dots,\gamma_d\}$. For $\alpha \in \{\gamma_1,\dots,\gamma_d\}$, we set

$$E^-(-\alpha,x) = \sum_{n\in\mathbb{N}} s_{\alpha,n}x^n \quad \in M(1)[[x]].$$

Note that $E^-(-\alpha,x)$ is an invertible element of $M(1)[[x]]$ as $s_{\alpha,0} = 1$. For a general element $\alpha = k_1\gamma_1 + k_2\gamma_2 + \cdots + k_d\gamma_d \in L$, where $k_1,\dots,k_d \in \mathbb{Z}$, we define

$$E^-(-\alpha,x) = \prod_{i=1}^{d} E^-(-\gamma_i,x)^{k_i} \quad \in M(1)[[x]].$$

Then for $\alpha,\beta \in L$,

$$E^-(\alpha,x)E^-(\beta,x) = E^-(\alpha+\beta,x),$$

$$E^{-}(0,x) = \mathbf{1}.$$

As $M(1)$ is isomorphic to the universal enveloping algebra of the abelian Lie algebra with basis $\{s_{\alpha,n} \mid \alpha \in \{\gamma_1, \dots, \gamma_d\}, n \in \mathbb{Z}_+\}$, we see $M(1)$ is naturally a bialgebra with

$$\varepsilon(E^{-}(-\alpha, x)) = \mathbf{1}, \tag{3.1}$$

$$\Delta(E^{-}(-\alpha, x)) = E^{-}(-\alpha, x) \otimes E^{-}(-\alpha, x) \tag{3.2}$$

for $\alpha \in L$.

Define a $\mathcal{B}$-action on $M(1)$ by $e^{z\mathcal{D}}\mathbf{1} = \mathbf{1}$ and

$$e^{z\mathcal{D}} \prod_{i=1}^{r} E^{-}(-\alpha_i, x_i) = \prod_{i=1}^{r} E^{-}(-\alpha_i, x_i + z) E^{-}(\alpha_i, z) \tag{3.3}$$

for $r \in \mathbb{Z}_+$, $\alpha_i \in \{\gamma_1, \dots, \gamma_d\}$. Then (3.3) holds for $r \in \mathbb{Z}_+$, $\alpha_i \in L$. It is straightforward to check

$$e^{(z+z_0)\mathcal{D}} = e^{z\mathcal{D}} e^{z_0\mathcal{D}} \quad \text{on } M(1),$$

$$e^{z\mathcal{D}}(ab) = (e^{z\mathcal{D}}a)(e^{z\mathcal{D}}b) \quad \text{for } a, b \in M(1).$$

Then $M(1)$ is a $\mathcal{B}$-module algebra. Furthermore, for $r \in \mathbb{Z}_+$, $\alpha_i \in L$, we have

$$\begin{aligned} \varepsilon e^{z\mathcal{D}} \prod_{i=1}^{r} E^{-}(-\alpha_i, x_i) &= \varepsilon \prod_{i=1}^{r} E^{-}(-\alpha_i, x_i + z) E^{-}(\alpha_i, z) = \mathbf{1} \\ &= \varepsilon \prod_{i=1}^{r} E^{-}(-\alpha_i, x_i), \end{aligned}$$

and

$$\begin{aligned} &\Delta e^{z\mathcal{D}} \prod_{i=1}^{r} E^{-}(-\alpha_i, x_i) \\ &= \Delta \prod_{i=1}^{r} E^{-}(-\alpha_i, x_i + z) E^{-}(\alpha_i, z) \\ &= \Big(\prod_{i=1}^{r} E^{-}(-\alpha_i, x_i + z) E^{-}(\alpha_i, z)\Big) \otimes \Big(\prod_{i=1}^{r} E^{-}(-\alpha_i, x_i + z) E^{-}(\alpha_i, z)\Big) \\ &= \Big(e^{z\mathcal{D}} \prod_{i=1}^{r} E^{-}(-\alpha_i, x_i)\Big) \otimes \Big(e^{z\mathcal{D}} \prod_{i=1}^{r} E^{-}(-\alpha_i, x_i)\Big) \\ &= (e^{z\mathcal{D}} \otimes e^{z\mathcal{D}}) \Delta \prod_{i=1}^{r} E^{-}(-\alpha_i, x_i). \end{aligned}$$

Therefore, $(M(1), \Delta, \varepsilon)$ is a $\mathcal{B}$-module bialgebra.

For $\alpha \in \{\gamma_1, \gamma_2, \dots, \gamma_d\}$, we inductively define linear operators $r_{\alpha,n}$ for $n \in \mathbb{N}$ on $M(1)$ by

$$r_{\alpha,n}\mathbf{1} = \delta_{n,0}\mathbf{1}, \tag{3.4}$$

$$r_{\alpha,n}\prod_{i=1}^{r}s_{\beta_i,m_i}=\sum_{j_1,\dots,j_r\in\mathbb{N}}\left(\prod_{i=1}^{r}(-1)^{j_i}\binom{\langle\alpha,\beta_i\rangle}{j_i}s_{\beta_i,m_i-j_i}\right)r_{\alpha,n-j_1-\cdots-j_r}\mathbf{1}\tag{3.5}$$

for $\beta_i\in\{\gamma_1,\gamma_2,\dots,\gamma_d\}$, $m_i\in\mathbb{N}$, where $r_{\alpha,m}$ is understood to be zero if $m<0$. From (3.5), we see

$$r_{\alpha,n}s_{\beta,m}=\sum_{i\in\mathbb{N}}(-1)^i\binom{\langle\alpha,\beta\rangle}{i}s_{\beta,m-i}r_{\alpha,n-i}\quad\text{on } M(1)\tag{3.6}$$

for $\beta\in\{\gamma_1,\gamma_2,\dots,\gamma_d\}$ and $m\in\mathbb{N}$. For $\alpha\in\{\gamma_1,\gamma_2,\dots,\gamma_d\}$, we set

$$E^+(-\alpha,x)=\sum_{n\in\mathbb{N}}r_{\alpha,n}x^{-n}\qquad\in(\operatorname{End}M(1))[[x^{-1}]].$$

Furthermore, for $\alpha=k_1\gamma_1+k_2\gamma_2+\cdots+k_d\gamma_d\in L$, where $k_1,\dots,k_d\in\mathbb{Z}$, we define

$$E^+(-\alpha,x)=\prod_{i=1}^{d}E^+(-\gamma_i,x)^{k_i}\qquad\in(\operatorname{End}M(1))[[x^{-1}]].$$

Then for $\alpha,\beta\in L$,

$$E^+(\alpha,x)E^+(\beta,x)=E^-(\alpha+\beta,x),$$
$$E^+(0,x)=1,$$

and furthermore,

$$E^+(-\alpha,x_1)E^-(-\beta,x_2)=\left(1-\frac{x_2}{x_1}\right)^{\langle\alpha,\beta\rangle}E^-(-\beta,x_2)E^+(-\alpha,x_1).\tag{3.7}$$

Lemma 3.1. *In the $\mathcal{B}$-module algebra $M(1)$, we have*

$$\Delta(E^-(-\alpha,x)E^+(-\alpha,x)u)=(E^-(-\alpha,x)E^+(-\alpha,x)\otimes E^-(-\alpha,x))\Delta u\tag{3.8}$$

for $\alpha\in L$ and $u\in M(1)$.

Proof. We use induction on m to show (3.8) holds for $u=s_{\alpha_1,n_1}s_{\alpha_2,n_2}\dots s_{\alpha_m,n_m}$ with $m\in\mathbb{N}$, $\alpha_i\in L$, $n_i\in\mathbb{N}$. For $u=\mathbf{1}$, we have

$$\begin{aligned}\Delta E^-(-\alpha,x)E^+(-\alpha,x)\mathbf{1}&=\Delta E^-(-\alpha,x)=E^-(-\alpha,x)\otimes E^-(-\alpha,x)\\&=(E^-(-\alpha,x)E^+(-\alpha,x)\otimes E^-(-\alpha,x))(\mathbf{1}\otimes\mathbf{1})\\&=(E^-(-\alpha,x)E^+(-\alpha,x)\otimes E^-(-\alpha,x))\Delta\mathbf{1}.\end{aligned}$$

The induction step is given by

$$\begin{aligned}&\Delta E^-(-\alpha,x)E^+(-\alpha,x)E^-(-\beta,z)u\\=&\left(1-\frac{z}{x}\right)^{\langle\alpha,\beta\rangle}\Delta E^-(-\beta,z)E^-(-\alpha,x)E^+(-\alpha,x)u\\=&\left(1-\frac{z}{x}\right)^{\langle\alpha,\beta\rangle}(\Delta E^-(-\beta,z))(\Delta E^-(-\alpha,x)E^+(-\alpha,x)u)\end{aligned}$$

$$
\begin{aligned}
&= \left(1 - \frac{z}{x}\right)^{\langle\alpha,\beta\rangle} (E^{-}(-\beta, z) \otimes E^{-}(-\beta, z))(E^{-}(-\alpha, x)E^{+}(-\alpha, x) \otimes E^{-}(-\alpha, x))\Delta u \\
&= \left(1 - \frac{z}{x}\right)^{\langle\alpha,\beta\rangle} (E^{-}(-\beta, z)E^{-}(-\alpha, x)E^{+}(-\alpha, x) \otimes E^{-}(-\beta, z)E^{-}(-\alpha, x))\Delta u \\
&= (E^{-}(-\alpha, x)E^{+}(-\alpha, x)E^{-}(-\beta, z) \otimes E^{-}(-\alpha, x)E^{-}(-\beta, z))\Delta u \\
&= (E^{-}(-\alpha, x)E^{+}(-\alpha, x) \otimes E^{-}(-\alpha, x))\Delta E^{-}(-\beta, z)u.
\end{aligned}
$$

This completes the induction. As $M(1)$ is spanned by elements of the form

$$s_{\alpha_1,n_1} s_{\alpha_2,n_2} \dots s_{\alpha_m,n_m},$$

our assertion follows. □

Set

$$B_{L,\epsilon} = \mathbb{F}_{\epsilon}[L] \otimes M(1), \tag{3.9}$$

an associative algebra.

Lemma 3.2. *For $\alpha \in L$ and $u \in M(1)$, define*

$$e^{x\mathcal{D}}(e_\alpha \otimes u) = e_\alpha \otimes E^{-}(-\alpha, x)e^{x\mathcal{D}}u.$$

Then $B_{L,\epsilon}$ is a $\mathcal{B}$-module algebra.

Proof. For $e_\alpha \otimes u, e_\beta \otimes v \in B_{L,\epsilon}$, we have

$$
\begin{aligned}
e^{x\mathcal{D}}((e_\alpha \otimes u)(e_\beta \otimes v)) &= \epsilon(\alpha, \beta)e^{x\mathcal{D}}(e_{\alpha+\beta} \otimes uv) \\
&= \epsilon(\alpha, \beta)e_{\alpha+\beta} \otimes E^{-}(-\alpha - \beta, x)e^{x\mathcal{D}}(uv) \\
&= e_\alpha e_\beta \otimes E^{-}(-\alpha, x)E^{-}(-\beta, x)(e^{x\mathcal{D}}u)(e^{x\mathcal{D}}v) \\
&= (e_\alpha \otimes E^{-}(-\alpha, x)(e^{x\mathcal{D}}u))(e_\beta \otimes E^{-}(-\beta, x)(e^{x\mathcal{D}}v)) \\
&= (e^{x\mathcal{D}}e_\alpha \otimes u)(e^{x\mathcal{D}}e_\beta \otimes v).
\end{aligned}
$$

Thus $B_{L,\epsilon}$ is a $\mathcal{B}$-module algebra. □

Set

$$B_L = \mathbb{F}[L] \otimes M(1), \tag{3.10}$$

a unital commutatively associative algebra. As in the case of characteristic zero (see [13]), we have the following universal property of B_L:

Lemma 3.3. *Let A be a $\mathcal{B}$-module algebra and let $f : \mathbb{F}[L] \to A$ be a homomorphism of algebras. Then f can be extended uniquely to a homomorphism of $\mathcal{B}$-module algebras from B_L to A.*

Proof. For $\alpha \in \{\gamma_1, \gamma_2, \dots, \gamma_d\}$, define

$$fE^{-}(-\alpha, x) = (fe^{-\alpha})e^{x\mathcal{D}}fe^{\alpha}. \tag{3.11}$$

Since $M(1)$ is freely generated by $s_{\alpha,n}$ for $\alpha \in \{\gamma_1, \gamma_2, \dots, \gamma_d\}$ and $n \in \mathbb{Z}_+$, it follows that f can be extended to a homomorphism of algebras.

Now, we show that in fact (3.11) holds for all $\alpha \in L$. Let P be the subset of L consisting of α such that (3.11) holds for all $n \in \mathbb{N}$. From definition, we have $\gamma_i \in P$ for $1 \le i \le d$. Assume $\alpha, \beta \in P$. Then we get

$$\begin{aligned} fE^-(-\alpha-\beta,x) &= f(E^-(-\alpha,x)E^-(-\beta,x)) \\ &= (fe^{-\alpha})(e^{x\mathcal{D}}fe^{\alpha})(fe^{-\beta})(e^{x\mathcal{D}}fe^{\beta}) \\ &= (fe^{-\alpha})(fe^{-\beta})(e^{x\mathcal{D}}fe^{\alpha})(e^{x\mathcal{D}}fe^{\beta}) \\ &= (fe^{-\alpha-\beta})e^{x\mathcal{D}}((fe^{\alpha})(fe^{\beta})) \\ &= (fe^{-\alpha-\beta})e^{x\mathcal{D}}(fe^{\alpha+\beta}), \end{aligned}$$

proving $\alpha+\beta \in P$.

Now, assume $\alpha \in P$. As $E^-(\alpha,x)E^-(-\alpha,x) = 1$ and $(e^{x\mathcal{D}}fe^{\alpha})(e^{x\mathcal{D}}fe^{-\alpha}) = e^{x\mathcal{D}}1 = 1$, we have

$$fE^-(\alpha,x) = (fE^-(-\alpha,x))^{-1} = ((fe^{-\alpha})e^{x\mathcal{D}}fe^{\alpha})^{-1} = (fe^{\alpha})e^{x\mathcal{D}}fe^{-\alpha},$$

proving $-\alpha \in P$. Thus $P = L$, that is, (3.11) holds for all $\alpha \in L$.

Next we show that f is a homomorphism of $\mathcal{B}$-modules. For $\alpha \in L$, we have

$$fe^{x\mathcal{D}}(e^{\alpha}) = f(e^{\alpha} \otimes E^-(-\alpha,x)) = (fe^{\alpha})(fe^{-\alpha})e^{x\mathcal{D}}(fe^{\alpha}) = e^{x\mathcal{D}}(fe^{\alpha}).$$

Using the equation above and (3.3), we have

$$\begin{aligned} e^{x\mathcal{D}}fE^-(-\alpha,z) &= e^{x\mathcal{D}}((fe^{-\alpha})e^{z\mathcal{D}}fe^{\alpha}) \\ &= (e^{x\mathcal{D}}fe^{-\alpha})(e^{x\mathcal{D}}e^{z\mathcal{D}}fe^{\alpha}) \\ &= (fe^{\alpha})(e^{x\mathcal{D}}fe^{-\alpha})(fe^{-\alpha})(e^{(z+x)\mathcal{D}}fe^{\alpha}) \\ &= f(E^-(\alpha,x))f(E^-(-\alpha,z+x)) \\ &= f(E^-(\alpha,x)E^-(-\alpha,z+x)) \\ &= fe^{x\mathcal{D}}E^-(-\alpha,z). \end{aligned}$$

Since B_L as an algebra is generated by $s_{\alpha,n}$ and e^{α} for $\alpha \in L$ and $n \in \mathbb{N}$, it follows that $fe^{x\mathcal{D}} = e^{x\mathcal{D}}f$. Thus f is a $\mathcal{B}$-module homomorphism. □

For $\alpha \in L$, we define $x^{\alpha} \in (\operatorname{End} V_L)[x,x^{-1}]$ by

$$x^{\alpha}(e^{\beta} \otimes u) = x^{\langle\alpha,\beta\rangle}(e^{\beta} \otimes u) \tag{3.12}$$

for $\beta \in L^{\circ}$ and $u \in M(1)$.

Lemma 3.4. *For $\alpha \in L$, we have*

$$E^+(-\alpha,x)x^{\alpha} \in \operatorname{PEnd}(B_{L,\epsilon}).$$

Proof. For $\beta \in L$, we have

$$\begin{aligned} &(e^{x_2\mathcal{D}}E^+(-\alpha,x_1-x_2)(x_1-x_2)^{\alpha}e_{\beta})E^+(-\alpha,x_1)x_1^{\alpha} \\ &= (x_1-x_2)^{\langle\alpha,\beta\rangle}(e^{x_2\mathcal{D}}e_{\beta})E^+(-\alpha,x_1)x_1^{\alpha} \end{aligned}$$

$$
\begin{aligned}
&= (x_1 - x_2)^{\langle\alpha,\beta\rangle} E^-(-\beta, x_2) e_\beta E^+(-\alpha, x_1) x_1^\alpha \\
&= (x_1 - x_2)^{\langle\alpha,\beta\rangle} E^-(-\beta, x_2) E^+(-\alpha, x_1) e_\beta x_1^\alpha \\
&= (x_1 - x_2)^{\langle\alpha,\beta\rangle} \left(1 - \frac{x_2}{x_1}\right)^{-\langle\alpha,\beta\rangle} E^+(-\alpha, x_1) E^-(-\beta, x_2) e_\beta x_1^\alpha \\
&= x_1^{-\langle\alpha,\beta\rangle} (x_1 - x_2)^{\langle\alpha,\beta\rangle} \left(1 - \frac{x_2}{x_1}\right)^{-\langle\alpha,\beta\rangle} E^+(-\alpha, x_1) E^-(-\beta, x_2) x_1^\alpha e_\beta \\
&= E^+(-\alpha, x_1) E^-(-\beta, x_2) x_1^\alpha e_\beta \\
&= E^+(-\alpha, x_1) x_1^\alpha e^{x_2 \mathcal{D}} e_\beta.
\end{aligned}
$$

Since L generates $V_{L,\epsilon}$ as a nonlocal vertex algebra, it follows from Lemma 2.10 that $E^+(-\alpha, x)x^\alpha$ is a homomorphism of nonlocal vertex algebras. By Lemma 2.13, we see that $E^+(-\alpha, x)x^\alpha \in \mathrm{PEnd}(B_{L,\epsilon})$. □

Lemma 3.5. *There exists a unique B_L-module structure Y_M on $B_{L,\epsilon}$ such that*

$$Y_M(e^\alpha, x) = E^+(-\alpha, x)x^\alpha \quad \text{for } \alpha \in L,$$

and $(B_{L,\epsilon}, Y_M)$ is a nonlocal vertex B_L-module-algebra.

Proof. Denote $\Phi_\alpha(x) = E^+(-\alpha, x)x^\alpha$ for $\alpha \in L$. By Lemma 3.4, we have $\Phi_\alpha(x) \in \mathrm{PEnd}(B_{L,\epsilon})$. Clearly $\Phi_0(x) = 1$ and

$$\Phi_\alpha(x)\Phi_\beta(x) = \Phi_{\alpha+\beta}(x) \quad \text{for } \alpha, \beta \in L.$$

Let A be the subalgebra of $B(B_{L,\epsilon})$ generated by $\partial_x^{(n)}\Phi_\alpha(x)$ for $n \in \mathbb{N}$, $\alpha \in L$. Clearly A is a commutative $\mathcal{B}$-module algebra. By Lemma 3.3, there exists a homomorphism f of $\mathcal{B}$-module algebras from B_L to A such that $f(e^\alpha) = \Phi_\alpha(x)$ for all $\alpha \in L$. Then by Lemma 2.18 we see that $B_{L,\epsilon}$ is a nonlocal vertex B_L-module-algebra. □

As $B_{L,\epsilon}$ is a nonlocal vertex B_L-module-algebra by Lemma 3.5, we have the nonlocal vertex algebra $B_{L,\epsilon}\sharp B_L$ by Theorem 2.19.

Theorem 3.6. *Denote*

$$U = \coprod_{\alpha \in L} \mathbb{F}(e_\alpha \otimes e^\alpha) \otimes \Delta(M(1)),$$

a subspace of $B_{L,\epsilon}\sharp B_L$. Then U is a vertex subalgebra of the nonlocal vertex algebra $B_{L,\epsilon}\sharp B_L$, and the linear map

$$
\begin{aligned}
\pi : \quad & V_L \to U, \\
& e_\alpha \otimes u \mapsto (e_\alpha \otimes e^\alpha) \otimes \Delta(u)
\end{aligned}
$$

for $\alpha \in L$ and $u \in M(1)$ is a vertex algebra homomorphism. Furthermore, if $\det(A_L) \not\equiv 0 \pmod{p}$*, the map π is an isomorphism.*

Proof. This is a slight modification of the proof in [13]. As Δ is a homomorphism from $M(1)$ to $M(1) \otimes M(1)$, we see that π is a linear homomorphism. We then show that π is a vertex algebra homomorphism. Let $\alpha, \beta \in L$ and $u \in M(1)$. Then we have

$$Y_{V_L}(e_\alpha, x)(e^\beta \otimes u) = x^{\langle\alpha,\beta\rangle}\epsilon(\alpha, \beta)(e_{\alpha+\beta} \otimes E^-(-\alpha, x)E^+(-\alpha, x)u).$$

By Lemma 3.1, we have

$$\begin{aligned}
&\pi(Y_{V_L}(e_\alpha, x)(e^\beta \otimes u)) \\
&= x^{\langle\alpha,\beta\rangle}\epsilon(\alpha,\beta)(e_{\alpha+\beta} \otimes e^{\alpha+\beta})\Delta(E^-(-\alpha, x)E^+(-\alpha, x)u) \\
&= x^{\langle\alpha,\beta\rangle}\epsilon(\alpha,\beta)(e_{\alpha+\beta} \otimes e^{\alpha+\beta})(E^-(-\alpha, x)E^+(-\alpha, x) \otimes E^-(-\alpha, x))\Delta(u).
\end{aligned}$$

Since $\Delta(e^\alpha) = e^\alpha \otimes e^\alpha$, by Lemma 3.2 we have

$$\begin{aligned}
Y^\sharp(e_\alpha \otimes e^\alpha, x) &= Y_{B_{L,\epsilon}}(e_\alpha, x)Y_M(e^\alpha, x) \otimes Y_{B_L}(e^\alpha, x) \\
&= E^-(-\alpha, x)e_\alpha E^+(-\alpha, x)x^\alpha \otimes E^-(-\alpha, x)e^\alpha.
\end{aligned}$$

Then

$$\begin{aligned}
&Y^\sharp(e_\alpha \otimes e^\alpha, x)\pi(e_\beta \otimes u) \\
&= E^-(-\alpha, x)e_\alpha E^+(-\alpha, x)x^\alpha \otimes E^-(-\alpha, x)e^\alpha(e_\beta \otimes e^\beta)\Delta(u) \\
&= x^{\langle\alpha,\beta\rangle}\epsilon(\alpha,\beta)(e_{\alpha+\beta} \otimes e^{\alpha+\beta})(E^-(-\alpha, x)E^+(-\alpha, x) \otimes E^-(-\alpha, x))\Delta(u).
\end{aligned}$$

Therefore

$$\pi(Y_{V_L}(e_\alpha, x)(e^\beta \otimes u)) = Y^\sharp(e_\alpha \otimes e^\alpha, x)\pi(e_\beta \otimes u)$$

for $\alpha,\beta \in L$ and $u \in M(1)$. Since L generates V_L as a vertex algebra by [10, Theorem 1], it follows from Lemma 2.10 that π is a nonlocal vertex algebra homomorphism. As V_L is a vertex algebra, we see that π is a vertex algebra homomorphism. If $\det(A_L) \not\equiv 0 \pmod p$, it follows from [10, Theorem 13] that V_L is a simple vertex algebra, then π is an isomorphism. □

Extend ϵ to a map from $L \times L^\circ$ to $\mathbb{F}^\times$ such that

$$\epsilon(\alpha,\beta)\epsilon(\alpha + \beta, \gamma) = \epsilon(\alpha, \beta + \gamma)\epsilon(\beta, \gamma)$$

for $\alpha,\beta \in L$ and $\gamma \in L^\circ$ (see [14]). Define an $\mathbb{F}_\epsilon[L]$-module structure on $\mathbb{F}[L^\circ]$ by

$$e_\alpha \cdot e^\gamma = \epsilon(\alpha, \gamma)e^{\alpha+\gamma} \quad \text{for } \alpha \in L, \gamma \in L^\circ.$$

Set

$$V_{L^\circ} = \mathbb{F}[L^\circ] \otimes M(1).$$

By the same proof of [13, Proposition 5.8], we have:

Proposition 3.7. *There exists a unique V_L-module structure on V_{L° such that*

$$Y(e_\alpha, x) = E^-(-\alpha, x)E^+(-\alpha, x)e_\alpha x^\alpha$$

for $\alpha \in L$.

Acknowledgments

The author was supported by the China NSF (grant 11571391) and the Heilongjiang Provincial NSF (grant JQ2020A002).

Conflict of interest

The authors declare there is no conflicts of interest.

References

1. I. Frenkel, J. Lepowsky, A. Meurman, *Vertex Operator Algebras and the Monster*, *Pure Appl. Math.*, 134. Academic Press, Inc., Boston, MA, 1988. https://doi.org/10.1142/9789812798411_0010
2. C. Dong, L. Ren, Representations of vertex operator algebras over an arbitrary field, *J. Algebra*, **403** (2014), 497–516. https://doi.org/10.1016/j.jalgebra.2014.01.007
3. L. Ren, Modular $A_n(V)$ theory, *J. Algebra*, **485** (2017), 254–268. https://doi.org/10.1016/j.jalgebra.2017.04.027
4. C. Dong, L. Ren, Vertex operator algebras associated to the Virasoro algebra over an arbitrary field, *Trans. Amer. Math. Soc.*, **368** (2016), 5177–5196. https://doi.org/10.1090/tran/6529
5. C. Dong, C. H. Lam, L. Ren, Modular framed vertex operator algebras, preprint, arXiv:1709.04167
6. R. E. Borcherds, Modular moonshine III, *Duke Math. J.*, **93** (1998), 129–154. https://doi.org/10.1215/S0012-7094-98-09305-X
7. R. E. Borcherds, A. Ryba, Modular moonshine II, *Duke Math. J.*, **83** (1996), 435–459. https://doi.org/10.1215/S0012-7094-96-08315-5
8. R. L. Griess Jr, C. H. Lam, Groups of Lie type, vertex algebras, and modular moonshine, *Int. Math. Res. Not. IMRN*, **2015** (2015), 10716–10755. https://doi.org/10.1093/imrn/rnv003
9. C. Dong, R. L. Griess Jr, Integral forms in vertex operator algebras which are invariant under finite groups, *J. Algebra*, **365** (2012), 184–198. https://doi.org/10.1016/j.jalgebra.2012.05.006
10. Q. Mu, Lattice vertex algebras over fields of prime characteristic, *J. Algebra*, **417** (2014), 39–51. https://doi.org/10.1016/j.jalgebra.2014.06.027
11. H.-S. Li, Axiomatic G_1-vertex algebras, *Commun. Contemp. Math.*, **5** (2003), 281–327. https://doi.org/10.1142/S0219199703000987
12. H.-S. Li, Nonlocal vertex algebras generated by formal vertex operators, *Selecta Math. (N.S.)*, **11** (2005), 349–397. https://doi.org/10.1007/s00029-006-0017-1
13. H.-S. Li, A smash product construction of nonlocal vertex algebras, *Commun. Contemp. Math.*, **9** (2007), 605–637. https://doi.org/10.1142/S0219199707002605
14. J. Lepowsky, H.-S. Li, *Introduction to Vertex Operator Algebras and Their Representations*, Progr. Math., 227. Birkhäuser Boston, Inc., Boston, MA, 2004. https://doi.org/10.1007/978-0-8176-8186-9
15. H.-S. Li, Q. Mu, Heisenberg VOAs over fields of prime characteristic and their representations, *Trans. Amer. Math. Soc.*, **370** (2018), 1159–1184. https://doi.org/10.1090/tran/7094