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Research article

$\bar{\partial}$-equation look at analytic Hilbert's zero-locus theorem

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Abstract: Stemming from the Pythagorean Identity $\sin^2 z + \cos^2 z = 1$ and Hörmander's L^2-solution of the Cauchy-Riemann's equation $\bar{\partial}u = f$ on $\mathbb{C}$, this article demonstrates a corona-type principle which exists as a somewhat unexpected extension of the analytic Hilbert's Nullstellensatz on $\mathbb{C}$ to the quadratic Fock-Sobolev spaces on $\mathbb{C}$.

Keywords: $\bar{\partial}u = f$; quadratic Fock-Sobolev space; analytic Hilbert's Nullstellensatz

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1. Description of Theorem 1.1

As one of the fundamentals of algebraic-complex geometry, the analytic Hilbert's Nullstellensatz (either theorem of zeros or zero-locus theorem) [1] on the finite complex plane $\mathbb{C}$ asserts that for finitely many analytic polynomials $\{p_j\}_{j=1}^n$ without common zeros in $\mathbb{C}$,

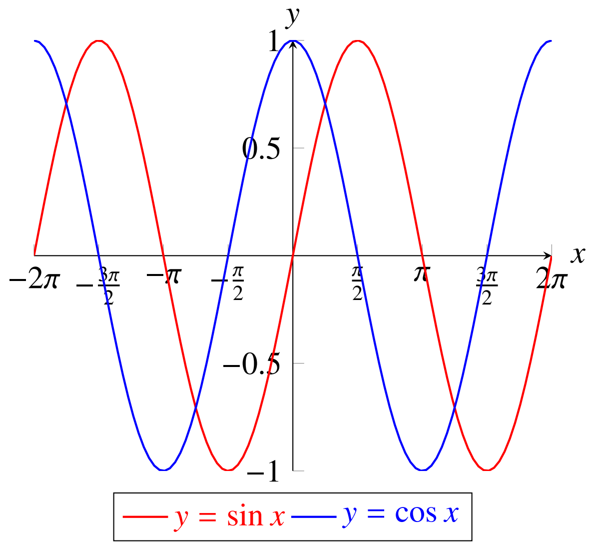
$$\exists \text{ finitely many analytic polynomials } \{q_j\}_{j=1}^n \text{ such that } \sum_{j=1}^n p_j q_j = 1. \tag{1}$$

This celebrated principle has been improved and extended for more than a century; see e.g., Hermann [2], Masser-Wüstholz [3], Brownawell [4], Kollár [5], and Kwon-Neryanun-Trent [6] whose Lemma 1.4 especially indicates that an entire function Y is a polynomial on $\mathbb{C}$ if and only if $\lim_{|z|\to\infty} |z|^{-m}|Y(z)| = 0$ for some positive integer m.

Meanwhile, in complex trigonometry, the Pythagorean Identity on $\mathbb{C}$ states that

$$\sin^2 z + \cos^2 z = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 = 1 \ \forall \ z \in \mathbb{C}, \tag{2}$$

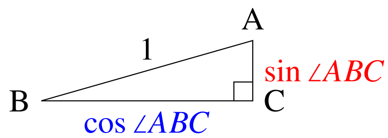
and $\sin z$ & $\cos z$ have no common zero as graphically shown below



Although the entire functions $\sin z$ & $\cos z$ are not analytic polynomials, they can be appropriately approximated by analytic polynomials and satisfy:

$$\begin{cases}(\sin z)g_1(z) + (\cos z)g_2(z) = 1;\\ \{g_1(z) = \sin z,\ g_2(z) = \cos z\};\\ \sup_{z\in\mathbb{C}}\left(|g_1(z)| + |g_2(z)|\right)e^{-|z|} < \infty;\\ |\sin z| + |\cos z| \geq 1,\end{cases}$$

where this last inequality is geometric due to the basic fact that the sum of the lengths of the adjacent & opposite sides BC & CA is not less than the length of the hypotenuse AB in the right triangle $\triangle ABC$ drawn below



$$\sin\angle ABC + \cos\angle ABC \geq 1$$

The previous two-fold observation actually inspires us to extend the analytic Hilbert's Nullstellensatz to some entire function spaces.

For $\alpha > 0$, let $\mathcal{F}^2_\alpha$ be the Fock-Hilbert space of all $L^2(\lambda_\alpha)$-integrable entire functions (or analytic functions on $\mathbb{C}$) with the inner product

$$\langle f, g\rangle_{\mathcal{F}^2_\alpha} = \int_{\mathbb{C}} f(z)\overline{g(z)}\,\mathrm{d}\lambda_\alpha(z)\ \ \forall\ \text{entire function pair}\ \{f, g\},$$

where

$$\mathrm{d}\lambda_\alpha(z) = \alpha\pi^{-1}e^{-\alpha|z|^2}\mathrm{dA}(z) = \alpha\pi^{-1}e^{-\alpha|z|^2}\mathrm{d}x\mathrm{d}y\ \ \forall\ \ z = x + iy \in \mathbb{C}.$$

Moreover, for a nonnegative integer m, let $\mathcal{F}^2_{\alpha,m}$ be the quadratic m-order Fock-Sobolev space of all entire functions obeying

$$\|f\|^2_{\mathcal{F}^2_{\alpha,m}} = \int_{\mathbb{C}} |z^m f(z)|^2 e^{-\alpha|z|^2}\mathrm{dA}(z) < \infty;$$

Evidently,

$$\alpha_1 < \alpha_2 \Longrightarrow \mathcal{F}^2_{\alpha_1,m} \subseteq \mathcal{F}^2_{\alpha_2,m};$$

see also Zhu's book [7] for more information.

Since all analytic polynomials are dense in $\mathcal{F}^2_{\alpha,m}$, as a somewhat unexpected variant of Eqs (1) and (2) we discover the following corona-type principle.

Theorem 1.1. *Let*

$$\begin{cases} \alpha \in (0,\infty); \\ m, n \in \{1,2,3,...\}; \\ f_1,...,f_n \in \mathcal{F}^2_{\alpha,1}. \end{cases}$$

If g is an entire function with

$$\sum_{j=1}^{n} |f_j| \geq |g|^m, \tag{3}$$

then

$$\exists \ g_1,...,g_n \in \mathcal{F}^2_{2m\alpha,1} \ \textit{such that} \ \sum_{j=1}^{n} f_j g_j = g^{3m}. \tag{4}$$

Especially, if

$$\sum_{j=1}^{n} |f_j| \geq 1, \tag{5}$$

then

$$\exists \ g_1,...,g_n \in \mathcal{F}^2_{2m\alpha,1} \ \textit{such that} \ \sum_{j=1}^{n} f_j g_j = 1. \tag{6}$$

2. Demonstration of Theorem 1.1

For $\alpha \in (0,\infty)$, let

$$\mathbb{C} \ni z \mapsto \begin{cases} f(z) = \sum_{n=0}^{\infty} a_n z^n \\ g(z) = \sum_{n=0}^{\infty} b_n z^n \end{cases} \quad \text{be two entire functions with their derivatives } \{f'(z), g'(z)\}.$$

Then some elementary calculations derive the following four formulae:

$$\begin{cases} \int_{\mathbb{C}} f(z)\overline{g(z)}\mathrm{d}\lambda_\alpha(z) = \sum_{n=0}^{\infty} \alpha^{-n} a_n \overline{b_n} n!; \\ \int_{\mathbb{C}} z f(z)\overline{z g(z)}\mathrm{d}\lambda_\alpha(z) = \sum_{n=0}^{\infty} \alpha^{-n-1} a_n \overline{b_n}(n+1)!; \\ \int_{\mathbb{C}} f'(z)\overline{g'(z)}\mathrm{d}\lambda_\alpha(z) = \sum_{n=1}^{\infty} \alpha^{1-n} a_n \overline{b_n} n^2 (n-1)! = \alpha \int_{\mathbb{C}} f(z)\overline{g(z)}(\alpha|z|^2 - 1)\mathrm{d}\lambda_\alpha(z); \\ \int_{\mathbb{C}} z f(z)\overline{z g(z)}\mathrm{d}\lambda_\alpha(z) = \alpha^{-2}\int_{\mathbb{C}} f'(z)\overline{g'(z)}\mathrm{d}\lambda_\alpha(z) + \alpha^{-1}\int_{\mathbb{C}} f(z)\overline{g(z)}\mathrm{d}\lambda_\alpha(z). \end{cases}$$

Consequently,

$$\mathcal{F}^2_{\alpha,1} \subseteq \mathcal{F}^2_\alpha \ \text{ with } \ \|f\|^2_{\alpha,1} = \int_{\mathbb{C}} |f'(z)|^2 \mathrm{d}\lambda_\alpha(z) \Longrightarrow \|f\|^2_{\mathcal{F}^2_{\alpha,1}} = \alpha^{-2}\|f\|^2_{\alpha,1} + \alpha^{-1}\|f\|^2_{\mathcal{F}^2_\alpha}.$$

Moreover, a modification of Cho-Zhu's statement on [8, p. 2496] gives the following pointwise estimation

$$f \in \mathcal{F}^2_{\alpha,1} \Longrightarrow |f(z)| \lesssim \|f\|_{\mathcal{F}^2_{\alpha,1}} (1+|z|)^{-1} e^{2^{-1}\alpha|z|^2} \ \ \forall \ z \in \mathbb{C}. \tag{1}$$

Lemma 2.1. *For*

$$\begin{cases} z = x + iy \in \mathbb{C}; \\ 1 < p < \infty; \\ p' = p(p-1)^{-1}; \\ C^2_c = \{\textit{all compactly-supported } C^2\textit{-functions: } \mathbb{C} \to \mathbb{C}\}; \\ A^p(e^{-2\phi}) = \{\textit{all analytic functions in } L^p(e^{-2\phi})\}, \end{cases}$$

let $\phi : \mathbb{C} \to \mathbb{R}$ & $g : \mathbb{C} \to \mathbb{C}$ *be* C^2*-smooth with*

$$\begin{cases} 0 \le \Delta\phi(z) = 4^{-1}(\partial_x^2 + \partial_y^2)\phi(z) = \partial_z \bar{\partial}_z \phi(z) = \partial\bar{\partial}\phi(z); \\ \partial = \partial_z = 2^{-1}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right); \\ \bar{\partial} = \partial_{\bar{z}} = 2^{-1}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right); \\ \bar{\partial}^*_{2\phi} g(z) = -e^{2\phi(z)}\partial\left(g(z)e^{-2\phi(z)}\right). \end{cases} \tag{2}$$

(i) *Weighted* $(L^p, \bar{\partial})$*-estimation - not only for a given function* f *on* $\mathbb{C}$ *there exists a weak solution* $u \in L^p(e^{-2\phi})$ *to* $\bar{\partial}u = f$ *in the sense of*

$$\int_{\mathbb{C}} u\overline{\bar{\partial}^*_{2\phi} g} e^{-2\phi} \mathrm{dA} = \int_{\mathbb{C}} f\bar{g} e^{-2\phi} \mathrm{dA} \ \ \forall \ g \in C^2_c \tag{3}$$

if and only if

$$\sup_{g \in C^2_c} \frac{\left|\int_{\mathbb{C}} f\bar{g}e^{-2\phi}\,\mathrm{dA}\right|}{\left(\int_{\mathbb{C}} |\bar{\partial}^*_{2\phi} g|^{p'} e^{-2\phi} \mathrm{dA}\right)^{\frac{1}{p'}}} < \infty, \tag{4}$$

but also Eq (4) *holds for all* $f \in L^p((\Delta(2\phi)e^{2\phi})^{-1})$ *if and only if*

$$\int_{\mathbb{C}} |g|^{p'} (\Delta(2\phi))^{\frac{p'}{p}} e^{-2\phi} \mathrm{dA} \le \int_{\mathbb{C}} |\bar{\partial}^*_{2\phi} g|^{p'} e^{-2\phi} \mathrm{dA} \ \ \forall \ g \in C^2_c. \tag{5}$$

(ii) *Uniqueness up to entire* $L^p(e^{-2\phi})$*-functions - arbitrary two solutions in (i) differ by a function* $h \in A^p(e^{-2\phi})$ *with*

$$\int_{\mathbb{C}} |h|^p e^{-2\phi} \mathrm{dA} \le 2^{p+1} \int_{\mathbb{C}} |f|^p (\Delta(2\phi)e^{2\phi})^{-1} \mathrm{dA}.$$

(iii) *Weighted* $(L^p, \bar{\partial})$*-Poincaré inequality - if* $u \in C^1$ *satisfies (i) above and the* $L^p(e^{-2\phi})$*-minimality below*

$$\int_{\mathbb{C}} |u|^p e^{-2\phi} \mathrm{dA} = \inf_{h \in A^p(e^{-2\phi})} \int_{\mathbb{C}} |u + h|^p e^{-2\phi} \mathrm{dA}, \tag{6}$$

then

$$\int_{\mathbb{C}} |u|^p e^{-2\phi} \mathrm{dA} \leq \int_{\mathbb{C}} |\bar{\partial} u|^p \left(e^{2\phi} \Delta(2\phi)\right)^{-1} \mathrm{dA}, \tag{7}$$

and consequently, if $u \in C^1$ *enjoys the case* $p = 2$ *of (i) and the* $L^2(e^{-2\phi})$*-orthogonality*

$$\int_{\mathbb{C}} u\bar{h} e^{-2\phi} \mathrm{dA} = 0 \ \ \forall \ \ h \in A^2(e^{-2\phi}), \tag{8}$$

then Eq (7) *holds for* $p = 2$.

(iv) *Weighted* $(L^2, \bar{\partial})$*-estimation is always available.*

(v) *Uniqueness - if*

$$\exists \ \ \epsilon \in (0, 2) \ \ \text{such that} \ \ \int_{\mathbb{C}} \left((1 + |z|)^{\epsilon - 2} e^{2\phi(z)}\right)^2 \mathrm{dA}(z) < \infty, \tag{9}$$

then the solution in (iv) is unique.

Proof. (i) This part is motivated by Berndtsson's [9, Theorems 2–3]. But, the argument comes from an adjustment of the case $p = 2$ presented in Berndtsson's [10, Proposition 1.1].

Suppose that for a given function f on $\mathbb{C}$ there exists a weak solution $u \in L^p(e^{-2\phi})$ to $\bar{\partial} u = f$ in the sense of Eq (3). Then the Hölder inequality derives

$$\left| \int_{\mathbb{C}} f \bar{g} e^{-2\phi} \mathrm{dA} \right| = \left| \int_{\mathbb{C}} u \overline{\bar{\partial}^*_{2\phi} g} e^{-2\phi} \mathrm{dA} \right| \leq \left(\int_{\mathbb{C}} |u|^p e^{-2\phi} \mathrm{dA} \right)^{\frac{1}{p}} \left(\int_{\mathbb{C}} |\bar{\partial}^*_{2\phi} g|^{p'} e^{-2\phi} \mathrm{dA} \right)^{\frac{1}{p'}} \ \ \forall \ \ g \in C^2_c.$$

Hence Eq (4) holds for the previous function f. Conversely, if Eq (4) is valid for a given function f on $\mathbb{C}$, then

$$S_\phi = \{\bar{\partial}^*_{2\phi} g : \ \ g \in C^2_c\}$$

is a subspace of $L^{p'}(e^{-2\phi})$, and hence the given function f induces the following bounded antilinear functional on S_ϕ:

$$\mathrm{L}_f(\bar{\partial}^*_{2\phi} g) = \int_{\mathbb{C}} f \bar{g} e^{-2\phi} \mathrm{dA}.$$

This, along with the Hahn-Banach extension theorem, ensures an extension of L_f from S_ϕ to $L^{p'}(e^{-2\phi})$. Consequently, the Riesz-type representation theorem for the dual of $L^{p'}(e^{-2\phi})$ produces a function

$$u \in [L^{p'}(e^{-2\phi})]^* = L^p(e^{-2\phi})$$

such that

$$\mathrm{L}_f(G) = \int_{\mathbb{C}} u \overline{G} e^{-2\phi} \mathrm{dA} \ \ \forall \ \ G \in L^{p'}(e^{-2\phi}).$$

Upon taking $G = \bar{\partial}^*_{2\phi} g$, we find that the last function $u \in L^p(e^{-2\phi})$ is a weak solution to $\bar{\partial} u = f$ in the sense of Eq (3).

Moreover, if Eq (4) holds for all $f \in L^p((\Delta(2\phi)e^{2\phi})^{-1})$, then an application of the duality

$$[L^p(e^{-2\phi})]^* = L^{p'}(e^{-2\phi})$$

under the pairing

$$\langle f, g\rangle_{2\phi} = \int_{\mathbb{C}} f\bar{g}e^{-2\phi}\mathrm{dA} = \int_{\mathbb{C}} \Big((\Delta(2\phi))^{-\frac{1}{p}} f\Big)\Big((\Delta(2\phi))^{\frac{1}{p}}\bar{g}\Big)e^{-2\phi}\,\mathrm{dA}$$

derives Eq (5). Evidently, if Eq (5) holds, then $\langle f, g\rangle_{2\phi}$ deduces that Eq (4) is valid for all $f \in L^p((\Delta(2\phi)e^{2\phi})^{-1})$.

As an aside of the preceding demonstration, we achieve that for any $f \in L^p((\Delta(2\phi)e^{2\phi})^{-1})$ there exists a weak solution $u \in L^p(e^{-2\phi})$ of $\bar{\partial} u = f$ with

$$\int_{\mathbb{C}} |u|^p e^{-2\phi}\mathrm{dA} \le \int_{\mathbb{C}} |f|^p (\Delta(2\phi))^{-1} e^{-2\phi}\mathrm{dA}$$

if and only if Eq (5) holds.

(ii) This follows from the fact that $\bar{\partial} u = 0$ if and only if u is analytic.

(iii) This comes from a modification of the argument for Berndtsson's [10, Corollary 1.4]. Indeed, without loss of generality, we may assume

$$\int_{\mathbb{C}} |\bar{\partial} u|^p (e^{2\phi}\Delta(2\phi))^{-1}\mathrm{dA} < \infty.$$

Now, the verification of (i) ensures that

$$\bar{\partial} v = \bar{\partial} u \in L^p((\Delta(2\phi)e^{2\phi})^{-1})$$

has a weak solution v enjoying the inequality

$$\int_{\mathbb{C}} |v|^p e^{-2\phi}\,\mathrm{dA} \le \int_{\mathbb{C}} |\bar{\partial} u|^p (\Delta(2\phi))^{-1} e^{-2\phi}\,\mathrm{dA}. \tag{10}$$

Note that (ii) produces a function $h_\dagger \in A^p(e^{-2\phi})$ such that $v = u + h_\dagger$. So Eqs (6) & (10) imply

$$\int_{\mathbb{C}} |u|^p e^{-2\phi}\,\mathrm{dA} \le \int_{\mathbb{C}} |u + h_\dagger|^p e^{-2\phi}\,\mathrm{dA} \le \int_{\mathbb{C}} |\bar{\partial} u|^p (\Delta(2\phi))^{-1} e^{-2\phi}\,\mathrm{dA},$$

as desired in Eq (7).

Especially, if Eq (8) is valid, then a combination of

$$\bar{\partial}(v + h) = \bar{\partial} u \ \ \forall \ \ h \in A^2_\phi$$

and the closedness of $A^2(e^{-2\phi})$ in $L^2(e^{-2\phi})$ yields a function $h_\ddagger \in A^2(e^{-2\phi})$ such that

$$\begin{cases} \int_{\mathbb{C}} |v + h_\ddagger|^2 e^{-2\phi}\mathrm{dA} = \inf_{h\in A^2_\phi} \int_{\mathbb{C}} |v + h|^2 e^{-2\phi}\mathrm{dA} \le \int_{\mathbb{C}} |v|^2 e^{-2\phi}\mathrm{dA}; \\ \int_{\mathbb{C}} (v + h_\ddagger - u)\bar{h} e^{-2\phi}\mathrm{dA} = 0 \ \ \forall \ \ h \in A^2(e^{-2\phi}). \end{cases} \tag{11}$$

Upon noticing $\bar\partial(v + h_{\ddagger} - u) = 0$, we obtain

$$v + h_{\ddagger} - u \in (A^2(e^{-2\phi})) \cap (A^2(e^{-2\phi}))^{\perp} = \{0\} \ \& \ v + h_{\ddagger} = u.$$

As a consequence, u is the $L^2(e^{-2\phi})$-minimal solution to $\bar\partial v = \bar\partial u$. Thus, the weighted $(L^2, \bar\partial)$-Poincaré inequality follows from the first inequality of Eq (11) and the case $p = 2$ of Eq (10).

Here it is appropriate to mention that as shown in [10, Theorem 3.3] the case $p = 2$ of Eq (10) can be used to establish the following Brunn-Minkowski-type concavity: if $\mathbb{D}$ is a convex open subset of the $(n+1)$-dimesnional Euclidean space $\mathbb{R}^n \times (-\infty, \infty)$ and $\mathbb{D}_t = \{x : (x,t) \in \mathbb{D}\}$ then the Lebesgue measure $\mathrm{M}_n(\mathbb{D}_t)$ satisfies $\partial_t^2 \log \mathrm{M}_n(\mathbb{D}_t) \le 0$ - i.e.,- the function $t \mapsto \log \mathrm{M}_n(\mathbb{D}_t)$ is concave - in particular - so is $t \mapsto \log \mathrm{A}(\mathbb{D}_t) = \log \mathrm{M}_2(\mathbb{D}_t)$.

(iv) This is a minor variant of [11, Theorem 1.1] - the well-known Hörmander L^2-estimate for the $\bar\partial$-equation presented in [12]. In fact, given $f \in L^2\big((e^{2\phi}\Delta(2\phi))^{-1}\big)$ the basic identity (cf. [10, Proposition 1.2])

$$\int_{\mathbb{C}} |g|^2(\Delta(2\phi))e^{-2\phi}\mathrm{dA} + \int_{\mathbb{C}} |\bar\partial g|^2 e^{-2\phi}\mathrm{dA} = \int_{\mathbb{C}} |\bar\partial_{2\phi}^* g|^2 e^{-2\phi}\mathrm{dA} \ \ \forall \ g \in C_c^2,$$

ensures the second iff-condition of (i) with $p = 2$

$$\int_{\mathbb{C}} |g|^2(\Delta(2\phi))e^{-2\phi}\mathrm{dA} \le \int_{\mathbb{C}} |\bar\partial_{2\phi}^* g|^2 e^{-2\phi}\mathrm{dA} \ \ \forall \ g \in C_c^2,$$

thereby reaching the existence of a weak solution u to $\bar\partial u = f$ with

$$\int_{\mathbb{C}} |u|^2 e^{-2\phi}\,\mathrm{dA} \le \int_{\mathbb{C}} |f|^2(\Delta(2\phi))^{-1} e^{-2\phi}\,\mathrm{dA}.$$

(v) Such a uniqueness is newly induced by Eq (1). Yet, its proof is similar to the argument for Hedenmalm's curvature-orientated uniqueness of the $\bar\partial$-equation in [11, Theorem 1.4]. As a matter of fact, if $u_1 \,\&\, u_2$ are two solutions in (iv), then $u_1 - u_2$ is an entire function on $\mathbb{C}$ due to (ii), and hence $\log|u_1 - u_2|$ is subharmonic on $\mathbb{C}$. This, plus Eq (2), deduces

$$\Delta \log\big(|u_1 - u_2|e^{\phi}\big) = \Delta \log|u_1 - u_2| + \Delta\phi > 0,$$

and so that

$$|u_1 - u_2|e^{\phi} = \exp\big(\log(|u_1 - u_2|e^{\phi})\big)$$

is subharmonic on $\mathbb{C}$. Now, for any

$$(z_0, r) \in \mathbb{C} \times (1 + |z_0|, \infty),$$

a combination of Eq (9), the mean-value-inequality for the subharmonic function $|u_1 - u_2|e^{\phi}$ and the Cauchy-Schwarz inequality derives

$$\begin{aligned} |u_1(z_0) - u_2(z_0)|e^{\phi(z_0)} &\le (\pi r^2)^{-1} \int_{|z - z_0| < r} |u_1 - u_2|e^{\phi}\,\mathrm{dA} \\ &\le (\pi r^2)^{-1} \int_{|z| < 2r - 1} |u_1 - u_2|e^{\phi}\,\mathrm{dA} \end{aligned}$$

$$
\begin{aligned}
&\le \pi^{-1}(2r)^{-\epsilon}\int_{|z|<2r-1}|u_1(z)-u_2(z)|e^{\phi(z)}(1+|z|)^{\epsilon-2}\,\mathrm{dA}(z)\\
&\le \pi^{-1}(2r)^{-\epsilon}\int_{\mathbb{C}}|u_1(z)-u_2(z)|e^{\phi(z)}(1+|z|)^{\epsilon-2}\,\mathrm{dA}(z)\\
&\le \pi^{-1}(2r)^{-\epsilon}\left(\int_{\mathbb{C}}|u_1-u_2|^2\frac{dA}{e^{2\phi}}\right)^{\frac12}\left(\int_{\mathbb{C}}((1+|z|)^{\epsilon-2}e^{2\phi(z)})^2\,\mathrm{dA}(z)\right)^{\frac12}\\
&\le 2^2\pi^{-1}(2r)^{-\epsilon}\left(\int_{\mathbb{C}}|f|^2\frac{\mathrm{dA}}{(\Delta\phi)e^{2\phi}}\right)^{\frac12}\left(\int_{\mathbb{C}}((1+|z|)^{\epsilon-2}e^{2\phi(z)})^2\,\mathrm{dA}(z)\right)^{\frac12}.
\end{aligned}
$$

Letting $r\to\infty$ in the last estimation gives

$$
\begin{cases}
|u_1(z_0)-u_2(z_0)|e^{\phi(z_0)}=0;\\
u_1(z_0)=u_2(z_0).
\end{cases}
$$

Since z_0 is arbitrary, the last equality ensures $u_1=u_2$ on $\mathbb{C}$. □

Argument for Theorem 1.1. Clearly, if $g\equiv 1$ in (3)–(4), then (5)$\Longrightarrow$(6) follows from Eq (3)$\Longrightarrow$(4) which is verified as below.

Suppose that (3) is valid. Let

$$
\begin{cases}
\varphi_j=\frac{g^m\overline{f_j}}{\sum_{l=1}^n|f_l|^2} & \forall\ \ j\in\{1,...,n\};\\
H_{j,k}=g^m\varphi_j\bar{\partial}\varphi_k & \forall\ \ j,k\in\{1,...,n\}.
\end{cases}
$$

If $b_{j,k}$ is a function solving pointwisely the $\bar{\partial}$-equation

$$
\bar{\partial}b_{j,k}=H_{j,k}, \tag{12}
$$

then each

$$
g_j=g^{2m}\varphi_j+\sum_{k=1}^n(b_{j,k}-b_{k,j})f_k \tag{13}
$$

is an entire function enjoying

$$
\sum_{j=1}^n g_jf_j=g^{2m}\sum_{j=1}^n f_j\varphi_j+\sum_{k,j=1}^n(b_{jk}-b_{kj})f_kf_j=g^{3m},
$$

and hence the equation in (4) is met.

Thanks to the smoothness of $f_1,...,f_n,g$, Lemma 2.1(iv) with

$$
\phi(z)=2^{-1}(2m-1)\alpha|z|^2
$$

produces a function $b_{j,k}$ such that (12) holds pointwisely with

$$\int_{\mathbb{C}} |b_{j,k}(z)|^2 e^{-(2m-1)\alpha|z|^2}\, dA(z) \le 2^{-1} \int_{\mathbb{C}} |H_{j,k}(z)|^2 \left(\frac{e^{-(2m-1)\alpha|z|^2}}{\Delta\phi(z)} \right) \mathrm{dA}(z)$$
$$= ((2m-1)\alpha)^{-1} \int_{\mathbb{C}} |H_{j,k}(z)|^2 e^{-(2m-1)\alpha|z|^2}\, \mathrm{dA}(z). \tag{14}$$

In order to achieve $g_j \in \mathcal{F}^2_{2m\alpha,1}$ in (4), in the sequel we employ (12)–(13) to prove

$$\begin{cases} \int_{\mathbb{C}} |H_{j,k}(z)|^2 e^{-(2m-1)\alpha|z|^2} \mathrm{dA}(z) < \infty; \\ \int_{\mathbb{C}} |zg_j(z)|^2 e^{-2m\alpha|z|^2} \mathrm{dA}(z) < \infty \end{cases} \tag{15}$$

▷ It is easy to get

$$\sup_{z\in\mathbb{C}} |\varphi_j(z)| = \sup_{z\in\mathbb{C}} \left| \frac{g^m(z)\overline{f_j(z)}}{\sum_{l=1}^n |f_l(z)|^2} \right| = \sup_{z\in\mathbb{C}} \left(\frac{|g(z)|^m}{(\sum_{l=1}^n |f_l(z)|^2)^{\frac{1}{2}}} \right) \left(\frac{|f_j(z)|}{(\sum_{l=1}^n |f_l(z)|^2)^{\frac{1}{2}}} \right) \lesssim 1.$$

In the above and below, $X \lesssim Y$ stands for $X \le cY$ for a positive constant c.

- For the case $m = 1$, we utilize Eqs (1) & (3) to derive

$$\begin{aligned} \int_{\mathbb{C}} |zg^{2m}(z)\varphi_j(z)|^2 e^{-2m\alpha|z|^2}\, \mathrm{dA}(z) &\lesssim \int_{\mathbb{C}} |zg^2(z)|^2 e^{-2\alpha|z|^2}\, \mathrm{dA}(z) \\ &\lesssim \|g\|^2_{\mathcal{F}^2_{\alpha,1}} \int_{\mathbb{C}} |z|^2 |g(z)|^2 \left(\frac{e^{2^{-1}\alpha|z|^2}}{1+|z|} \right)^2 e^{-2\alpha|z|^2}\, \mathrm{dA}(z) \\ &\lesssim \|g\|^2_{\mathcal{F}^2_{\alpha,1}} \int_{\mathbb{C}} |z|^2 |g(z)|^2 (1+|z|)^{-2}\, e^{-\alpha|z|^2} \mathrm{dA}(z) \\ &\lesssim \|g\|^2_{\mathcal{F}^2_{\alpha,1}} \int_{\mathbb{C}} |z|^2 |g(z)|^2\, e^{-\alpha|z|^2} \mathrm{dA}(z) \\ &\lesssim \|g\|^4_{\mathcal{F}^2_{\alpha,1}} \\ &\lesssim \sum_{j=1}^n \|f_j\|^4_{\mathcal{F}^2_{\alpha,1}} < \infty. \end{aligned}$$

- For the case $m > 1$, we utilize Eq (1) - the Hölder inequality - Eq (3) to derive

$$\begin{aligned} \int_{\mathbb{C}} |zg^{2m}(z)\varphi_j(z)|^2 e^{-2m\alpha|z|^2} \mathrm{dA}(z) &\lesssim \|g\|^{4m}_{\mathcal{F}^2_{\alpha,1}} \int_{\mathbb{C}} (1+|z|)^{-4m} |z|^2\, \mathrm{dA}(z) \\ &\lesssim \|g^m\|^4_{\mathcal{F}^2_{\alpha,1}} \int_{\mathbb{C}} (1+|z|)^{-2(2m-1)}\, \mathrm{dA}(z) \\ &\lesssim \sum_{j=1}^n \|f_j\|^4_{\mathcal{F}^2_{\alpha,1}} < \infty. \end{aligned}$$

In summary, we always have

$$\int_{\mathbb{C}} |zg^{2m}(z)\varphi_j(z)|^2 e^{-2m\alpha|z|^2} \mathrm{dA}(z) \lesssim \sum_{j=1}^n \|f_j\|^4_{\mathcal{F}^2_{\alpha,1}} < \infty. \tag{16}$$

Now, a straightforward computation gives

$$\bar{\partial}\varphi_j = \frac{g^m \sum_{l=1}^n f_l \left(\overline{f_l}\,\overline{\partial f_j} - \overline{f_j}\,\overline{\partial f_l}\right)}{\left(\sum_{l=1}^n |f_l|^2\right)^2}.$$

As evaluated in [13, 14], we have

$$|\bar{\partial}\varphi_j|^2 \lesssim \frac{|g|^{2m}\left(\sum_{l=1}^n |f_l|^2\right)^2 \sum_{l=1}^n |\partial f_l|^2}{\left(\sum_{l=1}^n |f_l|^2\right)^4} \lesssim \frac{\sum_{l=1}^n |\partial f_l|^2}{\sum_{l=1}^n |f_l|^2},$$

thereby producing

$$|H_{j,k}|^2 = \left|g^m \varphi_j \bar{\partial}\varphi_k\right|^2 \lesssim \sum_{l=1}^n |\partial f_l|^2.$$

Clearly, we get

$$\begin{aligned} \int_{\mathbb{C}} |H_{j,k}(z)|^2 e^{-(2m-1)\alpha|z|^2} \mathrm{dA}(z) &\lesssim \int_{\mathbb{C}} \sum_{l=1}^n |\partial f_l(z)|^2 e^{-(2m-1)\alpha|z|^2} \mathrm{dA}(z) \\ &\lesssim \sum_{l=1}^n \|f_l\|^2_{\mathcal{F}^2_{\alpha,1}} < \infty, \end{aligned} \tag{17}$$

whence verifying the first inequality of (15).

▷ According to Lemma 2.1(i), there exists $b_{j,k}$ classically resolving (12) with (14), and consequently, a combination of Eqs (1) & (17) yields

$$\begin{aligned} \int_{\mathbb{C}} |zb_{j,k}(z)|^2 |f_k(z)|^2 e^{-2m\alpha|z|^2} \mathrm{dA}(z) &\lesssim \|f_k\|^2_{\mathcal{F}^2_{\alpha,1}} \int_{\mathbb{C}} |b_{j,k}(z)|^2 |z|^2 (1+|z|)^{-2} e^{-(2m-1)\alpha|z|^2} \mathrm{dA}(z) \\ &\lesssim \|f_k\|^2_{\mathcal{F}^2_{\alpha,1}} \int_{\mathbb{C}} |b_{j,k}(z)|^2 e^{-(2m-1)\alpha|z|^2} \mathrm{dA}(z) \\ &\lesssim \|f_k\|^2_{\mathcal{F}^2_{\alpha,1}} \int_{\mathbb{C}} |H_{j,k}(z)|^2 e^{-(2m-1)\alpha|z|^2} \mathrm{dA}(z) \\ &\lesssim \|f_k\|^2_{\mathcal{F}^2_{\alpha,1}} \sum_{l=1}^n \|f_l\|^2_{\mathcal{F}^2_{\alpha,1}} < \infty. \end{aligned} \tag{18}$$

Since the comparable constants in (18) are independent of $\{j,k\}$, the formula (13), along with (16) & (18), validates the second inequality of (15).

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Conflict of interest

The authors declare there is no conflict of interest.

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