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Research article

Incompressible limit of Euler equations with damping

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Abstract: The Cauchy problem for the compressible Euler system with damping is considered in this paper. Based on previous global existence results, we further study the low Mach number limit of the system. By constructing the uniform estimates of the solutions in the well-prepared initial data case, we are able to prove the global convergence of the solutions in the framework of small solutions.

Keywords: incompressible limit; Mach number; Euler equations; damping; global solution

1. Introduction

We are concerned with the 3D compressible Euler equations with frictional damping

$$\begin{cases} \partial_t\rho + \nabla\cdot(\rho u) = 0, \\ \rho(\partial_t u + (u\cdot\nabla)u) + \frac{1}{\varepsilon^2}\nabla p(\rho) = -A\rho u. \end{cases} \tag{1.1}$$

Here, $x \in \mathbb{T}^3$, $t > 0$, the unknown functions ρ and u denote the density and velocity of the fluid respectively; the pressure-density function p is given by

$$p(\rho) = a\rho^\gamma, \qquad a > 0, \qquad \gamma > 1;$$

the constant $A > 0$ is the friction coefficient and ε is the Mach number. The system can be used to model the compressible fluid flow through a porous media.

The compressible Euler equations with damping can be used to simulate the motion for the compressible gas flow through a porous medium. The medium induces a friction force, proportional to the linear momentum in the opposite direction. Usually, this model has many variations. For example, the friction coefficient A may depend on time [1] and, more generally, one can consider the nonlinear damping [2]. Here, we only study the constant-coefficient linear damping case.

It is well-known that the compressible Euler equations will develop singularity in finite time for general initial data [3]. However, the damping effect will prevent the formation of singularities in small amplitude flows, but large solutions may still break down [31]. Due to strong physical background and significant mathematical challenge, system (1.1) with fixed Mach number has been studied by many researchers. The readers are referred to [30] for BV solutions and [4, 5] for L^∞ solution. For the existence of weak solutions, we refer to [6, 7]. While for global classical solution, see [8–12]. We also refer to [13–18] and references therein for further studies in this direction and related problems.

In particular, the global-in-time existence and asymptotic behavior of the classical solution were obtained by Wang and Yang [11]. By employing the Green's function method and energy estimate, they proved that if the suitable Sobolev norm of the initial data is small, then the global existence of classical solution can be obtained, and the L^2 norm of the solution decays at the rate of $t^{-3/4}$ in the whole space.

Physically, the Mach number represents the ratio of the typical speed of the flow to the speed of sound. In practical applications, the incompressible equations are often used when the Mach number is sufficiently small. So it is natural to think, under appropriate conditions on the initial data, that solutions of the compressible system converge to the solution of the incompressible system when the Mach number goes to zero. This is the low Mach number limit problem in fluid mechanics.

Over the past four decades, many results have been obtained about the incompressible limit of fluid dynamic equations, which is a special case of the low Mach number limit. In their ground-breaking works [19, 20], Klainerman and Majda setup a general framework for the study of singular limit of hyperbolic PDEs. With the general theory, they proved the incompressible limit of isentropic Euler equations in the well-prepared initial data case. For general initial data, Ukai [21] obtained the convergence of the solutions in the whole space. Finally by using the filtering method, Schochet [22] solved the incompressible limit problem for Euler equations in the torus case. Concerning the low Mach number limit of the non-isentropic Euler equations, we refer to [23–25].

Since these pioneering works, the mathematical analysis of the low Mach number limit to the isentropic Navier-Stokes equations also attracted a lot of attention. In the framework of weak solutions, the incompressible limit of Navier-Stokes equations in the periodic case and bounded domain was studied in [26] and [27] respectively. For global-in-time regular solutions, Bessaih [28] established the uniform estimates of the solutions with almost incompressible initial data and proved the convergence to the solution of incompressible system with no-slip boundary conditions. Ou [29] further extended this result to the slip-type boundary conditions.

For the local classical solutions, the incompressible limit of the system (1.1) can be established in a similar fashion as the compressible Euler equations. If the damping effect is considered, it is possible to study the incompressible limit of the system (1.1) for all time since we have global small solutions. To the best of our knowledge, there are no results studying the global incompressible limit of Euler equations with damping. Incompressible fluid flow differs from compressible fluid flow in that the continuity equation is replaced by the divergence-free condition on the velocity field. Mathematically, the low Mach number limit attempts to bridge the gap between those two different descriptions and, in some sense, it is relatively easy to study the incompressible system instead of the compressible equations.

In this paper, based on the previous global existence results of the Euler system with damping, we are going to study the global incompressible limit to system (1.1) in the framework of small amplitude solutions. Namely, we will show that as the Mach number goes to zero, the solutions to Eqs (1.1) will converge to the solution of the following system

$$\begin{cases} \bar{\rho}(\partial_t v + (v\cdot\nabla)v) + \nabla\pi = -A\bar{\rho}v, \\ \nabla\cdot v = 0, \end{cases} \tag{1.2}$$

where $\bar{\rho}$ is some constant and π can be formally obtained by applying $\nabla\cdot$ to the first equation.

The rest of the paper is arranged as follows. In Section 2, we reformulate the Cauchy problem for Eq (1.1) into a symmetric hyperbolic system and state the global existence of classical solutions to the system with fixed Mach number. In Section 3, the global uniform estimates of the solution are obtained. Finally with the uniform estimates, we will prove the convergence of the solutions of the original equations (1.1) to the solution of the limit system in Section 4.

Notation. Throughout the paper C will denote a positive constant whose value may be different in each occurrence, that may depend PDEs, domain, and Sobolev index, but are independent of the small parameter ε. The small letter c and its variants denote similar constants whose value is fixed. ∂_x^α or simply ∂^α with multi-index α stands for the usual spatial derivatives. For any integer $s \geq 0$, H^s denote the inhomogeneous Sobolev space $H^s(\mathbb{T}^3)$ with the norm $\|\cdot\|_{H^s}$. We denote $\|\cdot\| = \|\cdot\|_{L^2}$ for simplicity. Generally, the solution is dependent on the small parameter ε in this paper, so it is better to write the solution $u^\varepsilon(t,x)$ rather than $u(t,x)$ for example. But for simplicity, we always omit the superscript ε and use $u(t,x)$ instead when there is no confusion.

2. Reformulation of the problem

In this short section, we will give a reformulation of the problem. To simplify the presentation, we introduce

$$h(\rho) = \frac{a\gamma}{\gamma-1}\rho^{\gamma-1}. \tag{2.1}$$

Using h and u as the new unknown functions, we get from the original equation (1.1) that

$$\begin{cases} e(h)(\partial_t h + (u\cdot\nabla)h) + \nabla\cdot u = 0, \\ \partial_t u + (u\cdot\nabla)u + \frac{1}{\varepsilon^2}\nabla h = -Au, \end{cases} \tag{2.2}$$

where

$$e(h) = \frac{1}{(\gamma-1)h}$$

is a smooth functions of $h > 0$.

In this paper, we will consider the perturbative solutions of the above equations. Thus we choose the constant equilibrium state $(h^*, 0)$ with $h^* > 0$ and set

$$h = h^* + \varepsilon q.$$

The system satisfied by q and u is

$$\begin{cases} e(h)(\partial_t q + (u\cdot\nabla)q) + \frac{1}{\varepsilon}\nabla\cdot u = 0, \\ \partial_t u + (u\cdot\nabla)u + \frac{1}{\varepsilon}\nabla q = -Au. \end{cases} \tag{2.3}$$

Initial data is given by

$$q(t=0)=q_0(x), \qquad u(t=0)=u_0(x). \tag{2.4}$$

For fixed $\varepsilon>0$, it has been proved by many authors, c.f. [11,31], that (2.3)–(2.4) admits a global classical solution provided that the certain Sobolev norm of the initial data is sufficiently small. Namely, we have the following global existence theorem.

Theorem 2.1. *For any fixed $\varepsilon>0$, suppose that the initial data $(q_0(x),u_0(x))\in H^s(\mathbb{T}^3)$ with $s\geq 3$ and $\|(q_0(x),u_0(x))\|_{H^s(\mathbb{T}^3)}$ is sufficiently small. Then there exists a unique, global, classical solution $(q^\varepsilon(x,t),u^\varepsilon(x,t))$ to the Cauchy problem of* (2.3)–(2.4).

The aim of the present paper is to show that the sequences of the solution $(q^\varepsilon(x,t),u^\varepsilon(x,t))$ actually converge to the solution of a limit system for all time. The key step is to establish the uniform estimates both in Mach number and time.

3. Uniform estimates

In this part, we are going to prove the uniform estimates of the solution. To this end, we define a weighted norm

$$E(w)=\|w\|_{H^3}^2+\|\partial_t w\|_{H^2}^2+\varepsilon^2\|\partial_{tt}w\|_{H^1}^2+\varepsilon^4\|\partial_{ttt}w\|^2. \tag{3.1}$$

First, we have the following estimate.

Lemma 3.1. *If $U=(q,u)\in C([0,T];H^3)$ is a solution of the system* (2.3)–(2.4) *for any given $T>0$, then we have*

$$\frac{d}{dt}E(U)+AE(u)\leq CE(U)^{\frac{3}{2}}. \tag{3.2}$$

Proof. The proof is divided into two parts. First, we will show that

$$\frac{d}{dt}\|U\|_{H^3}^2+A\|u\|_{H^3}^2\leq CE(U)^{\frac{3}{2}}. \tag{3.3}$$

Second, we shall prove that

$$\frac{d}{dt}\sum_{k=1}^{3}\varepsilon^{2k-2}\|\partial_t^k U\|_{H^{3-k}}^2+A\sum_{k=1}^{3}\varepsilon^{2k-2}\|\partial_t^k u\|_{H^{3-k}}^2\leq CE(U)^{\frac{3}{2}}. \tag{3.4}$$

Thus, by the definition of the functional $E(U)$, it is obvious that (3.2) holds.

Part I: For $0\leq|\alpha|\leq 3$, applying ∂_x^α to (2.3) and testing by $\partial_x^\alpha U$ gives

$$\begin{aligned}\int_{\mathbb{T}^3}e(h)(\partial_t\partial_x^\alpha q+(u\cdot\nabla)\partial_x^\alpha q)\partial_x^\alpha q dx+\int_{\mathbb{T}^3}(\partial_t\partial_x^\alpha u+(u\cdot\nabla)\partial_x^\alpha u)\cdot\partial_x^\alpha u dx\\ +A\int_{\mathbb{T}^3}\partial_x^\alpha u\partial_x^\alpha u dx=\int_{\mathbb{T}^3}C_1\partial_x^\alpha q dx+\int_{\mathbb{T}^3}C_2\cdot\partial_x^\alpha u dx,\end{aligned} \tag{3.5}$$

where the singular terms are cancelled by integration by parts and the commutators C_1 and C_2 are given by

$$
\begin{aligned}
C_1 &= -[\partial_x^\alpha, e(h)]\partial_t q - [\partial_x^\alpha, e(h)u\cdot\nabla]q \\
C_2 &= -[\partial_x^\alpha, u\cdot\nabla]u.
\end{aligned}
$$

By direct calculation, we get

$$
\begin{aligned}
&\int_{\mathbb{T}^3} e(h)(\partial_t\partial_x^\alpha q + (u\cdot\nabla)\partial_x^\alpha q)\partial_x^\alpha q dx \\
&\qquad = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3} e(h)|\partial_x^\alpha q|^2 dx - \frac{1}{2}\int_{\mathbb{T}^3}\partial_t e(h)|\partial_x^\alpha q|^2 dx \\
&\qquad - \frac{1}{2}\int_{\mathbb{T}^3}\nabla\cdot(e(h)u)|\partial_x^\alpha q|^2 dx
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{T}^3}(\partial_t\partial_x^\alpha u + (u\cdot\nabla)\partial_x^\alpha u)\cdot\partial_x^\alpha u dx \\
&\qquad = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}|\partial_x^\alpha u|^2 dx - \frac{1}{2}\int_{\mathbb{T}^3}\nabla\cdot u|\partial_x^\alpha u|^2 dx.
\end{aligned}
$$

Combining the above two equalities with (3.5), one finds

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3} e(h)|\partial_x^\alpha q|^2 dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}|\partial_x^\alpha u|^2 dx + A\int_{\mathbb{T}^3}\partial_x^\alpha u\partial_x^\alpha u dx \\
&\quad\le \frac{1}{2}\Big|\int_{\mathbb{T}^3}\partial_t e(h)|\partial_x^\alpha q|^2 dx\Big| + \frac{1}{2}\Big|\int_{\mathbb{T}^3}\nabla\cdot(e(h)u)|\partial_x^\alpha q|^2 dx\Big| \\
&\quad + \frac{1}{2}\Big|\int_{\mathbb{T}^3}\nabla\cdot u|\partial_x^\alpha u|^2 dx\Big| + \Big|\int_{\mathbb{T}^3} C_1\partial_x^\alpha q dx + \int_{\mathbb{T}^3} C_2\cdot\partial_x^\alpha u dx\Big|. \qquad (3.6)
\end{aligned}
$$

Clearly, the first three terms on the right-hand side of (3.6) can be controlled by

$$
\begin{aligned}
&\|\partial_t e(h)\|_{L^\infty}\|\partial_x^\alpha q\|^2 + \|\nabla\cdot(e(h)u)\|_{L^\infty}\|\partial_x^\alpha q\|^2 + \|\nabla\cdot u\|_{L^\infty}\|\partial_x^\alpha u\|^2 \\
&\qquad \le CE(U)^{\frac{3}{2}}, \qquad (3.7)
\end{aligned}
$$

where standard Sobolev embedding inequalities are used here. The commutator estimates is given by

$$
\begin{aligned}
\Big|\int_{\mathbb{T}^3} C_1\partial_x^\alpha q dx\Big| &= \Big|\int_{\mathbb{T}^3}\Big\{[\partial_x^\alpha, e(h)]\partial_t q - [\partial_x^\alpha, e(h)u\cdot\nabla]q\Big\}\partial_x^\alpha q dx\Big| \\
&\le \sum_{\beta+\gamma=\alpha, |\beta|>0} C_\alpha^\beta\int_{\mathbb{T}^3}[|\partial_x^\beta e(h)\partial_x^\gamma\partial_t q| + |\partial_x^\beta(e(h)u)\partial_x^\gamma\nabla q|]\partial_x^\alpha q dx.
\end{aligned}
$$

We only consider the case $|\alpha| = 3$, because other cases are relatively easy to deal with. When $|\beta| = 1, |\gamma| = 2$, we have

$$
\int_{\mathbb{T}^3}[|\partial_x^\beta e(h)\partial_x^\gamma\partial_t q| + |\partial_x^\beta(e(h)u)\partial_x^\gamma\nabla q|]\partial_x^\alpha q dx
$$

$$\leq C[\|\partial_x^\beta e(h)\|_{L^\infty}\|\partial_x^\gamma\partial_t q\| + \|\partial_x^\beta(e(h)u)\|_{L^\infty}\|\partial_x^\gamma\nabla q\|]\|\partial_x^\alpha q\|$$
$$\leq CE(U)^{\frac{3}{2}}. \tag{3.8}$$

When $|\beta| = 2, |\gamma| = 1$,

$$\int_{\mathbb{T}^3}[|\partial_x^\beta e(h)\partial_x^\gamma\partial_t q| + |\partial_x^\beta(e(h)u)\partial_x^\gamma\nabla q|]\partial_x^\alpha q dx$$
$$\leq C[\|\partial_x^\beta e(h)\|_{L^3}\|\partial_x^\gamma\partial_t q\|_{L^6} + \|\partial_x^\beta(e(h)u)\|_{L^3}\|\partial_x^\gamma\nabla q\|_{L^6}]\|\partial_x^\alpha q\|$$
$$\leq C[\|\nabla\partial_x^\beta e(h)\|\|\nabla\partial_x^\gamma\partial_t q\| + \|\nabla\partial_x^\beta(e(h)u)\|\|\nabla\partial_x^\gamma\nabla q\|]\|\partial_x^\alpha q\|$$
$$\leq CE(U)^{\frac{3}{2}}. \tag{3.9}$$

When $|\beta| = 3, |\gamma| = 0$,

$$\int_{\mathbb{T}^3}[|\partial_x^\beta e(h)\partial_x^\gamma\partial_t q| + |\partial_x^\beta(e(h)u)\partial_x^\gamma\nabla q|]\partial_x^\alpha q dx$$
$$\leq C[\|\partial_x^\beta e(h)\|\|\partial_t q\|_{L^\infty} + \|\partial_x^\beta(e(h)u)\|\|\nabla q\|_{L^\infty}]\|\partial_x^\alpha q\|$$
$$\leq CE(U)^{\frac{3}{2}}. \tag{3.10}$$

Collecting (3.8)–(3.10), we get the following estimate

$$|\int_{\mathbb{T}^3} C_1\partial_x^\alpha q dx| \leq CE(U)^{\frac{3}{2}}.$$

In a similar fashion, one can show that

$$|\int_{\mathbb{T}^3} C_2\cdot\partial_x^\alpha u dx| \leq CE(U)^{\frac{3}{2}}.$$

Substituting the commutator estimates and (3.7) into (3.6) and taking summation over $0 \leq |\alpha| \leq 3$, we get

$$\frac{d}{dt}\|U\|_{H^3}^2 + A\|u\|_{H^3}^2 \leq CE(U)^{\frac{3}{2}}. \tag{3.11}$$

This completes the proof of the first part.
Part II: For $1 \leq k \leq 3$ and $k + |\alpha| \leq 3$, applying $\partial_t^k\partial_x^\alpha$ to (2.3) and testing by $\varepsilon^{2k-2}\partial_t^k\partial_x^\alpha U$ gives

$$\begin{aligned}\varepsilon^{2k-2}\int_{\mathbb{T}^3} & e(h)(\partial_t\partial_t^k\partial_x^\alpha q + (u\cdot\nabla)\partial_t^k\partial_x^\alpha q)\partial_t^k\partial_x^\alpha q dx \\ &+ \varepsilon^{2k-2}\int_{\mathbb{T}^3}(\partial_t\partial_t^k\partial_x^\alpha u + (u\cdot\nabla)\partial_t^k\partial_x^\alpha u)\cdot\partial_t^k\partial_x^\alpha u dx \\ &+ A\varepsilon^{2k-2}\int_{\mathbb{T}^3}\partial_t^k\partial_x^\alpha u\partial_t^k\partial_x^\alpha u dx = \varepsilon^{2k-2}\int_{\mathbb{T}^3} C_3\partial_t^k\partial_x^\alpha q dx \\ &+ \varepsilon^{2k-2}\int_{\mathbb{T}^3} C_4\cdot\partial_t^k\partial_x^\alpha u dx,\end{aligned} \tag{3.12}$$

where

$$C_3 = -[\partial_t^k \partial_x^\alpha, e(h)]\partial_t q - [\partial_t^k \partial_x^\alpha, e(h)u \cdot \nabla]q,$$
$$C_4 = -[\partial_t^k \partial_x^\alpha, u \cdot \nabla]u.$$

By integrating by parts, we still have

$$\varepsilon^{2k-2}\frac{d}{dt}\int_{\mathbb{T}^3}[e(h)|\partial_t^k\partial_x^\alpha q|^2 + |\partial_t^k\partial_x^\alpha u|^2]dx + A\varepsilon^{2k-2}\int_{\mathbb{T}^3}|\partial_t^k\partial_x^\alpha u|^2 dx$$
$$\leq \varepsilon^{2k-2}\int_{\mathbb{T}^3} C_3\partial_t^k\partial_x^\alpha q dx + \varepsilon^{2k-2}\int_{\mathbb{T}^3} C_4 \cdot \partial_t^k\partial_x^\alpha u dx + CE(U)^{\frac{3}{2}}. \tag{3.13}$$

It remains to give the estimates of the commutators. Basically, the commutator estimates is similar to (3.8)–(3.10) except that we need to guarantee there are enough powers of ε to balance the norms of the unknowns. We should be careful when we encountered with higher time derivatives. For example, the estimate of the first term in C_3 will involve

$$\|\partial_t^2 q \partial_x^\alpha e(h)\|,$$

with $|\alpha| = 2$. The definition of $E(U)$ suggests that there should be a ε to balance $\partial_t^2 q$. Fortunately, since $h = h^* + \varepsilon q$, taking derivative to $e(h)$ will give us an additional ε. The other terms in the commutator is similar, we just omit the details for the sake of simplicity.

Using the strategy we explained above to estimate the commutators, we get

$$\varepsilon^{k-1}\|C_3\| + \varepsilon^{k-1}\|C_4\| \leq CE(U). \tag{3.14}$$

Combined with (3.13), one has

$$\varepsilon^{2k-2}\frac{d}{dt}[\|\partial_t^k\partial_x^\alpha q\|^2 + \|\partial_t^k\partial_x^\alpha u\|^2] + A\varepsilon^{2k-2}\|\partial_t^k\partial_x^\alpha u\|^2 dx$$
$$\leq CE(U)^{\frac{3}{2}}. \tag{3.15}$$

By taking summation over $1 \leq k \leq 3$ and $k + |\alpha| \leq 3$, we get the desired results (3.4). This completes the proof of Lemma 3.1. □

Next, we have the following lemma.

Lemma 3.2. *If $U = (q, u) \in C([0, T]; H^3)$ is a solution of the system* (2.3)–(2.4)*, then*

$$\frac{1}{\varepsilon^2}\|q\|_{H^3}^2 \leq C(\|u_t\|_{H^2}^2 + \|u\|_{H^2}^2) + CE(U)^2 \leq CE(u) + CE(U)^2, \tag{3.16}$$

and

$$\frac{d}{dt}\sum_{0\leq|\alpha|+k\leq 2}\varepsilon^{2k}\int_{\mathbb{T}^3}[\partial_t^{k+1}\partial_x^\alpha u \cdot \partial_t^k\partial_x^\alpha u + (u\cdot\nabla)\partial_t^k\partial_x^\alpha u \cdot \partial_t^k\partial_x^\alpha u]dx$$
$$+ \sum_{0\leq|\alpha|+k\leq 2}\varepsilon^{2k}\|\partial_t^{k+1}\partial_x^\alpha q\|^2$$
$$\leq CE(u) + CE(U)^{\frac{3}{2}}. \tag{3.17}$$

Proof. For $0 \le |\alpha| \le 2$, taking ∂_x^α to the second equation of (2.3) yields

$$\partial_x^\alpha\Big[\partial_t u + (u\cdot\nabla)u\Big] + \frac{1}{\varepsilon}\nabla\partial_x^\alpha q = -A\partial_x^\alpha u. \tag{3.18}$$

This further gives

$$\frac{1}{\varepsilon^2}\|\nabla\partial_x^\alpha q\|^2 \le \|\partial_x^\alpha\Big[\partial_t u + (u\cdot\nabla)u\Big]\|^2 + A\|\partial_x^\alpha u\|^2. \tag{3.19}$$

Taking summation over $0 \le |\alpha| \le 2$ gives

$$\frac{1}{\varepsilon^2}\|q\|_{H^3}^2 \le C(\|u_t\|_{H^2}^2 + \|u\|_{H^2}^2) + CE(U)^2, \tag{3.20}$$

where we have used the Poincaré inequality to control $\|q\|_{L^2}$.

Next, for $0 \le k + |\alpha| \le 2$, taking $\partial_t^k\partial_x^\alpha$ to the first equation of (2.3) gives

$$e(h)(\partial_t^{k+1}\partial_x^\alpha q + (u\cdot\nabla)\partial_t^k\partial_x^\alpha q) + \frac{1}{\varepsilon}\nabla\cdot\partial_t^k\partial_x^\alpha u = C_5, \tag{3.21}$$

where the commutator C_5 is given by

$$C_5 = -[\partial_t^k\partial_x^\alpha, e(h)]\partial_t q - [\partial_t^k\partial_x^\alpha, e(h)u\cdot\nabla]q.$$

Then, multiplying the above equation by $\partial_t^{k+1}\partial_x^\alpha q$ and integrating over $\mathbb{T}^3$, we have

$$\begin{aligned}\int_{\mathbb{T}^3} e(h)|\partial_t^{k+1}\partial_x^\alpha q|^2 dx &+ \int_{\mathbb{T}^3} e(h)(u\cdot\nabla)\partial_t^k\partial_x^\alpha q\partial_t^{k+1}\partial_x^\alpha q dx \\ &+ \frac{1}{\varepsilon}\int_{\mathbb{T}^3}\nabla\cdot\partial_t^k\partial_x^\alpha u\partial_t^{k+1}\partial_x^\alpha q dx = \int_{\mathbb{T}^3} C_5\partial_t^{k+1}\partial_x^\alpha q dx.\end{aligned} \tag{3.22}$$

Applying $\partial_t^{k+1}\partial_x^\alpha$ to the second equation of (2.3) gives

$$\partial_t^{k+2}\partial_x^\alpha u + (u\cdot\nabla)\partial_t^{k+1}\partial_x^\alpha u + \frac{1}{\varepsilon}\nabla\partial_t^{k+1}\partial_x^\alpha q = -A\partial_t^{k+1}\partial_x^\alpha u + C_6, \tag{3.23}$$

with

$$C_6 = [\partial_t^{k+1}\partial_x^\alpha, u\cdot\nabla]u. \tag{3.24}$$

Multiplying the above equation by $\partial_t^k\partial_x^\alpha u$ and integrating over $\mathbb{T}^3$, one has

$$\begin{aligned}\int_{\mathbb{T}^3}\partial_t^{k+2}\partial_x^\alpha u\cdot\partial_t^k\partial_x^\alpha u dx &+ \int_{\mathbb{T}^3}(u\cdot\nabla)\partial_t^{k+1}\partial_x^\alpha u\cdot\partial_t^k\partial_x^\alpha u dx \\ &+ \frac{1}{\varepsilon}\int_{\mathbb{T}^3}\nabla\partial_t^{k+1}\partial_x^\alpha q\partial_t^k\partial_x^\alpha u dx \\ &= -A\int_{\mathbb{T}^3}\partial_t^{k+1}\partial_x^\alpha u\partial_t^k\partial_x^\alpha u dx + \int_{\mathbb{T}^3} C_6\partial_t^k\partial_x^\alpha u dx.\end{aligned} \tag{3.25}$$

Adding (3.22) to (3.25), then multiplying by ε^{2k} yields

$$\begin{aligned}\varepsilon^{2k}\int_{\mathbb{T}^3} e(h)|\partial_t^{k+1}\partial_x^\alpha q|^2dx &= -\varepsilon^{2k}\int_{\mathbb{T}^3} e(h)(u\cdot\nabla)\partial_t^k\partial_x^\alpha q\partial_t^{k+1}\partial_x^\alpha q dx\\ &-\varepsilon^{2k}\int_{\mathbb{T}^3}\partial_t^{k+2}\partial_x^\alpha u\cdot\partial_t^k\partial_x^\alpha u dx-\varepsilon^{2k}\int_{\mathbb{T}^3}(u\cdot\nabla)\partial_t^{k+1}\partial_x^\alpha u\cdot\partial_t^k\partial_x^\alpha u dx\\ &-A\varepsilon^{2k}\int_{\mathbb{T}^3}\partial_t^{k+1}\partial_x^\alpha u\partial_t^k\partial_x^\alpha u dx+\varepsilon^{2k}\int_{\mathbb{T}^3}C_5\partial_t^{k+1}\partial_x^\alpha q\\ &+\varepsilon^{2k}\int_{\mathbb{T}^3}C_6\partial_t^k\partial_x^\alpha u dx\equiv\sum_{j=1}^6 I_j. \end{aligned}\tag{3.26}$$

Each term on the right-hand side of the above equality need to be estimated now. For I_1, we have

$$|I_1|\le\varepsilon^{2k}\|e(h)u\|_{L^\infty}\|\partial_t^k\nabla\partial_x^\alpha q\|\|\partial_t^{k+1}\partial_x^\alpha q\|\le CE(U)^{\frac{3}{2}}.$$

The second term I_2 can be reformulated into

$$I_2=-\varepsilon^{2k}\frac{d}{dt}\int_{\mathbb{T}^3}\partial_t^{k+1}\partial_x^\alpha u\cdot\partial_t^k\partial_x^\alpha u dx+\varepsilon^{2k}\int_{\mathbb{T}^3}|\partial_t^{k+1}\partial_x^\alpha u|^2dx.$$

Similarly, we have

$$\begin{aligned}I_3&=-\varepsilon^{2k}\frac{d}{dt}\int_{\mathbb{T}^3}(u\cdot\nabla)\partial_t^k\partial_x^\alpha u\cdot\partial_t^k\partial_x^\alpha u dx\\ &+\varepsilon^{2k}\int_{\mathbb{T}^3}(\partial_t u\cdot\nabla)\partial_t^k\partial_x^\alpha u\cdot\partial_t^k\partial_x^\alpha u dx+\varepsilon^{2k}\int_{\mathbb{T}^3}(u\cdot\nabla)\partial_t^k\partial_x^\alpha u\cdot\partial_t^{k+1}\partial_x^\alpha u dx.\end{aligned}$$

While for I_4, it is easy to find that

$$|I_4|\le C\varepsilon^{2k}\|\partial_t^{k+1}\partial_x^\alpha u\|\|\partial_t^k\partial_x^\alpha u\|\le CE(u).$$

For the estimates of the commutators, we mention that there are enough powers of ε to balance the time derivatives of the solution. Thus, I_5 and I_6 can be handled in a similar way as Lemma 3.1. For simplicity, we omit the details of the estimates and give

$$|I_5|+|I_6|\le CE(U)^{\frac{3}{2}}.$$

Finally, putting the above estimates of I_j into (3.26) and taking summation over$0\le k+|\alpha|\le 2$, we get

$$\begin{aligned}&\frac{d}{dt}\sum_{0\le|\alpha|+k\le2}\varepsilon^{2k}\int_{\mathbb{T}^3}[\partial_t^{k+1}\partial_x^\alpha u\cdot\partial_t^k\partial_x^\alpha u+(u\cdot\nabla)\partial_t^k\partial_x^\alpha u\cdot\partial_t^k\partial_x^\alpha u]dx\\ &\quad+\varepsilon^{2k}\sum_{0\le|\alpha|+k\le2}\int_{\mathbb{T}^3}e(h)|\partial_t^{k+1}\partial_x^\alpha q|^2dx\\ &\quad\le CE(u)+CE(U)^{\frac{3}{2}}.\end{aligned}\tag{3.27}$$

This completes the proof of Lemma 3.2. □

Now, following the two lemmas above, we are ready to give the uniform estimates of the solutions. Multiplying (3.16) and (3.17) by a sufficiently small constant κ, adding them onto (3.2), we get

$$\begin{aligned}&\frac{d}{dt}\Big[E(U)+\kappa\sum_{0\le|\alpha|+k\le2}\varepsilon^{2k}\int_{\mathbb{T}^3}[\partial_t^{k+1}\partial_x^\alpha u\cdot\partial_t^k\partial_x^\alpha u+(u\cdot\nabla)\partial_t^k\partial_x^\alpha u\cdot\partial_t^k\partial_x^\alpha u]dx\Big]\\&\quad+AE(u)+\frac{\kappa}{\varepsilon^2}\|q\|_{H^3}^2+\kappa\sum_{0\le|\alpha|+k\le2}\varepsilon^{2k}\|\partial_t^{k+1}\partial_x^\alpha q\|^2\\&\quad\le C\kappa E(u)+CE(U)^{3/2}+CE(U)^2.\end{aligned}\tag{3.28}$$

Notice that the first term $C\kappa E(u)$ on the right-hand side of (3.28) can be absorbed by $AE(u)$ due to the smallness of κ, thus one has

$$\frac{d}{dt}\tilde{E}(U)+c_1E(U)+\frac{c_2}{\varepsilon^2}\|q\|_{H^3}^2\le CE(U)^{3/2}+CE(U)^2,\tag{3.29}$$

where

$$\begin{aligned}\tilde{E}(U)&=E(U)\\&\quad+\kappa\sum_{0\le|\alpha|+k\le2}\varepsilon^{2k}\int_{\mathbb{T}^3}[\partial_t^{k+1}\partial_x^\alpha u\cdot\partial_t^k\partial_x^\alpha u+(u\cdot\nabla)\partial_t^k\partial_x^\alpha u\cdot\partial_t^k\partial_x^\alpha u]dx.\end{aligned}$$

Now, assume a priori that

$$E(U)\le\delta,$$

with δ being sufficiently small. Then by the smallness of ε and κ, we find that $\tilde{E}(U)$ is equivalent to $E(U)$. Namely, there exist positive constant c_3 and c_4 such that

$$c_3E(U)\le\tilde{E}(U)\le c_4E(U).\tag{3.30}$$

Moreover, since δ is small, we find all the terms on the right-hand side of (3.29) can be absorbed by the $c_1E(U)$. This gives

$$\tilde{E}(U(t))+c_5\int_0^t[E(U(\tau))+\frac{1}{\varepsilon^2}\|q(\tau)\|_{H^3}^2]d\tau\le\tilde{E}(U_0).\tag{3.31}$$

Thus if we assume that the initial data is sufficiently small such that

$$E(U_0)\le\delta_0<\frac{c_3}{c_4}\delta,$$

we can get from (3.30) and (3.31) that

$$c_3E(U)\le\tilde{E}(U(t))\le\tilde{E}(U_0)\le c_4E(U_0)\le c_3\delta.$$

This closes the a priori assumption and we get the uniform estimates of the solution. In a word, we have proved:

Theorem 3.3. *Suppose that the initial data* $(q_0(x),u_0(x))\in H^3(\mathbb{T}^3)$. *Then there exist positive constants* δ_0 *and* ε_0 *such that if*

$$E(U_0)\le\delta_0$$

and $0<\varepsilon<\varepsilon_0$ *hold, system* (2.3)–(2.4) *admits a global classical solution* $(q(x,t),u(x,t))$ *satisfying*

$$E(U(t))+c\int_0^t[E(U(\tau))+\frac{1}{\varepsilon^2}\|q(\tau)\|_{H^3}^2]d\tau\le C\delta_0,\tag{3.32}$$

for all $t\in\mathbb{R}^+$.

4. Convergence

With the uniform estimates established in previous section, we shall study the convergence of the solution in this section.

Actually, from the uniform estimates (3.32) and the Aubin-Lions compactness lemma, we can find a limit function

$$u^0 \in L^\infty(\mathbb{R}^+; H^3(\mathbb{T}^3)) \cap C(\mathbb{R}^+; H^{3-\eta}(\mathbb{T}^3))$$

with $\eta > 0$ such that, by passing to a subsequence,

$$u^\varepsilon \to u^0, \qquad \text{weak}^* \text{ in } L^\infty(\mathbb{R}^+; H^3(\mathbb{T}^3)), \tag{4.1}$$

$$u^\varepsilon \to u^0, \qquad \text{strongly in } C(\mathbb{R}^+; H^{3-\eta}(\mathbb{T}^3)). \tag{4.2}$$

Let $\mathbb{P}$ be the orthogonal projection of $(L^2(\Omega))^3$ onto the subspace

$$H_\sigma = \Big\{u \in (L^2(\Omega))^3 \Big| \int_\Omega u \cdot \nabla\phi dx = 0, \forall \phi \in H^1(\Omega)\Big\}.$$

Applying $\mathbb{P}$ to the second equation of (2.3) yields

$$\mathbb{P}[\partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla)u^\varepsilon + Au^\varepsilon] = 0.$$

Letting ε goes to zero in the above equations and using (4.1), (4.2), one has

$$\mathbb{P}[\partial_t u^0 + (u^0 \cdot \nabla)u^0 + Au^0] = 0.$$

By the properties of $\mathbb{P}$, there exists a function $\pi^0 \in L^\infty(\mathbb{R}^+; H^3(\mathbb{T}^3))$ such that

$$\partial_t u^0 + (u^0 \cdot \nabla)u^0 + Au^0 = -\nabla\pi^0. \tag{4.3}$$

On the other hand, the first equation in (2.3) gives

$$\varepsilon e(h^\varepsilon)(\partial_t q^\varepsilon + (u^\varepsilon \cdot \nabla)q^\varepsilon) + \nabla \cdot u^\varepsilon = 0.$$

The uniform estimates (3.32) enables us to take $\varepsilon \to 0$ in the above equation to get

$$\nabla \cdot u^0 = 0. \tag{4.4}$$

Equations (4.3) and (4.4) constitute the limit system

$$\begin{cases} \partial_t u^0 + (u^0 \cdot \nabla)u^0 + Au^0 = -\nabla\pi^0, \\ \nabla \cdot u^0 = 0, \\ u^0|_{t=0} = u_0^0(x), \end{cases} \tag{4.5}$$

where the initial data $u_0^0(x)$ is given by $\|u_0^\varepsilon - u_0^0(x)\|_{H^3} \to 0$ as $\varepsilon \to 0$.

Conflict of interest

The author declares there is no conflict of interest.

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