Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 30(1): 1–34.
DOI: 10.3934/era.2022001
Received: 24 March 2021
Revised: 30 September 2021
Accepted: 12 October 2021
Published: 08 December 2021

Research article

Additive actions on hyperquadrics of corank two

Yingqi Liu*

Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

* **Correspondence:** Email: liuyingqi20@mails.ucas.ac.cn.

Abstract: For a projective variety X in $\mathbb{P}^m$ of dimension n, an additive action on X is an effective action of $\mathbb{G}_a^n$ on $\mathbb{P}^m$ such that X is $\mathbb{G}_a^n$-invariant and the induced action on X has an open orbit. Arzhantsev and Popovskiy have classified additive actions on hyperquadrics of corank 0 or 1. In this paper, we give the classification of additive actions on hyperquadrics of corank 2 whose singularities are not fixed by the $\mathbb{G}_a^n$-action.

Keywords: commutative unipotent group action; equivariant compactification; hyperquadric of corank two; symmetric bilinear form; simultaneous orthogonal similarity

1. Introduction

1.1. Main results

Throughout the paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic zero. Let $\mathbb{G}_a = (\mathbb{K}, +)$ be the additive group of the field and $\mathbb{G}_a^n = \mathbb{G}_a \times \mathbb{G}_a \times ... \times \mathbb{G}_a (n \text{ times})$ be the vector group. In this article we study additive actions on projective varieties defined as follows.

Definition 1.1. *Let X be a closed subvariety of dimension n in $\mathbb{P}^m$. An additive action on X is an effective algebraic group action $\mathbb{G}_a^n \times \mathbb{P}^m \to \mathbb{P}^m$ such that X is $\mathbb{G}_a^n$-invariant and the induced action $\mathbb{G}_a^n \times X \to X$ has an open orbit O. Two additive actions on X are said to be equivalent if one is obtained from another via an automorphism of $\mathbb{P}^m$ preserving X.*

In the following we represent an additive action on X by a pair $(\mathbb{G}_a^n, X)$ or a triple $(\mathbb{G}_a^n, X, \mathbf{L})$, where $\mathbf{L}$ is the underlying projective space. We define $X \backslash O$ to be the boundary of the action and define $l(\mathbb{G}_a^n, X)$ to be the maximal dimension of orbits in the boundary. For a group action of G on a set S, we define the set of fixed points under the action to be $Fix(S) = \{x \in S \mid g \cdot x = x, \text{for any } g \in G\}$. We say a subset in the projective space is non-degenerate if it is not contained in any hyperplane.

Recall that a $\mathbb{G}_a^n$-action on $\mathbb{P}^m$ is induced by a linear representation of $\mathbb{G}_a^n$, namely write $\mathbb{P}^m = \mathbb{P}V$ for an $(m+1)$-dimensional vector space V, then the action is given by:

$$\begin{aligned}\mathbb{G}_a^n \times \mathbb{P}V &\mapsto \mathbb{P}V\\ (g, [v]) &\mapsto [\rho(g)(v)]\end{aligned}$$

where $\rho : \mathbb{G}_a^n \mapsto GL(V)$ is a rational representation of the vector group $\mathbb{G}_a^n$. In [1] Hassett and Tschinkel showed that if the action is faithful and has a non-degenerate orbit O in $\mathbb{P}^m$, then the vector space V can be realized as a finite dimensional local algebra. They identified additive actions on projective spaces with certain finite dimensional local algebras. The simplest additive action on a projective space is the one with fixed boundary, it is unique and can be given explicitly as follows.

$$\begin{aligned}\mathbb{G}_a^m \times \mathbb{P}^m &\mapsto \mathbb{P}^m\\ (g_1, ..., g_m) \times [x_0 : x_1 : ... : x_m] &\mapsto [x_0 : x_1 + g_1x_0 : ... : x_m + g_mx_0]\end{aligned}$$

In [2] Arzhantsev and Popovskiy identified additive actions on hypersurfaces in $\mathbb{P}^{n+1}$ with invariant d-linear symmetric forms on $(n+2)$-dimensional local algebras. As an application they obtained classifications of additive actions on hyperquadrics of corank 0 and 1, where the corank of a hyperquadric Q is the corank of the quadratic form defining Q. Given an additive action on a hyperquadric Q, if $corank(Q) = 0$ (i.e., Q is smooth), then the action is unique up to equivalences (also cf. [6]) and $l(\mathbb{G}_a^n, Q) = 1$. If $corank(Q) = 1$, then the action is determined by a symmetric matrix up to an orthogonal transformation, adding a scalar matrix and a scalar multiplication (cf. [2, Proof of Proposition 7]), namely for two symmetric matrices Λ and Λ', they determine the same action if and only if there exist a nonzero $a \in \mathbb{K}$, $h \in \mathbb{K}$ and an orthogonal matrix A (i.e., $A^\intercal A = I$) such that $\Lambda' = A^\intercal(a\Lambda + hI)A$. In this case, the action has fixed singular locus and $l(\mathbb{G}_a^n, Q) = 2$.

In this paper, we study additive actions on hyperquadrics of corank 2. In this case the action is determined by two symmetric bilinear forms on a certain finite dimesnional local algebra. The singular locus, which is a projective line, is either fixed by the action or is the union of a orbit and a fixed point.

When the singular locus is fixed by the $\mathbb{G}_a^n$-action, it is a natural generalization of the case when $corank(Q) = 1$. In this case, using a similar method as in [2, Proposition 7] one can see that the action is determined by a pair of symmetric matrices up to a simultaneous orthogonal similarity and an affine transformation of pais of matrices, namely for two pairs of symmetric matrices (Λ_1, Λ_2) and (Λ_1', Λ_2'), they determine the same action if and only if there exist $a_{11}, a_{12}, a_{21}, a_{22}, h_1, h_2 \in \mathbb{K}$ with $a_{11}a_{22} - a_{12}a_{21} \neq 0$ and an orthogonal matrix A such that:

$$\begin{aligned}\Lambda_1' &= A^\intercal(a_{11}\Lambda_1 + a_{12}\Lambda_2 + h_1I)A\\ \Lambda_2' &= A^\intercal(a_{21}\Lambda_1 + a_{22}\Lambda_2 + h_2I)A\end{aligned}$$

Remark 1.2. *For $\mathbb{K} = \mathbb{C}$, the problem of classifying pairs of matrices under simultaneous similarity is solved explicitly by Friedland [4]. As an application, for almost all pairs of symmetric matrices (A, B), the characteristic polynomial $|\lambda I - (A + xB)|$ determines a finite number of similtaneous orthogonal similarities classes.*

In this paper we focus on the case when the action has unfixed singularities, our main observation is that under the identification, one of the bilinear forms vanishes on a certain hyperplane of the maximal

ideal. As a result, the action can be recovered from two kinds of simpler actions which has been classified before. One is an action on a projective space with fixed boundary, the other one is an action on a hyperquadric of corank $r \geqslant 2$, which can be simply recovered from an action on a hyperquadric of corank one as follows.

Definition 1.3. *Let Q be a hyperquadric of corank one in $\boldsymbol{P} = \mathbb{P}V$ with an additive action induced by $\rho : \mathbb{G}_a^n \mapsto GL(V)$. Choose an element α in the open orbit O. For any $r \geqslant 1$, viewing $\boldsymbol{P}$ as a subspace of $\boldsymbol{P'} = \mathbb{P}^{n+r}$ of codimension r and write the coordinate of $\boldsymbol{P}$ and $\boldsymbol{P'}$ to be $[v] = [x_0, x_1 : ... : x_n]$ and $[v, z] = [v : z_1 : ... : z_r]$ respectively, where $\alpha = [1 : 0 : ... : 0]$. Let $\boldsymbol{L} = \{v = 0\} \subseteq \boldsymbol{P'}$ and $\widetilde{Q}$ be the projective cone over Q with vertex being $\boldsymbol{L}$. Then we extend the action on Q to $\widetilde{Q}$ as follows.*

Write $\mathbb{G}_a^{n+r} = \mathbb{G}_a^n \times \mathbb{G}_a^r = \{(g, h) : g \in \mathbb{G}_a^n, h \in \mathbb{G}_a^r\}$, then the action $(\mathbb{G}_a^{n+r}, \widetilde{Q})$ is defined to be:

$$\begin{aligned} \mathbb{G}_a^{n+r} \times \widetilde{Q} &\mapsto \widetilde{Q} \\ (g, h) \times [v : z] &\mapsto [\rho(g)(v) : z + x_0 \cdot h] \end{aligned}$$

If we extend the action using another element $\alpha' \in O$, then the induced action on $\widetilde{Q}$ is equivalent to the previous action through an linear isomorphism ϕ of P' such that $\phi(P) = P, \phi(\alpha) = \alpha'$ and $\phi_{|_L} = id_L$. Hence the definition of the extended action on $\widetilde{Q}$ is unique up to equivalences. We call the extended action is simply recovered from the given action $(\mathbb{G}_a^n, Q)$.

Remark 1.4. *Geometrically the action on $\widetilde{Q}$ is extended by the action on Q through an action on a projective space with fixed boundary. Note that $\widetilde{Q}$ is contained in the linear span $< L_\alpha, D >$, where L_α is the cone over $\boldsymbol{L}$ with the vertex being α , D is the boundary $Q \backslash O$. Hence the action of $\mathbb{G}_a^{n+r} = \mathbb{G}_a^n \times \mathbb{G}_a^r$ on $\widetilde{Q}$ is determined by its action on L_α and D, which is rather simple: the action of $\mathbb{G}_a^n$ on L_α and the action of $\mathbb{G}_a^r$ on D are both trival while the action of $\mathbb{G}_a^r$ on L_α is an additive action on the projective space with fixed boundary.*

Now to recover a given action by a simpler action, we introduce an operation for any given additive action on a hyperquadric with unfixed singularities or an action on a projective space with unfixed boundary. We start with the following definition.

Definition 1.5. *Let X in $\mathbb{P}^m$ be a hyperquadric or a projective space with an additive action, O being the open orbit.*

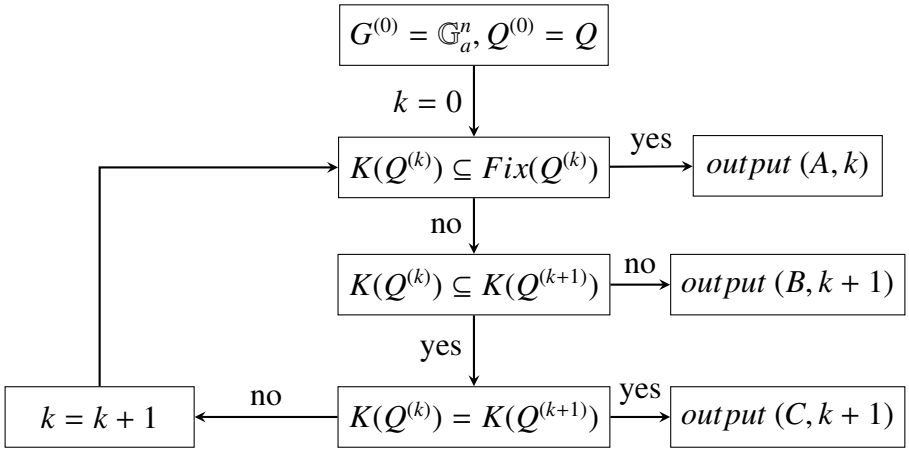
$$K(X) = \begin{cases} Sing(X) & \text{if } X \text{ is a hyperquadric} \\ X \backslash O & \text{if } X \text{ is a projective space} \end{cases}$$

Theorem 1.6. *For an additive action on X in $\mathbb{P}^m$, where X is either a hyperquadric or a projective space with open orbit O such that $K(X) \nsubseteq Fix(X)$. Choose $x_0 \in O$. Let $G^{(1)} = \cap_{x \in K(X)} G_x$ and let $\boldsymbol{L}^{(1)}$ be the linear span of $G^{(1)} \cdot x_0$, then:*
(i) $\boldsymbol{L}^{(1)} \subsetneq \mathbb{P}^m$.
(ii) $\boldsymbol{L}^{(1)}$ is $G^{(1)}$-invariant and the action of $G^{(1)}$ on $\boldsymbol{L}^{(1)}$ induces an additive action on $Q^{(1)} = \overline{G^{(1)} \cdot x_0} \subseteq \boldsymbol{L}^{(1)}$ with the open orbit $O^{(1)} = G^{(1)} \cdot x_0$, where $Q^{(1)}$ is either a non-degenerate hyperquadric or the whole projective space $\boldsymbol{L}^{(1)}$.

We furtherly define when such an operation is effective for our classification.

Definition 1.7. *Let Q be a hyperquadric with an additive action such that $Sing(X) \nsubseteq Fix(X)$, we say the operation obtained in Theorem 1.6: $(\mathbb{G}_a^n, Q, \mathbb{P}^{n+1}) \mapsto (G^{(1)}, Q^{(1)}, \boldsymbol{L}^{(1)})$ is effective if $K(Q) \subsetneq K(Q^{(1)})$.*

Starting from a given additive action on the hyperquadric Q with unfixed singularities, the operation defined in Theorem 1.6 and the effective condition in Definition 1.7 give a procedure of reducing the present action to a lower dimensional one, which has to terminate as the dimension of the underlying projective space decreases strictly by Theorem 1.6 (i). The procedure ends in three different ways, which we call Type A, Type B and Type C. We use the following flow chart to represent the procedure.



where for each k, if $K(Q^{(k)}) \nsubseteq Fix(Q^{(k)})$, let $(G^{(k)}, Q^{(k)}) \mapsto (G^{(k+1)}, Q^{(k+1)})$ be the operation obtained in Theorem 1.6.

We use $(x, t, G^{(t)}, Q^{(t)})$ to represent the final output of the flow chart, where (x, t) is the output of the flow chart and $(G^{(t)}, Q^{(t)})$ is the corresponding action.

In the case of corank two, the following theorem shows that the flow chart conversely gives the explicit process of recovering and together with $l(G_a^n, Q)$ the final output determines the action up to equivalences.

Theorem 1.8. *Let Q be an hyperquadric of corank two with an additive action, assume the action has unfixed singularities and $dim(Q) \geqslant 5$, let $(x, t, G^{(t)}, Q^{(t)})$ be the final output of the flow chart above. Then:*

(i) $(G^{(t)}, Q^{(t)})$ is either an action on a projective space with fixed boundary or an action on a hyperquadric given in Definition 1.3.

(ii) $l(\mathbb{G}_a^n, X) \leqslant 3$ and $codim(Q^{(k+1)}, Q^{(k)}) = 1$, for any $k \leqslant t - 1$.

(iii) if $(\mathbb{G}_a^n, \widetilde{Q})$ is another additive action on the hyperquadric of corank two $\widetilde{Q}$ with unfixed singularities and $dim(\widetilde{Q}) \geqslant 5$, let $(x', t', \widetilde{G}^{(t')}, \widetilde{Q}^{(t')})$ be the final output of the flow chart, then $(\mathbb{G}_a^n, Q)$ is equivalent to $(\mathbb{G}_a^n, \widetilde{Q})$ if and only if $l(\mathbb{G}_a^n, Q) = l(\mathbb{G}_a^n, Q'), x = x', t = t'$ and $(G^{(t)}, Q^{(t)})$ is equivalent to $(\widetilde{G}^{(t')}, \widetilde{Q}^{(t')})$.

Combining Remark 1.4 with classification of actions on hyperquadrics of corank one, we can determine the output action $(G^{(t)}, Q^{(t)})$ explicitly. Then by Theorem 1.8, we can give classification of additive actions on hyperquadrics of corank two with unfixed singularities in terms of the final output of the flow chart.

Theorem 1.9. *Let Q be a hyperquadric of corank two, then additive action on Q with unfixed singularities has equivalence type as follows:*

(a) $dim(Q) \geqslant 5$. Let the final output in the flow chart be $(x, t, G^{(t)}, Q^{(t)})$ then we separate it into 8 different types with respect to the value of x, t and whether $Q^{(t)}$ is a projective space or a hyperquadric:

(a.1) Type x_0: if $x \in \{B, C\}$ and $t = 1$.

(a.2) Type x_1: if $x \in \{A, B, C\}$, $t \geqslant 2$ when $x \in \{B, C\}$ and $Q^{(t)}$ is a projective space.

(a.3) Type x_2: if $x \in \{A, B, C\}$, $t \geqslant 2$ when $x \in \{B, C\}$ and $Q^{(t)}$ is a hyperquadric.

(b) $dim(Q) \leqslant 4$: there are 14 different types.

Remark 1.10. *Explicit classification result of each type will be given in Proposition 4.3,4.5,4.7 and Section 4.2 in terms of the algebraic structure of finite dimensional local algebras.*

The simplest types are Type B_0 and Type C_0, i.e., Type x_0 for $x \in \{B, C\}$. They can be directly recovered from an additive action on a hyperquadric of corank one. Here we describe actions of Type B_0 as an example.

Example 1.11. *Let Q be a hyperquadric of corank two in $\mathbb{P}^{n+1} = \mathbb{P}V$ with an additive action, assume $dim(Q) \geqslant 5$ and $Sing(Q) \nsubseteq Fix(Q)$, consider $(G^{(1)}, Q^{(1)}, \boldsymbol{L}^{(1)})$ obtained in Theorem 1.6. If it is of Type B_0 then:*

(i) $Q^{(1)}$ is a hyperquadric of corank one in $\boldsymbol{L}^{(1)}$.

(ii) choose any α in the open orbit O and any $\alpha' \in Sing(Q)\backslash Fix(Q)$ there exist suitable coordinate $\{x_0, x_1, x_2, ..., x_{n-1}, y_0, y_1\}$ of $\mathbb{P}^{n+1}$ w.r.t the basis of V, $\alpha_0, \alpha_1, .., \alpha_{n-1}, \beta_0, \beta_1$, such that $\alpha = [\beta_1]$, $\alpha' = [\alpha_1]$, $\boldsymbol{L}^{(1)} = \{x_1 = 0\}$, $Q^{(1)} = \boldsymbol{L}^{(1)} \cap Q$ and

$$Q = \{x_2^2 + ... + x_{n-1}^2 + y_0 \cdot y_1 = 0\}.$$

Moreover for $V' = \langle \alpha_0, \alpha_2, .., \alpha_{n-1}, \beta_0, \beta_1 \rangle$ such that $\boldsymbol{L}^{(1)} = \mathbb{P}V'$, let the action $(G^{(1)}, Q^{(1)})$ be given by:

$$G^{(1)} \times \boldsymbol{L}^{(1)} \mapsto \boldsymbol{L}^{(1)}$$
$$(a, [v']) \mapsto [\rho(a)(v')]$$

where $\rho : G^{(1)} \to GL(V')$ is a rational representation of $G^{(1)}$.

Then there is a decomposition of $\mathbb{G}_a^n = G^{(1)} \oplus \mathbb{G}_a$ such that if we write $a = (a_0, a_2, .., a_{n-1}) \in G^{(1)}$, $s \in G_a$, $v = (x_0, x_1, .., x_{n-1}, y_0, y_1) \in V$ and $v' = (x_0, 0, .., x_{n-1}, y_0, y_1) \in V'$ then the action $(\mathbb{G}_a^n, Q)$ is given by:

$$\mathbb{G}_a^n \times \mathbb{P}^{n+1} \mapsto \mathbb{P}^{n+1}$$
$$((a, s) \times [v]) \mapsto [v'']$$

where $v'' = \rho(a)(v') + (\frac{s^2 y_1}{2} + s x_1) \cdot \alpha_0 + (s y_1 + x_1) \cdot \alpha_1$.

1.2. Notation and conventions

Throughout the article, in a given finite dimensional local algebra R, we use $\alpha \cdot \beta$ to represent multiplication between two elements in R, where α can also be taken as a scalar in $\mathbb{K}$. Furthermore we define the following:

(a) if $\alpha \in R$, $V \subseteq R$, then $\alpha \cdot V \doteq \{\alpha \cdot \beta : \beta \in V\}$

(b) if $V, V' \subseteq R$, then $V \cdot V' \doteq \{\sum_{i=1}^{n} \alpha_i \cdot \alpha_i' : n \in \mathbb{N}, \alpha_i \in V, \alpha_i' \in V'\}$.

1.3. Outline of the classification

Given an additive action on a hyperquadric Q in $\mathbb{P}^{n+1}$, there is an $(n+2)$-dimensional local algebra R with a hyperplane W of the maximal ideal $\mathfrak{m}$ and a bilinear form F on R such that

$$\mathbb{P}^{n+1} = \mathbb{P}(R), Q = \mathbb{P}(\{r \in R : F(r,r) = 0\}),$$

and if we choose a basis of W, $w_1, ..., w_n$, then the action is given by (up to equivalences):

$$\mathbb{G}_a^n \times R \mapsto R$$
$$((a_1, a_2, ..., a_n), r) \mapsto r \cdot exp(a_1 w_1 + ... + a_n w_n).$$

Hence to classify additive actions is equivalent to classify algebraic structures of the triple (R, W, F). Note that $Sing(Q) = \mathbb{P}(Ker(F))$, furthermore we show that $Ker(F) \subseteq W$ and if we choose a basis of $Ker(F)$, $\mu_1, .., \mu_l$, and choose any $b_0 \in \mathfrak{m}^2 \backslash W$ then we can represent the multiplications of elements in $\mathfrak{m}$ as follows:

$$a \cdot a' = F(a, a')b_0 + V_1(a, a')\mu_1 + V_2(a, a')\mu_2 + ... + V_l(a, a')\mu_l, \tag{1.1}$$

for any $a, a' \in \mathfrak{m}$, where $\{V_i : 1 \leqslant i \leqslant l\}$ is a set of symmetric bilinear forms on R. When the corank equals one we have $l = 1$ and $\mu_1 \cdot \mathfrak{m} = 0$, also one can choose b_0 s.t. $b_0 \cdot \mathfrak{m} = 0$. Hence if we extend μ_1 to a basis of W namely $\mu_1, e_1, .., e_{n-1}$, s.t. $F(e_i, e_j) = \delta_{i,j}$ then the multiplication in $\mathfrak{m}$ depends on the matrix $\Lambda = (V_1(e_i, e_j))$. Also note that an orthogonal tranformation of the basis e_i's with respect to F leads to an orthogonal transformation of the matrix Λ, hence the classification of the action of corank one is reduced to normalize a symmetric bilinear forms under orthogonal transforamtions (cf. [2, Proposition 7] and [5, Chapter XI, §3]).

When the corank equals two, we can still choose b_0 s.t. $b_0 \cdot \mathfrak{m} = 0$. We note that in this case the condition $Sing(Q) \nsubseteq Fix(Q)$ enables us to use the idea of the case of corank one.

Firstly we show that $Sing(Q) \nsubseteq Fix(Q)$ is equivalent to $Ker(F) \cdot W \neq 0$. And if $Ker(F) \cdot W \neq 0$, then we can furtherly define a hyperplane $V^{(1)}$ in W:

$$V^{(1)} = \{\alpha \in W : \alpha \cdot Ker(F) = 0\}$$
$$V_{(1)} = Ker(F_{|_{V^{(1)}}})$$

By using the correspondence between additive actions and finite dimensional local algebras we show that if $V_{(1)} = V^{(1)}$ then the action $(G^{(1)}, Q^{(1)}, \mathbf{L}^{(1)})$ obtained in Theorem 1.6 is an action on a projective space and it corresponds to $(R^{(1)}, V^{(1)})$, where $R^{(1)} = V^{(1)} \oplus \langle 1_R \rangle$. If $V^{(1)} \neq V_{(1)}$ we show that the action $(G^{(1)}, Q^{(1)}, \mathbf{L}^{(1)})$ corresponds to the triple $(R^{(1)}, V^{(1)}, F^{(1)})$ where $R^{(1)} = V^{(1)} \oplus \langle b_0, 1_R \rangle$, $F^{(1)} = F_{|_{V^{(1)}}}$ and $Q^{(1)}$ is a hyperquadric.

Then our first key step is to show that after choosing suitable $\mu_1 \in Ker(F)$, we have

$$V^{(1)} \cdot W \subseteq \langle \mu_1, b_0 \rangle,$$

which shows that the bilinear form V_2 defined in (1.1) vanishes on $V^{(1)}$. For the obtained subspaces $V_{(1)} \subseteq V^{(1)} \subseteq W$, our second key step is that if $Ker(F) \nsubseteq V^{(1)}$ (resp. $V_{(1)} = Ker(F)$), which geometrically means $K(Q) \nsubseteq K(Q^{(1)})$ (resp. $K(Q) = K(Q^{(1)})$), then we can directly normalize the multiplications in $\mathfrak{m}$. As a result, we recover action $(\mathbb{G}_a^n, Q)$ from the action $(G^{(1)}, Q^{(1)})$, which is an action given in Definition 1.3. This corresponds to an output of Type B_0 or C_0 in the flow chart.

Otherwise we show that $codim(Ker(F), V_{(1)}) = 1$ and we furtherly consider the new action $(G^{(1)}, Q^{(1)}, L^{(1)})$, for which we separate into two more subcases.

(1) If $V_{(1)} \cdot V^{(1)} = 0$ then we are in a situation similar to the case of corank one: $V_2 = ... = V_l = 0$, $V_{(1)} \cdot V^{(1)} = 0$, $b_0 \cdot V^{(1)} = 0$. Hence we can normalize the multiplications in $\mathfrak{m}^{(1)} = V^{(1)} \oplus \langle b_0 \rangle$. As a result, we recover action $(\mathbb{G}_a^n, Q)$ from the action $(G^{(1)}, Q^{(1)})$, which is an action given in Definition 1.3. This corresponds to an output $(A, 1)$ in the flow chart.

(2) If $V_{(1)} \cdot V^{(1)} \neq 0$ then we are in a situation similar to $Ker(F) \cdot W \neq 0$, except that in this case the action is on a hyperquadric of corank three. On the other hand, we have $Ker(F) \cdot V^{(1)} = b_0 \cdot V^{(1)} = 0$ and $V_2 = V_3 = 0$, hence the uncertainity of multiplications in $\mathfrak{m}^{(1)}$ is still one dimensional. For this reason we furtherly define

$$V^{(2)} = \{\alpha \in V^{(1)} : \alpha \cdot V_{(1)} = 0\}$$
$$V_{(2)} = Ker(F_{|_{V^{(1)}}})$$

This corresponds to a new action $(G^{(2)}, Q^{(2)}, L^{(2)})$ in the flow chart, with $L^{(2)} \subsetneqq L^{(1)}$. Similarly we show that if $V_{(1)} \nsubseteq V_{(2)}$ or $V_{(1)} = V_{(2)}$ or $V_{(1)} \subsetneq V_{(2)}$ with $V_{(2)} \cdot V^{(2)} = 0$, then we can already normalize the multiplications in $\mathfrak{m}^{(1)}$. Otherwise we find $V_{(2)} \cdot V^{(2)} \neq 0$ then as before we can furtherly define $(V^{(3)}, V_{(3)})$ with $V^{(3)} \subsetneqq V^{(2)}$, and check whether it satisfies the conditions to be normalized. The discussion will be continued as above until we find that the present action satisfies the condition to be normalized i.e., to obtain an output in the flow chart, the procedure has to terminate as the dimension of $V^{(i)}$ decreases strictly. As a result we show that the output action is either an action on a projective space with fixed boundary or an action given in Definition 1.3. Moreover we obtain a chain of subspaces in W corresponding to the flow chart:

$$Ker(F) \subseteq V_{(1)} \subseteq ...V_{(s)} \subseteq V^{(s)} \subseteq ... \subseteq V^{(1)} \subseteq W,$$

where $s = t$ if $x = A$, $s = t - 1$ if $x = B$ or $x = C$.

Then it remains to normalize the multiplications between elements outside $V^{(s)}$. This is completed through more technical operations shown in Lemmas 4.2, 4.4 and 4.6. After the normalization of the structure of R, we show the uniqueness of the normalized structure up to equivalences, which proves Theorem 1.8 (iii). And the normalized structure of (R, W, F) gives the explicit result of our classification of actions when $dim(Q) \geqslant 5$. Finally when $dim(Q) \leqslant 4$ we give the classification case by case.

The article is organized as follows: in Section 2, we recall the correspondence between additive actions and finite dimensional local algebras; in Section 3 we first prove Theorem 1.6 to obtain the action $(G^{(1)}, Q^{(1)})$. Then we show that the existence of unfixed singularities will lead to $V^{(1)} \cdot W \subseteq \langle \mu_1, b_0 \rangle$ and we normalize the algebraic structure of (R, W, F) when the type is B_0 or C_0; in Section 4, we first normalize the structure of R, then we show the uniqueness of the normalized structure, which gives proof of Theorem 1.8 and also gives explicit result of our classification result shown in Theorem 1.9.

2. Additive actions and finite dimensional local algebras

As mentioned before an additive action $(\mathbb{G}_a^n, X, \mathbb{P}^m)$ is induced by a faithful rational linear representation $\rho : \mathbb{G}_a^n \to GL_{m+1}(\mathbb{K})$. Furtherly if X is non-degenerate in $\mathbb{P}^m$ then ρ becomes a cyclic representation i.e., $\langle \rho(g) \cdot v : g \in \mathbb{G}_a^n \rangle = \mathbb{K}^{m+1}$ for some nonzero $v \in \mathbb{K}^{m+1}$. Hassett and Tschinkel in [1] gave a complete characterization of such representations.

Theorem 2.1 ([1],Theorem 2.14)**.** *There is 1-1 correspondence between the following two classes: (1) equivalence classes of faithful rational cyclic representation $\rho : \mathbb{G}_a^n \mapsto GL_{m+1}(\mathbb{K})$; (2) isomorphism classes of (R, W), where R is a local (m+1)-dimensional algebra with maximal ideal $\mathfrak{m}$ and W is an n-dimensional subspace of $\mathfrak{m}$ that generates R as an alegbra with unit.*

Remark 2.2. *Under this correspondence a representation of $\mathbb{G}_a^n$ on $\mathbb{K}^{m+1}$ can always be viewed as an action on a local algebra $R \cong \mathbb{K}^{m+1}$. Moreover if we choose a $\mathbb{K}$-basis of W: $W = \langle w_1, ..., w_n \rangle$ then we can write down the action explicitly:*

$$\mathbb{G}_a^n \times R \mapsto R$$
$$(g_1, g_2, ..., g_n) \times r \mapsto r \cdot exp(g_1 w_1 + ... + g_n w_n).$$

And the induced action of the Lie algebra $\mathfrak{g}(G_a^n) = G_a^n$ on R is:

$$\mathfrak{g} \times R \mapsto R$$
$$(g_1, g_2, ..., g_n) \times r \mapsto r \cdot (g_1 w_1 + ... + g_n w_n),$$

we identify $\mathfrak{g} \cong W$ as vector spaces.

Moreover Hassett and Tschinkel proved in [1] and later Arzhantsev and Popovskiy proved in [2] the following 1-1 correspondences.

Theorem 2.3 ([1], Proposition 2.15)**.** *There's a bijection between the following two classes:*
(1) equivalence classes of additive actions on $\mathbb{P}^n$;
(2) equivalence classes of (n+1)-dimensional local commutative algebras.

Under the correspondence the action is given as in Remark 2.2, where the subspace W is the maximal ideal of the local algebra.

Theorem 2.4 ([2], Proposition 3)**.** *There's a bijection between the following two classes:*
(1) equivalence classes of additive actions on hypersurfaces H in $\mathbb{P}^{n+1}$ of degree at least two;
(2) equivalence classes of (R, W), where R is a local (n+2)-dimensional algebra with maximal ideal $\mathfrak{m}$ and W is a hyperplane of $\mathfrak{m}$ that generates the algebra R with unit.

Then in [2] they furtherly introduced the notion of *invariant multilinear form* for a pair (R, W).

Definition 2.5 ([2], Definition 3)**.** *Let R be a local algebra with maximal ideal $\mathfrak{m}$. An invariant d-linear form on R is a d-linear symmetric map*

$$F : R \times R \times ... \times R \mapsto \mathbb{K}$$

such that $F(1, 1, ..., 1) = 0$, the restriction of F to $\mathfrak{m} \times ... \times \mathfrak{m}$ is nonzero, and there exist a hyperplane W in $\mathfrak{m}$ which generates the algebra R and such that:

$$F(ab_1, b_2, ..., b_d) + F(b_1, ab_2, ..., b_d) + ... + F(b_1, b_2, ..., ab_d) = 0 \ \ \forall a \in W, b_1, ..., b_d \in R.$$

We say F is irreducible if it can not be represented as product of two lower dimensional forms.

Now given an additive action on a hypersurface $H = \{f(x_0, .., x_{n+1}) = 0\} \subseteq \mathbb{P}^{n+1}$, then under the correspondence in Theorem 2.4 the polarization F of f is an invariant multilinear form on (R, W), which induces the following more explicit correspondence.

Theorem 2.6 ([2], Theorem 2)**.** *There is a bijection between the following two classes:*
(1) equivalence classes of additive actions on hypersurface $H \subseteq \mathbb{P}^{n+1}$ of degree at least two;
(2) equivalence classes of (R,F), where R is a local algebra of dimension $n + 2$ and F is an irreducible invariant d-linear form on R up to a scalar.

Under the correspondence $\mathbb{P}^{n+1} = \mathbb{P}(R)$, $H = \mathbb{P}(\{r \in R : F(r, r, ..., r) = 0\})$, and the action on $\mathbb{P}^{n+1}$ corresponds to the action on R as shown in Remark 2.2, with the open orbit $O = \mathbb{P}(G_a^n \cdot 1_R)$. Moreover F is determined by (R, W) as follows.

Lemma 2.7 ([2], Lemma 1)**.** *Fix a $\mathbb{K}$-linear automorphism $\mathfrak{m}/W \cong \mathbb{K}$ with the projection $\pi : \mathfrak{m} \to \mathfrak{m}/W \cong \mathbb{K}$ then the corresponding invariant linear form is (up to a scalar):*

$$F_W(b_1, ..., b_d) = (-1)^k k!(d - k - 1)!\pi(b_1...b_d),$$

where k is the number of units among $b_1, ..., b_d$ and for $k = d$ let $F_W(1, 1, ..., 1) = 0$.

In the following we focus on additive actions on hyperquadrics, i.e., $d = 2$ and we use a triple (R, W, F) to represent an additive action on a hyperquadric Q where F is the bilinear form given in Theorem 2.6. By Lemma 2.7 we have the following.

Lemma 2.8. *Fix $b_0 \in \mathfrak{m}\backslash W$ and the projection $y_0 : R \to \mathbb{K}$ s.t. $y_0(1_R) = y_0(W) = 0$ and $y_0(b_0) = 1$. Then for $a, a' \in \mathfrak{m}$ and for $r \in W$ we have:*

$$F(a, a') = y_0(aa').$$
$$F(1, 1) = F(1, r) = 0,\ F(1, b_0) = -1.$$

As F is the polarization of the homogenous polynomial defining Q we have $Sing(Q) = \mathbb{P}(Ker(F))$. Moreover we have the following.

Lemma 2.9. *$Ker(F) \subseteq W$ and $Ker(F|_W) = Ker(F)$.*

Proof. By [3, Theorem 5.1], the degree of the hypersurface is the maximal exponent d such that $\mathfrak{m}^d \nsubseteq W$, for $d = 2$ we have $\mathfrak{m}^2 \nsubseteq W$ and $\mathfrak{m}^3 \subseteq W$. Hence we can take a $b_0 \in \mathfrak{m}^2\backslash W$ and the projection y_0 defined in Lemma 2.8.

For any $l \in Ker(F)$, write $l = a + tb_0 + l_W$ for some $a, t \in \mathbb{K}$ and $l_W \in W$, then $t = -F(1, l) = 0$ by Lemma 2.8. And

$$0 = F(b_0, l) = -a + F(b_0, l_W) = -a + y_0(b_0 l_W) = -a$$

as $b_0 l_W \in \mathfrak{m}^3 \subseteq W$, concluding that $l = l_W \in W$.

For any $l \in Ker(F|_W)$, then $F(1, l) = 0$ as $l \in W$ and $F(l, b_0) = y_0(lb_0) = 0$ as $b_0 l \in \mathfrak{m}^3 \subseteq W$ and $y_0(W) = 0$, concluding that $l \in Ker(F)$. □

Lemma 2.10. *For any $b_0 \in \mathfrak{m}^2\backslash W$, $\mathfrak{m}^2 \subseteq Ker(F) \oplus \langle b_0 \rangle$.*

Proof. Firstly choose a $b_0 \in \mathfrak{m}^2 \backslash W$. Given any $a, a' \in \mathfrak{m}$ we have

$$aa' = y_0(aa') \cdot b_0 + (aa')_W,$$

where y_0 is the projection defined in Lemma 2.8. Now for any $r \in \mathfrak{m}$ then

$$r \cdot (aa')_W = r \cdot (aa' - y_0(aa') \cdot b_0) \in \mathfrak{m}^3 \subseteq W,$$

as $b_0 \in \mathfrak{m}^2$. Hence by Lemma 2.8

$$F((aa')_W, r) = y_0((aa')_W \cdot r) = 0,$$

as $r \cdot (aa')_W \in W$. Note that $F(1, (aa')_W) = 0$ since $F(1, W) = 0$. It follows that $(aa')_W \in Ker(F)$, concluding the proof.

□

From above lemmas we can thus choose a $b_0 \in \mathfrak{m}^2 \backslash W$ such that $F(1, b_0) = -1$ and $\mathfrak{m}^2 \subseteq Ker(F) \oplus \langle b_0 \rangle$. Moreover if we fix a basis of $Ker(F) = \langle \mu_1, ..., \mu_l \rangle$ then we can represent the multiplications of elements in $\mathfrak{m}$ as follows.

$$aa' = F(a, a')b_0 + V_1(a, a')\mu_1 + V_2(a, a')\mu_2 + ... + V_l(a, a')\mu_l. \tag{2.1}$$

3. Unfixed singularities and vanishing of bilinear forms

In this section, we first prove Theorem 1.6. Then we show that in the case of corank two the existence of unfixed singularities leads to $V^{(1)} \cdot W \subseteq \langle b_0, \mu_1 \rangle$. Finally we show that if $K(Q) \nsubseteq K(Q^{(1)})$ or $K(Q) = K(Q^{(1)})$ then we can already normalize the algebraic structure of (R, W, F).

3.1. Operation for actions with unfixed singularities

We first give an algebraic characterization of related concepts. Given an additive action on hyperquadric Q represented by (R, W, F), recall that $Sing(Q) = \mathbb{P}(Ker(F))$, $G^{(1)} = \cap_{x \in K(X)} G_x$, $V^{(1)} = \{r' \in W \,|\, r' \cdot Ker(F) = 0\}$ and $V_{(1)} = Ker(F_{|_{V^{(1)}}})$. We furtherly define $S' = \{r \in R \,|\, r \cdot W = 0\}$. Then we have the following.

Proposition 3.1. *(i)* $Fix(Q) = \mathbb{P}(S')$.

(ii) $G^{(1)} = exp(\mathfrak{g}^{(1)})$*, where* $\mathfrak{g}^{(1)} \subseteq \mathfrak{g}(\mathbb{G}_a^n)$ *is a Lie subalgebra and* $\mathfrak{g}^{(1)} \cong V^{(1)}$ *under the identification* $\mathfrak{g}(\mathbb{G}_a^n) \cong W$ *given in Remark 2.2.*

(iii) $Ker(F) \cdot \mathfrak{m} \neq 0$ *if and only if* $V^{(1)} \neq W$ *if and only if* $Sing(Q) \nsubseteq Fix(Q)$.

Proof. (i) By Remark 2.2, the action of $\mathfrak{g} = \mathfrak{g}(\mathbb{G}_a^n)$ on R is given by multiplying elements of W to R. Hence we have:

$$S' = \{r \in R : r \cdot W = 0\} = \{r \in R : \mathfrak{g} \cdot r = 0\}$$

Also by Remark 2.2, the action of $\mathbb{G}_a^n$ on $\mathbb{P}^{n+1}$ is identified with the action on R. Hence we have:

$$Fix(Q) = \mathbb{P}(\{r \in R : g \cdot r = r, \ \forall g \in G_a^n\}) = \mathbb{P}(\{r \in R : x \cdot r = 0, \ \forall x \in \mathfrak{g}\}) = \mathbb{P}(S').$$

(ii) Similarly for the isotropy group $G^{(1)}$ of $Sing(Q)$ we have:

$$\begin{aligned} G^{(1)} &= \{g \in \mathbb{G}_a^n : g \cdot x = x,\ \forall x \in Sing(Q)\} = \{g \in \mathbb{G}_a^n : g \cdot r = r,\ \forall r \in Ker(F)\} \\ &= exp(\{x \in \mathfrak{g} : x \cdot r = 0,\ \forall r \in Ker(F)\}). \end{aligned}$$

Then by Remark 2.2, under the identification of $\mathfrak{g} \cong W$, we have $\{x \in \mathfrak{g} : x \cdot r = 0,\ \forall r \in Ker(F)\} \cong \{r' \in W : r' \cdot r = 0,\ \forall r \in Ker(F)\} = V^{(1)}$.

(iii) The first equivalence follows from the definition of $V^{(1)}$ and the fact that $\mathfrak{m}$ can be generated by W. For the second equivalence, we have $Sing(Q) \subseteq Fix(Q)$ if and only if $\mathbb{G}_a^n = G^{(1)}$ if and only if $\mathfrak{g} = \mathfrak{g}^{(1)}$ if and only if $V^{(1)} = W$, where the last equivalence follows from (ii).

□

Next we introduce a lemma to describe multiplications between elements in $\mathfrak{m}$ and elements in $Ker(F)$.

Lemma 3.2. *(i) $Ker(F) \cdot \mathfrak{m} \subseteq Ker(F)$ and there exist a $\mathbb{K}$-basis of $Ker(F)$, $\mu_1, \mu_2, ..., \mu_l$, such that $\mu_i \cdot \mathfrak{m} \subseteq \langle \mu_1, ..., \mu_{i-1} \rangle$. (ii) $V^{(1)} \neq 0$.*

Proof. (i) First note that $Sing(Q)$ is $\mathbb{G}_a^n$-stable. Then by Theorem 2.6 and $\mathbb{P}(Ker(F)) = Sing(Q)$, $Ker(F)$ is a $\mathbb{G}_a^n$-invariant subspace, hence by Remark 2.2 and the fact that $\mathfrak{m}$ is generated by W we conclude that $Ker(F) \cdot \mathfrak{m} \subseteq Ker(F)$

Now we choose a $\mathbb{K}$-basis of $\mathfrak{m}$ to be S_0, then for any $c \in S_0$ we can define a linear map induced by multiplications:

$$\begin{aligned} \phi_c : Ker(F) &\mapsto Ker(F) \\ r &\mapsto c \cdot r \end{aligned}$$

Note that R is a commutative Artinian local ring, hence $\{\phi_c : c \in S_0\}$ is a set of commutative nilpotent linear maps on $Ker(F)$. Therefore we can choose a basis of $Ker(F) = \langle \mu_1, ..., \mu_l \rangle$ s.t. $\phi_c(\mu_i) \subseteq \langle \mu_1, ..., \mu_{i-1} \rangle$, for any $c \in S_0$. As S_0 is a basis of $\mathfrak{m}$, (i) is proved.

(ii) If $Ker(F) = 0$ then $V^{(1)} = W \neq 0$ from the definition of $V^{(1)}$. If $Ker(F) \neq 0$ then by (i) there exist a $\mu_1 \neq 0$ s.t. $\mu_1 \cdot \mathfrak{m} = 0$ and hence $\mu_1 \in V^{(1)}$, concluding that $V^{(1)} \neq 0$. □

Now we use the correspondences given in Theorem 2.3 and Theorem 2.4 to obtain the operation described in Theorem 1.6.

Proof of Theorem 1.6. Firstly note that $Q^{(1)}$ is a non-degenerate variety in $\mathbf{L}^{(1)}$, hence it suffices to prove that there exist a linear space $\mathbf{L}^{(1)}$ satisfying Theorem 1.6 (i) and (ii). In the following we assume $dim(V^{(1)}) = m$ for some $m \leqslant n - 1$.

(a) If X is a hyperquadric, then we represent the action by (R, W, F) with $x_0 \in O$ s.t. $x_0 = [1_R]$ and define $(V^{(1)}, V_{(1)})$ as in Proposition 3.1. Also by Lemma 3.2 we have $0 \neq V^{(1)} \subsetneq W$.

Case 1. $V^{(1)} \cdot V^{(1)} \subseteq V^{(1)}$, then the induced action is an additive action on a projective space. From Lemma 2.8 we conclude that $V^{(1)} = V_{(1)}$.

In this case $R^{(1)} = V^{(1)} \oplus \langle 1_R \rangle$ is a well-defined subring of R. Furthermore it can be easily seen that $R^{(1)}$ is a finite dimensional $\mathbb{K}$-local algebra with maximal ideal $\mathfrak{m}^{(1)} = V^{(1)}$. Then by HT-correspondence (Theorem 2.3), the pair $(R^{(1)}, V^{(1)})$ gives an additive action of $\mathbb{G}_a^m$ on the projective space $\mathbb{P}(R^{(1)})$ with

open orbit $\mathbb{G}_a^m \cdot [1_R]$. On the other hand, by Remark 2.2, the action is given through identifying $\mathfrak{g}(\mathbb{G}_a^m)$ with $V^{(1)}$, hence from Proposition 3.1.(ii) we conclude that up to equivalences the corresponding action is exactly induced by the action of $G^{(1)}$ on $R^{(1)}$. Thus the action of $G^{(1)}$ on $\mathbb{P}(R^{(1)})$ is an additive action on the projective space with open orbit $G^{(1)} \cdot [1_R]$, and $\mathbb{P}(R^{(1)}) \subsetneq \mathbb{P}(R)$ as $V^{(1)} \subsetneq W$. Above all we have found the subspace $\mathbf{L}^{(1)} = \mathbb{P}(R^{(1)}) = Q^{(1)}$ of $\mathbb{P}^{n+1}$ satisfying Theorem 1.6 (i) and (ii):

$$\begin{array}{ccc} G^{(1)} \times \mathbb{P}(R^{(1)}) & \longrightarrow & \mathbb{P}(R^{(1)}) \supseteq G^{(1)} \cdot [1_R] \\ \downarrow & & \downarrow \\ \mathbb{G}_a^n \times \mathbb{P}^{n+1} & \longrightarrow & \mathbb{P}^{n+1} \supseteq \mathbb{G}_a^n \cdot [1_R] \end{array}$$

Case 2. $V^{(1)} \cdot V^{(1)} \nsubseteq V^{(1)}$, then the induced action is an additive action on a hyperquadric. From Lemma 2.8 we conclude that $V^{(1)} \neq V_{(1)}$.

First we can choose a suitable $b_0 \in \mathfrak{m}^2 \backslash W$ s.t. $V^{(1)} \cdot V^{(1)} \subseteq V^{(1)} \oplus \langle b_0 \rangle$ and $b_0 \cdot Ker(F) = 0$. In fact, from $V^{(1)} \neq V_{(1)}$, there exist $a, a' \in V^{(1)}$ with $F(a, a') = 1$. Now we define $b_0 = a \cdot a'$ then $b_0 \in \mathfrak{m}^2 \backslash W$ and $b_0 \cdot Ker(F) = 0$ as $a \in V^{(1)}$. Moreover for any $c, c' \in V^{(1)}$:

$$c \cdot c' = y_0(cc') \cdot b_0 + (c \cdot c')_{|_W},$$

hence from $c \cdot c' \cdot Ker(F) = b_0 \cdot Ker(F) = 0$ we have $(c \cdot c')_{|_W} \in V^{(1)}$, concluding that $V^{(1)} \cdot V^{(1)} \subseteq V^{(1)} \oplus \langle b_0 \rangle$.

Now we set $R^{(1)} = V^{(1)} \oplus \langle b_0 \rangle \oplus \langle 1_R \rangle$, $\mathfrak{m}^{(1)} = V^{(1)} \oplus \langle b_0 \rangle$. Then

$$\begin{aligned} &b_0 \in (\mathfrak{m}^{(1)})^2 \nsubseteq V^{(1)}, \\ &b_0 \cdot \mathfrak{m}^{(1)} \subseteq (\mathfrak{m}^{(1)})^3 \subseteq V^{(1)}, \end{aligned}$$

as $(\mathfrak{m}^{(1)})^3 \subseteq \mathfrak{m}^3 \subseteq W$ and $(\mathfrak{m}^{(1)})^3 \cdot Ker(F) = 0$, where $\mathfrak{m}^3 \subseteq W$ follows from [3, Theorem 5.1] and the fact that (R, W, F) reprensents an action on a hyperquadric.

Now it follows that $R^{(1)}$ is a finite dimensional local $\mathbb{K}$-algebra with maximal ideal $\mathfrak{m}^{(1)} = V^{(1)} \oplus \langle b_0 \rangle$, $V^{(1)}$ is a hyperplane of $\mathfrak{m}^{(1)}$ generating the algebra $R^{(1)}$ such that $(\mathfrak{m}^{(1)})^2 \nsubseteq V^{(1)}$ and $(\mathfrak{m}^{(1)})^3 \subseteq V^{(1)}$. Hence by Theorem 2.4 and [3, Theorem 5.1], $(R^{(1)}, \mathfrak{m}^{(1)}, V^{(1)})$ corresponds to an additive action of G_a^m on a hyperquadric $Q^{(1)}$ in $\mathbb{P}(R^{(1)})$ with open orbit $\mathbb{G}_a^m \cdot [1_R]$. Then similar to Case 1, by Remark 2.2 and Proposition 3.1.(ii) we conclude that the corresponding action (up to equivalences) is exactly induced by the action of $G^{(1)}$ on $R^{(1)}$. Thus the action of $G^{(1)}$ on $\mathbb{P}(R^{(1)})$ induces an additive action on a hyperquadric $Q^{(1)}$ with the open orbit $O^{(1)} = G^{(1)} \cdot [1_R]$, and $\mathbb{P}(R^{(1)}) \subsetneq \mathbb{P}(R)$ as $V^{(1)} \subsetneq W$. Moreover in the more explicit correspondence Theorem 2.6 we can easily see the corresponding bilinear form $F^{(1)}$ is just $F_{|_{R^{(1)}}}$.

Now $\mathbb{P}(R^{(1)})$ is already a subspace satisfying Theorem 1.6 (i) and (ii):

$$\begin{array}{ccccc} G^{(1)} \times \mathbb{P}(R^{(1)}) & \longrightarrow & \mathbb{P}(R^{(1)}) \supseteq Q^{(1)} \supseteq O^{(1)} = G^{(1)} \cdot [1_R] \\ \downarrow & & \downarrow \\ \mathbb{G}_a^n \times \mathbb{P}^{n+1} & \longrightarrow & \mathbb{P}^{n+1} \supseteq Q \supseteq O = \mathbb{G}_a^n \cdot [1_R] \end{array}$$

(b) If X is a projective space, following Theorem 2.3, we represent the action $(\mathbb{G}_a^n, \mathbb{P}^n)$ by a pair $(R, \mathfrak{m})$, where $x_0 = [1_R]$. We first show that $K(X) = \mathbb{P}(\mathfrak{m})$. In fact, for any $[\alpha]$ in the open orbit we have

α is invertible by Remark 2.2. Conversely, for any invertible element $r \in R$ we have $dim(\mathbb{G}_a^n \cdot [r]) = dim(\mathfrak{g} \cdot r) = dim(\mathfrak{m} \cdot r) = dim(\mathfrak{m}) = n$, concluding that $[r]$ lies in the open orbit. Now we define $V^{(1)} = \{\alpha \in \mathfrak{m} : \alpha \cdot \mathfrak{m} = 0\}$, then $Fix(X) = \mathbb{P}(V^{(1)})$. Since $K(X) \nsubseteq Fix(X)$ by assumption of Theorem 1.6, we have $V^{(1)} \subsetneq \mathfrak{m}$. Moreover as elements in $\mathfrak{m}$ are nilpotent, we conclude that $V^{(1)} \neq 0$ by a similar discussion as that in Lemma 3.2.

Now we consider $R^{(1)} = V^{(1)} \oplus \langle 1_R \rangle$ then similar to Case 1 of (a), $\mathbb{P}(R^{(1)})$ is $G^{(1)}$-stable and the induced action is an additive action on a projective space with open orbit $G^{(1)} \cdot [1_R]$, and $\mathbb{P}(R^{(1)}) \subsetneq \mathbb{P}(R)$ as $V^{(1)} \subsetneq \mathfrak{m}$. Thus $\mathbb{P}(R^{(1)})$ is already a subspace satisfying Theorem 1.6 (i) and (ii). □

Combining the above proof with Proposition 3.1, we have the following.

Proposition 3.3. *Given an additive action on a hyperquadric Q with unfixed singularities, we represent the operation obtained in Theorem 1.6 by $(R, W, F) \mapsto (R^{(1)}, V^{(1)}, F^{(1)})$, then:*

(i) $Q^{(1)}$ is a projective space if and only if $V^{(1)} \cdot V^{(1)} \subseteq V^{(1)}$ if and only if $V^{(1)} = V_{(1)}$.

(ii) $Sing(Q) \nsubseteq K(Q^{(1)})$ if and only if $Ker(F) \nsubseteq V_{(1)}$, $Sing(Q) = K(Q^{(1)})$ if and only if $Ker(F) = V_{(1)}$.

(iii) the operation is effective if and only if $Ker(F) \subsetneq V_{(1)}$.

Proof. (i) By part (a) in the proof of Theorem 1.6, it suffices to show $V^{(1)} \cdot V^{(1)} \subseteq V^{(1)}$ if $V^{(1)} = V_{(1)}$. In this case, for any $a, a' \in V^{(1)}$ we have $F(a, a') = 0$, hence $aa' \in W$ by Lemma 2.8 and $aa' \cdot Ker(F) = 0$ by the definition of $V^{(1)}$, concluding that $V^{(1)} \cdot V^{(1)} \subseteq V^{(1)}$.

(ii) and (iii). By our definition of effective operation 1.7 and $Sing(Q) = \mathbb{P}(Ker(F))$, it suffices to show $K(Q^{(1)}) = \mathbb{P}(V_{(1)})$. If $Q^{(1)}$ is a projective space then from part (b) in the proof of Theorem 1.6, for the action on $Q^{(1)}$ represented by $(R^{(1)}, \mathfrak{m}^{(1)})$ we have $K(Q^{(1)}) = \mathbb{P}(\mathfrak{m}^{(1)}) = \mathbb{P}(V^{(1)}) = \mathbb{P}(V_{(1)})$ since in this case $\mathfrak{m}^{(1)} = \mathbb{P}(V^{(1)}) = V_{(1)}$ by Case 1 of part (a) in the proof of Theorem 1.6. If $Q^{(1)}$ is a hyperquadric, then $K(Q^{(1)}) = Sing(Q^{(1)}) = \mathbb{P}(Ker(F^{(1)}))$ and $Ker(F^{(1)}) = Ker(F^{(1)}_{|V^{(1)}}))$ by Lemma 2.9. Finally as $F^{(1)} = F_{|R^{(1)}}$, we have $Ker(F^{(1)}) = Ker(F_{|V^{(1)}}) = V_{(1)}$ by the definition of $V_{(1)}$, concluding the proof. □

3.2. Unfixed singularities and vanishing bilinear form

Our main result of this section is the following.

Proposition 3.4. *For an additive action on a hyperquadric Q of corank 2 represented by (R, W, F). If $Sing(Q) \nsubseteq Fix(Q)$ and $dim(Q) \geqslant 5$, then for the operation obtained in Theorem 1.6 we have:*

(i) $codim(Q^{(1)}, Q) = codim(V^{(1)}, W) = 1$.

(ii) there exist $b_0 \in \mathfrak{m}^2 \backslash W$ with $F(1, b_0) = 1$ and a $\mathbb{K}$-basis of $Ker(F)$, μ_1, μ_2, such that:

$$\begin{aligned} b_0 \cdot \mathfrak{m} = \mu_1 \cdot \mathfrak{m} = 0 \\ \mu_2 \cdot \mathfrak{m} \subseteq \langle \mu_1 \rangle \\ V^{(1)} \cdot \mathfrak{m} \subseteq \langle \mu_1, b_0 \rangle \end{aligned}$$

(iii) if the operation is not effective, i.e., $Ker(F) = V_{(1)}$ or $Ker(F) \nsubseteq V_{(1)}$, then we can normalize the algebraic structrue of (R, W, F). (see Lemma 3.6 and Lemma 3.9 for details).

First applying Lemma 3.2 we have the following:

Lemma 3.5. *(i) there exist suitable basis of $Ker(F)$, μ_1, μ_2, s.t. $\mu_1 \cdot \mathfrak{m} = 0$ and $\mu_2 \cdot \mathfrak{m} \subseteq \langle \mu_1 \rangle$. (ii) $codim(V^{(1)}, W) = 1$.*

Proof. (i) Applying Lemma 3.2 when $l = 2$.

(ii) For any $r \in W$ we have $r \cdot \mu_2 = \lambda_r \cdot \mu_1$ for some $\lambda_r \in \mathbb{K}$, this induces a linear form on W:

$$\begin{aligned} \Phi : W &\mapsto \mathbb{K} \\ \alpha &\mapsto \lambda_\alpha \end{aligned}$$

Hence we have $V^{(1)} = Ker(\Phi)$ and $codim(V^{(1)}, W) = 1$. □

From now on we always choose a basis of $Ker(F)$ satisying Lemma 3.5.(i).

We prove Proposition 3.4 through a case-by-case argument on analyzing the relation between $Ker(F)$ and $V_{(1)}$. More precisely, we separate it into the following cases.

1. $Sing(Q) \subseteq K(Q^{(1)})$, i.e., $Ker(F) \subseteq V_{(1)}$. In this case we have nice inclusions between subspaces: $Ker(F) \subseteq V_{(1)} \subseteq V^{(1)} \subseteq W$, for which we furtherly consider two subcases:

(1.a). $Sing(Q) = K(Q^{(1)})$, i.e., $Ker(F) = V_{(1)}$. In this subcase, we can normalize the algebraic structure of (R, W, F).

(1.b). The operation on $(\mathbb{G}_a^n, Q)$ is effective, i.e., $Ker(F) \subsetneq V_{(1)}$. In this subcase, it remains to determine the multiplication between elements in $V_{(1)}$ and $V^{(1)}$, which leads to our definition of $(V^{(2)}, V_{(2)})$ and further discussions in Section 4.

2. $Sing(Q) \nsubseteq K(Q^{(1)})$, i.e, $Ker(F) \nsubseteq V_{(1)}$. In this case, we can normalize the algebraic structure of (R, W, F).

3.2.1. $Ker(F) = V_{(1)}$

Recall $V_{(1)} = Ker(F|_{V^{(1)}})$ and $Ker(F|_W) = Ker(F)$ by Lemma 2.9, hence we can have a decomposition of W as follows:

$$W = Ker(F) \oplus \langle e_1, ..., e_t \rangle \oplus \langle e_{t+1} \rangle, \tag{3.1}$$

where $t \geqslant 2$, $e_i \in V^{(1)}$ for $1 \leqslant i \leqslant t$, $e_{t+1} \in W \backslash V^{(1)}$ and $F(e_i, e_j) = \delta_{i,j}$. Then we can furtherly choose a suitable b_0 and e_i, e_{t+1} to give a normalization of this case:

Lemma 3.6. *If $Ker(F) = V_{(1)}$, then let $b_0 = e_1^2$ we have:*

(i) $b_0 \in \mathfrak{m}^2 \backslash W$ and $b_0 \cdot W = b_0 \cdot \mathfrak{m} = 0$, $V^{(1)} \cdot \mathfrak{m} \subseteq \langle \mu_1, b_0 \rangle$.

(ii) one can choose suitable e_i, e_{t+1} such that

$$e_{t+1} \cdot e_i = 0, e_{t+1} \cdot \mu_2 = \mu_1, e_{t+1}^2 = b_0 + \delta \cdot \mu_2,$$

where $1 \leqslant i \leqslant t$, $\delta = 1$ if $dim(\mathfrak{m}^2) = 3$ and $\delta = 0$ if $dim(\mathfrak{m}^2) = 2$.

Proof. (i) As $F(e_1, e_1) = 1 \neq 0$ we have $b_0 = e_1^2 \in \mathfrak{m}^2 \backslash W$ from Lemma 2.8. By formula (2.1) we can describe the multiplications in $\mathfrak{m}$ as follows:

$$aa' = F(a, a') \cdot b_0 + V_1(a, a') \cdot \mu_1 + V_2(a, a') \cdot \mu_2. \tag{3.2}$$

Note that from $e_1 \in V^{(1)}$ we have $b_0 \cdot Ker(F) = 0$, hence to show $b_0 \cdot W = b_0 \cdot \mathfrak{m} = 0$ it suffices to check $b_0 \cdot e_i = 0$ for $1 \leqslant i \leqslant t + 1$.

For any $1 \leqslant i \leqslant t$, we choose some $j \neq i$. Then from $e_i, e_j \in V^{(1)}$ we have:

$$\begin{aligned} b_0 \cdot e_i =&(e_j^2 - V_1(e_j, e_j) \cdot \mu_1 - V_2(e_j, e_j) \cdot \mu_2) \cdot e_i = e_j^2 \cdot e_i \\ =&e_j \cdot (\delta_{i,j} \cdot b_0 + V_1(e_i, e_j) \cdot \mu_1 + V_2(e_i, e_j) \cdot \mu_2) = 0. \end{aligned} \tag{3.3}$$

For e_{t+1} we have :

$$b_0 \cdot e_{t+1} = e_1^2 \cdot e_{t+1} = e_1 \cdot (\delta_{1,t+1} \cdot b_0 + V_1(e_1, e_{t+1}) \cdot \mu_1 + V_2(e_1, e_{t+1}) \cdot \mu_2) = 0.$$

Now for any $a \in V^{(1)}$ and any $a' \in W$, by multiplying e_{t+1} to both sides of equation (3.2) we have:

$$\begin{aligned} LHS =&e_{t+1} \cdot a \cdot a' = a \cdot (F(e_{t+1}, a') \cdot b_0 + V_1(e_{t+1}, a') \cdot \mu_1 + V_2(e_{t+1}, a') \cdot \mu_2) = 0. \\ RHS =&e_{t+1} \cdot (-F(a, a') \cdot b_0 + V_1(a, a') \cdot \mu_1 + V_2(a, a') \cdot \mu_2) = \lambda_{t+1} \cdot V_2(a, a') \cdot \mu_1 \end{aligned}$$

where $e_{t+1} \cdot \mu_2 = \lambda_{t+1} \cdot \mu_1$ with $\lambda_{t+1} \neq 0$ by $e_{t+1} \in W \backslash V^{(1)}$. Hence form $LHS = RHS$ we have $V_2(a, a') = 0$. Thus $V^{(1)} \cdot W \subseteq \langle b_0, \mu_1 \rangle$. Since $W \cdot \langle \mu_1, b_0 \rangle = 0$ by arguments above, $V^{(1)} \cdot W^{(k)} = 0$ for all $k \geqslant 2$. Since $\mathfrak{m}$ is generated by W, we conclude that $V^{(1)} \subseteq \langle b_0, \mu_1 \rangle$.

(ii) Firstly as $F(e_{t+1}, e_i) = 0$ for $1 \leqslant i \leqslant t$ and from (i) we have $e_{t+1} \cdot e_i \in \langle \mu_1 \rangle$. Thus if we replace e_i by $e_i - \lambda_{t+1}^{-1} V_1(e_i, e_{t+1}) \cdot \mu_2$ then $e_{t+1} \cdot e_i = 0$ and we still have $F(e_i, e_j) = \delta_{i,j}$. Furtherly by (3.2) we have:

$$e_{t+1}^2 = b_0 + V_1(e_{t+1}, e_{t+1}) \cdot \mu_1 + V_2(e_{t+1}, e_{t+1}) \cdot \mu_2.$$

then we can replace e_{t+1} by $e_{t+1} - \frac{V_1(e_{t+1}, e_{t+1})}{2\lambda_{t+1}} \cdot \mu_2$ to make $V_1(e_{t+1}, e_{t+1}) = 0$. Note that this will not affect the multiplication of e_{t+1} and e_i for $i \leqslant t$. Then by (i) and Lemma 2.10 we conclude that $V_2(e_{t+1}, e_{t+1}) \neq 0$ if and only if $dim(\mathfrak{m}^2) = 3$. Now if $V_2(e_{t+1}, e_{t+1}) \neq 0$, we replace μ_2 by $V_2(e_{t+1}, e_{t+1}) \cdot \mu_2$ to make $e_{t+1}^2 = b_0 + \delta \cdot \mu_2$ and then replace μ_1 by $e_{t+1} \cdot \mu_2$ to make $e_{t+1} \cdot \mu_2 = \mu_1$. □

3.2.2. $Ker(F) \subsetneq V_{(1)}$

In this subcase we start with the following observation.

Observation 3.7. $codim(Ker(F), V_{(1)}) = 1$.

Proof. As $Ker(F) \subseteq V_{(1)} \subseteq V^{(1)} \subseteq W$ and $Ker(F|_W) = Ker(F)$, we have a natural injective linear map:

$$\begin{aligned} V_{(1)}/Ker(F) &\overset{\sigma}{\mapsto} (W/V^{(1)})^* \\ \overline{\alpha} &\mapsto \sigma(\overline{\alpha}) : \overline{\beta} \to F(\alpha, \beta) \end{aligned}$$

hence $codim(Ker(F), V_{(1)}) \leqslant codim(V^{(1)}, W) = 1$, concluding the proof. □

Note that by the assumption of $dim(W) \geqslant 5$ we have $codim(V_{(1)}, V^{(1)}) \geqslant 1$. And by $Ker(F|_W) = Ker(F)$ we have a decomposition of W in this subcase:

$$W = \overbrace{\underbrace{\underbrace{\langle \mu_1, \mu_2 \rangle}_{Ker(F)} \oplus \langle g_1 \rangle}^{V_{(1)}} \oplus \langle e_1, e_2, ..., e_t \rangle \oplus \langle f_1 \rangle}^{V^{(1)}} \quad (t \geqslant 1) \tag{3.4}$$

We now can find a suitable b_0.

Lemma 3.8. *Let $b_0 = e_1^2$ then $b_0 \in \mathfrak{m}^2 \backslash W$ and $b_0 \cdot W = b_0 \cdot \mathfrak{m} = 0$, $V^{(1)} \cdot \mathfrak{m} \subseteq \langle \mu_1, b_0 \rangle$.*

Proof. First we check that $b_0 \cdot W = b_0 \cdot \mathfrak{m} = 0$. For $b_0 \cdot g_1 = 0$:

$$b_0 \cdot g_1 = e_1^2 \cdot g_1 = e_1 \cdot (F(e_1, g_1) \cdot b_0 + V_1(e_1, g_1) \cdot \mu_1 + V_2(e_1, g_1) \cdot \mu_2) = 0,$$

where the last equation follows from $F(e_1, g_1) = 0$ and $e_1 \in V^{(1)}$.

To show $b_0 \cdot e_i = 0$ for any $1 \leqslant i \leqslant t$. Firstly note that if $t \geqslant 2$ then we can prove it by using the same computation as (3.3).

Now we assume $t = 1$. As $g_1 \in V_{(1)} \backslash Ker(F)$ we can assume $F(g_1, f_1) = 1$ moreover we can assume $F(f_1, e_1) = 0$ up to replacing e_1 by $e_1 - F(e_1, f_1) \cdot g_1$. Then the calculation of $b_0 \cdot e_1$ follows:

$$\begin{aligned} b_0 \cdot e_1 &= (f_1 \cdot g_1 - V_1(f_1, g_1) \cdot \mu_1 - V_2(f_1, g_1) \cdot \mu_2) \cdot e_1 = g_1 \cdot f_1 \cdot e_1 \\ &= g_1 \cdot (F(f_1, e_1) \cdot b_0 + V_1(f_1, e_1) \cdot \mu_1 + V_2(f_1, e_1) \cdot \mu_2) = 0, \end{aligned}$$

where the last equation follows from $g_1 \in V^{(1)}$. Then the calculation of $b_0 \cdot f_1$ follows:

$$b_0 \cdot f_1 = e_1^2 \cdot f_1 = e_1 \cdot (F(e_1, f_1) \cdot b_0 + V_1(e_1, f_1) \cdot \mu_1 + V_2(e_1, f_1) \cdot \mu_2) = 0.$$

Finally we conclude that $V^{(1)} \cdot W \subseteq \langle b_0, \mu_1 \rangle$ by multiplyng f_1 to both sides of the formula (3.2). □

3.2.3. $Ker(F) \cdot Ker(F) \neq 0$

By Lemma 3.5 we have $\mu_1 \cdot Ker(F) = 0$ and $\mu_2 \cdot W \subseteq \langle \mu_1 \rangle$, hence we can assume $\mu_2^2 = \mu_1$. Now we have a decomposition of W:

$$W = \langle \mu_1, \mu_2 \rangle \oplus \langle e_1, ..., e_t \rangle, \tag{3.5}$$

with $F(e_i, e_j) = \delta_{i,j}$. Moreover for any e_i with $e_i \cdot \mu_2 = \lambda_i \cdot \mu_1$ we can replace e_i by $e_i - \lambda_i \cdot \mu_2$ to make $e_i \cdot \mu_2 = 0$, which does not affect the value of $F(e_i, e_j)$ as $\mu_2 \in Ker(F)$. Then $V^{(1)} = \langle \mu_1, e_1, ..., e_t \rangle$ and we can find suitable b_0 as before, which also gives a normalization of this subcase:

Lemma 3.9. *Let $b_0 = e_1^2$ then $b_0 \in \mathfrak{m} \backslash W$ and*
(i) $b_0 \cdot W = b_0 \cdot \mathfrak{m} = 0$, $V^{(1)} \cdot \mathfrak{m} \subseteq \langle \mu_1, b_0 \rangle$.
(ii) $\mu_2^2 = \mu_1$, $e_i \cdot \mu_2 = e_i \cdot \mu_1 = 0$.

Proof. It suffices to prove (i). First we note that we can use the same method in Lemma 3.6 to show $b_0 \cdot W = 0$. Then it suffices to prove $V^{(1)} \cdot \mathfrak{m} \subseteq \langle \mu_1, b_0 \rangle$. For any $a \in V^{(1)}$, $a' \in W$ equation (3.2) still holds and in this case we multiply it by μ_2:

$$\begin{aligned} LHS =& \mu_2 \cdot a \cdot a' = 0. \\ RHS =& \mu_2 \cdot (F(a, a') \cdot b_0 + V_1(a, a') \cdot \mu_1 + V_2(a, a') \cdot \mu_2) = V_2(a, a') \cdot \mu_1. \end{aligned}$$

Then from $LHS = RHS$ we have $V_2(a, a') = 0$, concluding the proof. □

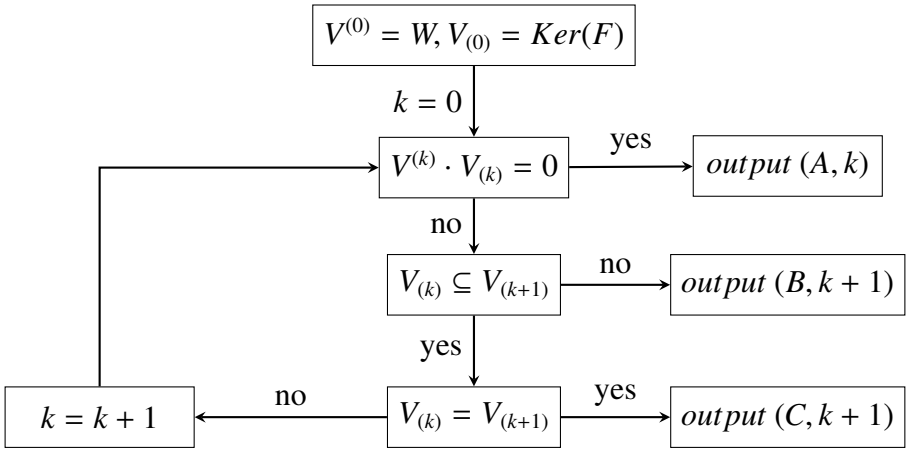
4. Classification of actions with unfixed singularities

4.1. Classification of actions with unfixed singularities (I): $dim(Q) \geqslant 5$

In this and next subsections we always consider additive actions on hyperquadrics of corank two with unfixed singularities. Firstly we give the algebraic version of the flow chart, which induces an *algebraic structure sequence* for a given triple (R, W, F). Then by analyzing the sequence we normalize the structure of (R, W, F). Finally we show the uniqueness of the normalized structure up to equivalences.

4.1.1. Algebraic version of the flow chart

Recall in the proof of Theorem 1.6, we have represented an operation $(\mathbb{G}_a^n, Q, \mathbb{P}^m) \to (G^{(1)}, Q^{(1)}, L^{(1)})$ by $(R, W, F) \to (R^{(1)}, V^{(1)}, F^{(1)})$ or $(R, W, F) \to (R^{(1)}, V^{(1)})$. In Proposition 3.3, we also gave the algebraic criterion for the output condition in the flow chart. Thus the algebraic version of the flow chart naturally arises as the following:



where for any $(V^{(k)}, V_{(k)})$ if $V^{(k)} \cdot V_{(k)} \neq 0$ we furtherly define:

$$\begin{aligned} V^{(k+1)} &= \{\alpha \in V^{(k)} : \alpha \cdot V_{(k)} = 0\} \\ V_{(k+1)} &= Ker(F_{|_{V^{(k+1)}}}) \end{aligned} \tag{4.1}$$

and we represent the final output by $(x, s, V^{(s)}, V_{(s)})$, where for a output (x, t) we set $s = t-1$ if $x \in \{B, C\}$ and $s = t$ if $x = A$.

Then for the final output we obtain an *algebraic structure sequence* as follows:

$$Ker(F) = V_{(0)} \subseteq ... \subseteq V_{(s)} \subseteq V^{(s)} \subseteq ... \subseteq V^{(0)} = W, \tag{4.2}$$

where $V^{(k)} \cdot V_{(k-1)} = 0$ for $1 \leqslant k \leqslant s$.

For the sequence, our first step is to generalize Proposition 3.4 (i) and Observation 3.7 to the following.

Proposition 4.1. *For an algebraic structure sequence:* $\{(V^{(k)}, V_{(k)}) : 0 \leqslant k \leqslant s\}$*:*
$A_{(k)}$: *if* $V^{(k+1)} \subsetneqq V^{(k)}$ *then* $codim(V^{(k+1)}, V^{(k)}) = 1$*;*
$B_{(k)}$: *if* $V_{(k)} \subsetneqq V_{(k+1)}$ *then* $codim(V_{(k)}, V_{(k+1)}) = 1$.

Proof. Firstly note that if $V^{(k+1)} \subsetneqq V^{(k)}$ then $V^{(i+1)} \subsetneqq V^{(i)}$ for any $i \leqslant k-1$, similarly if $V_{(k)} \subsetneqq V_{(k+1)}$ then $V_{(i)} \subsetneqq V_{(i+1)}$ for any $i \leqslant k-1$. Hence we can prove $A_{(k)}$and $B_{(k)}$ by induction on k.

For $k = 0$, $A_{(0)}$ follows from Lemma 3.5 and $B_{(0)}$ follows from Observation 3.7. Now assuming $A_{(k-1)}$ and $B_{(k-1)}$ is true for some $k \geqslant 1$, then for a given $(V_{(k)}, V^{(k)})$ in the process we already have $V_{(k-1)} \subsetneqq V_{(k)} \subseteq V^{(k)} \subsetneqq V^{(k-1)}$ with $V^{(k)} \cdot V_{(k-1)} = 0$ and $codim(V_{(k-1)}, V_{(k)}) = 1$ by induction. Now since (R, W, F) represents an action on a hyperquadric of corank two with unfixed singularities and $V_{(k)} \subseteq V_{(0)} = W, V^{(k)} \subseteq V^{(1)}$, we have $V^{(k)} \cdot V_{(k)} \subseteq V^{(1)} \cdot W \subseteq \langle \mu_1 \rangle$ by Proposition 3.4 (ii). Hence by the definition of $V^{(k+1)}$ we conclude that $codim(V^{(k+1)}, V^{(k)}) \leqslant 1$, implying $A_{(k)}$.

Now if $V_{(k)} \subsetneqq V_{(k+1)}$ then from the process we already have $V^{(k+1)} \subsetneqq V^{(k)}$ and $A_{(k)}$ holds. Moreover we have the chain $V_{(k)} \subsetneqq V_{(k+1)} \subseteq V^{(k+1)} \subsetneqq V^{(k)}$, which induces an injective map:

$$\begin{aligned} V_{(k+1)}/V_{(k)} &\overset{\sigma_k}{\longmapsto} (V^{(k)}/V^{(k+1)})^* \\ \overline{\alpha} &\mapsto \sigma(\overline{\alpha}) : \overline{\beta} \mapsto F(\alpha, \beta) \end{aligned}$$

It follows that $codim(V_{(k)}, V_{(k+1)}) \leqslant codim(V^{(k+1)}, V^{(k)}) = 1$, implying $B_{(k)}$.

□

4.1.2. Normalization

In this subsection we normalize the structure of (R, W, F) by analyzing the algebraic structure sequence case by case.

In the following, we always start with a $b_0 \in \mathfrak{m}^2 \backslash W$ and a basis of $Ker(F)$, μ_1, μ_2, satisfying Proposition 3.4. We furtherly define $V_{(-1)} = \langle \mu_1 \rangle$.

Case 1. $x = A$. In this case the sequence becomes

$$V_{(-1)} \subsetneqq Ker(F) = V_{(0)} \subsetneqq ... \subsetneqq V_{(s)} \subseteq V^{(s)} \subsetneqq ... \subsetneqq V^{(0)} = W,$$

with $V^{(k)} \cdot V_{(k-1)} = V^{(s)} \cdot V_{(s)} = 0$ for $1 \leqslant k \leqslant s$ (here $s \geqslant 1$ as we assume there exist unfixed singular points). Then we have the following normalization.

Lemma 4.2. *(i) If $V^{(s)} \neq V_{(s)}$ then there exist $f_i \in V^{(i-1)} \backslash V^{(i)}$, $g_i \in V_{(i)} \backslash V_{(i-1)}$ for $1 \leqslant i \leqslant s$, $g_0 \doteq \mu_2$ and $\{e_k : 1 \leqslant k \leqslant p\} \subseteq V^{(s)} \backslash V_{(s)}$ such that*

$$\begin{aligned} V^{(s)} &= V_{(s)} \oplus \langle e_1, ..., e_p \rangle, \\ e_k \cdot e_l &= \delta_{k,l} \cdot b_0 + V_1(e_k, e_l) \cdot \mu_1, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &e_k \cdot f_i = e_k \cdot g_i = f_i \cdot f_j = f_v \cdot g_{v'} = 0, \\ &f_i \cdot g_i = b_0,\ f_i \cdot g_{i-1} = \mu_1,\ f_1^2 = \delta \cdot \mu_2, \end{aligned} \tag{4.4}$$

for $1 \leqslant i \leqslant s$, $1 \leqslant k, l \leqslant p$, $v - v' \notin \{0, 1\}$, $2 \leqslant j \leqslant s$ when $s \geqslant 2$,

$$\delta = \begin{cases} 0 & \text{if } dim(\mathfrak{m}^2) = 2; \\ 1 & \text{if } dim(\mathfrak{m}^2) = 3 \end{cases}$$

and the matrix $\Lambda = (V_1(e_k, e_l) : 1 \leqslant k, l \leqslant p)$ is of the canonical form (see (4.6) below).

(ii) If $V^{(s)} = V_{(s)}$ then there exist $f_i \in V^{(i-1)} \backslash V^{(i)}$, $g_i \in V_{(i)} \backslash V_{(i-1)}$ for $1 \leqslant i \leqslant s$, $g_0 \doteq \mu_2$ such that

$$f_i \cdot f_j = f_v \cdot g_{v'} = 0,\ f_i \cdot g_i = b_0,\ f_i \cdot g_{i-1} = \mu_1,\ f_1^2 = \delta \cdot \mu_2,$$

for $1 \leqslant i \leqslant s$, $v - v' \notin \{0, 1\}$, $2 \leqslant j \leqslant s$ when $s \geqslant 2$, and δ is the same as in (i).

Proof. Recall by Proposition 3.4 (ii) we always have $V^{(0)} \cdot V^{(0)} \subseteq \langle \mu_1, \mu_2, b_0 \rangle$ and $V^{(1)} \cdot V^{(0)} \subseteq \langle \mu_1, b_0 \rangle$. Hence if choosing any nonzero $f_i \in V^{(i-1)} \backslash V^{(i)}$ and any nonzero $h_{i-1} \in V_{(i-1)} \backslash V_{(i-2)}$ then we can have $f_i \cdot h_{i-1} = c \cdot \mu_1$ for some nonzero c as $V^{(i-1)} \cdot V_{(i-2)} = 0$, $V^{(i-1)} \cdot V_{(i-1)} \neq 0$ and $codim(V_{(i-2)}, V_{(i-1)}) = 1$. Moreover choosing any nonzero $g_i \in V_{(i)} \backslash V_{(i-1)}$ we have $F(f_i, g_i) \neq 0$ from the definition of $V_{(i)}$.

(i) If $V_{(s)} \neq V^{(s)}$ we can choose e_k's satisfying (4.3), i.e., $F(e_k, e_l) = \delta_{k,l}$. Then we find f_i, g_i inductively. For $i = s$ we first choose $f_s \in V^{(s-1)} \backslash V^{(s)}$, $g_s \in V_{(s)} \backslash V_{(s-1)}$ and $h_{s-1} \in V_{(s-1)} \backslash V_{(s-2)}$ s.t. $f_s \cdot h_{s-1} = \mu_1$, $F(f_s, g_s) = 1$ and $F(f_s, f_s) = 0$. Then for the multiplications:

$$\begin{aligned} f_s \cdot g_s &= b_0 + V_1(f_s, g_s) \cdot \mu_1, \\ f_s \cdot e_k &= F(f_s, e_k) \cdot b_0 + V_1(f_s, e_k) \cdot \mu_1, \\ f_s^2 &= V_1(f_s, f_s) \cdot \mu_1 + V_2(f_s, f_s) \cdot \mu_2, \end{aligned}$$

we can normalize them through the following steps:

$$\begin{aligned} & g_s \mapsto g_s - V_1(f_s, g_s) \cdot h_{s-1} && \text{to make } g_s \cdot f_s = b_0 \\ & e_k \mapsto e_k + F(f_s, e_k) \cdot g_s - V_1(f_s, e_k) \cdot h_{s-1} && \text{to make } f_s \cdot e_k = 0 \\ & f_s \mapsto f_s - (V_1(f_s, f_s)/2) \cdot h_{s-1} && \text{to make } f_s^2 = \begin{cases} 0 & \text{if } s \geqslant 2 \\ d_0 \cdot \mu_2 & \text{if } s = 1 \end{cases} \end{aligned}$$

for some $d_0 \in \mathbb{K}$, where the arrow $A \mapsto B$ means to replace A by B.

Now if $s \geqslant 2$ and assuming we have found $S_{i_0} = \{f_i, g_i : i \geqslant i_0 + 1\}$ for some $1 \leqslant i_0 \leqslant s-1$ satisfying (4.4) except that if there exist i such that $i \geqslant i_0 + 2$ then $f_i \cdot g_{i-1} = c_i \cdot \mu_1$ for some nonzero $c_i \in \mathbb{K}$. Then we furtherly choose $f_{i_0} \in V^{(i_0-1)} \backslash V^{(i_0)}$, $g_{i_0} \in V_{(i_0)} \backslash V_{(i_0-1)}$ and $h_{i_0-1} \in V_{(i_0-1)} \backslash V_{(i_0-2)}$ s.t. $f_{i_0} \cdot h_{i_0-1} = \mu_1$, $F(f_{i_0}, g_{i_0}) = 1$ and $F(f_{i_0}, f_{i_0}) = 0$. And we normalize the multiplications through the following steps. Firstly:

$$f_{i_0} \mapsto f_{i_0} - \sum_{i=i_0+1}^{s} (F(f_{i_0}, f_i) \cdot g_i + F(f_{i_0}, g_i) \cdot f_i) - \sum_{k=1}^{p} F(e_k, f_{i_0}) \cdot e_k$$

to make $F(\alpha, f_{i_0}) = 0$ for all $\alpha \in S_{i_0} \cup \{e_k : 1 \leqslant k \leqslant p\}$. Then

$$\begin{aligned} & \alpha \mapsto \alpha - V_1(\alpha, f_{i_0}) \cdot h_{i_0-1} && \text{to make } \alpha \cdot f_{i_0} = 0 \\ & g_{i_0} \mapsto g_{i_0} - V_1(f_{i_0}, g_{i_0}) \cdot h_{i_0-1} && \text{to make } f_{i_0} \cdot g_{i_0} = b_0 \\ & f_{i_0} \mapsto f_{i_0} - (V_1(f_{i_0}, f_{i_0})/2) \cdot h_{i_0-1} && \text{to make } f_{i_0}^2 = \begin{cases} 0 & \text{if } i_0 \geqslant 2 \\ d_0 \cdot \mu_2 & \text{if } i_0 = 1 \end{cases} \end{aligned}$$

for some $d_0 \in \mathbb{K}$. Moreover from the discussion at the beginning we have $f_{i_0+1} \cdot g_{i_0} = c_{i_0+1} \cdot \mu_1$ with some nonzero $c_{i_0+1} \in \mathbb{K}$. And from $\mathfrak{m}^2 \subseteq \langle \mu_1, \mu_2, b_0 \rangle, b_0 \cdot \mathfrak{m} = 0, V^{(1)} \cdot \mathfrak{m} \subseteq \langle b_0, \mu_1 \rangle$ we have $d_0 = 0$ if and only if $dim(\mathfrak{m}^2) = 2$. Finally note that the symmetric matrix $\Lambda = (V_1(e_k, e_l))$ under orthogonal transformations on $\{e_1, ..., e_p\}$ transforms as the matrix of a bilinear form. And a such transformation will not affect our normalization on other elements, hence from [5, Chapter XI §3], $\Lambda = (V_1(e_i, e_j))$ can be transformed into a canonical symmetric block diagonal matrix (see (4.6) in Proposition 4.3).

To finish our normalization it suffices to make $f_i \cdot g_{i-1} = \mu_1$ and $f_1^2 = \delta \cdot \mu_2$. To do this we firstly replace f_i by $x_i \cdot f_i$ and replace g_i by $y_i \cdot g_i$. Then the condition $(f_i \cdot g_{i-1} = \mu_1, f_1^2 = \delta \cdot \mu_2, f_i \cdot g_i = b_0)$ gives a system of equations for $\{x_i, y_j : 1 \leqslant i \leqslant s, 0 \leqslant j \leqslant s\}$:

$$x_i \cdot y_i = 1,\ x_i \cdot y_{i-1} = c_i^{-1},\ x_1^2 \cdot d_0 = y_0 \cdot \delta \tag{4.5}$$

for which we have a solution to be calculated inductively:

$$(\delta = 1)\begin{cases} x_i = y_i^{-1} \\ y_i = y_{i-1} \cdot c_i \\ y_0 = (\frac{d_0}{c_1^2})^{\frac{1}{3}} \end{cases} \quad \text{and} \quad (\delta = 0)\begin{cases} x_i = y_i^{-1} \\ y_i = y_{i-1} \cdot c_i \end{cases}$$

concluding the normalization.
(ii) If $V^{(s)} = V_{(s)}$ then the process of normalization will be the same as in (i) except that we do not need to choose e_k at the beginning. □

Following our normalization we can thus determine the normalized structure of (R, W, F) in Case 1.

Proposition 4.3 (Classification of *Type A*)**.** *(R, W, F) can be transformed into the following:*
• *Type A_1: $Q^{(s)}$ is a projective space (equivalently $V_{(s)} = V^{(s)}$)*

$$M(F, TypeA_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 1 \\ 0 & 0 & \vdots & \dots & 0 & \vdots & \iddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & \vdots & \iddots & \vdots & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & \dots & 0 \end{pmatrix},$$

$W = \langle \mu_1, \mu_2 \rangle \oplus \langle g_1, ..., g_s \rangle \oplus \langle f_s, ..., f_1 \rangle$.
$R \cong \mathbb{K}[\mu_1, \mu_2, g_1, ..., g_s, f_1, ..., f_s]/(\mu_1 \cdot W, g_i \cdot \mu_2, f_l \cdot \mu_2, g_i \cdot f_i - g_v \cdot f_v, g_{l-1} \cdot f_l - \mu_1, f_1 \cdot \mu_2 - \mu_1, f_1^2 - \delta \cdot \mu_2, g_i \cdot g_v, f_h \cdot g_{h'}, f_l \cdot f_i, 1 \leqslant i, v \leqslant s,\ h - h' \notin \{0, 1\},\ 2 \leqslant l \leqslant s \text{ when } s \geqslant 2)$ *where*

$$\delta = \begin{cases} 0 & \text{if } dim(\mathfrak{m}^2) = 2; \\ 1 & \text{if } dim(\mathfrak{m}^3) = 3 \end{cases}$$

• *Type A_2: $Q^{(s)}$ is a hyperquadric (equivalently $V_{(s)} \neq V^{(s)}$).*

$$M(F, TypeA_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1 \\ 0 & 0 & \vdots & \dots & 0 & \vdots & \dots & \vdots & \vdots & \iddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \vdots & \dots & 0 & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \vdots & \iddots & \vdots & \vdots & \dots & \vdots & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix},$$

$W = \langle \mu_1, \mu_2 \rangle \oplus \langle g_1, ..., g_s \rangle \oplus \langle e_1, ..., e_p \rangle \oplus \langle f_s, ..., f_1 \rangle$.
$R \cong \mathbb{K}[\mu_1, \mu_2, g_1, ..., g_s, e_1, ..., e_p, f_1, ..., f_s]/(\mu_1 \cdot W, g_i \cdot \mu_2, e_{i'} \cdot \mu_2, f_l \cdot \mu_2, g_i \cdot f_i - e_{i'}^2 + \lambda_{i'i'}\mu_1, e_{i'} \cdot e_{i''} - \lambda_{i'i''}\mu_1, g_{l-1} \cdot f_l - \mu_1, f_1 \cdot \mu_2 - \mu_1, f_1^2 - \delta \cdot \mu_2, g_i \cdot g_v, f_l \cdot f_i, f_h \cdot g_{h'}, e_{i'} \cdot f_i, e_{i'} \cdot g_i, 1 \leqslant i, v \leqslant s, 1 \leqslant i' \neq i'' \leqslant p, h - h' \notin \{0, 1\}, 2 \leqslant l \leqslant s$ *when* $s \geqslant 2$*) where δ is the same as in Type A_1 and $\Lambda = (\lambda_{i'i''})$ is of the standard form, i.e., a symmetric block diagonal $t \times t$-matrix such that each block Λ_k is*

$$\lambda_k \begin{pmatrix} 1 & 0 & & 0 \\ 0 & \ddots & \ddots & \\ & \ddots & \ddots & 0 \\ 0 & & 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 0 \end{pmatrix} + \frac{i}{2} \begin{pmatrix} 0 & & 1 & 0 \\ & \ddots & \ddots & -1 \\ 1 & \ddots & \ddots & 0 \\ 0 & -1 & & 0 \end{pmatrix} \tag{4.6}$$

with some $\lambda_k \in \mathbb{K}$.

Case 2. $x = B$. In this case the sequence becomes

$$\langle \mu_1 \rangle = V_{(-1)} \subseteq Ker(F) = V_{(0)} \subseteq ... \subseteq V_{(s)} \subseteq V^{(s)} \subseteq ... \subseteq V^{(0)} = W,$$

with $V^{(k)} \cdot V_{(k-1)} = 0$ for $k \geqslant 0$ and $V_{(s)}^2 \neq 0$. We have the following normalization.

Lemma 4.4. *(i) If $V^{(s)} \neq V_{(s)}$ and $s \geqslant 1$ then there exist $f_i \in V^{(i-1)} \backslash V^{(i)}$, $g_i \in V_i \backslash V_{(i-1)}$ for $1 \leqslant i \leqslant s$, $g_0 \doteq \mu_2$ and $\{e_k : 1 \leqslant k \leqslant p\} \subseteq V^{(s)} \backslash V_{(s)}$ such that*

$$\begin{aligned} V^{(s)} &= V_{(s)} \oplus \langle e_1, ..., e_p \rangle, \\ e_k \cdot e_l &= \delta_{k,l} \cdot b_0 + V_1(e_k, e_l) \cdot \mu_1, \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} &e_k \cdot f_i = e_k \cdot g_i = f_i \cdot f_j = f_v \cdot g_{v'} = 0, \\ &f_i \cdot g_i = b_0,\ f_i \cdot g_{i-1} = \mu_1,\ f_1^2 = \delta \cdot \mu_2,\ g_s^2 = \mu_1, \end{aligned} \tag{4.8}$$

for $1 \leqslant i \leqslant s$, $1 \leqslant k, l \leqslant p$, $v - v' \notin \{0, 1\}$, $2 \leqslant j \leqslant s$ when $s \geqslant 2$,

$$\delta = \begin{cases} 0 & \text{if } dim(\mathfrak{m}^2) = 2; \\ 1 & \text{if } dim(\mathfrak{m}^3) = 3 \end{cases}$$

and the matrix $\Lambda = (V_1(e_k, e_l))$ *is of the canonical form (4.6).*

(ii) If $V^{(s)} = V_{(s)}$ *and* $s \geqslant 1$ *then there exists* $f_i \in V^{(i-1)}\backslash V^{(i)}$, $g_i \in V_{(i)}\backslash V_{(i-1)}$ *for* $1 \leqslant i \leqslant s$, $g_0 = \mu_2$ *such that*

$$f_i \cdot g_i = b_0,\ f_i \cdot g_{i-1} = \mu_1,\ f_i \cdot f_j = f_v \cdot g_{v'} = 0,\ f_1^2 = \delta \cdot \mu_2,\ g_s^2 = \mu_1,$$

for $1 \leqslant i \leqslant s$, $v - v' \notin \{0, 1\}$, $2 \leqslant j \leqslant s$ *when* $s \geqslant 2$, *and* δ *is the same as in (i).*

(iii) If $s = 0$ *then there exists a basis of* $Ker(F) = \langle \mu_1, \mu_2 \rangle$ *and* $\{e_k : 1 \leqslant k \leqslant p\}$ *such that* $\mu_2^2 = \mu_1, e_k \cdot \mu_2 = 0$ *and*

$$W = Ker(F) \oplus \langle e_1, ..., e_p \rangle$$
$$e_k \cdot e_l = \delta_{k,l} \cdot b_0 + V_1(e_k, e_l) \cdot \mu_1,$$

and the matrix $\Lambda = (V_1(e_k, e_l))$ *is of the canonical form (4.6).*

Proof. (i) As in Case 1.(i) we can first choose e_k satisfying (4.7). Also we can choose $f_s \in V^{(s-1)}\backslash V^{(s)}$, $g_s \in V_{(s)}\backslash V_{(s-1)}$ and $h_{s-1} \in V_{(s-1)}\backslash V_{(s-2)}$ s.t. $f_s \cdot h_{s-1} = \mu_1$, $F(f_s, g_s) = 1$ and $F(f_s, f_s) = 0$. Then as $V_{(s)} \cdot V_{(s)} \neq 0$, $V_{(s-1)} \cdot V_{(s)} \subseteq V_{(s-1)} \cdot V^{(s)} = 0$ and $codim(V_{(s-1)}, V_{(s)}) = 1$ we conclude that g_s^2 is a nonzero element in $\langle \mu_1 \rangle$ and hence we can assume $g_s^2 = \mu_1$. Now we normalize the multiplications between f_s, g_s, e_k through the following steps:

$$\begin{aligned}
&e_k \mapsto e_k - V_1(g_s, e_k) \cdot g_s && \text{to make } g_s \cdot e_k = 0\\
&f_s \mapsto f_s - \sum_{k=1}^{p} F(e_k, f_s) \cdot e_k && \text{to make } F(f_s, e_k) = 0\\
&e_k \mapsto e_k - V_1(f_s, e_k) \cdot h_{s-1} && \text{to make } f_s \cdot e_k = 0\\
&g_s \mapsto g_s - V_1(g_s, f_s) \cdot h_{s-1} && \text{to make } f_s \cdot g_s = b_0\\
&f_s \mapsto f_s - (V_1(f_s, f_s)/2) \cdot h_{s-1} && \text{to make } f_s^2 = \begin{cases} 0 & \text{if } s \geqslant 2 \\ d_0 \cdot \mu_2 & \text{if } s = 1 \end{cases}
\end{aligned}$$

for some $d_0 \in \mathbb{K}$.

After this note that we can still use previous inductive operations in Case 1 to find f_i, g_i for $i \leqslant s-1$, namely we can find suitable f_i, g_i satisfying (4.8) except that we have $f_i \cdot g_{i-1} = c_i \cdot \mu_1$ and $f_1^2 = d_0 \cdot \mu_2$ for some nonzero $c_i \in \mathbb{K}$ and $d_0 = 0$ if and only if $dim(\mathfrak{m}^2) = 2$. Also for the same reason as in Case 1 we can assume $\Lambda = (V_1(e_k, e_l))$ is of the canonical form.

Now to finish our normalization we replace f_i by $x_i \cdot f_i$, replace g_i by $y_i \cdot g_i$ and replace μ_1 by $z_0 \cdot \mu_1$. Then it suffices to satisfy the condition $(f_i \cdot g_{i-1} = \mu_1, f_1^2 = \delta \cdot \mu_2, f_i \cdot g_i = b_0, g_s^2 = \mu_1)$, which gives a system of equations for $\{x_i, y_j, z_0 \in \mathbb{K} : 1 \leqslant i \leqslant s, 0 \leqslant j \leqslant s\}$:

$$x_i \cdot y_i = 1,\ x_i \cdot y_{i-1} = z_0 \cdot c_i^{-1},\ x_1^2 \cdot d_0 = y_0 \cdot \delta,\ y_s^2 = z_0$$

for which we have a solution (where $c \doteq \prod_{i=1}^{s} c_i$) to be calculated inductively:

$$(\delta = 1) \begin{cases} x_i = y_i^{-1} \\ y_i = y_{i-1} \cdot c_i \cdot z_0^{-1} \\ z_0 = (\frac{c^3 \cdot d_0}{c_1^2})^{\frac{2}{6s-1}}, y_0 = (\frac{d_0 \cdot z_0^2}{c_1^2})^{\frac{1}{3}} \end{cases} \quad \text{and} \quad (\delta = 0) \begin{cases} x_i = y_i^{-1} \\ y_i = y_{i-1} \cdot c_i \\ y_0 = c^{-1}, z_0 = 1 \end{cases}$$

concluding our normalization.

(ii) If $V^{(s)} = V_{(s)}$ then the process will be the same as (i) except that we do not need to choose e_k at the beginning.

(iii) Note that $s = 0$ is equivalent to $Ker(F) \cdot Ker(F) \neq 0$, which is just the Case 2 in the proof of Proposition 3.4. Hence the assertion follows from Lemma 3.9. □

We can now determine the algebraic structure of (R, W, F) in Case 2.

Proposition 4.5 (Classification of $Type\ B$)**.** *(R, W, F) can be transformed into the following:*
• *$Type\ B_0$: $s = 0$ $(Ker(F) \cdot Ker(F) \neq 0)$*

$$M(F, TypeB_0) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \ W = \langle \mu_1, \mu_2 \rangle \oplus \langle e_1, ..., e_p \rangle$$

$R \cong \mathbb{K}[\mu_1, \mu_2, e_1, ..., e_p]/(\mu_1 \cdot W, \mu_2^2 - \mu_1, e_i \cdot \mu_2, e_i \cdot e_j - \lambda_{i,j} \cdot \mu_1, e_i^2 - e_j^2 - (\lambda_{i,i} - \lambda_{j,j}) \cdot \mu_1,\ 1 \leqslant i \neq j \leqslant p)$ *where $\Lambda = (\lambda_{i,j})$ is of the canonical form (4.6).*

• *$Type\ B_1$: $s \geqslant 1$ and $Q^{(s)}$ is a projective space (equivalently $V_{(s)} = V^{(s)}$).*

$$M(F, TypeB_1) = M(F, TypeA_1),$$

$W = \langle \mu_1, \mu_2 \rangle \oplus \langle g_1, ..., g_s \rangle \oplus \langle f_s, ..., f_1 \rangle$,
$R \cong \mathbb{K}[\mu_1, \mu_2, g_1, .., g_s, f_1, .., f_s]/(\mu_1 \cdot W, g_i \cdot \mu_2, f_l \cdot \mu_2, g_s^2 - \mu_1, g_i \cdot f_i - g_v \cdot f_v, g_{l-1} \cdot f_l - \mu_1, f_1 \cdot \mu_2 - \mu_1, f_1^2 - \delta \cdot \mu_2, f_h \cdot g_{h'}, f_i \cdot f_l, g_i \cdot g_{l'},\ 1 \leqslant i \neq v \leqslant s,\ h - h' \notin \{0, 1\},\ 2 \leqslant l \leqslant s,\ 1 \leqslant l' \leqslant s - 1$ *when* $s \geqslant 2$*) where δ is the same as in Type A_1.*

• *$Type\ B_2$: $s \geqslant 1$ and $Q^{(s)}$ is a hyperquadric (equiuvalently $V_{(s)} \neq V^{(s)}$).*

$$M(F, TypeB_2) = M(F, TypeA_2),$$

$W = \langle \mu_1, \mu_2 \rangle \oplus \langle g_1, ..., g_s \rangle \oplus \langle e_1, ..., e_p \rangle \oplus \langle f_s, ..., f_1 \rangle$,
$R \cong \mathbb{K}[\mu_1, \mu_2, g_1, ..., g_s, e_1, ..., e_p, f_1, ..., f_s]/(\mu_1 \cdot W, g_i \cdot \mu_2, e_{i'} \cdot \mu_2, f_l \cdot \mu_2, g_s^2 - \mu_1, g_i \cdot f_i - e_{i'}^2 + \lambda_{i'i'}\mu_1, g_{l-1} \cdot f_l - \mu_1, e_{i'} \cdot e_{i''} - \lambda_{i'i''}\mu_1, e_{i'} \cdot f_i, e_{i'} \cdot g_i, f_1 \cdot \mu_2 - \mu_1, f_1^2 - \delta \cdot \mu_2, g_i \cdot g_{l'}, f_i \cdot f_l, f_h \cdot g_{h'},\ 1 \leqslant i \leqslant s, 1 \leqslant i' \neq i'' \leqslant p, h - h' \notin \{0, 1\},\ 2 \leqslant l \leqslant s,\ 1 \leqslant l' \leqslant s - 1$ *when* $s \geqslant 2$*)*
where $\Lambda = (\lambda_{i'i''})$ is of the canonical form (4.6) and δ is the same as in Type A_1.

Case 3. $x = C$. In this case the algebraic sequence becomes

$$\langle \mu_1 \rangle = V_{(-1)} \subseteq Ker(F) = V_{(0)} \subseteq ... \subseteq V_{(s)} = V_{(s+1)} \subseteq V^{(s+1)} \subseteq V^{(s)} \subseteq ... \subseteq V^{(0)} = W,$$

with $V^{(k)} \cdot V_{(k-1)} = 0$ if $k \geqslant 0$ and $V_{(s)} \cdot V_{(s)} = 0$. We have the following normalization.

Lemma 4.6. *(i) If $V_{(s+1)} \neq V^{(s+1)}$ and $s \geqslant 1$ then there exist $f_i \in V^{(i-1)} \backslash V^{(i)}$, $g_j \in V_{(j)} \backslash V_{(j-1)}$ for $1 \leqslant i \leqslant s + 1, 1 \leqslant j \leqslant s$, $g_0 \doteq \mu_2$ and $\{e_k : 1 \leqslant k \leqslant p\} \subseteq V^{(s+1)} \backslash V_{(s+1)}$ such that*

$$\begin{aligned} V^{(s+1)} &= V_{(s+1)} \oplus \langle e_1, ..., e_p \rangle \\ e_k \cdot e_l &= \delta_{k,l} \cdot b_0 + V_1(e_k, e_l) \cdot \mu_1 \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &e_k \cdot f_i = e_k \cdot g_j = f_j \cdot f_{j'} = f_v \cdot g_{v'} = 0 \\ &f_j \cdot g_j = f_{s+1}^2 = b_0,\ f_i \cdot g_{i-1} = \mu_1,\ f_1^2 = \delta \cdot \mu_2, \end{aligned} \tag{4.10}$$

for $1 \leqslant i \leqslant s+1$, $1 \leqslant k, l \leqslant p$, $v - v' \notin \{0, 1\}$, $1 \leqslant j \leqslant s$, $2 \leqslant j' \leqslant s+1$,

$$\delta = \begin{cases} 0 & \text{if } dim(\mathfrak{m}^2) = 2; \\ 1 & \text{if } dim(\mathfrak{m}^3) = 3 \end{cases}$$

and the matrix $\Lambda = (V_1(e_k, e_l))$ *is of the canonical form (4.6).*
(ii) If $V^{(s+1)} = V_{(s+1)}$ *and* $s \geqslant 1$ *then there exist* $f_i \in V^{(i-1)} \backslash V^{(i)}$, $g_j \in V_{(j)} \backslash V_{(j-1)}$ *for* $1 \leqslant i \leqslant s+1, 1 \leqslant j \leqslant s$, $g_0 = \mu_2$ *such that*

$$f_j \cdot f_{j'} = f_v \cdot g_{v'} = 0,\ f_j \cdot g_j = f_{s+1}^2 = b_0,\ f_i \cdot g_{i-1} = \mu_1,\ f_1^2 = \delta \cdot \mu_2,$$

for $1 \leqslant i \leqslant s+1$, $v - v' \notin \{0, 1\}$, $1 \leqslant j \leqslant s$, $2 \leqslant j' \leqslant s+1$, *and* δ *is the same as in (i).*
(iii) If $s = 0$ *then there exist* $\{e_k : 1 \leqslant k \leqslant p+1\}$ *such that* $e_{p+1} \in W \backslash V^{(1)}$ *and:*

$$\begin{aligned} V^{(1)} &= \langle \mu_1, \mu_2 \rangle \oplus \langle e_1, ..., e_p \rangle, \\ e_k \cdot e_l &= \delta_{k,l} \cdot b_0 + V_1(e_k, e_l) \cdot \mu_1, \\ e_{p+1} \cdot \mu_2 &= \mu_1, e_{p+1} \cdot e_k = 0, e_{p+1}^2 = b_0 + \delta \cdot \mu_2, \end{aligned}$$

for $1 \leqslant k, l \leqslant p$ *and* δ *is the same as in (i).*

Proof. (i) Firstly we can choose e_k satisfying (4.9) and from $V_{(s)} = Ker(F_{|_{V(s)}})$ we can choose $f_{s+1} \in V^{(s)} \backslash V^{(s+1)}$ s.t. $F(e_k, f_{s+1}) = 0$ and $F(f_{s+1}, f_{s+1}) = 1$. Furtherly we can choose $f_s \in V^{(s-1)} \backslash V^{(s)}$, $g_s \in V_{(s)} \backslash V_{(s-1)}$ and $h_{s-1} \in V_{(s-1)} \backslash V_{(s-2)}$ s.t. $f_s \cdot h_{s-1} = \mu_1$, $F(f_s, g_s) = 1$ and $F(f_s, f_s) = 0$. Moreover we have $f_{s+1} \cdot g_s = c_{s+1} \cdot \mu_1$ for some nonzero $c_{s+1} \in \mathbb{K}$. Now we normalize the multiplications between e_k, f_s, g_s, f_{s+1} through the following steps:

$$\begin{aligned} e_k &\mapsto e_k - c_{s+1}^{-1} \cdot V_1(f_{s+1}, e_k) \cdot g_s && \text{to make } f_{s+1} \cdot e_k = 0 \\ f_{s+1} &\mapsto f_{s+1} - c_{s+1}^{-1} \cdot (V_1(f_{s+1}, f_{s+1})/2) \cdot g_s && \text{to make } f_{s+1}^2 = b_0 \\ f_s &\mapsto f_s - \sum_{k=1}^{p} F(e_k, f_s) \cdot e_k - F(f_s, f_{s+1}) \cdot f_{s+1} && \text{to make } F(f_s, e_k) = F(f_s, f_{s+1}) = 0 \end{aligned}$$

and for any $\alpha \in \{e_k, f_{s+1} : 1 \leqslant k \leqslant p\}$

$$\begin{aligned} \alpha &\mapsto \alpha - V_1(\alpha, f_s) \cdot h_{s-1} && \text{to make } f_s \cdot \alpha = 0 \\ g_s &\mapsto g_s - V_1(f_s, g_s) \cdot h_{s-1} && \text{to make } g_s \cdot f_s = b_0 \\ f_s &\mapsto f_s - (V_1(f_s, f_s)/2) \cdot h_{s-1} && \text{to make } f_s^2 = \begin{cases} 0 & \text{if } s \geqslant 2 \\ d_0 \cdot \mu_2 & \text{if } s = 1 \end{cases} \end{aligned}$$

After this as in Case 1 and 2, we can still inductively find suitable f_i, g_i satisfying (4.10) except that we have $f_i \cdot g_{i-1} = c_i \cdot \mu_1$ and $f_1^2 = d_0 \cdot \mu_2$ for some nonzero $c_i \in \mathbb{K}$ and $d_0 = 0$ if and only if $dim(\mathfrak{m}^2) = 2$. Also we can assume $\Lambda = (V_1(e_k, e_l))$ is of the canonical form (4.6).

Now to finish our normalization we again replace f_i by $x_i \cdot f_i$, replace g_j by $y_j \cdot g_j$ and replace μ_1 by $z_0 \cdot \mu_1$. Then the condition $(f_i \cdot g_{i-1} = \mu_1, f_1^2 = \delta \cdot \mu_2, f_i \cdot g_i = b_0, f_{s+1}^2 = b_0)$ gives a system of equations for $\{x_i, y_j, z_0, \in \mathbb{K} : 1 \leqslant i \leqslant s+1, 0 \leqslant j \leqslant s\}$:

$$x_k \cdot y_k = 1, x_k \cdot y_{k-1} = z_0 \cdot c_k^{-1}, x_{s+1}^2 = 1, d_0 \cdot x_1^2 = \delta \cdot y_0, x_{s+1} \cdot y_s = z_0 \cdot c_{s+1}^{-1}$$

where $1 \leqslant k \leqslant s$, and for which we have a solution:

$$(\delta = 1) \begin{cases} x_k = y_k^{-1} \\ y_k = y_{k-1} \cdot c_k \cdot z_0^{-1} \\ z_0 = (c^3 \cdot c_1^{-2} \cdot d_0)^{\frac{1}{3s+4}} \\ y_0 = (\frac{z_0^2 \cdot d_0}{c_1^2})^{\frac{1}{3}} \\ x_{s+1} = 1 \end{cases} \quad \text{and} \quad (\delta = 0) \begin{cases} x_k = y_k^{-1} \\ y_k = y_{k-1} \cdot c_k \cdot z_0^{-1} \\ z_0 = 1, y_0 = c^{-1}, x_{s+1} = 1 \end{cases}$$

where $c \doteq \prod_{i=0}^{s+1} c_i$, concluding the normalization.

(ii) If $V_{(s+1)} = V^{(s+1)}$ then similarly the process will be the same as (i) except that we do not need to choose e_k at the beginning.

(iii) If $s = 0$ then the assertion follows from Lemma 3.6. □

Proposition 4.7 (Classification of *Type C*)**.** *(R, W, F) can be transformed into the following:*
• *Type C_0: $s = 0$ (equivalently $V_{(1)} = Ker(F)$)*

$$M(F, TypeC_0) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \ W = \langle \mu_1, \mu_2 \rangle \oplus \langle e_1, ..., e_p, e_{p+1} \rangle$$

$R \cong \mathbb{K}[\mu_1, \mu_2, e_1, ..., e_p, e_{p+1}]/(\mu_1 \cdot W, \mu_2 \cdot e_i, e_i \cdot e_j - \lambda_{i,j} \cdot \mu_1, e_i^2 - e_j^2 - (\lambda_{i,i} - \lambda_{j,j}) \cdot \mu_1, e_{p+1} \cdot e_i, e_{p+1}^2 - \delta \cdot \mu_2 - e_i^2 + \lambda_{i,i} \cdot \mu_1, \ 1 \leqslant i \neq j \leqslant p)$
where $\Lambda = (\lambda_{i,i'})$ is of the canonical form (4.6) and δ is the same as in Type A_1.

• *Type C_1: $Q^{(s+1)}$ is a projective space (equivalently $V_{(s+1)} = V^{(s+1)}$)*

$$M(F, TypeC_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & \vdots & \dots & 0 & 0 & \vdots & \iddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \vdots & \iddots & \vdots & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

$W = \langle \mu_1, \mu_2 \rangle \oplus \langle g_1, ..., g_s \rangle \oplus \langle f_{s+1}, f_s, ..., f_1 \rangle$
$R \cong \mathbb{K}[\mu_1, \mu_2, g_1, ..., g_s, f_1, ..., f_s, f_{s+1}]/(\mu_1 \cdot W, g_i \cdot \mu_2, f_l \cdot \mu_2, g_i \cdot g_v, g_i \cdot f_i - g_v \cdot f_v, g_{l-1} \cdot f_l - \mu_1, f_l \cdot f_{l'}, f_h \cdot$

$g_{h'}, f_1\cdot\mu_2-\mu_1, f_1^2-\delta\cdot\mu_2, f_{s+1}^2-f_i\cdot g_i,\ 1\leqslant i,v\leqslant s, 2\leqslant l\leqslant s+1, 1\leqslant l'\leqslant s, h-h'\notin\{0,1\}$) *where* δ *is the same as in Type* C_0.

• *Type* C_2: $Q^{(s+1)}$ *is a hyperquadric (equivalently* $V_{(s+1)}\neq V^{(s+1)}$*).*

$$M(F,TypeC_2)=\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&\dots&0&0&\dots&0&0&0&\dots&1\\
\vdots&\vdots&\vdots&\dots&\vdots&\vdots&\dots&\vdots&\vdots&\vdots&\iddots&\vdots\\
0&0&0&\dots&0&0&\dots&0&0&1&\dots&0\\
0&0&0&\dots&0&1&\dots&0&0&0&\dots&0\\
\vdots&\vdots&\vdots&\dots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\dots&\vdots\\
0&0&0&\dots&0&0&\dots&1&0&0&\dots&0\\
0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&\dots&1&0&\dots&0&0&0&\dots&0\\
\vdots&\vdots&\vdots&\iddots&\vdots&\vdots&\dots&\vdots&\vdots&\vdots&\dots&\vdots\\
0&0&1&\dots&0&0&\dots&0&0&0&\dots&0
\end{pmatrix}$$

$W=\langle\mu_1,\mu_2\rangle\oplus\langle g_1,...,g_s\rangle\oplus\langle e_1,...,e_p\rangle\oplus\langle f_{s+1},f_s,...,f_1\rangle$
$R\cong\mathbb{K}[\mu_1,\mu_2,g_1,...,g_s,e_1,...,e_p,f_1,...,f_s,f_{s+1}]/(\mu_1\cdot W, g_i\cdot\mu_2, e_{i'}\cdot\mu_2, f_l\cdot\mu_2, g_i\cdot g_v, f_h\cdot g_{h'}, f_l\cdot f_{l'}, g_i\cdot f_i-e_{i'}^2+\lambda_{i'i''}\mu_1, g_{l-1}\cdot f_l-\mu_1, f_1\cdot\mu_2-\mu_1, e_{i'}\cdot e_{i''}-\lambda_{i'i''}\mu_1, e_{i'}\cdot f_{l'}, e_{i'}\cdot f_{s+1}, f_1^2-\delta\cdot\mu_2, f_{s+1}^2-f_i\cdot g_i,\ 1\leqslant i,v\leqslant s, 2\leqslant l\leqslant s+1, 1\leqslant i'\neq i''\leqslant p, 1\leqslant l'\leqslant s, h-h'\notin\{0,1\})$
where $\Lambda=(\lambda_{i'i''})$ *is of the canonical form (4.6) and* δ *is the same as in Type* C_0.

We now give a characterization of $dim(\mathfrak{m}^2)$.

Proposition 4.8. *Given an additive action on a hyperquadric* Q *of corank two with unfixed singularities and* $dim(Q)\geqslant 5$*, we represent it by* (R,W,F) *with* $\mathfrak{m}$ *the maximal ideal, then* $dim(\mathfrak{m}^2)=l(\mathbb{G}_a^n,Q)$.

Proof. Note that for any $\alpha\in R$, $dim(\mathbb{G}_a^n\cdot[\alpha])=dim(\mathfrak{g}(\mathbb{G}_a^n)\cdot\alpha)=dim(\alpha\cdot W)$. Thus the boundary $Q\backslash O=Q\cap\mathbb{P}(\mathfrak{m})$ as $dim(\mathbb{G}_a^n\cdot[x])=dim(\mathbb{G}_a^n\cdot[1_R])=dim(O)$ for any invertible element x in R. Then for any $x=[\alpha]$ in the boundary, as $\alpha\in\mathfrak{m}$, we have $dim(\mathbb{G}_a^n\cdot[\alpha])=dim(\alpha\cdot W))\leqslant dim(\mathfrak{m}^2)$, concluding that $l(\mathbb{G}_a^n,Q)\leqslant dim(\mathfrak{m}^2)$. On the other hand, by our normalization results of each type, we can find a suitable element in the boundary whose orbit has dimension $d=dim(\mathfrak{m}^2)$ as follows.

If the action is not of Type B_0 or C_0. By Lemma 4.2, Lemma 4.4 and Lemma 4.6, we have $[f_1]\in Q\backslash O$ and $dim(f_1\cdot W)=dim(\mathfrak{m}^2)$.

If the action is of Type B_0. By Lemma 4.4, we have $[e_1+i\cdot e_2+\mu_2]\in Q\backslash O$ and $dim((e_1+i\cdot e_2+\mu_2)\cdot W)=dim(\mathfrak{m}^2)$, where $i^2=-1$.

If the action is of Type C_0. By Lemma 4.6 (iii), we have $[e_{p+1}+i\cdot e_1]\in Q\backslash O$ and $dim((e_{p+1}+i\cdot e_1)\cdot W)=dim(\mathfrak{m}^2)$, where $i^2=-1$. □

In the following for a normalized structure of each type we call the normalized basis of W, i.e., $\{f_i,g_j,e_k,\mu_1,\mu_2\}$ a set of normalized elements and we call the matrix $\Lambda=(\lambda_{i,j})$ (if exists) to be the canonical matrix of the action.

4.1.3. Uniqueness

In this section we finish our classification by showing that the normalized structrue is determined by $l(G_a^n, Q)$ and the canonical matrix(if exists) up to certain elementary transformations.

Given two additive actions on hyperquadrics of corank two with unfixed singularities $(\mathbb{G}_a^n, Q)$ and $(\mathbb{G}_a^n, Q')$ for $n \geqslant 5$, we represent them by (R, W, F) and (R', W', F') respectively, and represent their final outputs in the algebraic version of flow chart by $(x, s, V^{(s)}, V_{(s)})$ and $(x', s', V'^{(s')}, V'_{(s')})$ respectively. Furthermore define $\{(V^{(k)}, V_{(k)}) : k \leqslant s\}$ and $\{(V'^{(k)}, V'_{(k)}) : k \leqslant s'\}$ to be the algebraic sturcture sequences of the two actions. Then we have the following.

Theorem 4.9. *(i) If the two actions are equivalent, i.e., there exist*

$$\Gamma : R \mapsto R'$$

such that Γ *is a local* $\mathbb{K}$*-algebra isomorphism and* $\Gamma(W) = W'$*. Then*

(i.a) (R, W, F) *and* (R', W', F') *are of the same normalized type with* $s = s'$ *and* $l(\mathbb{G}_a^n, Q) = l(\mathbb{G}_a^n, Q')$*.*

(i.b) if they are of Type A_1*,* B_1 *or* C_1*, then they have the same normalized structure.*

(i.c) if they are not of Type A_1*,* B_1 *or* C_1*, then their canonical matrices* Λ *and* Λ' *differ up to a permutation of blocks, a scalar multiplication, and adding a scalar matrix (which we call elementary transformations).*

Conversely

(ii) if the two actions are of the same type with $l(\mathbb{G}_a^n, Q) = l(\mathbb{G}_a^n, Q')$*,* $s = s'$ *and when they are not of Type* A_1*,* B_1 *or* C_1*, suppose that their canonical matrices differ up to above elementary transformations. Then the two actions are equivalent.*

We first prove (i.a) and (i.b).

Proof of Theorem 4.9 (i.a) and (i.b). (i.a) Firstly as Γ is an isomorphism, we conclude that $l(\mathbb{G}_a^n, Q) = l(\mathbb{G}_a^n, Q')$ by Proposition 4.8. By $\Gamma(W) = W'$ and Lemma 2.7 we have $F(a, b) = c \cdot F'(\Gamma(a), \Gamma(b))$ for some nonzero $c \in \mathbb{K}$, for any $a, b \in R$. Then from the algebraic version of the flow chart and our definition of $(V^{(k)}, V_{(k)})$ for each k, we conclude that $s = s'$, $\Gamma(V_{(k)}) = V'_{(k)}$ and $\Gamma(V^{(k)}) = V'^{(k)}$ for each k, implying that the two actions are of the same normalized type shown in Section 4.1.2.

(i.b) Note that the set of normalized elements of these types does not contain e_k hence the structure only depends on s and $l(\mathbb{G}_a^n, Q)$ by our normalization result, concluding the proof. □

To prove (i.c) and (ii), we separate it into two cases.

Case 1. If $s \geqslant 1$, let $\{\mu_1, \mu_2, e_k, g_i, f_1, b_0 : 1 \leqslant k \leqslant p, 1 \leqslant i \leqslant s\}$ and $\{\mu'_1, \mu'_2, e'_k, g'_i, f'_1, b'_0 : 1 \leqslant k \leqslant p, 1 \leqslant i \leqslant s\}$ be the associated elements in the normalized structures respectively. Then the isomorphism Γ gives :

$$\Gamma(b_0) = c_\Gamma \cdot b'_0 + f_{W'}, \tag{4.11}$$

$$\Gamma(f_1) = c_1 \cdot f'_1 + f_{1,W'}, \tag{4.12}$$

$$\Gamma(\mu_v) = f_{v,1} \cdot \mu'_1 + f_{v,2} \cdot \mu'_2, \tag{4.13}$$

$$\Gamma(e_k) = \sum_{l=1}^{p} a_{k,l} \cdot e'_l + \sum_{i=1}^{s} b_{k,i} \cdot g'_i + c_{k,1} \cdot \mu'_1 + c_{k,2} \cdot \mu'_2, \tag{4.14}$$

where $f_{W'} \in W'$, $f_{1,W'} \in V'^{(1)}$ and $v \in \{1,2\}$, $1 \leqslant k \leqslant p$, $c_\Gamma, c_1 \neq 0 \in \mathbb{K}$. Moreover we define $A = (a_{k,l})$ then we have the following.

Lemma 4.10. *(i)* $F'(\Gamma(a),\Gamma(b)) = c_\Gamma \cdot F(a,b)$.
(ii) $f_{W'} = \lambda^{(1)} \cdot \mu'_1 + \lambda^{(2)} \cdot \mu'_2 \in \langle \mu'_1, \mu'_2 \rangle$, $\lambda^{(2)} = f_{1,2} = 0$ *and* $f_{1,1}, f_{2,2} \neq 0$.
(iii) $A' \cdot A = c_\Gamma \cdot I_p$.

Proof. (i) Let $a \cdot b = F(a,b) \cdot b_0 + (a \cdot b)_{|_W}$. Then under the notation of Lemma 2.8 we have:

$$\begin{aligned} F'(\Gamma(a),\Gamma(b)) &= y'_0(\Gamma(a \cdot b)) = F(a,b) \cdot y'_0(\Gamma(b_0)) \\ &= F(a,b) \cdot y'_0(c_\Gamma \cdot b'_0 + f_{W'}) = c_\Gamma \cdot F(a,b). \end{aligned}$$

(ii) The first assertion follows from $b_0 \in \mathfrak{m}^2$ and $\Gamma(\mathfrak{m}^2) = (\mathfrak{m}')^2 \subseteq \langle \mu'_1, \mu'_2, b'_0 \rangle$. For $\lambda^{(2)}$, from $f_1 \cdot b_0 = 0$ we have:

$$0 = \Gamma(b_0) \cdot \Gamma(f_1) = (c_\Gamma \cdot b'_0 + \lambda^{(1)} \cdot \mu'_1 + \lambda^{(2)} \cdot \mu'_2) \cdot (c_1 \cdot f'_1 + f_{1,W'}) = c_1 \cdot \lambda^{(2)} \cdot \mu'_1,$$

concluding that $\lambda^{(2)} = 0$ as c_1 is nonzero in $\mathbb{K}$. For $f_{1,2}$, from $f_1 \cdot \mu_1 = 0$ we have

$$0 = \Gamma(\mu_1) \cdot \Gamma(f_0) = (f_{1,1} \cdot \mu'_1 + f_{1,2} \cdot \mu'_2) \cdot (c_1 \cdot f'_1 + f_{1,W'}) = c_1 \cdot f_{1,2} \cdot \mu'_1,$$

concluding that $f_{1,2} = 0$, hence $f_{1,1} \neq 0$ and $f_{2,2} \neq 0$.

(iii) Using (i) and (4.14) we have :

$$\begin{aligned} \delta_{k,k'} \cdot c_\Gamma = c_\Gamma \cdot F(e_k, e_{k'}) &= F'(\Gamma(e_k), \Gamma(e_{k'})) \\ &= \sum_{l,l'=1}^{p} \delta_{l,l'} \cdot a_{k,l} \cdot a_{k',l'} = \sum_{l=1}^{p} a_{k,l} \cdot a_{k',l}, \end{aligned}$$

concluding that $A' \cdot A = c_\Gamma \cdot I_p$. □

Now we are ready to prove Theorem 4.9 (i.c) and (ii) when $s \geqslant 1$.

Proof of Theorem 4.9 (i.c),(ii) when $s \geqslant 1$. (i.c) Computing $\Gamma(e_k \cdot e_{k'}) = \Gamma(e_k) \cdot \Gamma(e_{k'})$:

$$\begin{aligned} LHS &= \Gamma(\delta_{k,k'} \cdot b_0 + \lambda_{k,k'} \cdot \mu_1) = \delta_{k,k'} \cdot c_\Gamma \cdot b'_0 + (\delta_{k,k'} \cdot \lambda^{(1)} + \lambda_{k,k'} \cdot f_{1,1}) \cdot \mu'_1. \\ RHS &= (\sum_{l,l'=1}^{p} a_{k,l} \cdot a_{k',l'} \cdot \delta_{l,l'}) \cdot b'_0 + (\delta' \cdot b_{k,s} \cdot b_{k',s} + \sum_{l,l'=1}^{p} a_{k,l} \cdot \lambda'_{l,l'} \cdot a_{k',l'}) \cdot \mu'_1. \end{aligned}$$

where $\delta' \neq 0$ if $(g'_s)^2 = \mu_1$, i.e., the action is of Type B_2. And we claim in this case $b_{k,s} = 0$, implying $\delta' \cdot b_{k,s} \cdot b_{k',s'} = 0$. This follows from computing $\Gamma(e_k \cdot g_s) = \Gamma(e_k) \cdot \Gamma(g_s)$ from two sides.

Now from $LHS = RHS$ combined with $A' \cdot A = c_\Gamma \cdot I_p$ we have the equation:

$$\Lambda' = (\frac{\lambda^{(1)}}{c_\Gamma}) \cdot I_p + (\frac{f_{1,1}}{c_\Gamma^2}) \cdot A' \Lambda A$$

hence from [5, Chapter XI §3] we conclude that Λ and Λ' differ up to the listed elementary transformations.

(ii) If the two actions have the same normalized type with $l(\mathbb{G}_a^n, Q) = l(\mathbb{G}_a^n, Q')$, $s = s'$, Λ and Λ' differ up to elementary transformations, then by our normalization result, to give the isomorphism between actions it suffices to find a new set of normalized elements of (R, W) having the canonical matrix which equals Λ'. In the following we find the new normalized set $\{\mu_1^{(0)}, \mu_2^{(0)}, g_j^{(0)}, e_k^{(0)}, f_i^{(0)}, b_0^{(0)}\}$ case by case.

1). (up to a permutation of blocks)

Note that any permutation of blocks can be induced by a permutation of $\{e_k : 1 \leqslant k \leqslant p\}$. Hence the new set of normalized elements can be defined through a suitable permutation of e_k and identity on other elements.

2). (up to adding a scalar matrix) we assume $\Lambda' = \Lambda + h \cdot I_p$ for some nonzero $h \in \mathbb{K}$.

In this case it suffices to find a new set of normalized elements with $b_0^{(0)} = b_0 - h \cdot \mu_1$, $\mu_1^{(0)} = \mu_1$ and $e_k^{(0)} = e_k$. To find the set we run our normalization in Section 4.1.2 starting with $\mu_1, \mu_2, b_0^{(0)}$ and set e_k, f_i, g_j to be the initial elements we take at each step of the normalization. Then one can easily check that after running the normalization of each type, the new set of normalized elements meets our need.

3). (up to a scalar multiplication) we assume $\Lambda' = h \cdot \Lambda$ for some nonzero $h \in \mathbb{K}$.

In this case it suffices to find a new set of normalized elements with $b_0^{(0)} = c_\Gamma \cdot b_0, e_k^{(0)} = \sqrt{c_\Gamma} \cdot e_k, \mu_1^{(0)} = f_{1,1} \cdot \mu_1$ for some nonzero $c_\Gamma, f_{1,1} \in \mathbb{K}$ s.t. $c_\Gamma = h \cdot f_{1,1}$. To find the elements, we define $f_i^{(0)} = x_i \cdot f_i$, $g_j^{(0)} = y_j \cdot g_j$ and $\mu_2^{(0)} = y_0 \cdot \mu_2$. Then the condition $(f_i^{(0)} \cdot g_{i-1}^{(0)} = \mu_1^{(0)}, f_i^{(0)} \cdot g_i^{(0)} = b_0^{(0)})$ and extra conditions in different types shown in Section 4.1.2 gives a system of equations for each type:

$$(\text{Type } A_2)\begin{cases} x_i \cdot y_i = c_\Gamma \\ x_i \cdot y_{i-1} = f_{1,1} \\ x_1^2 = y_0 \end{cases} \qquad (\text{Type } B_2)\begin{cases} x_i \cdot y_i = c_\Gamma \\ x_i \cdot y_{i-1} = f_{1,1} \\ x_1^2 = y_0 \\ y_s^2 = f_{1,1} \end{cases}$$

$$(\text{Type } C_2)\begin{cases} x_i \cdot y_i = c_\Gamma \\ x_i \cdot y_{i-1} = f_{1,1} \\ x_1^2 = y_0 \\ x_{s+1}^2 = c_\Gamma \end{cases} \quad \text{with the condtion } c_\Gamma = h \cdot f_{1,1} \text{ for each type.}$$

For these equations one can easily check the existence of solutions, which enables us to find the set of normalized elements we need.

□

Case 2. If $s = 0$, i.e., they are of Type B_0 or C_0, then we can use similar method in Case 1 to prove (i.c) and also to prove (ii) when the two canonical matrices differ from a permutation of blocks or adding a scalar matrix. Hence it sufficies to prove (ii) when Λ and Λ' differ from a scalar multiplication.

As is Case 1, it sufficies to find a new set of normalized elements $b_0^{(0)} = c_\Gamma \cdot b_0, e_k^{(0)} = \sqrt{c_\Gamma} \cdot e_k, \mu_1^{(0)} = f_{1,1} \cdot \mu_1$ for some nonzero $c_\Gamma, f_{1,1} \in \mathbb{K}$ s.t. $c_\Gamma = h \cdot f_{1,1}$.

Now if the action is of Type B_0 we set

$$\mu_1^{(0)} = \mu_1, \mu_2^{(0)} = \mu_2, e_i = \sqrt{h} \cdot e_i, b_0^{(0)} = h \cdot b_0.$$

Then one can check $\{\mu_1^{(0)}, \mu_2^{(0)}, e_i^{(0)}, b_0^{(0)} : 1 \leqslant i \leqslant p\}$ is the normalized set we need.

If the action is of Type C_0 we set

$$\mu_1^{(0)} = \frac{\mu_1}{h^3}, \mu_2^{(0)} = \frac{\mu_2}{h^2}, e_i^{(0)} = \frac{e_i}{h}, e_{p+1}^{(0)} = \frac{e_{p+1}}{h}, b_0^{(0)} = \frac{b_0}{h^2}.$$

Then one can check $\{\mu_1^{(0)}, \mu_2^{(0)}, e_i^{(0)}, e_{p+1}^{(0)}, b_0^{(0)} : 1 \leqslant i \leqslant p\}$ is the normalized set we need, concluding the proof of Theorem 4.9.

As an application of our classification, we now prove Theorem 1.8.

Proof of Theorem 1.8. (i) Recall in Section 3.1 we have constructed $(R^{(1)}, \mathfrak{m}^{(1)})$ or $(R^{(1)}, V^{(1)}, F^{(1)})$ to be the corresponding local algebra (and invariant linear form on it) of the obtained action $(G^{(1)}, Q^{(1)})$ in Theorem 1.6. Hence combined with the algebraic version of the flow chart, for a normalized structure $\{(V^{(k)}, V_{(k)}) : k \leqslant s\}$ of an additive action (R, W, F), if the final output of the flow chart is $(x, t, G^{(t)}, Q^{(t)})$, then the output action $(G^{(t)}, Q^{(t)})$ is represented by

$$\begin{aligned} (R^{(t)}, \mathfrak{m}^{(t)}) &= (V^{(t)} \oplus \langle 1_R \rangle, V^{(t)}) && \text{if } Q^{(t)} \text{ is a projective space,} \\ (R^{(t)}, V^{(t)}, F^{(t)}) &= (V^{(t)} \oplus \langle b_0 \rangle \oplus \langle 1_R \rangle, V^{(t)}, F_{|_{V^{(t)}}}) && \text{if } Q^{(t)} \text{ is a hyperquadric,} \end{aligned}$$

where $t = s$ when $x = A$ and $t = s + 1$ when $x = B$ or C. Then (i) follows by Remark 2.1 and by checking the multiplications in $V^{(t)}$ in different types as shown in Lemma 4.2,4.4 and 4.6.

(ii) $l(\mathbb{G}_a^n, Q) \leqslant 3$ follows from Proposition 4.8 and our normalization result of each types. $codim(Q^{(k+1)}, Q^{(k)}) = 1$ follows from Proposition 4.1.

(iii) Now for two actions if they are equivalent induced by $\Gamma : R \mapsto R'$ then from Theorem 4.9 (i.a) they are of the same normalized type and $t = t', l(\mathbb{G}_a^n, Q) = l(\mathbb{G}_a^n, Q')$. Moreover as $\Gamma(V^{(k)}) = V'^{(k)}$ we conclude that $\Gamma_{|_{R^{(t)}}}$ induces an isomorphism between the output actions of the two actions, which proves the only if part of Theorem 1.8.

For the converse, it suffices to check the condition in Theorem 4.9 (ii). If $s = s' = 0$ then they are of the same type x_0. If $s = s' \geqslant 1$, then as the output action is equivalent, $Q^{(t)}$ and $\widetilde{Q}^{(t')}$ are either both hyperquadrics or projective spaces, hence they are of the same type x_1 or x_2.

Now if they are of Type A_2, B_2 or Type C_2, then consider the isomorphism between local algebras induced by the equivalence of the output actions:

$$\Gamma^{(t)} : (R^{(t)}, V^{(t)}, V_{(t)}) \mapsto (R'^{(t)}, V'^{(t)}, V'_{(t)}),$$

using the same method in the proof of Theorem 4.9 (i.c), $\Gamma^{(t)}$ will induce elementary transformations between the canonical matrices of the two actions. Therefore by Theorem 4.9 (ii) we conclude that the two actions are equivalent.

□

4.2. Classification of actions with unfixed singularities (II): $dim(Q) \leqslant 4$

In this subsection we consider the case when $dim(Q) \leqslant 4$. Equivalently for a triple (R, W, F) we have $dim(W) \leqslant 4$.

In the folowing we always take a basis of $Ker(F) = \langle \mu_1, \mu_2 \rangle$ satisfying Lemma 3.5 (i). We also take $(V^{(1)}, V_{(1)})$ defined in Section 3. Then we give the classification case by case.

Case 1. $dim(W) = 4$

Subcase (i): $Ker(F) \cdot Ker(F) \neq 0$. Note that in the proof of Case 2 of Proposition 3.4 we only need to assure the number of e_k is at least two, hence this case is just the 4-dimensional version of Type B_0.

Subcase (ii): $Ker(F) \cdot Ker(F) = 0$ and $Ker(F) = V_{(1)}$. We have the following:

$$Ker(F) = V_{(1)} \subsetneqq V^{(1)} \subsetneqq W,$$

with $codim(V_{(1)}, V^{(1)}) = 1$. In ths case, we can choose a $g_1 \in V^{(1)} \backslash V_{(1)}$ such that $F(g_1, g_1) = 1$ as $F(g_1, g_1) \neq 0$. Then for any $f_1 \in W \backslash V^{(1)}$, up to relplacing it by $f_1 - F(f_1, g_1)^{-1} \cdot g_1$, we can have $f(f_1, g_1) = 0$. Finally $F(f_1, f_1) \neq 0$ as $f_1 \notin Ker(F)$, hence we can have $F(f_1, f_1) = 1$. Then we divide it into two more subcases.

(*ii*.1) $dim(V^{(1)} \cdot W) = 3$ then there exist a basis of $Ker(F) = \langle \mu_1, \mu_2 \rangle$, $f_1 \in W \backslash V^{(1)}, g_1 \in V^{(1)} \backslash Ker(F)$ and $b_0 \in \mathfrak{m} \backslash W$ s.t.

$$g_1^2 = b_0,\ g_1 \cdot f_1 = \mu_2,\ f_1^2 = b_0 + \lambda \cdot \mu_2,\ f_1 \cdot \mu_2 = \mu_1,$$

for some $\lambda \in \mathbb{K}$.

To show this, from $g_1 \cdot g_1 \notin Ker(F)$, $g_1 \cdot Ker(F) = 0$, $f_1 \cdot \mu_2 \in \langle \mu_1 \rangle$ and our assumption $dim(V^{(1)} \cdot W) = 3$ we conclude that $g_1 \cdot f_1 = c_2 \cdot \mu_2 + c_1 \cdot \mu_1$ for some nonzero $c_2 \in \mathbb{K}$. Now we can normalize in the following steps:
First we define $b_0 = g_1^2$ then we replace μ_2 by $f_1 \cdot g_1$, replace f_1 by $f_1 - \frac{V_1(f_1, f_1)}{2V_1(f_1, \mu_2)} \cdot \mu_2$ and finally we replace μ_1 by $f_1 \cdot \mu_2$.

Then the classification of this case follows:

$$M(F) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ W = \langle \mu_1, \mu_2 \rangle \oplus \langle g_1, f_1 \rangle$$

and R is isomorphic to
$\mathbb{K}[\mu_1, \mu_2, g_1, f_1]/(\mu_1 \cdot W, \mu_2 \cdot \mu_2, \mu_2 \cdot g_1, f_1 \cdot \mu_2 - \mu_1, f_1 \cdot g_1 - \mu_2, f_1^2 - g_1^2 - \lambda \cdot \mu_2)$

Moreover for the coefficient $\lambda \in \mathbb{K}$ we have the following uniqueness result which is easy to check.

Proposition 4.11. *Two actions of Case (ii.1) with coefficients λ and λ' respectively are equivalent if and only if $\lambda = \pm\lambda'$.*

(*ii*.2) $dim(V^{(1)} \cdot W) = 2$. Then choosing $b_0 = g_1^2$ we have $V^{(1)} \cdot W \subseteq \langle \mu_1, b_0 \rangle$ and $b_0 \cdot W = 0$. Moreover we see $f_1 \cdot \mu_2 = c \cdot \mu_1$ for some nonzero $c \in \mathbb{K}$. And we set $f_1^2 = b_0 + V_1(f_1, f_1) \cdot \mu_1 + d_1 \cdot \mu_2$. Now we can normalize through the follwoing steps:

$$\begin{aligned} &g_1 \to g_1 - c^{-1} \cdot V_1(g_1, f_1) \cdot \mu_2 && \text{to make } g_1 \cdot f_1 = 0 \\ &f_1 \to f_1 - \frac{V_1(f_1, f_1)}{2c} \cdot \mu_2 && \text{to make } f_1^2 = b_0 + d_1 \cdot \mu_2 \end{aligned}$$

and if $d_1 \neq 0$ (i.e., $dim(\mathfrak{m}^2 = 3)$) we replace μ_2 by $d_1 \cdot \mu_2$ to make $f_1^2 = b_0 + \mu_2$ then replace μ_1 by $f_1 \cdot \mu_2$ to keep $f_1 \cdot \mu_2 = \mu_1$. This enables us to give the classification of this case:

$$M(F) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ W = \langle \mu_1, \mu_2 \rangle \oplus \langle g_1, f_1 \rangle$$

and R is isomorphic to
$\mathbb{K}[\mu_1, \mu_2, g_1, f_1]/(\mu_1 \cdot W, \mu_2 \cdot \mu_2, \mu_2 \cdot g_1, f_1 \cdot \mu_2 - \mu_1, f_1 \cdot g_1, f_1^2 - g_1^2 - \delta \cdot \mu_2)$
where δ is the same as we define in Section 4.1.2.

Subcase (iii): $Ker(F) \cdot Ker(F) = 0$ and $V_{(1)} = V^{(1)}$.

Then we can choose $f_1 \in W \backslash V^{(1)}, g_1 \in V^{(1)} \backslash Ker(F)$ s.t. $F(f_1, g_1) = 1$ and $F(f_1, f_1) = F(g_1, g_1) = 0$. And we divide it into two more subcases.

(*iii*.1) $dim(\mathfrak{m}^2) = 2$. We set $b_0 = g_1 \cdot f_1$ and replace μ_1 by $f_1 \cdot \mu_2$, then we replace f_1 by $f_1 - \frac{\mu_2}{2V_1(f_1, \mu_2)}$ to make $f_1^2 = 0$. Thus we have:

$$g_1 \cdot f_1 = b_0, f_1 \cdot \mu_2 = \mu_1, f_1^2 = 0, g_1^2 = h \cdot \mu_1$$

for some $h \in \mathbb{K}$. Now if $h \neq 0$ then we can furtherly make $g_1^2 = \mu_1$ through replacing elements as the following:

$$\mu_1 = \sqrt{h} \cdot \mu_1, \ \mu_2 = h^{\frac{1}{4}} \cdot \mu_2, \ f_1 = h^{\frac{1}{4}} \cdot f_1, \ g_1 = h^{-\frac{1}{4}} \cdot g_1, b_0 = b_0.$$

Then our classification of this case follows:

$$M(F) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ W = \langle \mu_1, \mu_2 \rangle \oplus \langle g_1, f_1 \rangle$$

and R is isomorphic to
$\mathbb{K}[\mu_1, \mu_2, g_1, f_1]/(\mu_1 \cdot W, \mu_2 \cdot \mu_2, \mu_2 \cdot g_1, f_1 \cdot \mu_2 - \mu_1, g_1^2 - \mu_1, f_1^2)$
or
$\mathbb{K}[\mu_1, \mu_2, g_1, f_1]/(\mu_1 \cdot W, \mu_2 \cdot \mu_2, \mu_2 \cdot g_1, f_1 \cdot \mu_2 - \mu_1, g_1^2, f_1^2)$, depending on whether $V^{(1)} \cdot V^{(1)}$ equals to zero or not.

(*iii*.2) $dim(\mathfrak{m}^2) = 3$. We divide it into two more subcases.
If $dim(V^{(1)} \cdot W) = 2$ then we have:

$$M(F) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \ W = \langle \mu_1, \mu_2 \rangle \oplus \langle g_1, f_1 \rangle$$

and R is isomorphic to
$\mathbb{K}[\mu_1, \mu_2, g_1, f_1]/(\mu_1 \cdot W, \mu_2 \cdot \mu_2, \mu_2 \cdot g_1, f_1 \cdot \mu_2 - \mu_1, g_1^2 - \mu_1, f_1^2 - \mu_2)$
If $dim(V^{(1)} \cdot W) = 3$ then we have:

$$M(F) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ W = \langle \mu_1, \mu_2 \rangle \oplus \langle g_1, f_1 \rangle$$

and R is isomorphic to
$\mathbb{K}[\mu_1, \mu_2, g_1, f_1]/(\mu_1 \cdot W, \mu_2 \cdot \mu_2, \mu_2 \cdot g_1, f_1 \cdot \mu_2 - \mu_1, g_1^2 - \mu_2, f_1^2 - \mu_2)$
or
$\mathbb{K}[\mu_1, \mu_2, g_1, f_1]/(\mu_1 \cdot W, \mu_2 \cdot \mu_2, \mu_2 \cdot g_1, f_1 \cdot \mu_2 - \mu_1, g_1^2 - \mu_2, f_1^2)$
where one can easily check these two actions are not equivalent.
Case 2. $dim(W) = 3$ Then we have two subcases as follows.
Subcase (i): $Ker(F) \cdot Ker(F) \neq 0$ then we have:

$$M(F) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ W = \langle \mu_1, \mu_2 \rangle \oplus \langle e \rangle$$

and R is isomorphic to
$\mathbb{K}[\mu_1, \mu_2, e]/(\mu_1 \cdot W, \mu_2 \cdot \mu_2 - \mu_1, \mu_2 \cdot e, e^3 - \mu_1)$, if $dim(\mathfrak{m}^2) = 2$ and $\mathfrak{m}^3 \neq 0$.
$\mathbb{K}[\mu_1, \mu_2, e]/(\mu_1 \cdot W, \mu_2 \cdot \mu_2 - \mu_1, \mu_2 \cdot e, e^3)$, if $dim(\mathfrak{m}^2) = 2$ and $\mathfrak{m}^3 = 0$.
$\mathbb{K}[\mu_1, \mu_2, e]/(\mu_1 \cdot W, \mu_2 \cdot \mu_2 - \mu_1, \mu_2 \cdot e, e^3 - \mu_2)$, if $dim(\mathfrak{m}^2) = 3$.

Subcase (ii): $Ker(F) \cdot Ker(F) = 0$ then we have:

$$M(F) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ W = \langle \mu_1, \mu_2 \rangle \oplus \langle e \rangle$$

and R is isomorphic to
$\mathbb{K}[\mu_1, \mu_2, e]/(\mu_1 \cdot W, \mu_2 \cdot \mu_2, \mu_2 \cdot e - \mu_1, e^3 - \mu_1)$, if $dim(\mathfrak{m}^2) = 2$ and $\mathfrak{m}^3 \neq 0$.
$\mathbb{K}[\mu_1, \mu_2, e]/(\mu_1 \cdot W, \mu_2 \cdot \mu_2, \mu_2 \cdot e - \mu_1, e^3)$, if $dim(\mathfrak{m}^2) = 2$ and $\mathfrak{m}^3 = 0$.
$\mathbb{K}[\mu_1, \mu_2, e]/(\mu_1 \cdot W, \mu_2 \cdot \mu_2, \mu_2 \cdot e - \mu_1, e^3 - \mu_2)$, if $dim(\mathfrak{m}^2) = 3$

Acknowledgments

I am grateful to Baohua Fu for introducing the problem, guidance and revising this paper. I am also grateful to Zhijun Luo for helpful discussions. I would like to thank the referee for many valuable comments and suggestions.

Conflict of interest

The author declares there is no conflict of interest.

References

1. B. Hassett, Y. Tschinkel, Geometry of equivariant compactifications of $\mathbf{G}_a^n$, *Int. Math. Res. Not.*, **22** (1999), 1211–1230. https://doi.org/10.1155/S1073792899000665
2. I. Arzhantsev, A. Popovskiy, Additive actions on projective hypersurfaces, in *Automorphisms in birational and affine geometry* (eds. J. McKernan, C. Ciliberto, M. Zaidenberg, I. Cheltsov), Springer Proceedings in Mathematics and Statistics, (2014), 17–33. https://doi.org/10.1007/978-3-319-05681-4_2
3. I. Arzhantsev, E. V. Sharoyko, Hassett-Tschinkel correspondence: modality and projective hypersurfaces, *J. Algebra*, **348** (2011), 217–232. https://doi.org/10.1016/j.jalgebra.2011.09.026
4. S. Friedland, Simultaneous similarity of matrices, *Adv. Math.*, **50** (1983), 189–265. https://doi.org/10.1016/0001-8708(83)90044-0
5. F. R. Gantmacher, *The theory of matrices. Vol. 1*, Translated from the Russian by K. A. Hirsch, Reprint of the 1959 translation, AMS Chelsea Publishing, Providence, RI, 1998.
6. E. V. Sharoyko, The Hassett-Tschinkel correspondence and automorphisms of a quadric, *Mat. Sb.*, **200** (2009), 145–160. https://doi.org/10.1070/SM2009v200n11ABEH004056

AIMS Press